A simple existence criterion for normal spanning trees in infinite graphs

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Halin proved in 1978 that there exists a normal spanning tree in every connected graph G that satisfies the following two conditions: (i) G contains no subdivision of a 'fat' $K_{\aleph_0}$, one in which every edge has been replaced by uncountably many parallel edges; and (ii) G has no $K_{\aleph_0}$ subgraph. We show that the second condition is unnecessary.

Introduction

A spanning tree of an infinite graph is *normal* if the endvertices of any chord are comparable in the tree order defined by some arbitrarily chosen root. (In finite graphs, these are their 'depth-first search' trees; see [3] for precise definitions.) Normal spanning trees are perhaps the most important single structural tool for analysing an infinite graph (see [4] for a good example), but they do not always exist. The question of which graphs have normal spanning trees thus is an important question.

All countable connected graphs have normal spanning trees [3]. But not all connected graphs do. For example, if T is a normal spanning tree of G and G is complete, then T defines a chain on its vertex set. Hence T must be a single path or ray, and G is countable.

For connected graphs of arbitrary order, there are three characterizations of the graphs that admit a normal spanning tree:

Theorem 1. *The following statements are equivalent for connected graphs G.*

(i) *G has a normal spanning tree*;

(ii) *$V(G)$ is a countable union of dispersed sets* (*Jung* [7, 2]);

(iii) *$|G|$ is metrizable* [1];

(iv) *G contains neither an $(\aleph_0, \aleph_1)$-graph nor an Aronszajn-tree graph as a minor* [5].

Here, a set of vertices in G is *dispersed* if every ray can be separated from it by some finite set of vertices. (The levels of a normal spanning tree are dispersed; see [3].) The dispersed vertex sets in a graph G are precisely those that are closed in the topological space $|G|$ of (iii), which consists of G and its ends [1]. The space $|G|$ will not concern us in this note, so we refer to [1] for the definition of the topology on $|G|$. But we shall use the equivalence of (i) and (iv) in our proof, and the forbidden minors mentioned in (iv) will be defined in Section 2.

Despite the variety in Theorem 1, it can still be hard in practice to decide whether a given graph has a normal spanning tree.[1] In most applications, none of these characterizations is used, but a simpler sufficient condition due to Halin. This condition, however, is much stronger, and hence does not always hold even if a normal spanning tree exists. It is the purpose of this note to show that this condition can be considerably weakened.

1. The result

Halin's [6] most-used sufficient condition for the existence of a normal spanning tree in a connected graph is that it does not contain a $TK_{\aleph_0}$. This is usually easier to check than the conditions in Theorem 1, but it is also quite a strong assumption. However, Halin [6] also proved that this assumption can be replaced by the conjunction of two independent much weaker assumptions:

- G contains no *fat* $TK_{\aleph_0}$: a subdivision of the multigraph obtained from a $K_{\aleph_0}$ by replacing every edge with $\aleph_1$ parallel edges;
- G contains no $K_{\aleph_0}$ (as a subgraph).

We shall prove that the second condition is unnecessary:

Theorem 2. *Every connected graph not containing a fat $TK_{\aleph_0}$ has a normal spanning tree.*

We remark that all the graphs we consider are simple, including our fat $TK_{\aleph_0}$s. When we say, without specifying any graph relation, that a graph G *contains* another graph H, we mean that H is isomorphic to a subgraph of G. Any other undefined terms can be found in [3].

2. The proof

Our proof of Theorem 2 will be based on the equivalence (i)↔(iv) in Theorem 1, so let us recall from [5] the terms involved here.

An *Aronszajn tree* is a poset $(T, \leqslant)$ with the following properties:[2]

- T that has a least element, its *root*;
- the down-closure of every point in T is well-ordered;
- T is uncountable, but all chains and all levels in T are all countable.

Here, the *down-closure* $\lceil t \rceil$ of a point $t \in T$ is the set $\{ x \mid x \leqslant t \}$; its *up-closure* is the set $\lfloor t \rfloor := \{ y \mid t \leqslant y \}$. More generally, if $x < y$ we say that x lies *below*

[1] In particular, the two types of graph mentioned in (iv) are not completely understood; see [5] for the – quite intriguing – problem of how to properly understand (or meaningfully classify) the $(\aleph_0, \aleph_1)$-graphs.

[2] Unlike the perhaps better known Suslin trees – Aronszajn trees in which even every antichain must be countable – Aronszajn trees can be shown to exist without any set-theoretic assumptions in addition to ZFC.

y and y *above* x. The *height* of a point $t \in T$ is the order type of the chain $\lceil t \rceil \setminus \{t\}$, and the *levels* of T are its maximal subsets of points of equal height.

An *Aronszajn-tree graph* or *AT-graph*, is a graph G on whose vertex set there exists an Aronszajn tree T such that

- the endvertices of every edge of G are comparable in T;
- for all $x < y$, the vertex y has a neighbour x' such that $x \leqslant x' < y$.

The second condition says that each vertex is joined cofinally to the vertices below it. The idea behind this is that if we were to construct any order tree T on $V(G)$ satifying the first condition, a tree satisfying also the second condition would be one that minimizes the level of each vertex.

Note that *intervals* in T, sets of the form $\{ t \mid x \leqslant t < y \}$ for some given points $x < y$, span connected subgraphs in G. This is because every $t > x$ has a neighbour t' with $x \leqslant t' < t$, by the second condition, and hence the interval contains for each of its elements t the vertices of a t–x path in G. Similarly, G itself is connected, because every vertex can be linked to the unique root of T.

An $(\aleph_0, \aleph_1)$-*graph with bipartition* (A, B) is a biparte graph with vertex classes A of size $\aleph_0$ and B of size $\aleph_1$ such that every vertex in B has infinite degree.

Replacing the vertices x of a graph X with disjoint connected graphs H_x, and the edges xy of X with non-empty sets of H_x–H_y edges, yields a graph that we shall call an IX (for 'inflated X'). More formally, a graph H *is an* IX if its vertex set admits a partition $\{ V_x \mid x \in V(X) \}$ into connected subsets V_x such that distinct vertices $x, y \in X$ are adjacent in X if and only if H contains a V_x–V_y edge. The sets V_x are the *branch sets of the* IX. Thus, X arises from H by contracting the subgraphs H_x, without deleting any vertices or edges (other than loops or parallel edges arising in the contraction). A graph X is a *minor* of a graph G if G contains an IX as a subgraph. See [3] for more details.

For our proof of Theorem 2 from Theorem 1 (i)↔(iv) it suffices to show the following:

Every IX, where X is either an $(\aleph_0, \aleph_1)$-graph or an AT-graph, contains a fat $TK_{\aleph_0}$ (as a subgraph). $(*)$

The rest of this section is devoted to the proof of $(*)$.

Lemma 3. *Let X be an $(\aleph_0, \aleph_1)$-graph, with bipartition (A, B) say.*

(i) *X has an $(\aleph_0, \aleph_1)$-subgraph X' with bipartition into $A' \subseteq A$ and $B' \subseteq B$ such that every vertex in A' has uncountable degree in X'.*

(ii) *For every finite set $F \subseteq A$ and every uncountable set $U \subseteq B$, there exists a vertex $a \in A \setminus F$ that has uncountably many neighbours in U.*

Proof. (i) Delete from X all the vertices in A that have only countable degree, together with their neighbours in B. Since this removes only countably many vertices from B, the remaining set $B' \subseteq B$ will still be uncountable. By the degree condition that X imposes on the vertices in B', the remaining set $A' \subseteq A$ is still infinite, and every vertex in A' still has uncountably many neighbours in B'. Thus, X' is an $(\aleph_0, \aleph_1)$-subgraph of X as required.

(ii) If there is no vertex $a \in A \setminus F$ as claimed, then each vertex $a \in A \setminus F$ has only countably many neighbours in U. As $A \setminus F$ is countable, this means that $U \setminus N(A \setminus F) \neq \emptyset$. But every vertex in this set has all its neighbours in F, and thus has finite degree. This contradicts our assumption that X is an $(\aleph_0, \aleph_1)$-graph. □

Lemma 4. *Let X be an $(\aleph_0, \aleph_1)$-graph with bipartition (A, B). Let $A' \subseteq A$ be infinite and such that for every two vertices a, a' in A' there is some uncountable set $B(a, a')$ of common neighbours of a and a' in B. Then A' is the set of branch vertices of a fat $TK_{\aleph_0}$ in X whose subdivided edges all have the form aba' with $b \in B(a, a')$.*

Proof. We have to find a total of $\aleph_0^2 \cdot \aleph_1 = \aleph_1$ independent paths in X between vertices in A'. Let us enumerate these desired paths as $(P_\alpha)_{\alpha<\omega_1}$; it is then easy to find them recursively on α, keeping them independent. □

Lemma 5. *Every IX, where X is an $(\aleph_0, \aleph_1)$-graph, contains a fat $TK_{\aleph_0}$.*

Proof. Let H be an IX for an $(\aleph_0, \aleph_1)$-graph X with bipartition (A, B), with branch sets V_x for vertices $x \in X$. Replacing X with an appropriate $(\aleph_0, \aleph_1)$-subgraph Y (and H with the corresponding $IY \subseteq H$) if necessary, we may assume by Lemma 3 (i) that every vertex in A has uncountable degree in X. We shall find our desired $TK_{\aleph_0}$ in two steps.

In a first step, we shall replace the given IX with an $IX' =: H' \subseteq H$ such that $X' \subseteq X$ is still an $(\aleph_0, \aleph_1)$-graph, with a bipartition into sets $A' \subseteq A$ and $B' \subseteq B$. For every $a \in A'$, its branch set V'_a in the new IX' will be a subset of its branch set V_a in the old IX, and V'_a will span a subdivided star $S(a)$ in H'. The vertices $b \in B'$ will have the same branch sets V_b for X' as they did for X.

In the second step we shall apply Lemma 4 to find a fat $TK_{\aleph_0}$ in X' with branch vertices in A'. This $TK_{\aleph_0}$ can then be turned into the desired fat $TK_{\aleph_0}$ in H by replacing its branch vertices $a \in A'$ with the centres of the stars $S(a)$, and its subdivided edges (which are paths P in X' of length 2) by paths in H through the branch sets of the three vertices of P.

For the first step, we have to construct an infinite subset A' of A. Let us pick its vertices $a_0, a_1, \dots$ inductively, as follows. Pick $a_0 \in A$ arbitrarily. For each of the uncountably many neighbours b of a_0 in B we can find a vertex $v_b \in V_b$ that sends an edge of H to V_{a_0}. For every b, pick one neighbour u_b of v_b in V_{a_0}. Consider a minimal connected subgraph H_0 of $H[V_{a_0}]$ containing all these vertices u_b, and add to it all the edges $u_b v_b$ to obtain the graph $T = T(a_0)$. By the minimality of H_0,

T is a tree in which every edge lies on a path between two vertices of the form v_b. (1)

Since there are uncountably many b and their v_b are distinct, T is uncountable and hence has a vertex s_0 of uncountable degree. For every edge e of T at s_0 pick a path in T from s_0 through e to some v_b; this is possible by (1). Let $S(a_0)$ be the union of all these paths. Then $S(a_0)$ is an uncountable subdivided star with centre s_0 all whose non-leaves lie in V_{a_0} and whose leaves lie in the branch sets V_b of distinct vertices $b \in B$. Let $B_0 \subseteq B$ be the (uncountable) set of these b, and rename the vertices v_b with $b \in B_0$ as v_b^0.

Assume now that, for some $n \geqslant 1$, we have picked distinct vertices $a_0, \ldots, a_{n-1}$ from A and defined uncountable subsets $B_0 \supseteq \ldots \supseteq B_{n-1}$ of B so that each a_i is adjacent in X to every vertex in B_i. By Lemma 3 (ii) there exists an $a_n \in A \smallsetminus \{a_0, \ldots, a_{n-1}\}$ which, in X, has uncountably many neighbours in B_{n-1}. As before, we can find an uncountable subdivided star $S(a_n)$ in H whose centre s_n and any other non-leaves lie in V_{a_n} and whose leaves v_b^n lie in the branch sets V_b of (uncountably many) distinct vertices $b \in B_{n-1}$. We let B_n be the set of those b. Then B_n is an uncountable subset of B_{n-1}, and a_n is adjacent in X to all the vertices in B_n, as required for n by our recursion.

By construction, every two vertices a_i, a_j in $A' := \{a_0, a_1, \ldots\}$ have uncountably many common neighbours in B: those in B_j if $i < j$. By Lemma 4 applied with $B(a_i, a_j) := B_j$ for $i < j$, we deduce that A' is the set of branch vertices of a fat $TK_{\aleph_0}$ in X whose subdivided edges $a_i \ldots a_j$ with $i < j$ have the form $a_i b a_j$ with $b \in B_j$. Replacing each of these paths $a_i b a_j$ with the concatenation of paths $s_i \ldots v_b^i \subseteq S(a_i)$ and $v_b^i \ldots v_b^j \subseteq H[V_b]$ and $v_b^j \ldots s_j \subseteq S(a_j)$, we obtain a fat $TK_{\aleph_0}$ in H with $s_0, s_1, \ldots$ as branch vertices. (It is important here that b is not just any common neighbour of a_i and a_j but one in B_j: only then do we know that $S(a_i)$ and $S(a_j)$ both have a leaf in V_b.) □

Let us now turn to the case of $(*)$ where X is an AT-graph. As before, we shall first prove that X itself contains a fat $TK_{\aleph_0}$, and later refine this to a fat $TK_{\aleph_0}$ in any IX. In this second step we shall be referring to the details of the proof of the lemma below, not just to the lemma itself.

Lemma 6. *Every AT-graph contains a fat $TK_{\aleph_0}$.*

Proof. Let X be an AT-graph, with Aronszajn tree T, say. Let us pick the branch vertices $a_0, a_1, \ldots$ of our desired $TK_{\aleph_0}$ inductively, as follows.

Let t_0 be the root of $T_0 := T$, and $X_0 := X$. Since X_0 is connected, it has a vertex a_0 of uncountable degree. Uncountably many of its neighbours lie above it in T_0, because its down-closure is a chain and hence countable, and all its neighbours are comparable with it (by definition of an AT-graph). As levels in T_0 are countable, a_0 has a successor t_1 in T_0 such that uncountably many X_0-neighbours of a_0 lie above t_1; let B_0 be some uncountable set of neighbours of a_0 in $\lfloor t_1 \rfloor_{T_0}$. (We shall specify B_0 more precisely later.)

Let T_1 be the down-closure of B_0 in $\lfloor t_1 \rfloor_{T_0}$. Since T_1 is an uncountable subposet of T_0 with least element t_1, it is again an Aronszaijn tree, and the subgraph X_1 it induces in X_0 is an AT-graph with respect to T_1.

Starting with t_0, T_0 and X_0 as above, we may in this way select for $n = 0, 1, \dots$ an infinite sequence $T_0 \supseteq T_1 \supseteq \dots$ of Aronszajn subtrees of T with roots $t_0 < t_1 < \dots$ satisfying the following:

- $X_n := X[T_n]$ is an AT-graph with respect to T_n;
- the predecessor a_n of t_{n+1} in T_n has an uncountable set B_n of X_n-neighbours above t_{n+1} in T_n;
- $T_{n+1} = \lfloor t_{n+1} \rfloor_{T_n} \cap \lceil B_n \rceil_{T_n}$.

By the last item above, there exists for every $b \in T_{n+1}$ a vertex $b' \in B_n \cap \lfloor b \rfloor$ (possibly $b' = b$). Applied to vertices b in $B_{n+1} \subseteq T_{n+1}$ this means that, inductively,

Whenever $i < j$, every vertex in B_j has some vertex of B_i in its up-closure. (2)

Let us now make $a_0, a_1, \dots$ into the branch vertices of a fat $TK_{\aleph_0}$ in X. As earlier, we enumerate the desired subdivided edges as one ω_1-sequence, and find independent paths $P_\alpha \subseteq X$ to serve as these subdivided edges recursively for all $\alpha < \omega_1$. When we come to construct the path P_α, beween a_i and a_j with $i < j$ say, we have previously constructed only the countably many paths P_β with $\beta < \alpha$. The down-closure D_α in T of all their vertices and all the a_n is a countable set, since the down-closure of each vertex is a chain in T and hence countable. We can thus find a vertex $b \in B_j$ outside D_α, and a vertex $b' \geqslant b$ in B_i by (2). The interval of T between b and b' thus avoids D_α, and since it is connected in X it contains the vertices of a b'–b path Q_α in $X - D_\alpha$. We choose $P_\alpha := a_i b' Q_\alpha b a_j$ as the αth subdivided edge for our fat $TK_{\aleph_0}$ in X. □

Lemma 7. *Every IX, where X is an AT-graph, contains a fat $TK_{\aleph_0}$.*

Proof. Let H be an IX with branch sets V_x for vertices $x \in X$, where X is an AT-graph with respect to an Aronszajn tree T. Rather than applying Lemma 6 to X formally, let us re-do its proof for X. We shall choose the sets B_n more carefully this time, so that we can turn the fat $TK_{\aleph_0}$ found in X into one in H.

Given n, the set B_n chosen in the proof of Lemma 6 was an arbitrary uncountable set of upper neighbours of a_n in T_n above some fixed successor t_n of a_n. We shall replace B_n with a subset of itself, chosen as follows. For every $b \in B_n$, pick a vertex $v_b^n \in V_b$ that sends an edge of H to a vertex $u_b^n \in V_{a_n}$. As in the proof of Lemma 5, there is a subdivided uncountable star S_n in H whose leaves are among these v_b^n and all whose non-leaves, including its centre s_n, lie in V_{a_n}. Let us replace B_n with its (uncountable) subset consisting of only those b whose v_b^n is a leaf of S_n.

Let $K \subseteq X$ be the fat $TK_{\aleph_0}$ found by the proof of Lemma 6 for these revised sets B_n. In order to turn K into the desired $TK_{\aleph_0}$ in H, we replace its branch vertices a_n by the centres s_n of the stars S_n, and its subdivided edges $P_\alpha = a_i b' Q_\alpha b a_j$ between branch vertices a_i, a_j by the concatenation of paths $s_i \dots v^i_{b'} \subseteq S_i$ and $Q'_\alpha = v^i_{b'} \dots v^j_b$ and $v^j_b \dots s_j \subseteq S_j$, where Q'_α is a path in H expanded from Q_α, i.e. whose vertices lie in the branch sets of the vertices of Q_α. These paths P'_α are internally disjoint for distinct α, because the P_α were internally disjoint. □

Proof of Theorem 2. Let G be a connected graph without a normal spanning tree; we show that G contains a fat $TK_{\aleph_0}$. By Theorem 1, G has an X-minor such that X is either an $(\aleph_0, \aleph_1)$-graph or an Aronszajn-tree graph. Equivalently, G has a subgraph H that is an IX, with X as above. By Lemmas 5 and 7, this subgraph H, and hence G, contains a fat $TK_{\aleph_0}$. □

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