

DEUTSCHES ELEKTRONEN-SYNCHROTRON **DESY**

DESY 83-052
June 1983



DIELECTRIC LATTICE GAUGE THEORY

by

Gerhard Mack

II. Institut für Theoretische Physik der Universität Hamburg

ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY behält sich alle Rechte für den Fall der Schutzrechtserteilung und für die wirtschaftliche Verwertung der in diesem Bericht enthaltenen Informationen vor.

DESY reserves all rights for commercial use of information included in this report, especially in case of filing application for or grant of patents.

To be sure that your preprints are promptly included in the
HIGH ENERGY PHYSICS INDEX ,
send them to the following address (if possible by air mail) :

**DESY
Bibliothek
Notkestrasse 85
2 Hamburg 52
Germany**

Introduction

The dielectric theory of quark confinement [1] is very appealing because it offers a physical picture which is easy to understand and predictive to some extent (Figure 1). It is therefore a challenging task to derive this theory from Quantum Chromodynamics (QCD) and to develop it into a full theory from which one can compute "everything". This should also shed light on the dynamics of gauge theories in general. Some work in this direction was begun by Nielsen and Patkos [2].

In (classical) electrodynamics of polarizable media, the dielectric constant embodies sufficient information about the polarizable medium if attention is restricted to static, spatially homogeneous situations. In an effective dielectric theory based on QCD one restricts attention to long distance properties only. The role of the "polarizable medium" is played by the high frequency parts of the gauge fields (and possibly of the matter fields, when they are treated as dynamical fields), and information about it is embodied in a slowly varying dielectric field $\epsilon(x)$, or related fields $\sigma(x)$. One seeks to describe the long distance behavior of QCD by an effective (Euclidean) action L which incorporates an Ultraviolet (UV) cutoff M and depends on the variables $\sigma(x)$ in addition to the gauge fields (and matter fields) that were present in the original QCD action.

It is expected that the effective action L will have a unique maximum at $\epsilon = 0$ (i.e. $\sigma(x) \equiv 0$) when the UV-cutoff M is low enough (of the order of the ultimate physical mass scale). This is the basic hypothesis of the dielectric theory of confinement, and confinement is supposed to follow it (see Figure 1).

* This is in contrast with Adler's approach where the dielectric field is a function of the electromagnetic field strength [2].

Dielectric Lattice Gauge Theory *

Gerhard Mack

II. Institut für Theoretische Physik der Universität Hamburg

Abstract: Dielectric lattice gauge theory models are introduced. They involve variables $\phi(b) \in \mathcal{G}$ that are attached to the links $b = (x+\mu, x)$ of the lattice and take their values in the linear space \mathcal{G} which consists of real linear combinations of matrices in the gauge group G . The polar decomposition $\phi(b) = U(b)\sigma_\mu(b)$ specifies an ordinary lattice gauge field $U(b)$ and a kind of dielectric field $\epsilon_{ij} \propto \sigma_i \sigma_j^* \delta_{ij}$. A gauge invariant positive semidefinite kinetic term for the ϕ -field is found, and it is shown how to incorporate Wilson fermions in a way which preserves Osterwalder Schrader positivity. Theories with $G = SU(2)$ and without matter fields are studied in some detail. It is proved that confinement holds, in the sense that Wilson loop expectation values show an area law decay, if the Euclidean action has certain qualitative features which imply that $\phi = 0$ (i.e. dielectric field $\equiv 0$) is the unique maximum of the action.

* Work supported in part by Deutsche Forschungsgemeinschaft.

Following Nielsen and Patkos one may imagine that block spins in a (pure) gauge theory are defined as superpositions of parallel transporters $U(\omega)$ along paths with fixed end points

$$\phi(y, x) = \sum_{\omega: x \rightarrow y} \rho(\omega) U(\omega) \quad (1.1)$$

Here, x and y are centers of neighbouring block cells of side length M^{-1} , $\omega: x \rightarrow y$ are paths from x to y on the original lattice (or continuum) and $\rho(\omega)$ is a non-negative real weight function.

For instance one might choose

$$\phi(y, x) = \frac{\partial^r}{\partial \mu^r} (-\Delta_U + \mu^2)^{-1} (y, x), \quad \mu = O(M), \quad r = d-2 \quad (1.2)$$

in d dimensions. Δ_U is the covariant Laplacian in the fundamental representation. Eq. (1.1) is obtained by random walk expansion of the propagator in (1.2). On a lattice it gives

$$\rho(\omega) = a^{2(|\omega|)} (2d + \mu^2 a^2)^{-|\omega|-1} \quad a = \text{lattice spacing}$$

if the path ω consists of $|\omega|$ links $\phi_1 \dots \phi_{|\omega|}$ of the original lattice. The parallel transporter is $U(\omega) = U(\phi_{|\omega|}) \dots U(\phi_1)$. Such random walk representations were studied by Fröhlich and Durhuus, Brydges and Spencer, and Aizenmann, following the pioneering work of Symanzik [4,5,6]. Similar expansions will be used extensively in the present paper.

Evidently, ϕ will take their values in the linear space \mathcal{U} which consists of real linear combinations of matrices in the gauge group G . Following Drouffe [7], one may write down a polar decomposition of ϕ (b), this defines the variables $\sigma_\mu(x)$ and an ordinary lattice gauge field on the block lattice.

In the present paper I introduce models of effective actions $L(\phi)$. This seems worthwhile since it will permit to study the dielectric confinement mechanism in

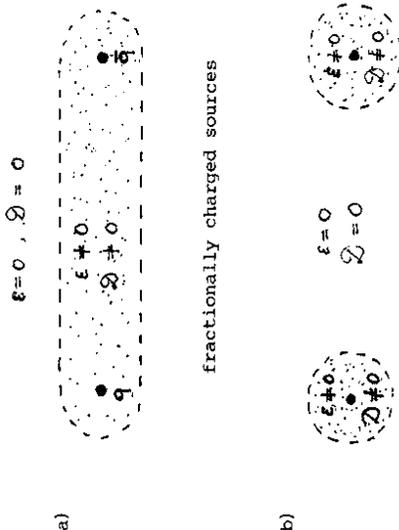


Figure 1 The dielectric theory of confinement [1]. Coloured sources are the source of an electric induction \mathcal{Q} which is related to the colour electric field \vec{E} by a dielectric constant (or rather field) ϵ , $\mathcal{Q} = \epsilon \vec{E}$. Therefore \mathcal{Q} can only be nonzero where $\epsilon \neq 0$. If $\epsilon = 0$ is the unique classical vacuum then $\epsilon \neq 0$ costs energy and this prevents the \mathcal{Q} -field from spreading. If the sources transform nontrivially under the center Γ of the gauge group then Gauss law for the abelian group Γ forces the \mathcal{Q} -field to extend to infinity or to another source, because the gauge field carries no Γ -charge itself [3]. As a result, a string will form whose energy is proportional to its length (Fig. 1a). If the sources transform trivially under the center of the gauge group, the string can break and the source gets screened by the gauge field (Fig. 1b).

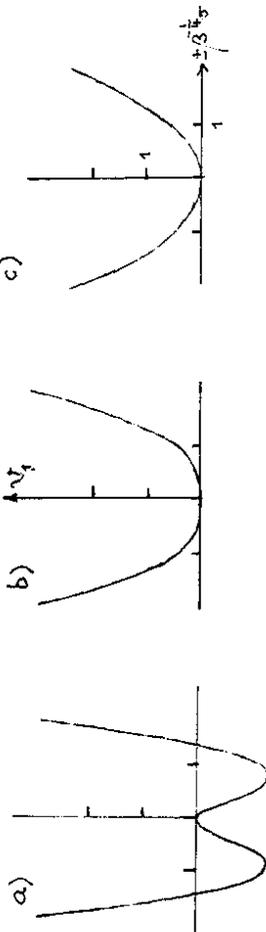


Figure 2 Potential V_1 . The drawing shows expression (2.61) for $G = SU(2)$. a) $\lambda = .1 < \lambda_c$ b) $\lambda = .25 = \lambda_c$ c) $\lambda = .5 > \lambda_c$

some detail. The models are local. A gauge invariant positive semidefinite kinetic term of the ϕ -field is found. As a byproduct, a "natural" definition of field strengths in lattice gauge theory emerges. It is shown how to incorporate dynamical Wilson fermions in a way which preserves Osterwalder Schrader positivity. Theories with $G = SU(2)$ and without matter fields are studied in some detail. It is proved that confinement holds, in the sense that Wilson loop expectation values show an area law decay, if the action $L(\phi)$ has certain qualitative features which imply that $\phi = 0$ is the unique maximum of the action (theorem 1).

The models are described in section 2. The Wilson loop criterion for confinement [8] is also discussed there, theorem 1 is stated, and some qualitative expectations concerning the dependence of effective actions on the cutoff are formulated. Gauss' law and the Hamiltonian limit are discussed in section 3. Section 4 spells out the dependence of the action on variables attached to links of a particular direction. In section 5 random walk representations of generalized covariant propagators are derived and estimated. In section 6, theorem 1 is proven, using the material of sections 4,5. Section 7 contains the proof of Osterwalder Schrader positivity in the presence of fermions [9]. It is self contained and does not use the material of sections 3 - 6.

2. The models

The models live on hypercubic lattices Λ in d dimensions with sites x , links b , plaquettes p , etc. Unless indicated otherwise, the lattice spacing in all directions is the same and shall be set equal to 1. e_μ is the lattice vector in μ -direction, $e_{-\mu} = -e_\mu$. If $b = (x + \epsilon_\mu, x)$ then $-b = (x, x + \epsilon_\mu)$ is the same link with reversed direction. We shall use a symmetric summation convention $v_\mu v_\mu \equiv \frac{1}{2} \sum_{\mu=1 \dots d} v_\mu v_\mu$.

The models shall possess local gauge invariance under a gauge group G , for instance $G = SU(N)$ or $U(N)$.

Let \mathcal{Q} be the linear space of all matrices ϕ that admit a representation of the form $\phi = \sum_i r_i U_i$, with $U_i \in G$, r_i real. According to Drouffe [7], \mathcal{Q} consists of all complex $N \times N$ matrices if $G = SU(N)$ or $U(N)$ with $N \geq 3$, while it consists of all real multiples of elements of G if $G = SU(2)$ or $U(2)$.

The variables $\phi(b) \equiv \phi_\mu(x) \in \mathcal{Q}$ of dielectric lattice gauge theory are attached to the links $b = (x + \epsilon_\mu, x)$ of the lattice. They take their values in the linear space \mathcal{Q} , and

$$\phi(-b) = \phi(b)^* \quad , \quad \text{i.e.} \quad \phi_\mu(x)^* = \phi_{-\mu}(x + \epsilon_\mu) \quad (2.0)$$

They will be integrated over using Lebesgue measure $d\phi$ on \mathcal{Q} . They transform under gauge transformations in the standard way

$$\phi_\mu(x) \rightarrow V(x + \epsilon_\mu) \phi_\mu(x) V(x)^{-1} \quad , \quad V(z) \in G \quad (2.1a)$$

They admit a polar decomposition [7]

$$\phi_\mu(x) = U(x + \epsilon_\mu, x) \sigma_\mu(x) \quad \text{with} \quad U(b) \in G \quad (2.2a)$$

and

$$\begin{aligned} \sigma_\mu(x) &\geq 0, \text{ real, if } G = SU(2) \text{ or } U(2) \\ \sigma_\mu(x) &= \text{a positive hermitean } N \times N \text{ matrix for } U(N), N \geq 3 \quad (2.2b) \\ \sigma_\mu(x) &= e^{i\theta} \cdot \text{(positive hermitean } N \times N \text{ matrix) for } SU(N), N \geq 3 \end{aligned}$$

The lattice gauge field $U(b)$ transforms under gauge transformations in the same way (2.1a) as $\phi(b)$. $\sigma_\mu(x)$ is gauge invariant if $G = SU(2)$ or $U(2)$, and transforms covariantly according to

$$\sigma_\mu(x) \rightarrow V(x) \sigma_\mu(x) V(x)^{-1} \quad (2.1b)$$

in general. It is convenient to define generalized parallel transporters. If the path C consists of links $b_1 \dots b_n$ we set

$$\phi(C) = \phi(b_n) \dots \phi(b_1) \quad (2.4)$$

Note that $\phi(-b) \neq \phi(b)^{-1}$ in general; therefore the parallel transporter along a path $-C \circ C$ which runs forth and back is not unity, and spikes in a path C affect $\phi(C)$.

Given ϕ , we define a kind of covariant derivative \mathcal{D}_μ which acts on \mathcal{Q} -valued functions on links according to

$$\begin{aligned} \mathcal{D}_\mu \psi_\nu(x) &= \psi_\nu(x + \epsilon_\mu) \phi_\mu(x) - \phi_\mu(x + \epsilon_\nu) \psi_\nu(x) \\ &\equiv \psi(x + \epsilon_\mu + \epsilon_\nu, x + \epsilon_\mu) \phi(x + \epsilon_\mu, x) - \phi(x + \epsilon_\nu + \epsilon_\mu, x + \epsilon_\nu) \psi(x + \epsilon_\nu, x) \end{aligned} \quad (2.5)$$

Under gauge transformations, $\mathcal{D}_\mu \psi_\nu$ does not transform like ψ_ν itself, but rather

$$\mathcal{D}_\mu \psi_\nu(x) \rightarrow V(x + \epsilon_\mu + \epsilon_\nu) \psi_\nu(x) V(x)^{-1} \quad (2.1c)$$

Evidently, in the special case $\psi = \phi$ we have antisymmetry

$$\mathcal{F}_{\mu\nu}(x) \equiv \mathcal{D}_\mu \phi_\nu(x) - \mathcal{D}_\nu \phi_\mu(x) \quad (2.6)$$

$\mathcal{F}_{\mu\nu}$ is the difference of the parallel transporters along the two paths from x to $x + \epsilon_\mu + \epsilon_\nu$ shown in figure 3. It is a generalized field strength. As a

consequence of its definition it satisfies a Bianchi-identity (2nd Maxwell equation). Extend the definition (2.5) of \mathcal{D}_μ to

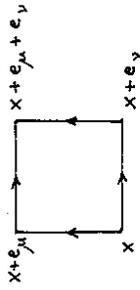


Figure 3. Field strength (see text)

$$\mathcal{D}_\mu \psi_{\nu_1 \dots \nu_k}(x) = \psi_{\nu_1 \dots \nu_k}(x + \epsilon_\mu) \phi_\mu(x) - \phi_\mu(x + \epsilon_{\nu_1} + \dots + \epsilon_{\nu_k}) \psi_{\nu_1 \dots \nu_k}(x) \quad (2.5')$$

Then the Bianchi identity reads

$$\mathcal{D}_\lambda \mathcal{F}_{\mu\nu} + \mathcal{D}_\nu \mathcal{F}_{\lambda\mu} + \mathcal{D}_\mu \mathcal{F}_{\nu\lambda} = 0 \quad (2.7)$$

Square integrable functions $\psi_{\nu_1 \dots \nu_k}$ form a Hilbert space \mathcal{H}_k , and covariant differentiation \mathcal{D} specifies a map $\mathcal{H}_k \rightarrow \mathcal{H}_{k+1}$. Its adjoint $\mathcal{D}^* : \mathcal{H}_{k+1} \rightarrow \mathcal{H}_k$ takes $\psi_{\mu\nu \dots \nu_k}$ into $\mathcal{D}_\mu^* \psi_{\mu\nu \dots \nu_k}$. Explicit computation shows that

$$\mathcal{D}_\mu^* = \mathcal{D}_{-\mu} \quad (2.5'')$$

These formulae can also be used for standard lattice gauge theory where $\phi(b) = U(b) \in G$. In particular, definition (2.6) of $\mathcal{F}_{\mu\nu}$ is a natural definition of a field strength on a lattice, in spite of the unusual transformations law (2.1c). In the limit of zero lattice spacing, $\epsilon_\mu \rightarrow 0$, and the standard transformation law in the continuum is recovered.

The hermitean conjugate of $\mathcal{F}_{\mu\nu}$ is

$$\mathcal{F}_{\mu\nu}^*(x) = \mathcal{F}_{-\mu, -\nu}(x + \epsilon_\mu + \epsilon_\nu) \quad (2.9)$$

We shall use a norm $\| \cdot \|$ of $N \times N$ matrices A which is defined by

$$\|A\|^2 = \frac{1}{N} \text{tr } A^* A \quad (2.10)$$

Now we are ready to write down candidates for dielectric lattice gauge theory actions. As a kinetic term for the dielectric gauge field ϕ we take *

$$L_{kin}(\phi) = -\frac{1}{16} \sum_x \sum_{\substack{\mu, \nu = \pm 1, \dots, \pm d \\ \mu \neq \pm \nu}} \|\mathcal{F}_{\mu\nu}(x)\|^2 \quad (2.11)$$

Evidently, $L_{kin}(\phi) \leq 0$. The absolute maximum $L_{kin}(\phi) = 0$ is assumed at $\phi = 0$ but also for some more general field configurations.

$L_{kin}(\phi) = 0$ if $U(b) = \text{pure gauge}$, and

either $\sigma_\mu(x) = 0$ except possibly for a single value of μ , (2.12a)

or $\sigma_\mu(x)$ depends only on x^μ . (2.12b)

This degeneracy is removed when mass terms are added.

* One may consider adding a term

$$L_{deg} = -\frac{x^2}{2} \sum_x \sum_{\mu, \nu = \pm 1, \dots, \pm d} \|\mathcal{D}_{\mu\nu} \phi_{\mu\nu}(x)\|^2$$

In particular, inclusion of the term with $\mu = -\nu$ in the sum (2.11) would amount to that. The degeneracy (2.12b) would be removed by such a term. However, such a term wrecks the proof of Osterwalder Schrader positivity (for reflections in planes with sites) because it is basically a next to nearest neighbour interaction.

In the special case when ϕ is a constant multiple of a standard lattice gauge field, $\phi(b) = \beta^{1/4} U(b)$ with $U(b) \in G$, the standard Wilson form of the lattice gauge theory action [8] is recovered

$$L_{kin}(\beta^{1/4} U) = \frac{\beta^2}{2N} \sum_p \text{tr} [U(\partial p) - 1] \quad (2.13a)$$

As usual $U(\partial p) \equiv U(b_1) \dots U(b_4)$ where $b_1 \dots b_4$ are the four links in the boundary of the oriented plaquette p .

$L_{kin}(\phi)$ is invariant under gauge transformations in G . In the case that $G = SU(N)$, $N \geq 3$, its actual gauge symmetry is larger, $U(N)$ rather than $SU(N)$ (but not $GL(N, C)$, since $\phi(-b) = \phi(b)^*$). It could be broken down to $SU(N)$ by adding terms involving $\det \phi(b)$ to the action. If $N > 4$ they have dimension > 4 and the question arises whether such terms are irrelevant and whether this could lead to spontaneous creation of a $U(N)$ symmetry.*

L_{kin} is a sum of products of four ϕ 's. We call it biquadratic because it is only quadratic in the fields ϕ_μ that are attached to links of a particular direction μ . ($\mathcal{F}_{\mu\nu}^* \mathcal{F}_{\mu\nu}$ involves two factors ϕ_μ and two factors ϕ_ν). This biquadratic character will be crucial in the proof of theorem 1.

We may add mass terms to the action. They can be either quadratic or biquadratic

$$\begin{aligned} \text{mass terms} = & - \sum_x \left\{ \frac{m^2}{4} \sum_{\mu = \pm 1, \dots, \pm d} \|\phi_\mu(x)\|^2 \right. \\ & \left. + \frac{x^2}{8} \sum_{\mu \neq \pm \nu} \|\phi_\mu(x)\|^2 \|\phi_\nu(x)\|^2 \right\} \end{aligned} \quad (2.11b)$$

If $m^2 > 0$ then the action $L_{kin} + \text{mass terms}$ has $\phi = 0$ as its unique maximum.

* One speaks of spontaneous symmetry generation if the long distance behavior of a theory shows a larger symmetry than its action. A celebrated example due to Fröhlich and Spencer [10] is the sine Gordon representation of a 2-dimensional Coulomb-gas in its low temperature (Kosterlitz Thouless, or dipole-) phase. Its symmetry at long distances is \mathbb{R} rather than \mathbb{Z} .

** This is in contrast with Sharatchandras work [24]

Finally we may add further local interactions, for instance

$$\begin{aligned}
 -V_1(\phi) &= - \sum_{x,\mu} \psi_1(\sigma_\mu(x)) \\
 -V_2(\phi) &= - \sum_{x,\mu,\nu} \psi_2(\sigma_\mu(x), \sigma_\nu(x)) \quad (1 \leq \mu < \nu \leq 1)
 \end{aligned}
 \tag{2.11c}$$

σ_ν are determined by ϕ through the polar decomposition (2.2). These expressions are automatically gauge invariant if $G = SU(2)$ or $U(2)$. If $G = SU(N)$ or $U(N)$ with $N \geq 3$ we require that ψ_1 and ψ_2 depend only on the eigenvalues of the matrices $\sigma_\nu(x)$.

One may wish to consider anisotropic lattices or anisotropic actions. In this case one may admit real functions $\psi_{\mu\nu}, \psi_{2\mu\nu}$ which depend on directions μ, ν in place of ψ_1, ψ_2 , and similarly for the mass parameters m^i, κ .

Adding the terms (2.11a-c) one obtains an action

$$L(\phi) = L_{kin}(\phi) + \text{mass terms} - V_1(\phi) - V_2(\phi)
 \tag{2.14}$$

Partition functions and expectation values in the pure dielectric lattice gauge theory model with action (2.14) are defined as usual

$$\begin{aligned}
 Z &= \int \prod_b d\phi(b) e^{L(\phi)} \\
 \langle O \rangle &= Z^{-1} \int \prod_b d\phi(b) O(\phi) e^{L(\phi)}
 \end{aligned}
 \tag{2.15}$$

$d\phi$ is Lebesgue measure on the real-linear space \mathcal{G} , it is invariant under gauge transformations (2.1a). We assume that the potentials are bounded below, so that the integrals are absolutely convergent (on a finite lattice). We adopt free boundary conditions unless otherwise indicated.

An interesting 2-parameter family of models of this kind is obtained by putting

$$\begin{aligned}
 m^2 = \kappa^2 = 0, \quad \psi_2 = 0 \\
 e^{-\psi_1(\sigma_\mu)} = \text{const} \cdot \int_{\mathcal{G}} dU \exp \left[-\frac{1}{2\lambda} \text{tr} \beta^{-1/4} \phi_\mu - U \right]^2
 \end{aligned}
 \tag{2.16}$$

dU is Haar measure on G . ψ_1 depends on $\phi_\mu \in \mathcal{G}$ only through the factor σ_μ in its polar decomposition (2.2), because of invariance of Haar measure dU . If $\lambda \rightarrow 0$, ψ_1 tends to a δ -function concentrated on $\beta^{1/4} G$. As a result, one obtains the standard lattice gauge theory with Wilson action (2.13a) as a limiting case. By computing the second derivative at $\phi_\mu = 0$ one sees that ψ_1 has the qualitative shape shown in Figure 2a if $\lambda < \lambda_c$, of Figure 2b if $\lambda = \lambda_c$, and of Figure 2c if $\lambda > \lambda_c$ (See p.2). $\lambda_c = \frac{1}{4}$ if $G = SU(2)$. The main result of this paper, theorem 1 below, implies that static quarks are confined in these models, for $G = SU(2)$, if $\lambda > \lambda_c$, i.e. if the potential ψ_1 has the qualitative shape of Figure 2c. [A quadratic mass term can be extracted from such a potential without affecting validity of the hypotheses of theorem 1.]

Imagine now that renormalization group transformations [11] map the two parameter family of models with action (2.16) into itself, for a suitable choice of blockspins, and in such a way that an initial point $\lambda = 0, \beta$ arbitrary, moves along a trajectory which reaches $\lambda > \lambda_c$ eventually. Then we could conclude that the standard lattice gauge theory model with Wilson action (= our model with $\lambda = 0$) shows confinement of static quarks for arbitrary β . Of course such a scenario is unrealistically simple - it could at the very best be approximately true - but it illustrates the general idea. Dielectric lattice gauge theory models are candidates for effective actions for Yang Mills theory, and it is hoped that renormalization group transformations produce such an effective action with a single non-degenerate maximum at $\phi = 0$ when the UV-cutoff is brought down far enough (to the order of the ultimate physical mass), in 4 or fewer dimensions, when asymptotic freedom is true in perturbative theory.

In contrast, in 5 or more dimensions, the model (2.16) is expected to undergo a deconfining phase transition as λ is lowered, if β is large enough.

Theorem 1 is for local actions. The exact effective action will be nonlocal but it ought to be possible to write it as a sum of a local action plus irrelevant terms which can hopefully be (neglected or) treated as a perturbation. The method of

sections 4 - 6 appears capable of generalization to include nonlocal terms, provided they are small and decay fast with distance. In place of the Gaussian integration which leads to eq. (4.12) one would have to use cluster expansions as in refs. [12].

I will now state some general properties of the models with action (2.14). The models satisfy reflection positivity for reflections θ in lattice planes through sites, assuming the boundary conditions are θ -invariant (compare section 7). As a result, the models admit a quantum mechanical interpretation. The Hilbert space \mathcal{H} of physical states consists of gauge invariant square integrable wave functions $\Psi(\{\phi(b)\}_{b \in \Sigma})$ which depend on variables $\phi(b) \in \mathcal{G}$ attached to links b in the time $x_d = 0$ hyperplane Σ . The scalar product is defined by integration with Lebesgue measure $\prod_{b \in \Sigma} d\phi(b)$. Furthermore, the transfer matrix T is hermitean and its square is therefore positive (semi) definite and can be used to define a Hamiltonian [13,14] so that $T^4 = e^{-2H_T}$.

Next the Wilson criterium for confinement of static quarks [8] will be adapted. Consider a rectangular closed loop C composed of straight pieces C_1, C_2, C_3, C_4 of length T, L, T, L respectively, as in figure 4, and define

$$W(C) = \int \phi(C_4) U(C_3) \phi(C_2) U(C_1) \quad (2.17)$$

The parallel transporter $\phi(C)$ is defined by eq. (2.4), and $U(C)$ is defined by the same equation with U substituted for ϕ . U is determined by ϕ through the polar decomposition (2.2). (The set of field configurations ϕ where either $U(C_1)$ or $U(C_3)$ is not well defined has measure zero. This

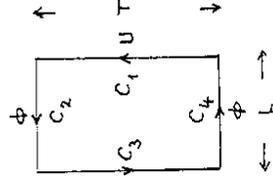


Figure 4 Wilson loop

remains true in the infinite volume limit, because of the Markov property [15]. In the limit $T \rightarrow \infty$

$$\langle W(C) \rangle = c(L) e^{-TV(L)} \quad (2.18)$$

and $V(L)$ can be interpreted as potential energy of a pair of static quarks at a distance L . Therefore static quarks will be confined by a linearly rising potential $V(L)$ if $\langle W(C) \rangle$ shows an area law behavior.

Instead of $W(C)$ one can use $\text{tr}U(C)$, but for our purposes $W(C)$ is more convenient. The argument for the validity of this criterion is the standard one. One considers the Hilbert space \mathcal{H}_{x_1, x_2} of states with static quarks (of opposite charge) at positions x_1 and x_2 , and specifies a trial state $\Psi \in \mathcal{H}_{x_1, x_2}$. Then one considers matrix elements of powers of the transfer matrix T .

$$(\Psi, T^{2n} \Psi) / (\Psi, \Psi) = e^{-2nV(x_1, x_2)} \quad \text{as } n \rightarrow \infty \quad (2.19)$$

assuming Ψ does not happen to be orthogonal to the state of lowest energy (highest eigenvalue of T) in \mathcal{H}_{x_1, x_2} . This is a standard assumption which always goes into the justification of a Wilson criterium. As a trial state we take

$$\Psi(\{\phi(b)\}_{b \in \Sigma}) = \phi(C_4) \int_{b > 0} d\phi(b) e^{L_+(\phi)} \quad (2.20)$$

$b > 0$ are the links in the $x^d > 0$ half space, and $L_+(\phi)$ is the sum of those terms in L which depend on some $\phi(b)$ with $b > 0$, plus $\frac{1}{2}$ those which depend only on $\phi(b)$ with $b \in \Sigma$. Then $(\Psi, T^{2n} \Psi) = Z \langle W(C) \rangle$ and $(\Psi, \Psi) = Z \langle \phi(C_4) \phi(C_4) \rangle$ if $T = 2n$. This justifies eq. (2.18) with c independent of T . (For simplicity we dropped some colour indices and sums over them.)

Finally the main result of this paper can be stated. I will only prove it for $G = SU(2)$. Roughly speaking it says that static quarks which transform according to the fundamental representation of $SU(2)$ are confined by a linearly rising potential in dielectric lattice gauge theories with a local action L (of the form (2.16)) provided $\phi = 0$ is the only maximum of $L(\phi)$ and it is nondegenerate (i.e. $\frac{d^2}{dt^2} L(t\phi) \neq 0$ for all $\phi = \{\phi(b) \in \mathcal{G}\}_{b \in \Lambda}$). This is in agreement with the intuitive argument of figure 1. The theorem is actually not quite as strong as that, though: It does not cover the case where the potential V has "valleys", such as $\psi_1(\sigma_1, \sigma_2) = \lambda(\sigma_1^2 - \sigma_2^2)$ (which would favor isotropic σ).

Theorem 1 Consider the dielectric lattice gauge theory model (without matter

fields) with gauge group $G = SU(2)$ and action $L(\phi)$ given by eqs. (2.14), (2.11a,b,c), with $m^2 > 0$, $\kappa^2 \geq 0$, ψ_1 and ψ_2 bounded below. If

$$\psi_1(\sigma_1) + \sum_{\nu=2}^d \psi_\nu(\sigma_1, \sigma_\nu)$$

is a nondecreasing function of $\sigma_1 \in \mathbb{R}_+$, for arbitrary $\sigma_\nu \in \mathbb{R}_+$, $\nu = 2 \dots d$, then the Wilson loop expectation value obeys an area law

$$\langle W(C) \rangle \leq c(L) e^{-\alpha LT} \quad \text{with } \alpha > 0. \quad (2.21)$$

\mathbb{R}_+ is the set of nonnegative real numbers and $W(C)$ was defined in eq. (2.17) and figure 4.

The theorem remains valid for anisotropic lattices or actions, provided m_1^2 , κ_1^2 , ψ_1 , ψ_{21} is substituted for m^2 , κ^2 , ψ_1 , ψ_2 in its statement.

Finally I will discuss how one may put quark fields into the action. One could of course use one of the standard forms of matter action in ordinary lattice gauge theory, with gauge field $U(b)$ that are obtained from ϕ (b) by the polar decomposition (2.2a). This satisfies the requirement of physical positivity (= Osterwalder-Schrader, or reflection positivity). But it is nonpolynomial in the ϕ 's.

A possible local matter action which is polynomial in the ϕ 's and in the quark fields ψ, ψ^\dagger is as follows (in 4 dimensions)

$$L_{\text{matter}}(\psi, \psi^\dagger, \phi) = \sum_x \left\{ -\bar{\psi}(x) \psi(x) + \sum_{\mu=1,2,3,4} \left[K_1 \bar{\psi}(x + e_\mu) (1 - \gamma_\mu) \phi_\mu(x) \psi(x) - K_2 \bar{\psi}(x) (1 - \gamma_\mu) \phi_\mu^*(x) \psi(x) \right] \right\} \quad (2.22)$$

Here γ_μ are Euclidean Dirac-Matrices, and $\bar{\psi} = \psi^\dagger \gamma_4$
 $\{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu}$ for $\mu, \nu = 1 \dots 4$, $\gamma_\mu^* = \gamma_\mu = -\gamma_{-\mu}$.

For practical calculations we use the hermitean matrices (7.4). Colour and spinor indices α are suppressed in (2.22).

The computation of expectation values involves integrals over anticommuting variables $\psi^\dagger(x), \psi(x)$ [16].

$$\langle 0 \rangle = Z^{-1} \int \prod_x \left[\prod_\mu d\phi_\mu(x) \prod_\alpha d\psi_\alpha^*(x) d\psi_\alpha(x) \right] \mathcal{O}(\psi, \psi^\dagger, \phi) \exp \left[L(\phi) + L_{\text{matter}}(\psi, \psi^\dagger, \phi) \right] \quad (2.22')$$

It will be shown in section 7 that physical positivity is satisfied provided

$$K_1^2 < \frac{2}{3} K \quad (\text{sufficient condition}) \quad (2.23)$$

In the special case of an ordinary lattice gauge theory, where $\phi_\mu(x) \in G$ is unitary, the last term in (2.22) is proportional to $\bar{\psi} \psi$, because $\phi_\mu^* \phi_{\mu+1}$ and the terms with γ_μ and $\gamma_{-\mu}$ cancel. The matter action (2.22) reduces therefore to the standard Wilson action with hopping parameter $K = K_1(1+8K)^{-1}$ after suitable rescaling of the fields. For given K the constraint (2.23) can be satisfied by a suitable choice of K_1 if $K < \frac{1}{6}$. This reproduces Lüscher's result for standard lattice gauge theory [14].

3. Hamiltonian Limit and Gauss' Law

The natural way to construct models in continuous time is to start with models on an anisotropic lattice, and to require that the variables $\mathcal{G}_i^*(x)$ attached to time-like links $(x+e_d, x)$ get frozen to a constant multiple $\sigma_0 > 0$ of $\mathbb{1}$ in the limit when the lattice spacing a_t in time direction tends to zero. This can be achieved for instance by taking $\psi_{i\mu}^*$ (which depends on direction μ on an anisotropic lattice) to be of the form (2.16) for $\mu = d$ with $\lambda \rightarrow 0$ as $a_t \rightarrow 0$. Integration over the remaining variables $U(x+e_d, x) \in G$ projects on the physical state space as in ordinary gauge theories. The limiting theory admits a conventional Hamiltonian description [18].

Let $\lambda^a, a = 1 \dots \dim G$ be a complete set of antihermitean generators of the gauge group G , and let $B^a = \text{tr}(B\lambda^a)$ for $B \in \mathcal{G}$. Gauss' law takes the form (in 4 dimensions)

$$(\mathcal{D}_i^* \mathcal{F}_{4i}(x))^a = \rho^a(x) \quad a = 1 \dots \dim G \quad (3.1a)$$

with

$$\rho^a = 2K_1 \psi_{4i}^*(x) \times^a \psi_{4i}(x) \quad (3.1b)$$

if the matter action has the form (2.22). The number of equations obtained in this way equals the number of generators of the gauge group, as it must be. Eq. (3.1) can be obtained, for instance, as the equation of motion that is obtained by varying U_d , cp. [19], and setting $U_d = 1$ in the end.

Let the electric field \mathcal{E}_i be defined like \mathcal{F}_{4i} , but with U 's in place of \mathcal{F} 's. Inserting the definition and expanding to leading order in the lattice spacing a_s in space direction gives, formally

$$\sum_{i=1}^3 ((\partial_i - A_i) \sigma_i \sigma_i^* \sigma_0 \mathcal{E}_i)^a + O(a_s) = \rho^a$$

This suggests to identify

$$\mathcal{E}_{ij} \propto \sigma_i \sigma_i^* \delta_{ij} \quad (3.2)$$

as a dielectric (tensor) field.

4. The ϕ_1 -dependent part of the action

We return to the consideration of models with discrete Euclidean time. The action $L(\phi)$ given by eqs. (2.14), (2.11) can be split into two pieces

$$L(\phi) = L^{\parallel}(\phi_1, \phi^{\perp}) + L^{\perp}(\phi^{\perp}) \quad (4.1)$$

The first piece contains all the dependence on the variables $\phi_1(x)$ that are attached to links in the 1-direction, and the second piece depends therefore only on the remaining variables

$$\phi^{\perp} = (\phi_2, \dots, \phi_d) \quad (4.2)$$

If \mathcal{G} consists of $N \times N$ matrices we set

$$(\phi_1, \psi_1) = \sum_x \frac{1}{N} \text{tr} \phi_1^*(x) \psi_1(x) \quad (4.3)$$

After a partial integration (summation), L^{\parallel} takes the form

$$L^{\parallel}(\phi) = -\frac{1}{2}(\phi_1, [-\Delta^{\perp} + x^2 \mathcal{K} + m^2] \phi_1) - \sum_x \psi_x^* (\sigma_1^2, \phi^{\perp}) \quad (4.4)$$

ψ_x^* is given by

$$\psi_x^* (\sigma_1^2, \phi^{\perp}) = \psi_1^* (\sigma_1) + \sum_{\nu=2, \dots, d} \psi_{\nu}^* (\sigma_1, \sigma_{\nu}(x)) \quad (4.5)$$

The multiplication operator \mathcal{K} and the "transverse covariant Laplacian" Δ^{\perp} depend on ϕ^{\perp} and are defined as follows

$$\begin{aligned} \mathcal{K} \psi_1(x) &= \frac{1}{2} \sum_{\mu=\pm 1, \dots, \pm d} [\phi_{\mu}^*(x+\epsilon_1) \phi_{\mu}^*(x) \psi_1(x) + \psi_1(x) \phi_{\mu}^*(x) \phi_{\mu}^*(x)] \\ &\equiv \mathcal{K}(x) \psi_1(x) \end{aligned} \quad (4.6)$$

If $G = \text{SU}(2)$, $\phi_{\mu}^* \phi_{\mu}(x)$ is a multiple of the unit matrix and commutes with matrices $\psi_1(x) \in \mathcal{G}$. Finally

If the hypotheses of theorem 1 are fulfilled, we may subtract a small fraction of the quadratic mass term αm^2 and add it to α_x^2 . The monotonicity assumption guarantees then that the Fourier representation (4.11) exists. Since

$$\sigma_1(x)^2 = N^{-1} \text{tr} \phi_1(x) \phi_1^*(x) \quad \text{we obtain, upon inserting (4.11)}$$

$$\begin{aligned} \langle W(C) \rangle &= \int \prod_x d\phi^{\pm}(x) (\alpha_1 a b m \alpha) \\ &= \int \prod_x \left[d\alpha(x) g_x(\alpha(x), \phi^{\pm}(x)) d\phi_1^{\pm}(x) \right] \exp \left\{ -\frac{1}{2} \left(\phi_1, [-\Delta^{\pm} + x^2 \mathcal{K} + m^2 - 2i\alpha] \phi_1 \right) \right\} \\ &\quad \prod_{s=1}^{L-1} \left[\phi_1(s, \nu_1) \alpha_{s+1} \alpha_s \bar{\phi}_1^{\pm}(s, \nu_2) / \beta_{s+1} \beta_s \right] \end{aligned}$$

The operator a is multiplication with $\alpha(x) \in \mathbb{R}$.

The ϕ_1^{\pm} -integration is now Gaussian, and can be done after interchange of the order of the integrations. As a result

$$\begin{aligned} \langle W(C) \rangle &= \frac{1}{Z} \int \prod_x d\phi^{\pm}(x) e^{L(\phi^{\pm})} U(C_3) / \beta_3 \alpha_0 U(C_1) / \beta_1 \alpha_L \\ &\quad \cdot \int \prod_x \left[d\alpha(x) g_x(\alpha(x), \phi_1^{\pm}) \right] \det \left(-\Delta^{\pm} + x^2 \mathcal{K} + m^2 - 2i\alpha \right)^{-\frac{1}{2}} \\ &\quad \prod_{s=0}^{L-1} \left(-\Delta^{\pm} + x^2 \mathcal{K} + m^2 - 2i\alpha \right)^{-1} \alpha_{s+1} \alpha_s / \beta_{s+1} \beta_s (s, \nu_1; s, \nu_2) \end{aligned} \quad (4.12)$$

For $G = \text{SU}(2)$.

The inverse of $(-\Delta^{\pm} + x^2 \mathcal{K} + m^2)^{-1}$ exists if $m^2 > 0$, because $-\Delta^{\pm}$ and \mathcal{K} are positive (s. below). It is an integral operator whose kernel is the propagator $(-\Delta^{\pm} + x^2 \mathcal{K} + m^2 - 2i\alpha)^{-1}(x; y)$. The decay properties of such propagators will be studied in the next section.

The partition function Z is represented by the same integral as appears in (4.12), without the last factor $\prod_s(\dots)$ in the integrand, and without the factors $U(C_1)$.

$$-\Delta^{\pm} = \mathcal{K} - \tilde{\mathcal{R}},$$

$$\begin{aligned} \tilde{\mathcal{R}} \psi_1(x) &= \sum_{\mu=\pm 2, \dots, \pm d} \phi_{\mu}(x + e_{\mu}) \psi_1(x + e_{\mu}) \psi_{\mu}(x) \\ &\equiv \sum_y \mathcal{R}(x, y) \psi_1(y) \end{aligned} \quad (4.7)$$

$\mathcal{R}(x, y)$ is a multiplication operator which is only nonzero if y is a nearest neighbour of x . Δ^{\pm} is a covariant generalization of the ordinary lattice Laplacian which is defined by $\Delta f(x) = \sum_y [f(y) - f(x)]$, sum over nearest neighbour y of x .

Let us now consider the expectation value $\langle W(C) \rangle$ of a Wilson loop operator (2.17), with \tilde{C} positioned as in figure 4, so that its "short legs" C_4, C_2 point in 1 direction.

$$W(C) = \text{tr} (\phi(C_4) U(C_3) \phi(C_2) U(C_1)) \quad (4.9)$$

$U(C_1)$ and $U(C_3)$ are determined by ϕ^{\pm} .

We start with a finite lattice Λ and derive bounds which are uniform in the lattice size. They remain therefore valid in the infinite volume limit.

We write $x^1 = s$, $x = (s, \mathcal{N})$, and denote the endpoints of the path C_3 by $(0, \mathcal{N}_1)$ and $(0, \mathcal{N}_2)$. Their distance is $|\mathcal{N}_1 - \mathcal{N}_2| = T$. Thus

$$\begin{aligned} \langle W(C) \rangle &= \frac{1}{Z} \int \prod_x d\phi^{\pm}(x) e^{L(\phi^{\pm})} U(C_3) / \beta_3 \alpha_0 U(C_1) / \beta_1 \alpha_L \\ &\quad \cdot \int \prod_x \left[d\phi_1^{\pm}(x) e^{-2\mathcal{W}_x(\sigma_1^{\pm}(x), \phi_1^{\pm})} \right] \exp \left\{ -\frac{1}{2} \left(\phi_1, [-\Delta^{\pm} + x^2 \mathcal{K} + m^2] \phi_1 \right) \right\} \\ &\quad \prod_{s=1}^{L-1} \left[\phi_1(s, \mathcal{N}_1) \alpha_{s+1} \alpha_s \bar{\phi}_1^{\pm}(s, \mathcal{N}_2) / \beta_{s+1} \beta_s \right] \end{aligned} \quad (4.10)$$

From now on we restrict attention to gauge group $G = \text{SU}(2)$ so that $\sigma_1^{\pm}(x)$ is real.

We insert the Fourier-integral representations of $e^{-2\mathcal{W}_x}$

$$e^{-2\mathcal{W}_x(\sigma_1^{\pm}, \phi_1^{\pm})} = \int_{-\infty}^{+\infty} d\alpha g_x(\alpha, \phi_1^{\pm}) e^{i\alpha \sigma_1^{\pm}} \quad (\sigma \in \mathbb{R}_+) \quad (4.11)$$

5. Random walk representations

Random walk representations for propagators are well known and were studied in detail by Brydges, Fröhlich and Spencer [5]. In this section I will present a self-contained derivation of the random walk representation for the propagators that appeared at the end of the last section, and derive some elementary bounds on them, for $x^1 > 0$.

We restrict attention to gauge group $G = SU(2)$ in the sequel. In this case, $\mathcal{A}(x)$ is multiplication with a ϕ^4 -dependent nonnegative real number, see eq. (4.6). We assume that $\mathcal{A}(x) > 0$ to begin with. This restriction can be dropped at the end by a limiting argument. $m^1 > 0$ is always assumed.

$-\Delta^1$ is a positive operator because $-\Delta^1 = \frac{1}{2} \sum_{y \neq x \pm 1} D_y^* D_y > 0$. Writing $-\Delta^1 = \mathcal{A} - \tilde{\mathcal{R}}$ as in (4.7) it follows that $\tilde{\mathcal{R}} \leq \mathcal{A}$. Call a site x odd if $\sum_{\mu=1}^d x^\mu$ is odd, and even otherwise. Define a unitary operator \mathcal{U} by $\mathcal{U}\psi_1(x) = \pm \psi_1(x)$ with $-$ for odd sites x and $+$ for even sites. Then $\mathcal{U}\mathcal{A}\mathcal{U}^* = \mathcal{A}$ and $\mathcal{U}\tilde{\mathcal{R}}\mathcal{U}^* = -\tilde{\mathcal{R}}$ because the nearest neighbours $x \pm e_\mu$ of an even site x are odd and vice versa. Of course, $-\mathcal{U}\Delta^1\mathcal{U}^*$ is also positive. Therefore $\mathcal{A} + \tilde{\mathcal{R}} > 0$, i.e. $\tilde{\mathcal{R}} \geq -\mathcal{A}$. Both inequalities together imply that

$$\tilde{\mathcal{R}} \approx \mathcal{A} \mathcal{R} \quad \|\mathcal{R}\| \leq 1 \tag{5.1}$$

Here $\|\cdot\|$ is the operator norm in the Hilbert space with scalar product (4.3).

Consider now the propagators that enter into eq. (4.12). Inserting $-\Delta^1 = \mathcal{A} - \mathcal{R}$ we have

$$(-\Delta^1 + x^1 \mathcal{A} + m^2 - 2ia)^{-1} = (1 - [1 + x^2 + (m^2 - 2ia)\mathcal{A}^{-1}]^{-1} \mathcal{R})^{-1} (m^2 - 2ia + [1 + x^2] \mathcal{A})^{-1} \tag{5.2}$$

a and \mathcal{A} are operators of multiplication with real numbers $a(x)$ and $\mathcal{A}(x) > 0$

Therefore

$$\| [1 + x^2 + (m^2 - 2ia)\mathcal{A}^{-1}]^{-1} \| = \sup_x \left| [1 + x^2 + (m^2 - 2ia(x))\mathcal{A}(x)^{-1}]^{-1} \right| \leq (1 + x^2)^{-1} \tag{5.3}$$

As a consequence of the bounds (5.1), (5.3) we have that

$$A \equiv [1 + x^2 + (m^2 - 2ia)\mathcal{A}^{-1}]^{-1} \mathcal{R} \quad \text{has norm } \|A\| \leq (1 + x^2)^{-1}, \tag{5.4}$$

and the first factor on the right hand side of eq. (5.2) may be expanded into a convergent Neumann series.

$$\begin{aligned} (-\Delta^1 + x^1 \mathcal{A} + m^2 - 2ia)^{-1} &= \sum_{n=0,1,\dots}^{\infty} A^n (m^2 - 2ia + (1 + x^2)\mathcal{A})^{-1} & (5.5a) \\ &= \sum_{n=0}^{\ell-1} A^n (m^2 - 2ia + (1 + x^2)\mathcal{A})^{-1} + A^\ell (-\Delta^1 + x^1 \mathcal{A} + m^2 - 2ia)^{-1}. & (5.5b) \end{aligned}$$

Because of eq. (4.8) we have

$$\begin{aligned} A\psi_1(x) &= [(1 + x^2)\mathcal{A} + m^2 - 2ia]^{-1} \tilde{\mathcal{R}}\psi_1(x) \\ &= \sum_z [(1 + x^2)\mathcal{A}(x) + m^2 - 2ia(y)]^{-1} \mathcal{R}(x,z)\psi_1(z) \end{aligned} \tag{5.6}$$

$\mathcal{R}(x,z)$ is only nonzero if x and z are nearest neighbours and $x^1 = z^1$. Therefore, eq. (5.5a) produces a random walk representation for the kernel. If $y^1 = x^1 = s$ we obtain

$$\begin{aligned} &(-\Delta^1 + x^1 \mathcal{A} + m^2 - 2ia)^{-1}(y, x) = \\ &= \sum_{\omega_s: x \rightarrow y} \prod_{(z, z') \in \omega_s} [m^2 - 2ia(z) + (1 + x^2)\mathcal{A}(z)]^{-1} \mathcal{R}(z, z') \} \\ &\quad \cdot [m^2 - 2ia(x) + (1 + x^1)\mathcal{A}(x)]^{-1} \end{aligned} \tag{5.6'}$$

Summation is over all paths ω_s from x to y which consist of links (z_1, z_2) in the $z^1 = s$ hyperplane. If $y^1 \neq x^1$ the kernel is 0. Upon inserting the explicit expression (4.8) for \mathcal{R} , we obtain the formula

$$\begin{aligned}
 & (-\Delta^1 + x^2 \mathcal{K} + m^2 - 2i\alpha)^{-1} \delta_{\delta}^{\epsilon} \alpha_{\beta} (y, x) = \\
 & = \sum_{\omega_3: x \rightarrow y} \phi(\omega_3 + \epsilon_1)_{\alpha\beta}^* \phi(\omega_3)_{\beta\alpha} \prod_{z \in \omega_3} [m^2 - 2i\alpha(z) + (1+x^2)\mathcal{K}(z)]^{-1} \\
 & \text{for } x^1 = y^1 = s. \tag{5.7}
 \end{aligned}$$

If the path ω_s consists of links $(z_0, z_1), (z_1, z_2), \dots, (z_{n-1}, z_n)$ the product over $z \in \omega_s$ is to be read as product over $z_i, i = 0 \dots n$. The number of visits $n_{\omega_s}(z)$ of the path ω_s to site z equals the number of values of $i, i = 0 \dots n$, with $z_i = z$; it can be ≥ 1 . $\omega_s + e_1$ is path ω_s shifted by one lattice spacing in 1-direction. The parallel transporters $\phi(\omega)$ were defined in eq. (2.4)

From eq. (5.5a) we will obtain a bound on the propagator (for $a(x) \equiv 0$).

$$\begin{aligned}
 & (-\Delta^1 + x^2 \mathcal{K} + m^2)^{-1} \delta_{\delta}^{\epsilon} \alpha_{\beta} (y, x) \\
 & = (\delta_{\delta, \gamma}^{\epsilon}, (-\Delta^1 + x^2 \mathcal{K} + m^2)^{-1} f_{\alpha\beta, x}) \tag{5.8}
 \end{aligned}$$

where $f_{\alpha\beta, x}$ is a \mathcal{G} -valued function defined by $[f_{\alpha\beta, x}(z)]_{\gamma\delta} = \delta_{\alpha\gamma}^{\epsilon} \delta_{\beta\delta}^{\epsilon} \delta_{xz} \sqrt{N}$. The scalar production (4.3) is in use. Let us use a distance $|x-y| =$ length of the shortest path from x to y .

$$|x-y| = \sum_{\mu=1}^d |x^{\mu} - y^{\mu}|$$

According to (5.6), the operator A makes only "single steps". Therefore, the first term in eq. (5.5b) makes no contribution to expression (5.8) if $\mathcal{L} \leq |x-y|$. Consequently the left hand side (l.h.s.) of (5.8) obeys

$$\begin{aligned}
 |l.h.s. \text{ of } (5.8)| & \leq \prod_{\delta, \gamma} f_{\alpha\beta, x} \prod_{\omega} \|A\| \prod_{z \in \omega} \|(-\Delta^1 + x^2 \mathcal{K} + m^2)^{-1}\| \\
 & \leq (1+x^2)^{-l} m^{-2},
 \end{aligned}$$

by (5.4) and positivity of the operators $-\Delta^1, \mathcal{K}$. Thus finally

$$|(-\Delta^1 + x^2 \mathcal{K} + m^2)_{\delta\beta}^{\alpha\epsilon} (y, x)| \leq m^{-2} (1+x^2)^{-|x-y|} \tag{5.9}$$

This is valid for arbitrary ϕ^1 . It shows that the propagator decays exponentially if $x^2 > 0$.

6. Generalized Fröhlich Durhuus (random surface) representation and proof of theorem 1

We start from expression (4.12) for the Wilson loop expectation value. We reinsert the integral representation for the determinant

$$\begin{aligned}
 \det & (-\Delta^1 + x^2 \mathcal{K} + m^2 - 2i\alpha)^{-1/2} \\
 & = \int \prod_x d\phi_1(x) \exp \left\{ i(\phi_1, \alpha \phi_1) - \frac{1}{2}(\phi_1, [-\Delta^1 + x^2 \mathcal{K} + m^2] \phi_1) \right\} \tag{6.1}
 \end{aligned}$$

We also insert the random walk representation (5.7) for the propagators in the last factor of the integrand in (4.12). As a result there will appear sums over L -tuples of paths $\omega = (\omega_0, \dots, \omega_{L-1})$, where ω_s is a path from (s, n_2) to (s, n_1) in the $z^1 = s$ hyperplane. Let

$$n_{\omega}(z) = n_{\omega_s}(z) \quad \text{if } z^1 = s \tag{6.2}$$

= number of visits of a path ω_s to z

Then the result takes the form

$$\begin{aligned}
 \langle W(C) \rangle & = \frac{1}{Z} \int \prod_{x, y} d\phi_x(x) e^{-\frac{1}{2}(\phi_x, [-\Delta^1 + x^2 \mathcal{K} + m^2] \phi_x) + L^1(\phi^1)} \\
 & = \sum_{\omega} \prod_z \left\{ d\alpha(z) g_z(\alpha(z), \phi^1) [\eta(z) - 2i\alpha(z)]^{n_{\omega}(z)} e^{i\alpha(z)\sigma_1(z)^2} \right\} \tag{6.3}
 \end{aligned}$$

with

$$\eta(x) = m^2 + (1+x^2)\mathcal{K}(x) > 0 \tag{6.3'}$$

Now the α -integrations can be done again. Upon inserting the integral representation

$$(\eta - 2i\alpha)^{-n} = \frac{1}{\Gamma(n)} \int_0^{\infty} d\xi \xi^{n-1} e^{-(\eta - 2i\alpha)\xi}$$

it follows from the definition of $g_x(a, \phi^1)$ (after a variable transformation $\eta^1 \rightarrow \xi$) that

$$\begin{aligned} & \int_{-\infty}^{+\infty} da g_x(a, \phi^{\perp}) e^{i a \sigma_1^{\perp}} [\eta - 2ia]^{-n} \\ &= \frac{\eta^{-n}}{\Gamma(n)} \int_0^{\infty} d\zeta \zeta^{n-1} e^{-\zeta} e^{-\mathcal{W}_x(\sigma_1^{\perp} + 2\zeta\eta^{-1}, \phi^{\perp})} \end{aligned} \quad (6.4)$$

Set

$$dV_n(\zeta) = \frac{1}{\Gamma(n)} d\zeta \zeta^{n-1} e^{-\zeta} \text{ if } n \geq 1, \quad dV_0(\zeta) = \delta(\zeta) \quad (6.5)$$

This is a probability measure on the positive real line, i.e. $\int_0^{\infty} dV_n(\zeta) = 1$. The result of the a-integrations is

$$\begin{aligned} \langle W(C) \rangle &= \frac{1}{Z} \int \prod d\phi_{\mu}^{\perp}(x) e^{-\frac{1}{2}(\phi_1^{\perp} [-\Delta^{\perp} + x^2 \mathcal{U} + m^2] \phi_1^{\perp}) + L^{\perp}(\phi^{\perp})} \\ &= \sum_{\omega} \int \prod (\eta(z)^{n_{\omega}(z)} dV_{n_{\omega}(z)}(z)) e^{-\mathcal{W}_x(\sigma_1^{\perp}(z)^2 + 2\zeta(z)\eta(z)^{-1}, \phi^{\perp})} \\ &\quad \cdot \prod_{s=0}^{L-1} \phi(\omega_s + e_1) / \beta_{s+1} \alpha_s \beta_s \phi(\omega_s) \alpha_s \beta_s U(C_s) / \beta_{s+1} \alpha_s \beta_s U(C_s) / \beta_{s+1} \alpha_s \beta_s \end{aligned} \quad (6.6)$$

Given $\phi^{\perp} = (\phi_x^{\perp}(x) = U(x+e_1, x) \sigma_y^{\perp}(x))_{y=2 \dots d}$ define $\hat{\phi}^{\perp} = (\hat{\phi}_y^{\perp}(x))$ by

$$\hat{\phi}_y^{\perp}(x) = \sigma_y^{\perp}(x) \mathbb{1} \quad (6.7)$$

Then

$$\phi(\omega_s) = U(\omega_s) \hat{\phi}(\omega_s) \quad (6.8)$$

and similarly for $\phi(\omega_s + e_1)^*$.

The L-tuples of paths $\omega = (\omega_s)$ can be looked upon as defining a surface with boundary C. Eq. (6.6) exhibits the Wilson loop expectation value as a sum of contributions from (random) surfaces ω . For the standard SU(2) lattice gauge theory model, this representation had been derived by Fröhlich and Durhuus [4].

In the standard lattice gauge theory, an area law decay is supposed to come about through destructive interference when the phase factors $U(\omega_s)$ from different paths ω_s are added up. In the present approach, part of this effect (at the level of the fundamental action) is supposed to have been taken care of in the course of the computation of the effective action $L(\phi)$, when the block spin variables $\phi(b)$ were constructed as linear superpositions of parallel transporters along different paths. Destructive interference among these will lead to favoring variables $\phi(b)$ with small modulus $\sigma_{\mu}^{\perp}(x)$.

In accordance with this physical intuition, we will now proceed to inequalities which are obtained from eq. (6.6) when all phase factors $U(\cdot)$ downstairs are dropped. This amounts to replacing ϕ by $\hat{\phi}$ in the last factor.

Since \mathcal{W}_x is a nondecreasing function of σ_1^{\perp} by hypothesis, and dV_n is a probability measure on the positive real axis (i.e. "it averages"), we have the inequality

$$\int dV_n(\zeta) e^{-\mathcal{W}_x(\sigma_1^{\perp} + 2\zeta\eta^{-1}, \phi^{\perp})} \leq e^{-\mathcal{W}_x(\sigma_1^{\perp}, \phi^{\perp})} \quad (6.9)$$

As a result

$$\begin{aligned} |\langle W(C) \rangle| &\leq \frac{1}{Z} \int \prod_{x, \mu} d\hat{\phi}_{\mu}^{\perp}(x) e^{L(\phi)} \delta_{\beta_s \alpha_s} \delta_{\beta_{s+1} \alpha_{s+1}} \\ &\quad \cdot \sum_{\omega} \left\{ \prod_z [m^2 + (1+x^2)\mathcal{U}(z)]^{-n_{\omega}(z)} \prod_{s=0}^{L-1} \hat{\phi}(\omega_s + e_1) / \beta_{s+1} \alpha_s \beta_s \right\} \end{aligned} \quad (6.10)$$

When $\hat{\phi}^{\perp}$ is substituted for $\hat{\phi}$, \mathcal{U} remains unchanged whereas $-\Delta^{\perp}$ gets replaced by $-\hat{\Delta}^{\perp}$. We can therefore now go backwards and use the random walk representation (5.7) again to do the summations over L-tuples of paths ω in (6.10).

In order to handle the special case $x^2 = 0$ later on, we extract a factor first.

We split

$$m^2 + (1+x^2) \mathcal{K}(x) = \rho(x) \left[\frac{1}{2} m^2 + (1 + \frac{1}{2} x^2) \mathcal{K}(x) \right]$$

with

$$\rho(x) = \frac{1 + \frac{1}{2} x^2 + \frac{1}{2} m^2 \mathcal{K}(x)^{-1}}{1 + x^2 + m^2 \mathcal{K}(x)^{-1}} \quad (6.11)$$

We extract a factor

$$\prod_{x \in \omega} \rho(x)$$

from the sum over ω in (6.10). After that we do the resummation with the result

$$|\langle W(C) \rangle| \leq \frac{1}{2} \int \prod_{x \in \mu} d\phi_\mu(x) e^{-L(\phi)} \delta_{\beta_0 \alpha_0} \delta_{\beta_1 \alpha_1} \left\{ \sup_{\omega} \prod_{x \in \omega} \rho(x) \right\} \cdot \prod_{s=0}^{L-1} \left(-\hat{\Delta}^{-1} + \frac{1}{2} x^2 \mathcal{K} + \frac{1}{2} m^2 \right)^{-1} \alpha_{s+1 \alpha_s} / \beta_{s+1} \beta_s (s, \mu_s; s, \mu_s)$$

Only terms with $\alpha_s = \beta_s$ contribute because $\hat{\phi}(\omega_s)$ are multiples of \mathbb{Z} .

Now we insert the bound (5.9) on the propagators. It is valid for arbitrary ϕ^{\perp} hence in particular for $\hat{\phi}^{\perp}$. This gives

$$|\langle W(C) \rangle| \leq 2 (4m^{-2})^L (1 + \frac{1}{2} x^2)^{-L \cdot T} < \sup_{\omega} \prod_{x \in \omega} \rho(x) > \quad (6.12)$$

In case $x^2 > 0$, the area law follows from this right away. Since $\mathcal{K}(z) > 0$, it follows that $0 \leq \rho(z) \leq (1 + \frac{1}{2} x^2)^{-1} (1+x^2)$. Therefore the expectation value is $\leq \left[(1 + \frac{1}{2} x^2) / (1+x^2) \right]^{L \cdot T}$. Therefore

$$|\langle W(C) \rangle| \leq 2 (4m^{-2})^L e^{-\alpha L \cdot T} \quad (6.13)$$

with $\alpha = \beta_m (1+x^2) > 0$.

The case $x^2 = 0$ is more complicated. Inequality (6.12) tells us that

$$|\langle W(C) \rangle| \leq 2 (4m^{-2})^L < \sup_{\omega} \prod_{x \in \omega} \left(\frac{1 + \frac{1}{2} m^2 \mathcal{K}(x)^{-1}}{1 + m^2 \mathcal{K}(x)^{-1}} \right) > \quad (6.14)$$

Since

$$\mathcal{K}(x) = \frac{1}{2} \sum_{\mu=2}^d \left[\sigma_{\mu}^2(x)^2 + \sigma_{\mu}^2(x+\epsilon_{\mu})^2 \right] \geq 0$$

$\mathcal{K}(x)^{-1}$ is nonnegative but it can become arbitrarily small. This requires, however, that $\sigma_{\mu}^2(x)^2$ or $\sigma_{\mu}^2(x+\epsilon_{\mu})^2$ is very large for some μ . Because of the hypothesis of theorem 1, the probability that this is the case is very small. The potential is bounded below and the quadratic mass term suppresses large σ_{μ}^2 by a factor $\exp \left[-\frac{1}{2} m^2 \sigma_{\mu}^2 \right]$. By standard procedure one can prove that the probability

$$\text{Prob} \left(\mathcal{K}(x_1)^{-1} < \epsilon, \dots, \mathcal{K}(x_n)^{-1} < \epsilon \right) < e^{-n\lambda/\epsilon} \quad (6.15)$$

for some $\lambda > 0$, if $\epsilon > 0$ is small enough and $x_1 \dots x_n$ are distinct points. This inequality can be derived by using either chessboard estimates [20] (for a similar application see e.g. section 8 of ref. 21), or superstability [22].

The sup over ω is over all L-tuples of paths ω_s , $s = 0 \dots L-1$, with fixed initial and final points. These paths visit at least T+1 distinct points. Each point $x \in \omega_s$ visited by ω_s contributes a factor < 1 to $\prod_{x \in \omega} (\dots)$ in (6.14).

Therefore the sup can only be increased if we abandon the requirement that the end points be fixed and strip the path ω_s down to a selfavoiding walk of length precisely T by removing loops and cutting away a piece at the end if necessary.

Therefore inequality (6.14) remains valid when the sup is read as a sup over

L-tuples ω of selfavoiding walks ω_s of length T with prescribed initial point. In the following we only consider ω 's with these properties, and sup's and sums over ω are restricted to these.

Set $n = L(T+1) =$ numbers of sites $x \in \omega$. All of these sites are distinct because ω_s are selfavoiding. Then

$$\begin{aligned}
 & \langle \sup_{\omega} \prod_{x \in \omega} \frac{1 + \frac{1}{2} \varepsilon m^2 \mathcal{K}(x)^{-1}}{1 + m^2 \mathcal{K}(x)^{-1}} \rangle \\
 & \leq \left(\frac{1 + \frac{1}{2} \varepsilon m^2}{1 + \varepsilon m^2} \right)^{-\frac{1}{2}n} + \sum_{\substack{k \\ n > k > \frac{1}{2}n}} \left(\frac{1 + \frac{1}{2} \varepsilon m^2}{1 + \varepsilon m^2} \right)^{-n+k} \text{Prob} \left\{ \begin{array}{l} \text{there exists } \omega \text{ such} \\ \text{that } \mathcal{K}(x) > \varepsilon^{-1} \text{ for} \\ k \text{ (distinct) sites } x \in \omega \end{array} \right\} \\
 & \leq \left(\frac{1 + \frac{1}{2} \varepsilon m^2}{1 + \varepsilon m^2} \right)^{-\frac{1}{2}n} + \sum_{\substack{k \\ n > k > \frac{1}{2}n}} \left(\frac{1 + \frac{1}{2} \varepsilon m^2}{1 + \varepsilon m^2} \right)^{-n+k} \sum_{\omega} \text{Prob} \left\{ \begin{array}{l} \mathcal{K}(x) > \varepsilon^{-1} \text{ for} \\ k \text{ sites on } \omega \end{array} \right\}
 \end{aligned}$$

The number of all sets I of k sites x on ω is $\binom{n}{k}$, and the number of all paths ω_s of length T with given initial point is $\varepsilon^{2d} T^T$. Together with the bound (6.15) this implies that

$$\begin{aligned}
 \sum_{\omega} \text{Prob} \left\{ \begin{array}{l} \mathcal{K}(x) > \varepsilon^{-1} \text{ for} \\ k \text{ sites on } \omega \end{array} \right\} & \leq \sum_{\substack{I \subset \omega \\ |I|=k}} \text{Prob} \left\{ \mathcal{K}(x) > \varepsilon^{-1} \text{ for all sites } x \in I \right\} \\
 & \leq (2d)^n \binom{n}{k} e^{-k\lambda\varepsilon^{-1}}
 \end{aligned}$$

Setting $k = \frac{1}{2}n + \ell$ gives

$$\langle \sup_{\omega} \prod_{x \in \omega} \frac{1 + \frac{1}{2} \varepsilon m^2 \mathcal{K}(x)^{-1}}{1 + m^2 \mathcal{K}(x)^{-1}} \rangle \leq \left(\frac{1 + \frac{1}{2} \varepsilon m^2}{1 + \varepsilon m^2} \right)^{-\frac{1}{2}n} \left\{ 1 + (2d)^n e^{-\frac{1}{2}n\lambda\varepsilon^{-1}} \sum_{\substack{\ell \\ \frac{1}{2}n > \ell > 0}} \binom{n}{\ell + \frac{1}{2}n} \left[(1 + \varepsilon m^2) e^{-\lambda\varepsilon^{-1}} \right]^{\ell} \right\}$$

Now we choose ε so small that

$$(1 + \varepsilon m^2) e^{-\lambda\varepsilon^{-1}} < 1 \quad \text{and} \quad e^{-\lambda\varepsilon^{-1}} < (8d)^{-1}$$

The sum over ℓ is then bounded by $\sum_{k=0}^n \binom{n}{k} = 2^n$. As a result

$$\langle \sup_{\omega} \prod_{x \in \omega} \frac{1 + \frac{1}{2} \varepsilon m^2 \mathcal{K}(x)^{-1}}{1 + m^2 \mathcal{K}(x)^{-1}} \rangle \leq \left(\frac{1 + \frac{1}{2} \varepsilon m^2}{1 + \varepsilon m^2} \right)^{-\frac{1}{2}n} [1 + 2^{-n}]$$

We insert this into inequality (6.14) and remember that $n = L(T+1)$. In this way we obtain the desired area law (2.21) with $\alpha = \frac{1}{2} \ln \left[(1 + \varepsilon m^2) / (1 + \frac{1}{2} \varepsilon m^2) \right] > 0$.

7. Physical positivity

Physical positivity can be expressed in terms of Euclidean expectation values (2.22'). This is the celebrated Osterwalder Schrader positivity condition [9]. It states that if O is any polynomial of positive time ($x^4 \gg 1$) fields we should find

$$\langle \theta(O^+) O \rangle \geq 0 \quad (7.1)$$

Here, θ denotes Euclidean time reflection and O^+ is the complex conjugate of O (e.g. $\bar{\psi}^+ = \psi^+ \psi$). In 4 dimensions

$$\begin{aligned}
 \theta \psi(x) &= \psi(\theta x), \text{ etc.} \\
 (\theta x)^4 &= -x^4 \quad ; \quad (\theta x)^i = x^i \quad (i=1,2,3) \quad (7.2)
 \end{aligned}$$

We assume that our lattice is positioned so that $x^4 = 0$ is a lattice plane containing sites, and that the boundary conditions are θ -invariant. It will be shown that the expectation values of a theory with action $L(\phi) + L_{\text{Matter}}(\psi, \psi^+, \phi) \in L_{\text{tot}}$ as defined by eqs. (2.11), (2.14), (2.22) satisfy the positivity condition (7.1) if the condition (2.23), viz. $K_1^2 \leq \frac{2}{3}k$, is fulfilled by the parameters in the matter action.

With any O as described above a wave function Ψ_O will be associated. A positive (semi) definite scalar product $(,)$ of such wave functions will be defined so that

$$(\Psi_{O_1}, \Psi_{O_2}) = \langle \theta(O_1^+) O_2 \rangle \quad (7.3)$$

Inequality (7.1) follows from positive semidefiniteness of the scalar product. The scalar product makes the space of all wave functions with finite norm $(\Psi, \Psi)^{1/2}$ into a Hilbert space \mathcal{H} (division by the subspace of all null vectors and completion is understood). \mathcal{H} is the quantum mechanical Hilbert space of states and Ψ are its states in "Schrödinger representation" [23].

It is convenient to use the following hermitean representation of γ -matrices

$$(\gamma_\mu^* = \gamma_\mu^* = -\gamma_\mu) \quad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_j^* = i \begin{pmatrix} 0 & \tau_j \\ -\tau_j & 0 \end{pmatrix} \quad (\tau_j: \text{Pauli matrices}, j=1,2,3) \quad (7.4)$$

We decompose the Dirac spinors into upper and lower components

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \psi^* = (\psi_+^*, \psi_-^*) \quad (7.5)$$

For pedagogical reasons, I will consider the case of free boundary conditions first. Let $\underline{\Sigma}$ be the time zero hyperplane $x^4 = 0$ ("space"). In canonical quantum field theory, ψ and ψ^* are conjugate variables. Therefore, in a Schrödinger representation the wave function should only depend on half of these variables. (In quantum mechanics, position Q and momentum P are conjugate variables, and the Schrödinger wave function depends only on Q .)

Our wave functions ψ will be functions of the variables $\phi(\underline{b}), \psi^+(\underline{x}), \psi_-(\underline{x})$ attached to spacelike links \underline{b} and sites \underline{x} in space $\underline{\Sigma}$.

We make a split of the total action so that

$$L_{\text{tot}}(\psi, \psi^*, \phi) = L_+(\psi, \psi^*, \phi) + L_-(\psi, \psi^*, \phi) + \mathcal{A} \quad (7.6)$$

with

$$\mathcal{A} = \sum_{\underline{x} \in \underline{\Sigma}} \left\{ -\bar{\psi}(\underline{x}) \psi(\underline{x}) + \sum_{i=1,2,3} \left[K_i \bar{\psi}(x+e_i) \phi_i(x) \psi(x) - k \bar{\psi}(x) \phi_i^*(x) \phi_i(x) \psi(x) \right] \right\} \quad (7.6')$$

and L_+ depends only on variables $\phi(\underline{b}), \psi(\underline{x}), \psi^+(\underline{x})$ attached to links \underline{b} and sites \underline{x} in the positive time halfspace (excluding those in $\underline{\Sigma}$, but including

links $(x+e_4, x)$ with $\underline{x} \in \underline{\Sigma}$, and on variables $\phi(\underline{b}), \psi^+(\underline{x}), \psi_-(\underline{x})$ attached to links \underline{b} and sites \underline{x} in $\underline{\Sigma}$. Furthermore it is required that

$$L_- = \Theta(L_+)^+ \quad (7.7)$$

The possibility of a split (7.6) with these properties is the crucial feature which will allow to define wave functions ψ , by eq. (7.9) below, which depend only on the variables listed above. There are three things to be checked to see that such a split is possible - the rest is obvious.

i) The contribution to L_{Matter} from the links $(x+e_4, x)$ with $\underline{x} \in \underline{\Sigma}$ should admit the split. Referring to eq. (2.22) it is seen that those contributions which do not contain matrices γ_j^+ make up the term \mathcal{A} in (7.6). Because of the off-diagonal form of matrices γ_j^+ the remaining terms take the form

$$\sum_{\underline{x}} \sum_{j=1,2,3} \left\{ -i K_1 \psi_+^*(x+e_j) \tau_j \phi_j(x) \psi_-(x) + i k \psi_+^*(x) \tau_j \phi_j(x) \psi_-(x) \right. \\ \left. + i K_1 \psi_-^*(x+e_j) \tau_j \phi_j(x) \psi_+(x) - i k \psi_-^*(x) \tau_j \phi_j(x) \psi_+(x) \right\} \quad (7.8)$$

The first term depends only on ϕ_j, ψ_+^* and ψ_- , and is incorporated into L_+ , and the second one in L_- .

ii) The contribution from links $(x+e_4, x)$ with $\underline{x} \in \underline{\Sigma}$ should not depend on $\psi_-(\underline{x})$ or $\psi_+(\underline{x})$. The presence of the projection operator $(1-\gamma_4)$ assures this.

This is the reason why the projection operator $(1-\gamma_4)$ in the last term in (2.22) was included.

iii) Eq. (7.7) needs to be checked. The $+$ operation is an antiautomorphism of the Grassmann algebra which takes c-numbers into their complex conjugate, ψ_α into ψ_α^+ , ψ_α^+ into ψ_α , and $(AB)^+ = B^+A^+$. Using these properties, the required equality is straightforward to check. For instance, the second line in (7.8) is the $+$ -conjugate of the first (when summed over \underline{x}, i). To see this one needs to use that $\phi_{\cdot j}(\underline{x}) = \phi_j^*(x-\epsilon_j^*)$ by (2.0).

Now we are ready to define the wave functions

$$\begin{aligned} \Psi_0 \left(\{ \phi_i(\underline{x}), \psi_+^+(\underline{x}), \psi_-(\underline{x}) \}_{\underline{y} \in \Sigma} \right) & \quad (7.9) \\ = Z^{-1/2} \int \prod'_{x, \mu, \alpha} [d\phi_{\mu, \alpha}(\underline{x}) d\psi_\alpha^+(\underline{x}) d\psi_\alpha(\underline{x})] O(\phi, \psi^+, \psi) e^{L_+(\psi, \psi^+, \phi)}. \end{aligned}$$

The product \prod' runs over x, μ with $x^4 > 0$, $\mu = 1 \dots 4$ or $x^4 = 0$, $\mu = 4$, and over Dirac- and colour indices α . The variables $\psi_+^+(\underline{x}), \psi_-(\underline{x})$ for $\underline{x} \in \Sigma$ are not integrated, instead the wave function Ψ_0 depends on them.

Next, the scalar product of two wave functions will be defined. We introduce a new notation

$$\psi_-(x) = \chi_-^+(x)\epsilon, \quad \psi_+^+(x) = \epsilon^{-1}\chi_-(x) \quad (7.10)$$

ϵ = antisymmetric tensor in 2 dimensions. (It acts on spinor indices and converts a column vector into a row vector in a rotation covariant way). With this notation, the wave functions Ψ depend on ψ_+^+, χ_-^+ , and their adjoint ψ^+ are functions of ψ_+ and χ_- . Expression (7.6') for \mathcal{A} takes the form

$$\mathcal{A} = - \sum_x \left[\psi_+^+(x) (\mathcal{A}\psi_+)(x) + \chi_-^+(x) (\bar{\mathcal{A}}\chi_-)(x) \right] \quad (7.11a)$$

with

$$(\mathcal{A}\psi_+)(x) = \left[1 + k \sum_{i=1, \dots, 2, 3} \phi_i^*(x) \phi_i(x) \right] \psi_+(x) - K_1 \sum_{i=1, \dots, 2, 3} \phi_i(x-\epsilon_i) \psi_+(x-\epsilon_i) \quad (7.11b)$$

The matrix $\bar{\mathcal{A}}$ has complex conjugate entries compares to \mathcal{A} . The scalar product of two wave functions is defined by

$$\begin{aligned} (\Psi_1, \Psi_2) &= \frac{1}{Z} \int \prod_{b \in \Sigma} d\phi(b) \prod_{\underline{x} \in \Sigma} [d\psi_+^+(\underline{x}) d\psi_+(\underline{x}) d\chi_-^+(\underline{x}) d\chi_-(\underline{x})] \\ & e^{\mathcal{A}(\Psi_1(\{\phi_i(\underline{x}), \psi_+^+(\underline{x}), \chi_-^+(\underline{x})\})^+, \Psi_2(\{\phi_i(\underline{x}), \psi_+^+(\underline{x}), \chi_-^+(\underline{x})\})^+)} \end{aligned} \quad (7.12)$$

with \mathcal{A} given by eq. (7.11).

With these definition, relation (7.3) between scalar products of wave functions and expectation values is satisfied. To see this one needs only insert the definition (7.9) of Ψ_0 , relabel the variables of integration $\psi(x) \rightarrow \psi(\theta x)$ in the integral representation for Ψ_0^+ , and refer to (7.6).

It remains to examine whether the scalar product (7.12) is positive definite. An answer is provided by the following well known lemma [16,14,19]

Lemma Let the Grassmann algebra \mathcal{O} be generated by the totally anticommuting objects $a_1^+, a_1^-, \dots, a_n^+, a_n^-$, and consider functions Ψ on the subalgebra \mathcal{O}^+ which is generated by $a_1^+ \dots a_n^+$. The scalar product

$$(\Psi_1, \Psi_2) = \int da_n^+ da_n^- \dots da_1^+ da_1^- e^{-\sum a_i^+ A_{ij} a_j^-} (\Psi_1(\alpha^+)) \Psi_2(\alpha^+)$$

is positive definite if the matrix $A = (A_{ij})$ is positive definite.

[Integrals over Grassmann variables admit linear changes of variables. Therefore it suffices to have validity of the lemma for $\sum a_i^+ A_{ij} a_j^- = \sum a_j^+ a_j^-$. In this case its validity follows from the well known isomorphism of \mathcal{O}^+ with a Fock space.]

The matrix A given by eq. (7.6) is positive (semi-) definite if $K_1^2 \leq \frac{3}{2}k$. This is seen as follows. Define the covariant shift operator S_i by

$$S_i \psi_+^*(x) = \phi_i^*(x-e_i) \psi_+^*(x-e_i)$$

Its adjoint $S_i^* = S_{-i}$ because of (2.0), and

$$A = \sum_{i=1, \dots, 3} \left[\frac{1}{6} (1 - 3K_1 S_i^*) (1 - 3K_1 S_i) + (k - \frac{3}{2} K_1^2) S_i^* S_i \right]$$

This is manifestly positive if $K_1^2 \leq \frac{3}{2}k$. This completes the proof of Osterwalder Schrader positivity, for free boundary conditions.

Osterwalder Schrader positivity holds also for antiperiodic boundary conditions in time direction and either periodic or antiperiodic boundary conditions in space directions. In this case, reflection θ leaves a pair $\Sigma = (\Sigma_+, \Sigma_-)$ of hyperplanes $x^4 = \text{const.}$ invariant, see figure 5.

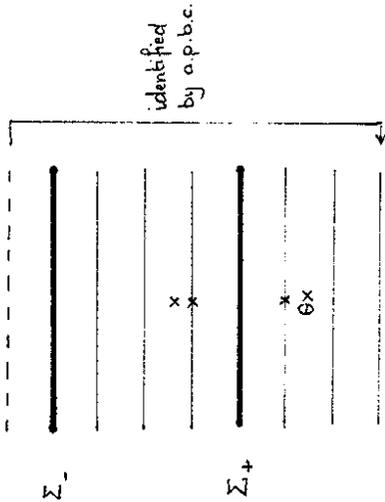


Figure 5 Reflections in a lattice with antiperiodic boundary conditions.

The Schrödinger wave function Ψ depends on ψ_+^*, ψ_-^* on Σ_+ and on ψ_+^*, ψ_-^* on Σ_- . We make the substitution

$$\begin{aligned} \psi_-^+ &= \chi_-^+ \epsilon & \psi_-^+ &= \epsilon^{-1} \chi_-^- & \text{on } \Sigma_+ \\ \psi_+^+ &= \chi_+^+ \epsilon & \psi_+^+ &= \epsilon^{-1} \chi_+^- & \text{on } \Sigma_- \end{aligned}$$

For antiperiodic boundary conditions in time direction, eq. (7.3) with (7.12) gets replaced by

$$\begin{aligned} \langle \theta(0^+) 0 \rangle &= \frac{1}{Z} \int_{b \in \Sigma_+} \prod_{x \in \Sigma_+} d\phi(b) \prod_{x \in \Sigma_+} [d\psi_+^*(x) d\psi_-(x) d\chi_-^+(x) d\chi_-(x)] \\ &\quad \cdot \prod_{x \in \Sigma_-} [d\psi_-^*(x) d\psi_+(x) d\chi_+^*(x) d\chi_+(x)] e^{-\beta} \\ &= \left(\Psi \left(\{ \phi, \psi_+^+, \chi_+^+ \}_{\Sigma_+}, \{ \phi, \psi_-^+, \chi_-^+ \}_{\Sigma_-} \right) \Psi \left(\{ \phi, \psi_+^+, \chi_+^+ \}_{\Sigma_+}, \{ \phi, \psi_-^+, \chi_-^+ \}_{\Sigma_-} \right) \right) \end{aligned}$$

(7.13)

in obvious notation, with

$$\begin{aligned} -\beta &= \sum_{x \in \Sigma_+} \left[\psi_+^*(x) (A \psi_+^*)(x) + \chi_-^+(x) (\bar{A} \chi_-^+)(x) \right] \\ &\quad - \sum_{x \in \Sigma_-} \left[\psi_-^*(x) (\bar{A} \psi_-^*)(x) + \chi_+^+(x) ({}^t A \chi_+^+)(x) \right] \end{aligned}$$

The lemma can be applied again. Since ψ_+, ψ_+ are independent integration variables, they can be transformed independently. By a variable transformation $\psi_{\pm} \rightarrow \psi_{\pm}, \chi_{\pm} \rightarrow -\chi_{\pm}$ the minus sign in the second term of $-\beta$ can be cancelled against the - sign in the argument $(-\psi_-, -\chi_-)$ of the wave function Ψ^+ which came from the use of antiperiodic boundary conditions. As a result, positivity of expression (7.13) is obtained. This completes the proof of Osterwalder Schrader positivity in the case of antiperiodic boundary conditions in the time direction.

We note, finally, that one can also write down a hermitean transfer matrix. It is given by a formula which involves an integration over variables $\phi_4(x)$ attached to timelike links.

References

1. J. Kogut and L. Susskind, Phys. Rev. D9 (1974) 3501;
T.D. Lee, Particle physics and introduction to field theory, Harwood, Chur 1981.
2. H.B. Nielsen, A. Patkos, Nucl. Phys. B195 (1982) 137;
S.L. Adler, Phys. Letters 110B (1982) 302;
S.L. Adler and T. Piran, Phys. Letters B113 (1982) 405, B117 (1982) 91.
3. G. Mack, Phys. Letters 78B (1978) 263;
Ch. Borgs, E. Seiler, Commun. Math. Phys. (in press).
4. B. Durhuus, J. Fröhlich, Commun. Math. Phys. 75 (1980) 103.
5. D.C. Brydges, J. Fröhlich, T. Spencer, Commun. Math. Phys. 83 (1982) 123.
6. M. Aizenmann, Phys. Rev. Letters 47 (1981) 1, Commun. Math. Phys. 86 (1982) 1-48;
K. Symanzik, J. Math. Phys. 7 (1966) 510.
7. J.M. Drouffe, A model regularizing gauge fields on the strong coupling region. TH 3472-CERN (Nov. 82).
8. K.G. Wilson, Phys. Rev. D10 (1974) 2445.
9. K. Osterwalder, R. Schrader, Helv. Phys. Acta 46 (1973) 277, Commun. Math. Phys. 42 (1975) 281.
10. J. Fröhlich, T. Spencer, Commun. Math. Phys. 83 (1982) 411.
11. K.G. Wilson, Phys. Rev. D3 (1971) 1818;
K.G. Wilson and J. Kogut, Phys. Reports 12C (1974) 75.
12. D. Brydges, P. Federbush, Commun. Math. Phys. 73 (1980) 197;
M. Göpfert, G. Mack, Commun. Math. Phys. 82 (1982) 545.
13. K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) 110 (1978) 440.
14. M. Lüscher, Commun. Math. Phys. 54 (1977) 283
15. J. Glimm, Cargèse lectures 1979, in: Recent developments in gauge theories, G. 't Hooft et al., eds., Plenum Press New York (1980).
16. F.A. Berezin, The method of second quantization, Academic Press New York 1966.
17. D. Brydges, J. Fröhlich, E. Seiler, Ann. Phys. 121 (1979) 227;
E. Seiler, Gauge Theories as a problem of constructive Quantum Field Theory and Statistical Mechanics. Lecture Notes in Physics, vol. 159, Springer, Heidelberg 1982.
18. J. Kogut, L. Susskind, Phys. Rev. D11 (1975) 395.
19. L.D. Faddeev, A.A. Slavnov, Gauge Fields, Introduction to quantum theory. Benjamin, Reading, Mass 1980.
20. J. Fröhlich, R. Israel, E.H. Lieb, B. Simon, Commun. Math. Phys. 62 (1982) 1;
J. Fröhlich, E. Lieb, Commun. Math. Phys. 60 (1978) 233.
21. G. Mack, V. Pečkova, Ann. Phys. 123 (1979) 442.
22. D. Ruelle, Commun. Math. Phys. 50 (1976) 189.
23. J. Glimm, A. Jaffe, Quantum Physics, Springer Verlag, Heidelberg 1981;
K. Symanzik, Nucl. Phys. B190 [FS3] (1981) 1.
24. H.S. Sharatchandra, talk presented at DESY.

