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Estimate of the effect of virtual quark loops on meson masses in lattice QCD

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Abstract The 0^- and 1^- meson masses are calculated in SU(2) lattice gauge theory at $\beta = 2.3$ with Wilson fermions on a 10^4 lattice. The calculation is based on a 32^{nd} order hopping parameter expansion. The fermion determinant can be taken into account for quark masses heavier than $\sim$ 600 MeV. For lighter quarks only an estimate is given due to the limitations of statistics.

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An important step in the numerical evaluation of lattice QCD [1] is the calculation of hadron masses. In the first attempts of hadron spectrum calculation [2-4] the effect of dynamical (virtual) quark loops was neglected ("quenched"- or "valence"-approximation). From the point of view of numerical computation this is an important simplification, because the virtual quark loops are contained in the "fermion determinant" which is very hard to take into account in the Monte Carlo updating procedure. It turned out that the results in the quenched approximation are quite reasonable, especially in the recent calculations done on larger lattices [5, 6]. This implies that the effect of dynamical quarks on the hadron masses cannot be very large. Nevertheless, it is obviously important to check this point in a calculation taking into account the fermion determinant. A first step in this direction was a recent calculation at negative flavour numbers (that is, replacing fermionic quarks by bosonic quarks in the determinant) [7] , which showed only small changes in the hadron mass ratios, indeed. The extrapolation to the physical flavour number is, however, non-trivial and was, up to now, not attempted.

The quark determinant can be taken into account by several methods like the pseudo-fermion method [8], stochastic method [9] or the method of hopping parameter expansion [10, 11]. In fact, in the 10^{th} order hopping parameter expansion of Ref. [11] the quark determinant was included. Its effect on the hadron masses was shown to be small, but a 10^{th} order expansion is obviously too low for drawing a firm conclusion. By the use of the iterative method [12, 6] the hopping parameter expansion can, however, be extended to much higher orders.

In the present letter a first calculation of meson masses is reported including the effect of virtual quark loops in a high (32^{nd}) order hopping parameter expansion. For dealing with the fermion determinant the experience gained in a recent calculation of the screened quark-antiquark potential [13] was important. In fact, the 20 SU(2) gauge configurations on 10^4 lattices are identical to those used in Ref. [13] for the measurement of Wilson-loops.

The hopping parameter expansion coefficients of the meson propagators were determined in the "periodic box" as described in Ref. [12], that is, imposing periodic boundary conditions on both the gauge- and quark fields. This does not allow to search directly for particle poles at real energy values as in the "copied gauge field method" [11, 12], because the periodicity of the quark fields is strongly reflected in the expansion coefficients and this restricts the momentum values to discrete euclidean points. Therefore, I first calculated the values of meson propagators for euclidean momenta $ap_{1,2,3} = 0$, $ap_4 = \pi\ell/5$ ($\ell = 0, \pm 1, \ldots, \pm 5$) corresponding to periodicity. At a given hopping parameter (K) the Padé-table obtained from the expansion coefficients was used to calculate the amplitude and the resulting values were Fourier-transformed to configuration space in order to determine the time slices of meson propagators. The masses were then extracted, as usual, from the exponential fall-off at the largest distances. Because of the statistical errors on time slices, it turned out necessary to average first order the 24th to 32nd order Padé-approximants and determine the masses from the averages. (A similar method was used also in Ref. [14].) The hopping parameter expansion coefficients of meson propagators were calculated upon 32nd order from 5 initial points on each of the 20 configurations and, in order to increase the lower statistical accuracy of the low order coefficients, up to 24th order from another 20 initial points. These 25 initial points per configuration were selected randomly from the points of the 10^4 lattice. The SU(2) gauge configurations themselves were created by the Metropolis updating method. In 5 independent streams of configurations the consecutive ones were separated by 50 sweeps with 3 hits per link.

The calculation of the quark determinant in the 32nd order hopping parameter expansion was done in the same way as in Ref. [13]. Let us recall that the quark part of the Wilson-action is

$$S_f = \sum_x \{ \tilde{\psi}_x \psi_x - K \sum_\mu \tilde{\psi}_{x+\hat{\mu}} U[x,\mu](1+\gamma_\mu)\psi_x \equiv \qquad (1)$$

$$\equiv \sum_{x,y} \tilde{\psi}_x Q_{xy} \psi_y \quad .$$

Here ψ_x, $\tilde{\psi}_x$ denote the anticommuting quark fields on the lattice point x, $U[x, \mu]$ is the gauge field variable sitting on the link $x \to x + \hat{\mu}$ ($\mu = \pm 1, \pm 2, \ldots, \pm 4$) and $\gamma_\mu = -\gamma_{-\mu}$ are hermitian Dirac-matrices. Writing the "quark matrix" as $Q \equiv 1 - KM$, the hopping parameter expansion of the quark determinant can be written (for one quark flavour) as

$$\det Q = \det(1-KM) = \exp\left\{ -\sum_{j=4}^{\infty} \frac{K^j}{j} \sum_x \mathrm{Tr}\left(M^j_{xx}\right) \right\} . \qquad (2)$$

Here the trace-sum over lattice points is explicitly written out, therefore Tr(....) means only colour-and Dirac-traces. The hopping matrix $M_{x_1 x_2}$ is according to Eq. (1)

$$M_{x_1 x_2} = \sum_{x,\mu} (1+\gamma_\mu) U[x,\mu] \, \delta_{x_1, x+\hat{\mu}} \, \delta_{x, x_2} \quad . \qquad (3)$$

The first non-vanishing term (j=4) in the hopping parameter expansion of the effective action $S_q^{eff} \equiv -\ln\det Q$ has the same form as the pure gauge Wilson-action. After performing the Dirac-trace calculation this term looks like

$$S_q^{eff}\ (j=4) = -16\, K^4 \sum_{\square} \mathrm{Tr}\, U_\square \quad . \qquad (4)$$

Here $\sum_\square$ denotes a sum over plaquettes. S_q^{eff} (j=4) can be included in the pure gauge action by a shift $\Delta\beta = 32\ K^4$ ($\beta = 4/g^2$). The Monte Carlo updating of the gauge configurations was done in our case at a fixed value $\beta = 2.3$, therefore the omission of the j = 4 term in S_q^{eff} means, that the calculation is performed at a K-dependent value of the gauge coupling constant

$$\beta_K = 2.3 - 32\ K^4 \qquad (5)$$

At $\beta = 2.3$ $S_q^{eff}(j=4)$ takes into account a substantial part of the quark effective action S_q^{eff}. (A further improvement in this direction would be to include also the j=6 term in Eq. (2) in the updating.)

The sum $\sum_x$ over the lattice points for the rest

$$S_q^{eff}\ (j \geq 6) \equiv S_q^{eff} - S_q^{eff}\ (j=4) \qquad (6)$$

of the effective action is evaluated only approximately by choosing a random sample of 300 points (out of the 10000) on every configuration. As noted in Ref. [13], in the expectation values only the deviation of S_q^{eff} $(j \geq 6)$ from its average $(\bar{S}_q^{eff}$ $(j \geq 6)$) appears, therefore one has to determine the expansion coefficients of

$$\Delta S_q^{eff} \; (j \geq 6) = S_q^{eff} \; (j \geq 6) - \bar{S}_q^{eff} \; (j \geq 6) \quad . \tag{7}$$

On a given configuration these are then used to calculate, for different hopping parameters, the value of

$$\det_6 \; (\Delta Q) \equiv \exp \{ \; -\Delta S_q^{eff} \; (j \geq 6) \} \; . \tag{8}$$

In the interesting range $K \leq 0.15$ usually a good value is obtained for ΔS_q^{eff} $(j \geq 6)$ already from the 32^{nd} order series. This can, however, be further improved by taking, instead of the series, some of the highest order central Padè-approximants. The expectation value of some quantity F is then given as a weighted average over the configurations:

$$\langle F \rangle = \frac{\sum_U F[U] \; \det_6 \; (\Delta Q[U])}{\sum_U \det_6 \; (\Delta Q[U] \;)} \tag{9}$$

The main limitation of the present method comes from the fact that the range of values of $\det_6$ (ΔQ) becomes larger for increasing hopping parameters (and hence for lighter quarks). Therefore, in Eq. (8) at large K only very few configurations give a substantial contribution. As a consequence, the statistics becomes very poor. In the present case this limits the hopping parameter values to $K \lesssim 0.145$, where the pion mass is greater than ~ 1 (in lattice units).

Before discussing the effects of virtual quark loops, let us consider the 0^- and 1^- meson masses for $\beta = 2.3$ in the quenched approximation (that is, replacing $\det Q$ by 1). The $20 \cdot 5 = 100$ 32^{nd} order initial points plus the $20 \cdot 20 = 400$ 24^{th} order initial points represent, in this case, a very good statistics. The obtained masses are shown in Fig. 1. For the critical value of the hopping parameter (K_{cr}), where the pion mass vanishes, a linear extrapolation of $(am_\pi)^2$ to zero gives

$$K_{cr} = 0.156 \pm 0.002 \tag{10}$$

At this K-value the ρ-meson mass is

$$am_\rho = 0.62 \pm 0.12 \tag{11}$$

From the physical ρ-meson mass it follows

$$a(N_f = 0, \; \beta = 2.3) = (0.80 \pm 0.16)\,\mathrm{GeV}^{-1} \cong 0.16 \text{ fermi} \tag{12}$$

This value has to be compared to the one obtained from the string tension: $a \cong 1.1 \; \mathrm{GeV}^{-1}$ [15, 16]. Therefore, the meson masses give a smaller physical lattice spacing than the string tesion, similarly to the recent calculations in SU(3) on large lattices [5, 6]. (Note that the correlation length at $\beta = 2.3$ is roughly equal to 1a, similarly to $\beta = 5.7$ in SU(3) considered in Ref. [6]. An earlier SU(2) calculation on a smaller $(5^3 \times 10)$ lattice [17] gave at $\beta = 2.3$ $K_{cr} = 0.162^{+0.002}_{-0.001}$ and $am_\rho = 0.80 \pm 0.08$.

The result for the π- and ρ-meson masses, which is obtained by applying Eq. (9) to the hopping parameter expansion coefficients, is shown in Fig. 2. As discussed before, the gauge coupling constant depends on the hopping parameter according to Eq. (5). In order to see the net effect of the quark determinant one should compare the results to the masses in the quenched approximation at the same coupling constant value, that is at $\beta = 2.3 - 32 \; K^4$. This can be done by shifting the $\beta = 2.3$ curves in Fig. 1 by an amount corresponding to the small shift in β. A possibility is to use the results at $\beta = 2.2$, 2.3, 2.4 obtained by Fukuta et al. [17]. (See Fig. 2.)

The present method does not allow to go with the quark determinant calculation to K-values above $K \cong 0.145$. This roughly corresponds to a bare quark mass $m_q \cong 0.6$ GeV. The only possibility is to try some extrapolation. An (ad hoc) linear extrapolation of the deviation from the $N_f = 0$, $\beta = 2.3$ curve gives for the critical K-value

$$K_{cr} \; (N_f = 1, \; \beta = \beta_K) \cong 0.154 \tag{13}$$

The ρ-π mass splitting (in lattice units) is in the measured points almost equal to the quenched ρ-π mass splitting at $\beta = 2.3$ (see Fig. 2). This gives for the ratios of lattice spacings (extrapolated to zero quark mass):

$$a(N_f = 1,\ \beta = \beta_K) / a(N_f = 0,\ \beta = 2.3) = 0.9 \pm 0.4 \qquad (14)$$

Using the perturbative ratios of SU(2) Λ-parameters [18, 19] :

$$\frac{\Lambda_{mom}^{(N_f=0)}}{\Lambda_{latt}^{(N_f=0)}} = 57.4 \qquad \frac{\Lambda_{mom}^{(N_f=1)}}{\Lambda_{latt}^{(N_f=1)}} = 61.4 \qquad (15)$$

Eq. (14) implies, with large errors

$$\frac{\Lambda_{mom}^{(N_f=1)}}{\Lambda_{mom}^{(N_f=0)}} \cong 0.7 \qquad (16)$$

The direction of shift of the critical K-value given by Eqs. (10, 13) is opposite to the one observed by Daffy et al. [7] at $N_f = -2$ (in SU(3)). This is, therefore, consistent with the simplest flavour extrapolation, although a linear extrapolation in N_f seems generally not allowed [13]. The other observation in Ref. [7] is the antiscreening of the wave functions at the origin for negative flavour numbers. The expected screening for positive N_f would mean that, for instance, the vector meson coupling constant f_V^{-1}, defined by

$$\langle 0|\bar{\psi}(0)\gamma_\mu \psi(0)|V(p,\sigma)\rangle = \frac{m_V^2}{f_V}\, \varepsilon_\mu(p,\sigma) \qquad (17)$$

should become smaller. This tendency is confirmed by the present calculation, although the measured effect (for $K \leq 0.145$) is not large. For instance, at $N_f = 0$, $\beta = 2.3$ and $K = 0.145$ we have $f_\rho^{-1} \cong 0.19$, whereas for $N_f = 1$, $\beta = \beta_K$ and $K = 0.145$ the result is $f_\rho^{-1} \cong 0.16$. Due to the increase of f_V^{-1} as a function of K, if extrapolated to $K = K_{cr}$, this value is still somewhat (about a factor 1.3) larger than the measured one: f_ρ^{-1} (experimental) = 0.19.

All the above estimates apply, of course, to the theoretical world with SU(2) colour and $N_f = 1$ light flavour. Using the experience gained during these calculations, the physical case with SU(3) colour and $N_f = 2$ or 3 seems, however, entirely within reach. I hope to complete this calculation in the near future.

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Figure captions

Fig. 1 The π- and ρ-meson masses as a function of the hopping parameter K in the quenched approximation at $\beta = 2.3$. The dashed line with open squares is the linear extrapolation of $(am_\pi)^2$ to zero. The ρ-meson mass is extrapolated linearly from the last measurable points to $K = K_{cr}$.

Fig. 2 The π- and ρ-meson masses for one flavour ($N_f=1$) in the quark determinant at a K-dependent β-value $\beta=\beta_K \equiv 2.3 - 32\ K^4$ (full lines). For comparison, the curves in Fig. 1 are shown by dashed lines ($N_f = 0$ and $\beta = 2.3$). The dashed-dotted line is an estimate of the ρ-meson mass at $\beta = \beta_K$. (The shift from $\beta = 2.3$ is estimated by using the β-dependence measured in Ref. [17].)

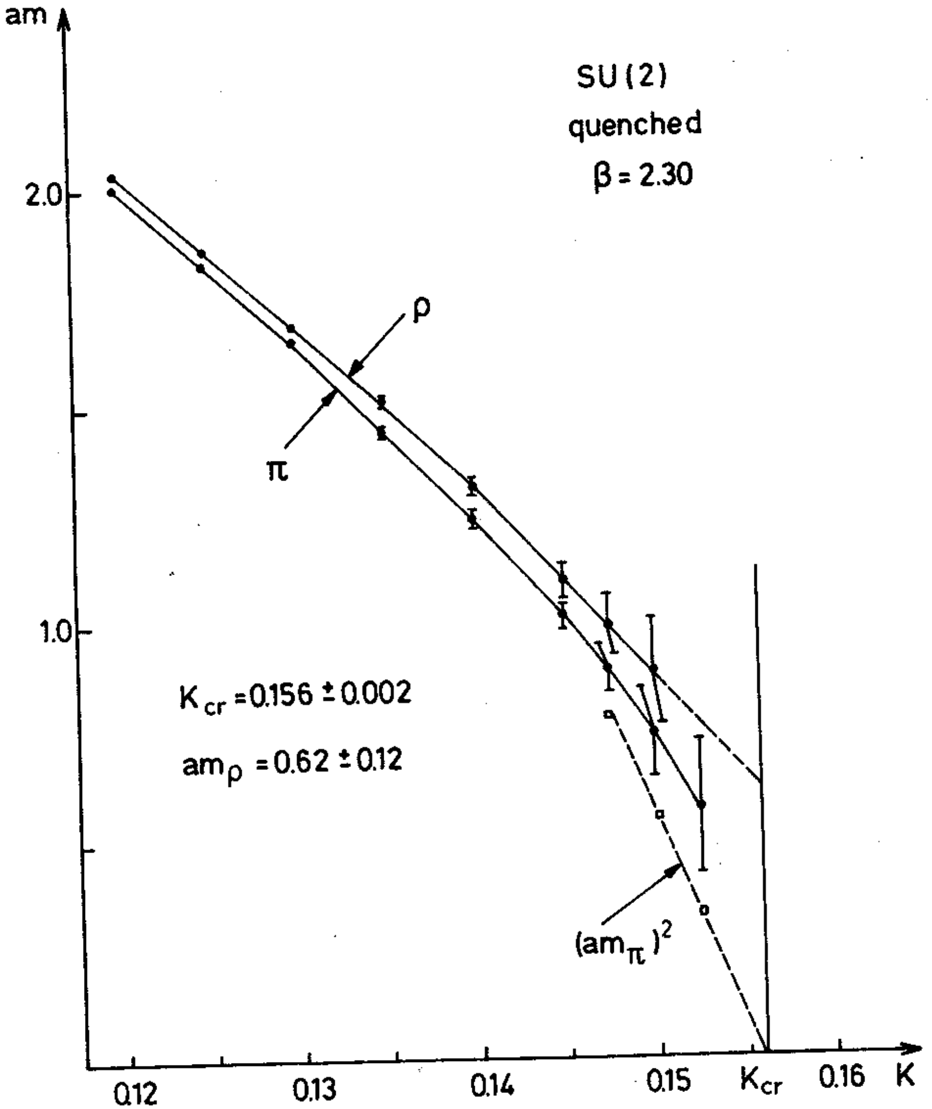

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Fig.1

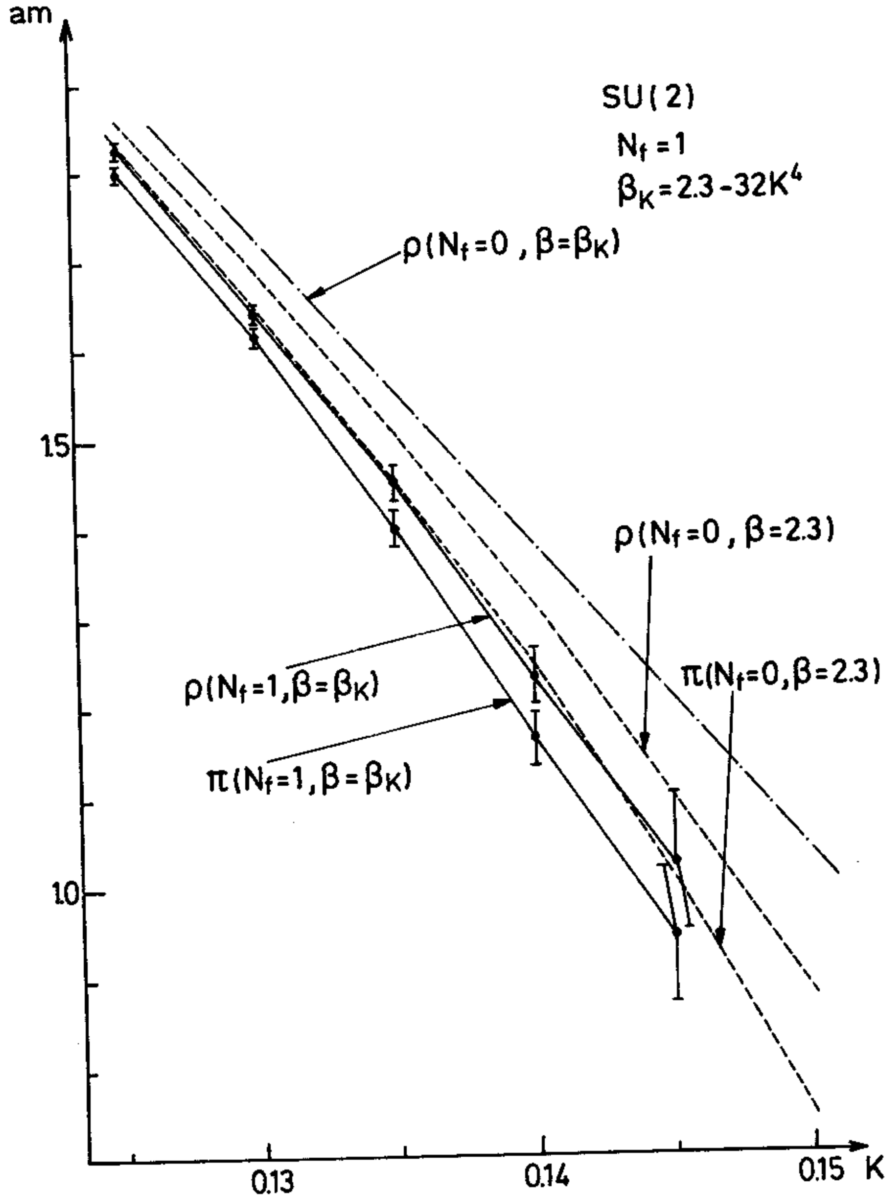

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Fig.2

Taking $R \simeq \frac{3}{4}$ fm, $m_q \simeq \frac{1}{3}\, m_p$, $\alpha_s(\mu^2) \approx \frac{1}{2}$, $\alpha_{GUM} \approx \frac{1}{50}$, and $f = 6$, we find

$$\tau(p \to \bar{\ell} + \text{any}) \approx O(10^3 \text{ to } 10^4)\, m_X^4/m_p^5 \tag{2.17}$$

which is at the lower end of the ball-park estimate (2.3).

The experimental limit of 2×10^{30} years on the proton lifetime actually refers to decay modes producing a muon[11]. In our context, muons could be produced either by Cabibbo mixing; or by $(\text{any})^0 \to \pi^+\pi^-\overline{(\text{any})}^0$, $\pi^\pm \to \mu^\pm\nu$; or by $(\text{any})^+ \to \pi^+\overline{(\text{any})}^0$, $\pi^+ \to \mu^+\nu$. From the ratios of rates in (2.16) we guess that about $\frac{1}{3}$ or $\frac{1}{2}$ of the proton decay modes may contain a muon. The lifetime (2.17) is therefore quite relevant to the experimental limit. The factor $\geq 10^3$, which is probably more general than the specific SU(5) model discussed here, suggests that m_X may in fact be pushed as low as 10^{14} GeV without making the proton too unstable, while models with $m_X \ll 10^{14}$ GeV should probably be excluded. We will subsequently estimate m_X to be $O(2 \times 10^{16})$ GeV in the Georgi-Glashow SU(5) model, which would correspond to a proton lifetime of $(10^{37} \sim 10^{38})$ years, out of reach of presently conceivable experiments.

3. RENORMALIZATION EFFECTS

Since any grand unified symmetry must be very badly broken, all symmetry predictions will be modified[3], only becoming exact way above the grand unification mass scale GUM. The obvious example is the unified coupling constant which separates into strong and weak/electromagnetic coupling constants which are greatly different at present energies. The energy dependence of these couplings has been calculated using the appropriate renormalization group β-functions, of which only the $O(g^3)$ terms are really signifcant for $Q > 10$ GeV [3]. The appropriate renormalization group equations can be used to calculate the energy dependence of other quantities at energies below GUM. An example is a fermion mass, whose renormalization is governed by the anomalous dimension of the fermion mass operator. This was pointed out in Ref. 4, where it was also shown that if $m_b = m_\tau$ at the grand unification mass scale GUM, as in the SU(5) model with a $\underline{5}$-plets of Higgs fields to give fermion masses, then restrictions on GUM from other considerations led to a mass estimate $m_b \sim (4$ to $10)$ GeV if there were six quark flavours in all. The first purpose of this section is to extend the renormalization considerations of Ref. 4 by including the renormalization due to the weak interactions as well as those due to the strong interactions discussed previously, the effect of finite mass corrections in the anomalous dimension of the mass operator, and the consequences of extra heavy quark thresholds. The resulting mass renormalization formalism is probably accurate to $O(10)$%. In the second part of this section, we apply the previous analysis to the specific SU(5) model of Georgi and Glashow with a $\underline{5}$-plet of Higgs mesons giving fermion masses. We start with $\alpha_s(Q^2 = 10\ \mathrm{GeV}^2)$ determined from charmonium[13] ($\alpha_s = 0.19$) or electroproduction[14] ($\alpha_s = 0.32$) analyses, determine the corresponding grand unification mass (1 to 4 × 10^{16} GeV) and the corresponding value of $\sin^2\theta_W$ at present energies ($\simeq 0.20$). We then calculate the strange and bottom quark masses using the inputs $m_s = m_\mu$, $m_b = m_\tau$ at GUM, and defining the quark mass at present energies[22]:

$$m_q \equiv m_q(Q = 2m_q) \tag{3.1}$$

We find from $m_\mu = 0.105$ GeV, $m_\tau = 1.9$ GeV that

$$m_s \approx (0.4 \text{ to } 0.5)\ \mathrm{GeV} \quad ; \quad m_b \approx (5.0 \text{ to } 5.9)\ \mathrm{GeV} \tag{3.2}$$

if there are six quark flavours, and $m_b \geq 6$ GeV if there are eight or more flavours. The results (3.2) may not be specific to the SU(5) model discussed here, but could apply to any model with a similar grand unification mass scale and the same starting point of equality between quark and lepton masses.

3.1 Mass renormalization

Suppose that a grand unified theory predicts the mass of a fermion f at the grand unification mass scale M: $m_f(M)$. The renormalization of this mass at lower energies will be governed by the anomalous dimension of the fermion mass operator, which is to second order

$$\gamma_{m_f} \simeq \gamma^{(3)}_{m_f} + \gamma^{(2)}_{m_f} + \gamma^{(1)}_{m_f} \tag{3.3}$$

We have separated γ_{mf} into pieces coming from the low energy SU(3), SU(2), and U(1) subgroups, but Eq. (3.3) can easily be modified if there are more (or different) sub-unification groups. In the second order approximation $\gamma^{(3)}_{mf}$ is[22]:

$$\gamma^{(i)}_{m_f}(Q^2) = \bar{\gamma}^{(i)}_{m_f}\left[1 - \frac{m_f^2}{Q^2}\ln\left(1 + \frac{Q^2}{m_f^2}\right)\right] \tag{3.4}$$

where the square bracket is a mass correction factor relevant at momentum scales Q close to m_f. The values of $\bar{\gamma}^{(3)}_{mf}$ and the $\gamma^{(2,1)}_{mf}$ are:

$$\left.\begin{aligned} \bar{\gamma}^{(3)}_{m_f} &= \begin{cases} -1/2\pi^2\, g_3^2 & \text{for quarks} \\ 0 & \text{for leptons} \end{cases} \qquad \gamma^{(2)}_{m_f} = -\frac{9}{32\pi^2}\, g_2^2 \\ \gamma^{(1)}_{m_f} &= -\frac{3}{8\pi^2 C^2}\,(T_3 - Q)_{f_L}\,(T_3 - Q)_{f_R}\; g_1^2 \end{aligned}\right\} \tag{3.5}$$

and we have discarded mass correction factors in the anomalous dimensions of $\gamma^{(2,1)}_{mf}$.

In (3.5) C is the constant of proportionality[3] between the weak interaction U(1) and the normalized generator T_0 of the grand unification group:

$$Q = T_3 + CT_0 \tag{3.6}$$

As an example, in the SU(5) model $C^2 = 5/3$ and the usual Weinberg-Salam[23] assignments of $T_{3L,R}$ apply so that

$$\left.\begin{aligned} \gamma^{(1)}_{m_{u,c,t,\ldots}} &= -\frac{1}{40\pi^2} g_1^2 \;; & \gamma^{(1)}_{m_{d,s,b,\ldots}} &= \frac{1}{80\pi^2} g_1^2 \\ \gamma^{(1)}_{m_{\nu_e,\nu_\mu,\nu_\tau,\ldots}} &= 0 \;, & \gamma^{(1)}_{e,\mu,\tau,\ldots} &= -\frac{9}{80\pi^2} g_1^2 \end{aligned}\right\} \tag{3.7}$$

If we introduce the Callan-Symanzik β-functions to $O(g^3)$ in a form similar to (3.4) and (3.5), we have[22]:

$$\beta^{(3)}(Q^2) \approx -\frac{g_3^3}{16\pi^2}\left\{ 11 - \frac{2}{3}\sum_q \left[1 - 6\frac{m_q^2}{Q^2} + \frac{12 m_q^4/Q^4}{(1+4m_q^2/Q^2)^{\frac{1}{2}}} \ln \frac{(1+\frac{4m_q^2}{Q^2})^{\frac{1}{2}}+1}{(1+\frac{4m_q^2}{Q^2})^{\frac{1}{2}}-1} \right] - \ldots \right\} \tag{3.8}$$

and

$$\beta^{(2)} \approx -\frac{g_2^3}{16\pi^2}\left\{\frac{22}{3} - \frac{2}{3}f - \ldots\right\} \;; \quad \beta^{(1)} \approx \frac{-g_1^3}{16\pi^2}\left\{-\frac{2}{3}f - \ldots\right\} \tag{3.9}$$

the square bracket is again a mass correction factor relevant at finite Q^2, and the dots indicate possible contributions from Higgs bosons, etc. We define $\bar{\beta}^{(3)}$ analogously to the $\bar{\gamma}^{(i)}_{m_f}$ of Eq. (3.4):

$$\bar{\beta}^{(3)} \equiv \lim_{m_q/Q \to 0} \beta^{(3)}/g_3^3 \tag{3.10}$$

A precise calculation of the mass renormalization cannot be done analytically, because of the complicated forms of the square-braketted mass correction factors in (3.4) and (3.8). If these are neglected the SU(5) model has

$$\begin{aligned} \ln\left[m_{u,c,t,\ldots}(\mu)\right] \approx \ln\left[m_{u,c,t,\ldots}(M)\right] &+ \left[\frac{4}{11-\frac{2}{3}f}\right]\ln\left[\frac{\alpha_S(\mu)}{\alpha_{GUM}}\right] \\ &+ \left[\frac{27}{88-8f}\right]\ln\left[\frac{\alpha_2(\mu)}{\alpha_{GUM}}\right] - \left[\frac{3}{10f}\right]\ln\left[\frac{\alpha_1(\mu)}{\alpha_{GUM}}\right] \end{aligned} \tag{3.11a}$$

$$\begin{aligned} \ln\left[m_{d,s,b,\ldots}(\mu)\right] \approx \ln\left[m_{d,s,b,\ldots}(M)\right] &+ \left[\frac{4}{11-\frac{2}{3}f}\right]\ln\left[\frac{\alpha_S(\mu)}{\alpha_{GUM}}\right] \\ &+ \left[\frac{27}{88-8f}\right]\ln\left[\frac{\alpha_2(\mu)}{\alpha_{GUM}}\right] + \left[\frac{3}{20f}\right]\ln\left[\frac{\alpha_1(\mu)}{\alpha_{GUM}}\right] \end{aligned} \tag{3.11b}$$

$$\ln\left[m_{e,\mu,\tau,\ldots}(\mu)\right] \approx \ln\left[m_{e,\mu,\tau,\ldots}(M)\right] + \left[\frac{27}{88-8f}\right]\ln\left[\frac{\alpha_2(\mu)}{\alpha_{GUM}}\right] - \left[\frac{27}{20f}\right]\ln\left[\frac{\alpha_1(\mu)}{\alpha_{GUM}}\right] \tag{3.11c}$$

where we have kept separate the SU(3), SU(2), and U(1) contributions. We will in fact only be interested here in the ratio of charge $-\frac{1}{3}$ quark (d, s, b, ...) to charge -1 lepton (e, μ, τ, ...) masses:

$$\ln\left[\frac{m_{d,s,b,\ldots}(\mu)}{m_{e,\mu,\tau,\ldots}(\mu)}\right] = \ln\left[\frac{m_{d,s,b,\ldots}(M)}{m_{e,\mu,\tau,\ldots}(M)}\right] + \left[\frac{4}{11-\frac{2}{3}f}\right]\ln\left[\frac{\alpha_s(\mu)}{\alpha_{GUM}}\right] + \left[\frac{3}{2f}\right]\ln\left[\frac{\alpha_1(\mu)}{\alpha_{GUM}}\right] + \begin{pmatrix}\text{computable non-analytic}\\ \text{finite mass corrections}\end{pmatrix} \tag{3.12}$$

The consequences of (3.12) for the strange and bottom quark masses are shown in Table 1. In the previous analysis[4] only the SU(3) term without mass corrections

$$\left[\frac{4}{11-\frac{2}{3}f}\right]\ln\left[\frac{\alpha_s(\mu)}{\alpha_{GUM}}\right]$$

was taken into account in calculating the quark to lepton mass ratio. In fact, for typical parameters of the SU(5) model to be discussed in the next section, the U(1) and finite mass correction factors in SU(3) each reduce m_q/m_ℓ by $O(10\%)$, whereas finite mass correction factors in SU(2) and U(1) are insignificant. We can think of three more contributions to (3.12) which might be included.

- Higher order contributions to γ_{m_f} and $\beta^{(i)}$ might be important, particularly $O(g_3^4,\ g_3^5)$, respectively. We have not calculated γ_{m_f} to $O(g^4)$, but have investigated numerically the effect of the g_3^5 term in $\beta^{(3)}$. It only has a 1% effect on the estimate of the bottom quark mass, but may be significant for the strange quark mass estimate, depending on the value of the strong interaction coupling constant at $Q^2 = 10\ \mathrm{GeV}^2$ which is used.

- Higgs meson contributions to the $\beta^{(i)}$ may be unsymmetric, because of the Higgs' role in breaking the SU(5) symmetry. As discussed in Section 4, the SU(5) model with a single $\underline{5}$-plet of Higgs mesons giving fermion masses may be expected to have a light SU(3) singlet and SU(2) doublet (some physical, some eaten by $W^\pm$, Z^0), and a "heavy" SU(3) triplet and SU(2) singlet. In this case the b quark mass estimate is $\sim$ 1% smaller than would be obtained with a symmetric $\underline{5}$-plet of Higgs mesons.

- The continuation from space-like to time-like momenta may be non-trivial. We follow Georgi and Politzer[22] in defining the "physical" quark mass to be $m_q\ (\mu_0)$ where

$$\mu_0 = 2m_q(\mu_0) \tag{3.13}$$

which should be threshold for producing naked $q\bar{q}$ pairs. But the renormalization group formalism only applies directly to space-like Q^2. Moorhouse, Pennington and Ross (MPR)[24] have pointed out that if (as suggested by the renormalization group equations)

$$g^2(q^2) \propto \frac{1}{\ln(-q^2/\Lambda^2)} \quad \text{for large negative } q^2 \tag{3.14}$$

then for $q^2 = |q^2|\, e^{i\theta}$ one should use

$$g^2(q^2) \propto \frac{1}{\ln(|q^2|/\Lambda^2) + i(\pi-\theta)} \tag{3.15}$$

Furthermore, any quantity such as a quark mass which the renormalization group tells us is $\sim [g^2(q^2)]^{\delta}$: $\delta \neq 0$ will also inherit the complex phase in (3.15). The difference between (3.14) and (3.15) is unimportant near the grand unification mass scale M, but potentially significant close to the quark thresholds we are interested in. For example, if $\alpha_s \simeq 0.32$ at $-q^2 = 10$ GeV^2 as suggested by leading order electroproduction analyses, the MPR[24] analysis would yield $|\alpha_s(q^2 = 10\ \text{GeV}^2)|_{MPR} \simeq 0.27$. If the MPR correction were substituted into (3.12), it would reduce the estimate of m_b by $\sim$ 4%. It is not clear to us what the correct procedure for continuing to the threshold region should be, but the MPR analysis suggests there is a (5 to 10)% slop in our estimates of m_b.

Table 2 summarizes our studies of corrections to the mass ratio renormalization formula (3.12). We would conclude that an estimate of m_b should be accurate to $O(10)$%, while an estimate of m_s is probably less precise [$\pm O(25)$%?], principally because of the higher order and time-like continuation uncertainties.

3.2 Renormalization calculations in SU(5)

To estimate these effects we have assumed a range of values for $\alpha_s(Q^2 = 10\ \text{GeV}^2)$. Analyses of the charmonium system using the 3-gluon annihilation model for the total hadronic decay rate yield[13] $\alpha_s \simeq 0.19$. If one writes this in a form motivated by the renormalization group:

$$\alpha_s(Q^2) \approx \frac{12\pi}{25\ln(Q^2/\Lambda^2)} \tag{3.16}$$

it corresponds to $\Lambda^2 = 0.005$ GeV^2. On the other hand, analyses of electroproduction[14] suggest a larger value of Λ^2, typically $\Lambda \simeq 0.3$ GeV, $\Lambda^2 = 0.09$ GeV^2 if only g^2 terms in anomalous dimensions and g^3 terms in the β function are retained, as we also do here. We will quote results for these extreme values, and for $\Lambda^2 = 0.03$ GeV^2.

We determined the grand unification mass scale M [$= M_X$ in the SU(5) model] using α_s and α as inputs, by allowing α_s, α_2 and α_1 to evolve independently, and assuming they are equal at M. This is a slight over-simplification, since there are in principle finite mass corrections close to the grand unification mass, due to the X and Y bosons and in principle Higgs bosons, which we have neglected. As a possible test of this assumption, we have compared the transition of α_s across a new quark threshold using the complete formula (3.8) with a naive procedure where α_s is allowed to evolve with $f = f_0$ up $Q^2 = m_f^2$, and then made to evolve with $f = f_0 + 1$ for $Q^2 > m_f^2$, keeping the coupling constant continuous at the transition, which involves changing Λ at the new quark threshold[25]. The difference between the approximate and exact values of α_3 above threshold was $\sim \frac{1}{4}\%$. While the discrepancy would be bigger for the transition to grand unification, we cannot believe that it would be a very big effect.

We calculated $\sin^2 \theta_W$ at present energies using the evolution of g_2^2 and g_1^2 given by (3.8) and including the effect of a light SU(2) × U(1) Higgs doublet:

$$\Delta\beta^{(3)} = 0 \quad ; \quad \Delta\beta^{(2)} = -\frac{1}{6}\frac{g_2^2}{16\pi^2} \quad ; \quad \Delta\beta^{(1)} = -\frac{1}{10}\frac{g_1^2}{16\pi^2} \tag{3.17}$$

Their effects on $\sin^2 \theta_W$ are actually only O(1 to 2)%. We chose to evaluate $\sin^2 \theta_W$ at $Q^2 = 10^4$ GeV^2, since this is to a good approximation the energy at which the weak and electromagnetic interactions are unified in the Weinberg-Salam[23] model.

Finally we calculated m_b [using $m_\tau = 1.9$ GeV [26] at present energies, and $m_b = m_\tau$ at M as with $\underline{5}$-plets of Higgs fields in SU(5)] and m_s [using $m_\mu = 0.105$ GeV at present energies, and $m_s = m_\mu$ at M] using the definitions (3.13). In principle we could also have calculated m_d from m_e, but the definition (3.13) is unusable for the u and d quarks. Actually, since

$$\frac{m_d(\mu)}{m_s(\mu)} = \frac{m_e}{m_\mu} \sim \frac{1}{200} \tag{3.18}$$

in this approach, the d quark mass is probably underestimated, since people usually prefer $\sim \frac{1}{20}$ [27] for m_d/m_s at short distances*).

The results are listed in Table 1. We see that the grand unification mass $M \sim 2 \times 10^{16}$ GeV to within a factor of 2 or 3 corresponding to a proton lifetime $O(10^{38\pm1})$ years, while $\alpha_{GUM} \sim \frac{1}{45}$. The value of $\sin^2 \theta_W$ is relatively stable at 0.20, which is to be compared with the latest CDHS experimental estimate of 0.24 ± 0.02 [21]. Experiment has drifted closer to theory since the calculations of Georgi, Quinn and Weinberg[3] and CEG[4]. The estimate of m_s is quite successful. If the Υ(9.4 to 10.4) states[15] turn out to be bottomonium, then the estimate of m_b will have been correct within the expected theoretical error of 10%. The naked $b\bar{b}$ threshold is probably at (10.4 to 10.6) GeV, corresponding to

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*) Long distance ("constituent") quark masses have additional terms generated dynamically. These are largest for the u and d quarks, non-negligible for s quarks, and negligible for c, b, t, ... quarks. See, for example, Ref. 22.

m_b = (5.2 to 5.3) GeV within the definition (3.13). The sensitivity of the m_b estimate to various modifications of our assumptions and approximations is set out in Table 2, and some of these effects were discussed in Section 3.1. We note with regret the insensitivity to the mass m_t of the sixth quark. We are happy to see the sensitivity to the inclusion of eight or ten quarks. But even if you were to believe all this nonsense, then you would perhaps not immediately conclude that there were only six quarks!

Actually, the estimate of the bottom quark mass is probably not very sensitive to the precise details of the SU(5) model. It only requires knowledge of the evolution of the SU(3), SU(2), and U(1) coupling constants below the grand unification mass, and not really how they are unified above it. The one sensitivity to the unification scheme lies in the parameter C of Eq. (3.6). This could be $\sqrt{n/n-2}$ in a more general SU(n) unifying group[28]. However, C only enters in the U(1) factor in m_b/m_τ, which is only a 9% effect anyway, so that the likely effect on m_b would be very small. Any grand unification scheme with a mass scale $O(10^{16})$ GeV at which $m_b = m_\tau$ and which broke down to SU(3) × $[SU(2)]^N$ × U(1) at present energies would have a prediction for m_b similar to the SU(5) model.

Before leaving this section, we should note one implication of using a group factor bigger than SU(2) for the weak interactions, such as SU(3) [19]. You will not be able to bring together the strong and weak interactions unless you also increase the strong gauge group from SU(3) to a bigger group like SU(4) [2], since the rate of approach of the couplings is controlled by the difference between the corresponding $\bar{\beta}^{(\text{strong})}$ and $\bar{\beta}^{(\text{weak})}$:

$$\frac{1}{g^2_{\text{strong}}(\mu)} - \frac{1}{g^2_{\text{weak}}(\mu)} \approx \left[\bar{\beta}^{(\text{strong})} - \bar{\beta}^{(\text{weak})}\right] \ln\left(M/\mu\right) \tag{3.19}$$

In order to achieve grand unification before the Planck mass, any such augmentation of the strong group must take place before $Q \sim 10^8$ GeV. Unfortunately, the interactions building SU(3) up to SU(4) could have a four-fermi coupling strength [cf. Eq. (2.8)] as small as $G_{3\to 4} \sim 10^{-17}/m_p^2$. The analogue of the Eqs. (3.11) for a bigger "strong interaction" group clearly implies a smaller renormalization of m_q/m_ℓ in such a model, unless the number of flavours is increased correspondingly: SU(3) → SU(4) requires an increase $O(6)$ in the number of flavours.

4. SYMMETRY BREAKING IN SU(5)

It would surprise us if SU(5) in fact describes the real world, but it is useful to work out the model in order to establish an "existence proof" for grand unified theories. All these models need a gauge hierarchy in which the grand unified group is first broken down on a very large mass scale m_X to a product of strong and weak/electromagnetic groups, and then the weak/electromagnetic group is broken down to U(1) on a much smaller mass scale m_W. Two questions then arise: can such a hierarchy of symmetry breakdown ($m_X >> m_W$) be achieved at all, and can it be made "natural"? In this section we study these questions in the SU(5) model, where the breakdown pattern should be

$$SU(5) \to SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1) \tag{4.1}$$

As discussed by Georgi and Glashow[1], the minimal Higgs system which can be considered involves an adjoint $\underline{24}$ representation of SU(5) for the superstrong breaking, and a spinorial $\underline{5}$ representation for the weak breaking.

We consider first the superstrong breaking in isolation. Introducing a matrix notation $\Phi = \sum_{a=1}^{24} \phi^a(\lambda^a/\sqrt{2})$ for the adjoint representation, the most general Φ potential is

$$V(\Phi) = -\frac{\mu^2}{2}\mathrm{Tr}(\Phi^2) + \frac{a}{4}\mathrm{Tr}((\Phi^2)^2) + \frac{b}{2}\mathrm{Tr}(\Phi^4) + \frac{c}{3}\mathrm{Tr}(\Phi^3) \tag{4.2}$$

but we will impose a discrete symmetry $\Phi \leftrightarrow -\Phi$ so that $c = 0$. The potential (4.2) permits many patterns of symmetry breakdown, but as pointed out by Li[29], if

$$b > 0 \quad \text{and} \quad a > -\frac{7}{15}b \tag{4.3}$$

the lowest vacuum is the asymmetric one corresponding to the desired breakdown of $SU(5) \to SU(3) \times SU(2) \times U(1)$, with a Higgs vacuum expectation matrix

$$\langle 0|\Phi|0\rangle = \begin{pmatrix} v & & & & \\ & v & & 0 & \\ & & v & & \\ & 0 & & -\frac{3v}{2} & \\ & & & & -\frac{3v}{2} \end{pmatrix} \tag{4.4}$$

with v determined by

$$\mu^2 = \frac{15}{2}av^2 + \frac{7}{2}bv^2 \tag{4.5}$$

The superheavy X and Y bosons then have masses[30]:

$$m_X^2 = m_Y^2 = \frac{25}{8}g^2v^2 \tag{4.6}$$

If we parametrize $\Phi - \langle 0|\Phi|0\rangle$ in the form

$$\Phi - \langle 0|\Phi|0\rangle = \left(\begin{array}{c|c|c} H_8 + \frac{2H_0}{\sqrt{30}} I & \underset{\sim}{H}_X & \underset{\sim}{H}_Y \\ \hline \underset{\sim}{H}_X^+ & \frac{H_Z}{\sqrt{2}} - \frac{3}{\sqrt{30}} H_0 & H^+ \\ \hline \underset{\sim}{H}_Y^+ & H^- & -\frac{H_Z}{\sqrt{2}} - \frac{3}{\sqrt{30}} H_0 \end{array}\right) \tag{4.7}$$

then $\underset{\sim}{H}_X$ and $\underset{\sim}{H}_Y$ are massless and eaten by the X and Y bosons, and

$$m_{H_8}^2 = \frac{5b}{2} v^2 \; ; \; m_{H_Z}^2 = m_{H^\pm}^2 = 10 b v^2 \; ; \; m_{H_0}^2 = 15 a v^2 + 7 b v^2 = 2\mu^2 \tag{4.8}$$

so that all the physical Higgs bosons have very large masses $O(m_{X,Y})$.

Now we add in a $\underline{5}$ of Higgs fields $\underline{H}$ for which the most general potential is

$$V(\underline{H}) = -\frac{\nu^2}{2}(\underline{H}^+ \cdot \underline{H}) + \frac{\lambda}{4}(\underline{H}^+ \cdot \underline{H})^2 \tag{4.9}$$

If the component of $\underline{H}$ which develops a vacuum expectation value lies in the SU(2) left behind by the superstrong breaking, then we may parametrize $\underline{H}$ in the form

$$\underline{H} = \begin{pmatrix} \underset{\sim}{H} \\ H_4 \\ \left(\frac{v_0 + \rho}{\sqrt{2}}\right) e^{i\zeta/v_0} \end{pmatrix} \tag{4.10}$$

with v_0 determined by

$$2\nu^2 = \lambda v_0^2 \tag{4.11}$$

The SU(2) symmetry is broken in the desired way by an isodoublet of Higgs fields, and the $W^\pm$ mass is:

$$m_W^2 = \frac{g^2 v_0^2}{4} \tag{4.12}$$

The mystery why $m_W \ll m_X$ now becomes the mystery why $v_0 \ll v$, or why $\nu^2 \ll \mu^2$. The $\underset{\sim}{H}$, H_4 and ζ Higgs fields are massless, H_4 and ζ being eaten by the W and Z bosons. The ρ field is a physical Higgs boson with

$$m_\rho^2 = \frac{\lambda v_0^2}{2} = \nu^2 \tag{4.13}$$

The construction of the Higgs system is by no means complete because only one combination of the massless Higgs fields $\underset{\sim}{H}_Y$ and $\underset{\sim}{H}$ can be eaten to give the Y bosons masses. The other combination would remain massless, which would be phenomenologically unacceptable because it would form light bound states with quarks, which would have strong interactions -- it would be natural to call them quiggs particles. However, there is no reason to exclude cross-coupling of the Φ and $\underline{H}$ fields from the Higgs potential, and indeed such terms are generated by renormalization. The most general $(\Phi, \underline{H})$ coupling terms respecting the discrete symmetry $\Phi \leftrightarrow -\Phi$ are

$$V(\Phi, \underline{H}) = \alpha (\underline{H}^{+} \cdot \underline{H}) \mathrm{Tr}(\Phi^2) + \beta \underline{H}^{+} \Phi^2 \underline{H} \tag{4.14}$$

and we should look for an extremum of the combined potential (4.2), (4.9), and (4.14) which removes the embarrassing massless Higgs without destroying the desirable features of (4.2) and (4.9). When Φ and $\underline{H}$ are coupled, $\langle 0|\Phi|0\rangle$ may also break SU(2), and we should look for solutions with

$$\langle 0|\Phi|0\rangle = \begin{pmatrix} v & & & & \\ & v & & 0 & \\ & & v & & \\ & 0 & & (-\frac{3}{2} - \frac{\epsilon}{2})v & \\ & & & & (-\frac{3}{2} + \frac{\epsilon}{2})v \end{pmatrix} \tag{4.4'}$$

There will be an extremum of the combined potential which connects up with the previous partial solutions in the limit as $\alpha, \beta \to 0$. For sufficiently small α and β this will still be the absolute minimum, and hence that chosen in the "real world". For this extremum ε should $\to 0$ as $\alpha, \beta \to 0$. The solution with these properties in fact has

$$\epsilon = \frac{3}{20} \frac{\beta v_0^2}{b v^2} + O\left(\left(\frac{v_0}{v}\right)^4\right) \tag{4.15}$$

so that the SU(2) breaking by the Φ Higgs fields is much less than that due to the $\underline{H}$ fields, in accord with the experimental preference for I = ½ SU(2) Higgs dominance[21,31]. In the combined potential the conditions (4.5) and (4.11) get replaced by

$$\mu^2 = \frac{15}{2} a v^2 + \frac{7}{2} b v^2 + \alpha v_0^2 + \frac{9}{30} \beta v_0^2 \tag{4.5'}$$

$$\nu^2 = \frac{\lambda v_0^2}{2} + 15 \alpha v^2 + \frac{7}{2} \beta v^2 - 3 \epsilon \beta v^2 \tag{4.11'}$$

The first of the conditions is just a minor modification of (4.5), whereas (4.11') looks rather "unnatural" in that it requires a very strong cancellation between different *a priori* large terms $O(v^2)$ in order that v_0^2 be kept very small. Thus the mystery why $m_W \ll m_X$ persists, and is not resolved by the coupling of Φ and $\underline{H}$. On the other hand, from a strictly logical point of view, the situation is no worse either -- you always need just one mass parameter to be much less than another. We should note in passing that m_X^2, m_Y^2, $m_{H_8}^2$, and $m_{H^\pm}^2$ are only altered by $O(v_0^2)$ in the combined Higgs system, while one combination of ρ, H_z and H_0 keeps a mass $O(v_0)$ and the others keep masses $O(v)$. The combination of $\underset{\sim}{H}_Y$ and $\underset{\sim}{H}$ which is massless and eaten by the Y boson is

$$\underset{\sim}{H}_Y - \frac{\sqrt{2}v_0}{5v}\underset{\sim}{H} + O\left(\frac{v_0^2}{v^2}\right) \tag{4.16}$$

whereas the orthogonal combination

$$H_3 \equiv \underset{\sim}{H} + \frac{\sqrt{2}v_0}{5v}\underset{\sim}{H}_Y + O\left(\frac{v_0^2}{v^2}\right) \tag{4.17}$$

is physical, with a mass

$$m_{H_3}^2 = -\frac{5}{2}\beta v^2 + O(v_0^2) \tag{4.18}$$

The H_3 boson therefore has a mass comparable with m_X, though the condition that the extremum studied here is in fact the absolute minimum presumably requires that β is in some sense "small" compared with a, b, λ, but not necessarily much smaller. In principle, baryon number violation could be mediated by H_3 exchange, but this seems likely to be much smaller than that mediated by X or Y exchange. We see from (4.17) that H_3 is mainly $\underset{\sim}{H}$, which has fermion couplings

$$g_{\underset{\sim}{H}f\bar{f}'} = O\left(g\,\frac{m_f}{m_W}\right) \tag{4.19}$$

like the familiar SU(2) Higgs. We therefore expect that

$$\frac{\Gamma(p \underset{H_3}{\to} \ell + \text{any})}{\Gamma(p \underset{X}{\to} \ell + \text{any})} = O\left(\frac{m_{X,Y}^4}{m_{H_3}^4}\frac{m_f^4}{m_W^4}\right) = O\left(\frac{g^4}{\beta^2}\frac{m_f^4}{m_W^4}\right) \tag{4.20}$$

which is plausibly much less than 1. We therefore conclude that the combined potential (4.2), (4.9), and (4.14) seems to have all the desired properties, at the continued price of the one "unnatural" condition (4.11′).

As was mentioned above, renormalization effects will couple the Φ and $\underline{H}$ systems even if they are independent in the tree approximation to the potential[21]. Following Coleman and Weinberg[20], the first order radiative corrections to the Higgs potential arising from vector boson loops, are

$$V'(\Phi, \underline{H}) = \frac{3}{64\pi^2}\,\mathrm{Tr}\left[\mathcal{M}^2 \ln\left(\mathcal{M}^2/M_0^2\right)\right] \tag{4.21}$$

where $\mathcal{M}^2_{ab}$ is the Higgs contribution to the vector boson mass^2 matrix:

$$\mathcal{M}^2_{ab} = g^2_{GUM}\left[-\frac{1}{4}\mathrm{Tr}\left\{[\lambda^a,\Phi][\lambda^b,\Phi]\right\} + \frac{1}{4}\underline{H}^\dagger\{\lambda^a,\lambda^b\}\underline{H}\right] \qquad (4.22)$$

In Appendix B we display the expression for $V'(\Phi, H)$ obtained by expanding the fields around the "zeroth" order vacuum, Eq. (4.4). This is sufficient for extracting the linear and quadratic terms in the Higgs fields, and is valid if the true vacuum indeed satisfies ε, $v_0/v << 1$. As an exercise to show that a solution with the desired properties can exist, we assume simplified forms for the zeroth order potential so that we can easily check *a posteriori* that neglect of the Higgs loop contributions is justified. These correspond to $\beta = b = 0$ and either $\alpha = 0$ or $\alpha = a = \lambda/4$. Then the extremum conditions are

$$\alpha^2_{GUM}\left[2\ln\frac{25v^2}{8M_0^2} + 1\right] = \frac{16}{375}\left[\frac{\mu^2}{v^2} - \frac{15}{2}a\right] + O\left(\frac{v_0^2}{v^2}\right) \equiv -\omega \qquad (4.5')$$

which determines v as a function of the parameters of the potential, and

$$\epsilon = \frac{3}{20}\frac{v_0^2}{v^2}\left[\frac{2\alpha^2_{GUM} - \omega}{\alpha^2_{GUM} - 3\omega}\right] + O\left(\frac{v_0^3}{v^3}\ln v^2/v_0^2\right) \qquad (4.15')$$

which shows that the desired form of SU(2) breaking persists "naturally", as well as the "unnatural" condition

$$\frac{\nu}{2v^2} = \frac{15\alpha}{2} + \frac{3}{10}\left[\frac{\mu^2}{v^2} - \frac{15}{2}a\right] + O\left(\frac{v_0^2}{v^2}\ln v_0^2/v^2\right) \qquad (4.11')$$

for the gauge hierarchy $m_X >> m_W$. Equation (4.11′) involves a complicated interplay between the parameters appearing in the zero- and one-loop potentials (which must be readjusted for each order of perturbation theory!). The condition for a minimum is

$$0 < \frac{15}{2}a - \frac{\mu^2}{v^2} < \frac{375}{32}\alpha^2_{GUM} \approx 5\times 10^{-3} \qquad (4.23)$$

and the SU(3) triplet of physical Higgs bosons will be sufficiently massive for proton stability if

$$\frac{15a}{2} - \frac{\mu^2}{v^2} \gtrsim 10^{-3} \qquad (4.24)$$

The constraint (4.24) ensures that Eq. (4.11′) can only be achieved by a strong cancellation. However, our limited investigation does not rule out the possibility that a less contrived looking solution might emerge from a more clever choice of potential.

We therefore conclude this section with the belief that an SU(5) Higgs system can be set up which has the desired phenomenological properties, and that these are not destroyed by first order radiative corrections. The "observed" pattern of SU(5) and SU(2) symmetry breaking is in many respects "natural", but the hierarchy $m_X >> m_W$, while possible, is unexplained and rather "unnatural"[21]. For convenience, the spectrum of physical bosons is listed in Table 3, together with some of their characteristics. The list is long, but shorter than in any other grand unified model known to us.

5. CONCLUSIONS

In this paper we have mainly studied three aspects of grand unified theories of the strong, weak, and electromagnetic interactions.

- The proton lifetime, which seems to be $O(10^3$ to $10^4)$ times longer than the $O(m_X^4/m_p^5)$ expected from simple dimensional arguments. Higgs boson exchanges do not seem to dominate proton decay.
- Mass renormalization effects, which in a class of models including SU(5) give realistic estimates (1.2) for the strange and bottom quark masses. On the other hand, $\sin^2 \theta_W \simeq 0.20$ in the SU(5) model, which is somewhat[16] low.
- The Higgs structure of SU(5), which is able to accommodate the symmetry-breaking pattern $SU(5) \to SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1)$ quite naturally, but does not naturally[20] give the large ratio observed between the two mass scales of spontaneous symmetry breaking.

We hope that the considerations under the first two of these headings have more general validity than in just the Georgi-Glashow[1] SU(5) model. This model has the experimental problems of the $SU(2)_L \times U(1)$ Weinberg-Salam[23] model, such as the absence of parity violation in atomic physics[32] and the presence of exotic trimuon events[33] in neutrino scattering, as well as a possible problem with the value of $\sin^2 \theta_W$ [16]. It is also theoretically unattractive because it affords no understanding of different mass scales and mixing angles, and gives no understanding of the number of fundamental fermion fields, which can just be added sequentially in the model. On the other hand, the SU(5) model cannot yet be rigorously excluded as a logical possibility, and may serve as a useful existence proof and prototype for grand unified models, just as the Weinberg-Salam model[23] has been a useful starting point for gauge theories of the weak and electromagnetic interactions. Also, we should remember that the SU(5) model is much less complicated than other grand unified models on the market. For example, the E_7 model[34] apparently[35] requires (at the least) $\underline{912}$, $\underline{133}$, and $\underline{1463}$ Higgs representations to get an appropriate pattern of symmetry breaking, and the appropriate renormalization of the strong relative to the weak coupling constant seems difficult to realize, for the reasons discussed at the end of Section 3. Also, the E_7 model in the form proposed is apparently ruled out by recent neutrino scattering experiments[36]. Another group proposed is SO(10) [12], which needs (at the least) $\underline{10}$, $\underline{16}$, $\underline{120}$, and $\underline{126}$ Higgs representations to get the appropriate pattern of symmetry breaking. However, it has[4] no problem with the renormalization of the strong relative to the weak coupling constant. The bleak comments of this paragraph may betoken the bankruptcy of the simple-minded simple group philosophy[6]. But even the wrong answer to the right question may be instructive.

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Table 1

$\alpha_s(Q^2 = 10)$	Λ^2	m_b	m_s	m_X	α_{GUM}	$\sin^2\theta_W$
0.32	0.09	5.9	0.50	3.7×10^{16}	0.022	0.20
0.26	0.03	5.5	0.45	2.1×10^{16}	0.022	0.20
0.19	0.005	5.0	0.38	0.9×10^{16}	0.022	0.20

Λ^2 is defined by $\alpha_s(Q^2) = \dfrac{12\pi}{25 \ln (Q^2/\Lambda^2)}$, all masses are in GeV.

Results are calculated with $m_t = 8$ GeV and neglecting Higgs contributions to the Callan-Symanzik β-functions. Finite mass corrections in the SU(3) renormalization (3.12) have been evaluated numerically.

Table 2

	f = 8	f = 10	Higgs included in β-functions	MPR time-like correction	$m_t \to 100$ GeV
$\dfrac{\Delta m_b}{m_b}$	+10%	+30%	$\pm\frac{1}{2}$%	-4%	-3%

Results are calculated with $\Lambda^2 = 0.03$.

Table 3

Physical boson content of the SU(5) model

Particle	Charge	Spin	Mass	Remarks
γ, G	0	1	0	The usual photon and gluons
$W^{\pm}$	± 1	1	$\approx \frac{gv_0}{2}$ $\approx$ 84 GeV	Usual vector bosons of Weinberg-Salam model, with mass relations essentially as given by I = ½ Higgs doublet and $\sin^2 \theta_W = 0.20$, because $\varepsilon = O(v_0^2/v^2)$.
Z^0	0	1	$\approx \frac{gv_0}{2 \cos \theta_W}$ $\approx$ 94 GeV	
X	$\pm {}^4/_3$	1	$\approx \frac{5gv}{2\sqrt{2}}$ $\approx 2 \times 10^{16}$ GeV	Violate baryon and lepton number conservation. Mass degeneracy broken in $O(gv_0^2/v)$.
Y	$\pm {}^1/_3$	1		
H	0	0	$O(\lambda^{1/2} v_0)$	Lightest state resembles the Higgs boson of the usual Weinberg-Salam model with an I = ½ Higgs multiplet. The others presumably have masses close to those of the X and Y vector bosons.
H′	0	0	$O(b^{1/2} v)$	
H″	0	0		
H_8	0	0	$O(b^{1/2} v)$	Colour octet with masses close to m_X, m_Y
$H^{\pm}$	± 1	0		Uncoloured and heavy charged Higgs particles
H_3	$\pm {}^1/_3$	0	$O(b^{1/2} v)$?	Colour triplet with mass perhaps somewhat less than m_X, m_Y. Violates baryon and lepton number conservation, but at a low level.

"Close in mass" probably means within a factor of 10. Note that the only low-mass [$\lesssim O(100)$ GeV] bosons are those in the usual Weinberg-Salam[23] model.

Figure captions

Fig. 1 : Baryon and lepton number violating exchanges in the SU(5) model due to (a) vector gauge bosons, and (b) Higgs bosons.

Fig. 2 : Lowest order gluonic corrections to the qqq vertex which determine the anomalous dimension relevant to the short distance enhancement factor.

Fig. 3 : Model for proton decay in which two quarks annihilate freely into a lepton and an antiquark.

APPENDIX A

THE ANOMALOUS DIMENSION OF THE TRI-QUARK OPERATOR

There are three independent operators appearing in the effective Lagrangian (2.7):

$$\begin{aligned} O^{e}_{L,R} &\equiv \left(\epsilon_{ijk}\, \bar{u}^{c}_{k_L} \gamma_\mu u_{j_L}\right)\left(\bar{e}^{+}_{L,R}\gamma^\mu d_{i_{L,R}}\right) \\ O^{\nu} &\equiv \left(\epsilon_{ijk}\, \bar{u}^{c}_{k_L}\gamma_\mu d_{j_L}\right)\left(\bar{\nu}^{c}_{e_R}\gamma^\mu d_{i_R}\right) \end{aligned} \tag{A.1}$$

Since we want only the leading contributions which are mass independent, helicity is a good quantum number. Then it is easy to convince oneself that the operators (A.1) are multiplicatively renormalized in leading order since they are uniquely characterized by the helicity $[q_L^C \equiv C(q_R)]$ and/or flavour of the external quarks, and there is only one colour singlet combination of dimension six. Furthermore, they are renormalized identically because gluon exchange is helicity and flavour independent.

We must evaluate, for example, the contributions of the diagrams of Fig. 2 to the matrix element

$$\begin{aligned} \langle O^\nu \rangle &\equiv \langle T\left(O^\nu (u^c_L)_{k,\gamma} (\bar{d}_L)_{j,\beta} (\nu^c_{e_R})_\delta (\bar{d}_R)_{i,\alpha}\right)\rangle \\ &= \epsilon_{ijk}\,(\gamma_\mu)_{\gamma\beta}(\gamma^\mu)_{\delta\alpha}\left[1+\frac{\alpha_s}{8\pi} d \ln\left(\frac{\mu^2}{-p^2}\right) + O(\alpha_s^2)\right] \end{aligned} \tag{A.2}$$

where μ^2 is the renormalization point and $p^2 < 0$ is the value of the external momenta with some suitable convention for fixing the ratios s/p^2, t/p^2. The constant d, related to the anomalous dimension of the operator by

$$\gamma_0 = \frac{\partial}{\partial\mu}\langle O^\nu\rangle\Big|_{-p^2=\mu^2} = \frac{\alpha_s}{4\pi} d \tag{A.3}$$

can be extracted from the coefficient of the logarithmically divergent term in the Feynman integrals of Fig. 2. Setting the external momenta to zero, and using the quark gluon coupling

$$\bar{q}\gamma^\mu \frac{\lambda^a}{2} q\, B^a_\mu = -\bar{q}^c \gamma^\mu \frac{\lambda^a}{2} q\, B^a_\mu \tag{A.4}$$

the integrals take the form:

$$\frac{d}{2}\epsilon_{ijk}(\gamma_\mu)_{\gamma\beta}(\gamma^\mu)_{\delta\alpha}\frac{-ig^2}{(2\pi)^4}\int\frac{d^4p}{p^4} \cdot \frac{\alpha_s}{4\pi}\frac{d}{2}\epsilon_{ijk}(\gamma_\mu)_{\gamma\beta}(\gamma^\mu)_{\delta\alpha}\ln\Lambda^2 \tag{A.5}$$

where Λ is the ultraviolet cut-off.

Using the relation

$$\epsilon_{i'j'k} \frac{\lambda^a_{i'i}}{2} \frac{\lambda^a_{j'j}}{2} = -\frac{2}{3} \epsilon_{ijk} \tag{A.6}$$

we obtain the factors

$$d_c^{(2a)} = -\frac{2}{3} \quad ; \quad d_c^{(2b)} = d_c^{(2c)} = +\frac{2}{3} \tag{A.7}$$

from contraction of colour SU(3) indices. Working in the t'Hooft-Feynman gauge, we find for Fig. 2a the Dirac algebra factor

$$d_D^{(3a)} (\gamma_\mu)_{\beta\gamma} (\gamma^\mu)_{\delta\alpha} = \frac{1}{p^2} (\gamma_\mu \not{p} \gamma_\nu)(\gamma^\mu (-\not{p}) \gamma^\nu)$$

giving

$$d_D^{(2a)} = -4 \tag{A.8a}$$

and in a similar way

$$d_D^{(2b)} = d_D^{(2c)} = +1 \tag{A.8b}$$

Putting together the results of (A.7) and (A.8):

$$\frac{d}{2} = \sum d_c^i d_D^i = 4 \tag{A.9}$$

The renormalization of the effective coupling constant is given by

$$G_{GV}^{eff} = A\, G_{GV} \quad : \quad A = \left[\frac{\alpha_s(\mu^2)}{\alpha_s(M_X^2)}\right]^{\frac{d-3d_\psi}{2b}} \tag{A.10}$$

where b and d are related to the usual γ and β functions by

$$\beta = -\frac{g^3}{16\pi^2} b \quad : \quad b = 11 - \frac{2}{3} f$$

$$\gamma = \frac{\alpha_s}{4\pi} d_\psi \quad : \quad d_\psi = \frac{4}{3}$$

in the 't Hooft-Feynman gauge. Putting these numbers together we obtain the result of Eq. (2.10). (We have also checked our result in the Landau gauge where the wave function renormalization γ vanishes.)

APPENDIX B

ONE-LOOP CORRECTIONS TO THE HIGGS POTENTIAL

We shall expand the Coleman-Weinberg potential

$$V'(\Phi, \underline{H}) \equiv V(\mathcal{M}^2) \tag{B.1}$$

defined in Eqs. (4.21) and (4.22) around the point

$$V'(\Phi_0, 0) \equiv V(\mathcal{M}_0^2) \tag{B.2}$$

where

$$\mathcal{M}^2_{0ab} = -\frac{g^2_{SUN}}{4} \mathrm{Tr}\left\{ [\lambda^a, \Phi_0][\lambda^b, \Phi_0] \right\} \tag{B.3}$$

and Φ_0 is the matrix defined in Eq. (4.4). The expansion is done by noting that the matrix (B.3) is a projection operator onto the X, Y subspace of the vector boson mass matrix

$$\mathcal{M}_0^2 = g^2 \frac{25}{8} v^2 P \equiv g^2 M^2 P \quad ; \quad P^2 = P \tag{B.4}$$

Defining the matrices

$$\Delta \equiv \mathcal{M}^2 - \mathcal{M}_0^2 \quad ; \quad \Delta_P = (1-P)\Delta(1-P) \tag{B.5}$$

we obtain the expression

$$\begin{aligned} V'(\Phi, \underline{H}) = \frac{3\alpha^2_{SUN}}{4} \Big\{ & 12 M^4 \ln M^2/M_0^2 + \mathrm{Tr}(\Delta_P^2 \ln \Delta_P/M_0^2) \\ & + (2\ln M^2/M_0^2 + 1)(M^2 \mathrm{Tr}(P\Delta) + \mathrm{Tr}(\Delta^2 P)) - (\ln M^2/M_0^2 - \tfrac{1}{2})\mathrm{Tr}((P\Delta)^2) \\ & + O(\Delta^3/M^2 \ln(\Delta_P/M^2)) \Big\} \end{aligned} \tag{B.6}$$

Expressed in terms of the Higgs fields:

$$\underline{H} \equiv \begin{pmatrix} \hat{\underline{H}} \\ \underset{\sim}{H}_2 \end{pmatrix} ; \quad \Phi - \Phi_0 = \left(\begin{array}{c|c|c} \hat{H}_8 + \frac{2H_0}{\sqrt{30}} I & \underset{\sim}{H}_X & \underset{\sim}{H}_Y \\ \hline \underset{\sim}{H}_X^+ & \frac{\hat{H}_2}{\sqrt{2}} - \frac{3}{\sqrt{30}} H_0 & H^+ \\ \hline \underset{\sim}{H}_Y^+ & H^- & -\frac{\hat{H}_2}{\sqrt{2}} - \frac{3}{\sqrt{30}} H_0 \end{array} \right) \tag{B.9}$$

the potential (B.6) takes the form:

$$
\begin{aligned}
V'(\Phi, \underline{H}) \simeq \frac{3\alpha^2_{GUM}}{4} \Big\{ & 12 M^4 \ln M^2/M_0^2 \; + \\
& + \left(2 \ln M^2/M_0^2 + 1\right)\Big[12 M^3 \sqrt{\tfrac{5}{3}}\, H_0 + 2|\hat{\underset{\sim}{H}}|^2 + 3|\underset{\sim}{H}_2|^2 + 10 M^2 (\underset{\sim}{H}_x^2 + \underset{\sim}{H}_y^2) - \frac{M}{\sqrt{2}} \hat{\underset{\sim}{H}}^+ (\underset{\sim}{H}_x \,\vdots\, \underset{\sim}{H}_y) \underset{\sim}{H}_z \\
& \qquad\qquad - \frac{M}{\sqrt{2}} \underset{\sim}{H}_2^+ \begin{pmatrix} \underset{\sim}{H}_x^+ \\ \overline{\underset{\sim}{H}_y^+} \end{pmatrix} \underset{\sim}{H}_3 \Big] \\
& + M^2 \left(6 \ln M^2/M_0^2 + 7\right)\Big[5 H_0^2 + 3(\hat{H}_z^2 + 2 H^+ H^-) + 2\,\mathrm{Tr}(\hat{H}_8^2)\Big] \\
& + \left(\ln M^2/M_0^2 + \tfrac{3}{2}\right)\Big[8\, \hat{\underset{\sim}{H}}^+ \frac{M \hat{H}_8}{\sqrt{2}} \hat{\underset{\sim}{H}} - 12\, \underset{\sim}{H}_2^+ M \begin{pmatrix} \hat{H}_z/\sqrt{2} & H^+ \\ H^- & -\hat{H}_z/\sqrt{2} \end{pmatrix} \underset{\sim}{H}_2 \\
& \qquad\qquad + M \sqrt{\tfrac{5}{3}}\, H_0 \left(2|\hat{\underset{\sim}{H}}|^2 + 3|\underset{\sim}{H}_2|^2\right) + |\hat{\underset{\sim}{H}}|^4 + \tfrac{3}{2} |\underset{\sim}{H}_2|^4 \Big] \\
& + \tfrac{2}{5}\left(9 + 13 \ln M^2/M_0^2\right) |\underset{\sim}{H}_2|^2 |\hat{\underset{\sim}{H}}|^2 + \lambda_+^2 \ln \lambda_+/M_0^2 + \lambda_-^2 \ln \lambda_-/M_0^2 \\
& \qquad\qquad + 2\lambda_2^2 \ln \lambda_2/M_0^2 + 4 \lambda_3^2 \ln \lambda_3/M_0^2 \Big\}
\end{aligned}
\tag{B.10}
$$

with

$$\lambda_\pm = \frac{2}{5}\left\{|\underset{\sim}{H}_2|^2 + |\hat{\underset{\sim}{H}}|^2 \pm \left[(|\underset{\sim}{H}_2|^2 + |\hat{\underset{\sim}{H}}|^2)^2 - \frac{15}{4}|\underset{\sim}{H}_2|^2 |\hat{\underset{\sim}{H}}|^2\right]^{\frac{1}{2}}\right\} ; \quad \lambda_2 = \frac{1}{2}|\underset{\sim}{H}_2|^2 ; \quad \lambda_3 = \frac{1}{2}|\hat{\underset{\sim}{H}}|^2$$

Assuming a symmetry breaking of the form (4.10) and (4.4′), i.e. setting

$$\left.\begin{aligned} & \hat{\underset{\sim}{H}} = \underset{\sim}{H} \;, \quad \hat{H}_8 = H_8 \;, \quad H_2 = \begin{pmatrix} H_4 \\ \left(\frac{\rho + v_0}{\sqrt{2}}\right) e^{i\zeta/v_0} \end{pmatrix} \\ & \hat{H}_z = H_z - \frac{\epsilon}{\sqrt{2}} v \end{aligned}\right\} \tag{B.11}$$

it is straightforward to extract the linear and quadratic terms in (B.11). However, since we are seeking a solution with $m_H^2 \gg m_{W,Z}^2$, it is not *a priori* obvious that terms arising from the Higgs boson loops are negligible. To avoid this prolem, we consider two prototype models with

$$\beta = b = 0 \tag{B.12}$$

and

$$\text{a)} \qquad \alpha = 0 \tag{B.13a}$$

$$\text{b)} \qquad \alpha = a = \lambda/4 \tag{B.13b}$$

Case (a) corresponds to a potential which is SU(24) ⊗ SU(10) symmetric and case (b) to SU(34). For an SU(N) symmetric coupling

$$\frac{\lambda}{4} S^2 \qquad : \qquad S = \sum_{i=1}^{N} \phi_i^2 \tag{B.14}$$

the one-loop contribution to the potential takes the form:

$$\frac{\lambda^2 s^2}{64\pi^2}\left[(N+8) \ln s/M_0^2 \;+\; O(\ln \lambda) \right] \tag{B.15}$$

Then (B.15) is a small correction to (B.14) as long as

$$\frac{\lambda}{16\pi^2} (N+8) \ln s/M_0^2 \quad \ll 1 \tag{B.16}$$

Taking the potential as defined by (B.10), (B.12), and (B.13), we obtain the constraints of Eqs. (4.15′), (4.5″), and (4.11″).

As long as the quantity in Eq. (4.23) is $\gg v_0^2/v^2$, the SU(3) triplet which does not get eaten is approximately the five-plet one, as in the zero-loop case, with mass

$$m^2_{H_3} = \frac{v^2}{10}\left[\frac{15a}{2} - \frac{\mu^2}{v^2} \right] + O\left(v_0^2 \ln v_0^2 \right) \tag{B.17}$$

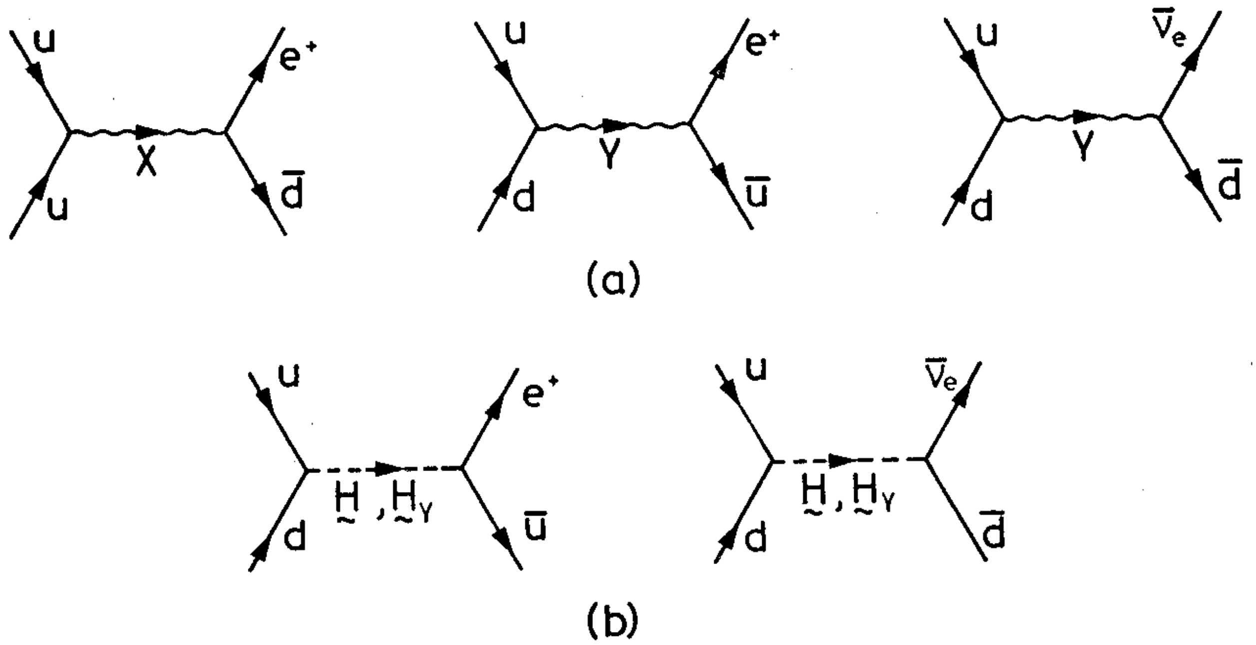
The masses of the physical 24-plet Higgs are approximately:

$$\begin{aligned} m^2_{H_8} &\approx \frac{75}{2}\alpha^2_{GUM} v^2 - \frac{v^2}{5}\left[\frac{15}{2}a - \frac{\mu^2}{v^2} \right] \\ m^2_{H^\pm, H_Z} &\approx \frac{225}{8}\alpha^2_{GUM} v^2 - \frac{4v^2}{5}\left[\frac{15}{2}a - \frac{\mu^2}{v^2} \right] \\ m^2_{H_0} &\approx 2\mu^2 + \frac{375}{4}\alpha^2_{GUM} v^2 \end{aligned} \tag{B.18}$$

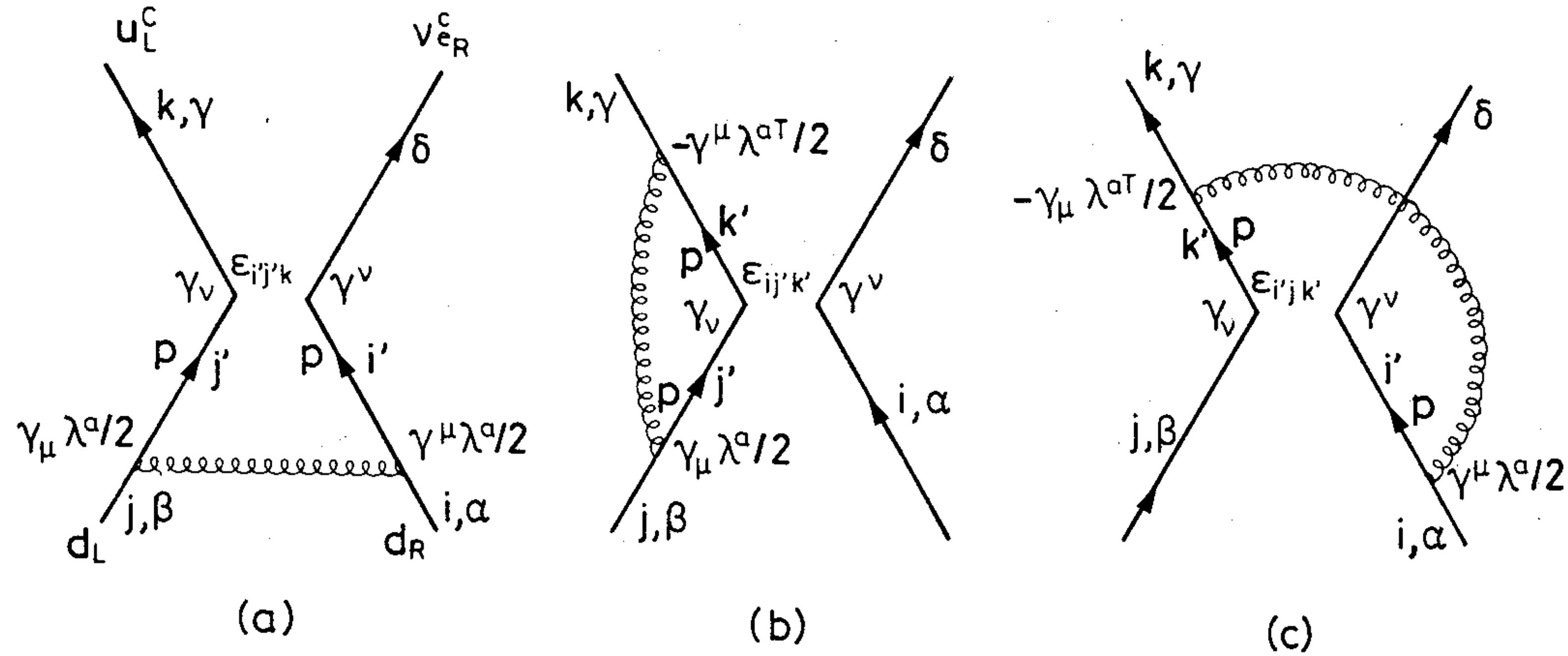
Positivity of the masses gives the constraint (4.23). The absence of a lower bound for the masses of H_8, $H^\pm$, H_Z reflects the fact that in our simple model they become massless goldstone bosons in the tree approximation. From (4.5″) and (4.23) we find that the condition (B.16) is satisfied if $a \ll \pi^2/4$.

We cannot show for a given choice of parameters that in the presence of radiative corrections to the Higgs potential the solution which we assume corresponds to the lowest vacuum. Imposing (4.11″), while at the same time keeping (B.17) of order v^2 means that if with the same set of parameters we looked for a solution, e.g. $\langle H \rangle = 0$, $\langle \hat{H} \rangle \neq 0$, we would find $\langle \hat{H} \rangle = O(v^2)$ and our perturbation expansion of V' would not be valid.

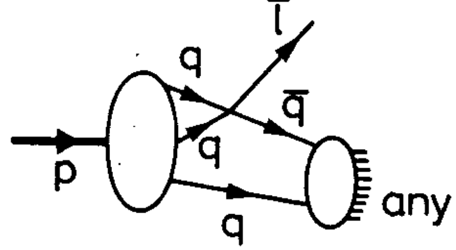
Finally, it is conceivable, although it appears unlikely, that a potential with say $\mu = a = 0$ could account more "naturally" for $v_0^2/v^2 \sim 10^{-30}$ and not destroy proton stability if the $\underset{\sim}{H}$, $\underset{\sim}{H}_Y$ mass matrix (which in this case has elements of *a priori* equal magnitude) chose a physical Higgs which is mostly $\underset{\sim}{H}_Y$. We cannot answer this question without going to higher order terms in the expansion (B.6).



- Figure 1 -



- Figure 2 -



- Figure 3 -