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O(N) NONLINEAR SIGMA MODEL IN PERTURBATION THEORY

by

K. Symanzik

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Continuum limit and improved action in lattice theories II

O(N) nonlinear sigma model in perturbation theory

K. Symanzik

Deutsches Elektronen-Synchrotron DESY, Hamburg

Abstract

The method of paper I of this series is applied to the O(N) nonlinear sigma model. Due to use of non-manifestly-invariant perturbation theory the improvement part of the action, computed explicitly to one-loop order, is not manifestly O(N) invariant. It can be brought into manifestly O(N) invariant form by use of linear identities among dimension-four operators, which follow from the field equations of the unimproved action. The adequacy of the resulting two-parameter family of manifestly O(N) invariant improved actions is verified to one-loop order.

1. Introduction

The nonlinear sigma model is defined by a linear continuous symmetry, here O(N), and a nonlinear constraint. To apply perturbation theory this constraint must be solved, to write the action in canonically independent fields. Hereby the linear symmetry turns into a nonlinear one which is not manifest in the perturbation expansion. In two dimensions where the O(N) symmetry cannot be broken /1/, perturbation theory leads to IR divergences that cancel only in invariant linear combinations /2/ after, to define perturbation theory for the model, e.g. an external symmetry-breaking field H is introduced and then let approach zero at the end.

In $2+\varepsilon$ dimensions ($\mathrm{Re}\,\varepsilon>0$), convenient for the improvement technique of paper I of this series /3/, perturbation theory does not require an IR regulator but leads into the phase of broken symmetry due to spontaneous magnetization at low temperature. The subtractions described in sect. 4 of I lead to the action $A_o + \Delta A_o + a^2 A_1$ where ΔA_o and $a^2 A_1$ contain not-manifestly-O(N) invariant terms. In ΔA_o, the nonsymmetry is related to renormalization of the coupling constant /4/ which occurs in perturbation theory in a nonsymmetric way, and the analog of the transformation described in sect. 3.3 of I removes ΔA_o. The remaining improvement part $a^2 A_1{}'$ (cp. sect. 4.2 of I) is still not manifestly symmetric though of the form found by Brézin, Zinn-Justin, and Le Guillou /4/ to be compatible with the O(N) Ward identities. The nonsymmetric form reflects here merely the freedom of choice concerning that part, due to several identities among dimension 4 operators derivable from the field equations of A_o, analogous to the identities in ϕ^4 theory described in sect. 5.2 of I. By use of these identities, $a^2 A_1{}'$ can be brought into manifestly symmetric form, with still two parameters free due to two identities among manifestly invariant dimension 4 operators.

In sect. 2 we construct perturbatively to one-loop order the subtracted action, which is to this order related to the local effective action (LEL) in $2+\varepsilon$ dimensions by merely change of sign. The dim$2+\varepsilon$ and dim$4+\varepsilon$ terms we find are indeed the ones allowed by Ward identities /4/. Their coefficients to this order are computed in appendix A and B. In sect. 3, identities among dim 4 operators are derived. We also show that selfcontractions that make the difference between normal and ordinary operator products affect only the next loop-order, with some algebra relegated to appendix C. The two-parameter family of improved and manifestly O(N) invariant actions, with coefficients as follow from sects. 2 and 3, is in sect. 4 directly derived by requiring improvement of Green functions to one-loop order and extended to all orders. Sect. 5 contains concluding remarks.

2. Perturbatively subtracted lattice action

2.1 Strategy

We apply the prescriptions of sect. 4 of I. Since the field has dimension zero, or $\varepsilon/2$ in $2+\varepsilon$ dimensions, we need to subtract all VFs four times. This is best done as follows: For the action subtracted up to $\mathcal{L}-1$ loop order, one constructs the effective action in the sense of Coleman and Weinberg /5/ up to 4^{th} order in momenta, with $\mathrm{Re}\,\varepsilon > 2\mathcal{L}^{-1}$ for the 4^{th} order term, to $\mathcal{L}$-loop order (the $\mathcal{L}-1$ loop orders vanish by construction). This effective action, rewritten for the lattice and with the "classical field" replaced by the

lattice field operator, is with change of sign the $\mathcal{L}$-loop order subtraction. In the present model, it is advisable to perform the reparametrization described in sect. 4.2 of I and a further transformation described here in sect. 4.3 in each loop order separately, however, before proceeding to the next loop order.

2.2 General form of momentum-expanded effective action

We use the functional method /6/. The generating functional $\Gamma(\pi)$ of VFs (in our case, π has N-1 components) can, upon Taylor-expanding all VFs to 4^{th} order in momenta at all momenta zero, be written in the continuum

$$\begin{aligned}\Gamma(\pi) = -\int dx \Big[& V(\pi) + \tfrac{1}{2}\partial_\mu\pi\cdot Z_{,\mu\cdot\nu}(\pi)\,\partial_\nu\pi + \\ & + \tfrac{1}{2}\partial_\mu\partial_\nu\pi\cdot\partial_\kappa\pi\cdot S_{,\mu\nu\cdot\kappa}(\pi) + \\ & + \tfrac{1}{8}\partial_\mu\partial_\nu\pi\cdot\partial_\kappa\partial_\lambda\pi\cdot T_{,\mu\nu\cdot\kappa\lambda}(\pi) + \\ & + \tfrac{1}{4}\partial_\mu\partial_\nu\pi\cdot\partial_\kappa\pi\cdot\partial_\lambda\pi\cdot U_{,\mu\nu\cdot\kappa\cdot\lambda}(\pi) + \\ & + \tfrac{1}{24}\partial_\mu\pi\cdot\partial_\nu\pi\cdot\partial_\kappa\pi\cdot\partial_\lambda\pi\cdot W_{,\mu\cdot\nu\cdot\kappa\cdot\lambda}(\pi) + \\ & + O(mom^5)\Big].\end{aligned} \tag{2.1}$$

In the presence of hypercubic lattice symmetry (called military symmetry by R. Jost), Z is proportional to $\delta_{\mu\nu}$, $S \equiv 0$, and T, U, W consist of terms proportional to the tensors $\delta_{\mu\nu}\delta_{\kappa\lambda}$, $\delta_{\mu\kappa}\delta_{\nu\lambda}$, $\delta_{\mu\lambda}\delta_{\nu\kappa}$, and $\delta_{\mu\nu}\delta_{\mu\kappa}\delta_{\mu\lambda}$.

For the action

$$A_0 = \sum[-\tfrac{1}{2}\pi K \pi + S(\pi)]a^{2+\varepsilon} - \text{measure term}(\pi) \qquad (2.2)$$

the functions in (2.1) are obtained /6/ by momentum-expanding one-particle-irreducible lattice graphs made of propagators $[K-S_{..}(\pi)]^{-1}$ and vertices that are third and higher derivatives of $S(\pi)$, with π space-"time" independent throughout the graph. Uncontracted π-indices at vertices are external arguments, at which infinitesimal momenta, to first or higher power, enter these graphs. Thus, in (2.1) the coefficient functions are obtained by momentum-expanding graphs as just described with the indicated number (up to four) of external arguments.

2.3 Second order in momenta

For our model, (2.2) is

$$A_0 = a^{2+\varepsilon}\sum_{n\in\mathbb{Z}^{2+\varepsilon}}[-\tfrac{1}{2}\vec{\pi}_n\cdot(\tilde{K}\vec{\pi})_n - \tfrac{1}{2}g^{-1}\sigma_n(\tilde{K}\sigma)_n - \\ - a^{-2-\varepsilon}\ln\sigma_n] \qquad (2.3)$$

where $\vec{\pi} = (\pi_i,\ i = 1\ldots N-1)$, $\sigma \equiv (1-g\vec{\pi}^2)^{1/2}$ and (see sect. 4.1 of I)

$$\tilde{K} = \sum_{\mu=1}^{2+\varepsilon}[-\partial_\mu\partial_\mu^+ + \tfrac{1}{12}a^2(\partial_\mu\partial_\mu^+)^2]. \qquad (2.4)$$

A constant magnetic field H would play the rôle of Δm_B^2 in the ϕ^4 model of I, and a source term $h\sigma$ would bring in also VFs with composite-operator arguments to which the subtraction technique extends in an obvious way. For simplicity, we omit such terms in this section. (See, however, sect. 3.1 below).

Comparing (2.3) with (2.2) we have

$$S''_{ij}(\pi) = -g\sigma^{-1}\pi_i\tilde{K}(\sigma^{-1}\pi_j) + \\ +\sigma^{-1}[\delta_{ij} + g\sigma^{-2}\pi_i\pi_j]\tilde{K}\sigma \qquad (2.5)$$

whereof the last term vanishes for π space-"time" independent. The propagator is, therefore,

$$[K-S''_{..}]^{-1}_{ij} = \tilde{K}^{-1}[\delta_{ij} - g\pi_i\pi_j]. \qquad (2.6)$$

In

$$\Gamma^{1\text{Loop}}(\pi) = -\tfrac{1}{2}\mathrm{Tr}\ln[K-S''_{..}(\pi)] - \text{measure term}(\pi) \qquad (2.7)$$

the last term already effects the subtraction at zero momenta /7/.

Differentiating (2.7) twice leads to the one-loop contributions to Z.. and T.... in (2.1). The vertices follow from (2.5), and we obtain the graphs shown in Fig. 1. Hereby, a broken line denotes $\tilde{K}$ of (2.4) and a dashed solid line the propagator (2.6). The number of derivatives of σ is indicated by strokes, and the two lettered arguments are the external ones carrying infinitesimal external momentum, to second order yielding Z, to 4th order T in (2.1). Dots indicate dummy arguments contracted by the propagator (2.6).

Diagrams A and B of Fig. 1 contribute to both Z and T in (2.1) and can, using $\sigma'_i = -g\sigma^{-1}\pi_i$, be combined:

$$\tilde{K} - g\tilde{K}\sigma^{-1}\pi_r\tilde{K}^{-1}(\delta_{rs} - g\pi_r\pi_s)\sigma^{-1}\pi_s\tilde{K} = \\ = \sigma^2\tilde{K}. \qquad (2.8)$$

Furthermore, noting that (2.6) contains the inverse of the matrix structure of

$$\sigma''_{ij} = -g\sigma^{-1}(\delta_{ij} + g\sigma^{-2}\pi_i\pi_j), \qquad (2.9)$$

we have

$$\sigma^2 \sigma''_{ir} (\delta_{rs} - g \pi_r \pi_s) \sigma''_{sj} = \\ = g^2 (\delta_{ij} + g \sigma^{-2} \pi_i \pi_j) \tag{2.10}$$

such that

(A+B) contribution to (2.1) to 2^{nd} order in momenta =

$$= -\tfrac{1}{2} g \sum_\mu (\partial_\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} + g^{-1} \partial_\mu \sigma \partial_\mu \sigma) \cdot \tag{2.11}$$

$\cdot$ (p^2 coeff. of fig. 2a).

(A solid line denotes the bare improved propagator $\tilde{K}^{-1}$). Due to $\tilde{K}^{-1}\tilde{K} = 1$, diagrams C and D contribute from $\tilde{K}$ only to second order in momenta since the triangle reduces to a tadpole, and give

$$\text{(C+D) contribution to (2.1)} = \sigma^{-1} \Delta \sigma \cdot \text{(diagram fig. 2b)} \tag{2.12}$$

where Δ is the Laplacian. Diagram E contributes only to T in (2.1). Diagram F, again due to $\tilde{K}^{-1}\tilde{K} = 1$, is momenta independent and gives zero. Finally, upon simple algebra,

$$\text{(G+H) contribution to (2.1)} = -\tfrac{1}{2} (N+1) \sigma^{-1} \Delta \sigma \cdot \text{(diagram fig. 2b)}. \tag{2.13}$$

Collecting, we have

$$\Gamma^{0\text{loop}}(\pi) + \Gamma^{1\text{loop}}(\pi) \text{ to second order in momenta} = \\ = -\tfrac{1}{2} \int dx \, (\partial_\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} + g^{-1} \partial_\mu \sigma \partial_\mu \sigma) \cdot \\ \cdot \left[1 + g \cdot (p^2 \text{ term of fig. 2a})\right] - \\ - \tfrac{1}{2} (N-1) \int dx \, \sigma^{-1} \Delta \sigma \cdot \text{fig. 2b} \;. \tag{2.14}$$

The two coefficients are evaluated in Appendix A, (A.2) and (A.4). They have

first-order poles at $\varepsilon = 0$.

It is now seen that the terms obtained are, as they must, the two terms found by Brêzin, Zinn-Justin, and Le Guillou /4/ to be compatible with the O(N) Ward identities. Noting that

$$\partial / \partial g \left[-\tfrac{1}{2} g^{-1} \int \sigma \Delta \sigma \right] = \tfrac{1}{2} g^{-2} \int \sigma^{-1} \Delta \sigma \tag{2.15}$$

we can rewrite

RHS of (2.14) =

$$= -\tfrac{1}{2} \int (\partial_\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} + \hat{g}^{-1} \partial_\mu \hat{\sigma} \partial_\mu \hat{\sigma}) Z_3 , \tag{2.16a}$$

$$\hat{g} = g + (N-1) g^2 a^{-\varepsilon} \cdot (A.4) + \dots \equiv g Z_1 Z_3^{-1}, \\ \hat{\sigma} \equiv (1 - \hat{g} \vec{\pi}^2)^{1/2} , \tag{2.16b}$$

$$Z_3 = 1 + g a^{-\varepsilon} \cdot (A.2) + \dots , \tag{2.16c}$$

Our subtraction prescription requires to subtract the lattice forms (where $-\Delta$ becomes $\tilde{K}$) of the 1-loop terms in (2.14) from A_o, however, the analog of the transformation in sect. 4.2 of I brings them with opposite sign to the LEL side again, such that (2.16) is, to one-loop order and, by the arguments of ref. /4/, to all orders the form of L_o' of equ. (4.4) of I.

The finite parts of the one-loop terms in the coefficients in (2.16) differ from the ones obtained with unimproved propagator. This difference determines the "Λ-ratio" $\Lambda_{imp}/\Lambda_{stan}$ that shows up when comparing computer results for the improved action with corresponding ones for the unimproved (standard) action, at the same value of g. This ratio is given in (A.7).

2.4 Fourth order in momenta

Diagrams A and B (which combine as before) contribute to T in (2.1) by expanding fig. 2a to fourth order in momentum. Diagram E contributes to T with fig. 3a taken at zero momentum. We shall not write in full all diagrams contributing to U and W in (2.1). It is immediately seen, however, that as far as momentum structure is concerned, in addition to figs. 2a and 3a only the expansion of the diagram fig. 3b (cp. figs. 1C,D) to second order in momenta, and of the diagram fig. 3c (cp. figs. 1B,F) to fourth order in momenta are needed, whereby only the terms linear in p and q, resp. linear in p, q, r, and s are relevant since each of these momenta should appear at least to first order. The calculation is given in Appendix B.

Doing the trivial but lengthy O(N) algebra (which can be streamlined using the method of Brown and Duff /5/) and collecting all results, we find

$\Gamma^{1\text{loop}}(\pi)$ to 4^{th} order in momenta = (2.17)

$$= a^{2-\epsilon}\int dx\Big[-\frac{1}{48}c_1(\epsilon)\,g\sum_\mu\left(\partial_\mu^2\vec{\pi}\cdot\partial_\mu^2\vec{\pi}+g^{-1}\partial_\mu^2\sigma\,\partial_\mu^2\sigma\right)+$$

$$+\frac{1}{72}c_2(\epsilon)\,g^2\sum_{\mu\nu}\left(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi}+g^{-1}\partial_\mu\sigma\,\partial_\nu\sigma\right)^2-$$

$$-\frac{1}{48}c_2(\epsilon)\,g^2\Big[\sum_\mu\left(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+g^{-1}\partial_\mu\sigma\,\partial_\mu\sigma\right)\Big]^2+$$

$$+\frac{1}{16}c_3(\epsilon)\,g^2\sum_\mu\left(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+g^{-1}\partial_\mu\sigma\,\partial_\mu\sigma\right)^2+$$

$$+\frac{1}{4}c_4(\epsilon)\,g\sum_\mu\left(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+g^{-1}\partial_\mu\sigma\,\partial_\mu\sigma\right)\cdot\sigma^{-1}\Delta\sigma+$$

$$+\frac{1}{4}(N-1)\,c_5(\epsilon)\left(\sigma^{-1}\Delta\sigma\right)^2\Big].$$

The coefficients $c_i(\epsilon)$ are defined in (B.1 - 5). Again one notes that precisely the dim4 terms compatible with Ward identities found by Brézin, Zinn-Justin, and Le Guillou /4/ appear, considering that under lattice symmetry also the structures of the first and fourth term in (2.17) are allowed. In one-loop order, the term $\Delta\vec{\pi}\cdot\Delta\vec{\pi}+g^{-1}\Delta\sigma\Delta\sigma$ of ref. /4/ has zero coefficient.

The terms in (2.17), rewritten as lattice terms and with opposite sign, yield the improvement part a^2A_1' in (4.4) of I to one-loop order. When transferring (2.17) onto the lattice, its terms should, by virtue of the derivation of (2.17), be interpreted as normal products in the sense of Zimmermann /8/. While massless self contractions are zero in dimensional integration, they are not zero on the lattice and ought to be put in. We shall show in sect. 3.3, however, that these extra terms are two-loop ones and thus outside the one-loop approximation. Therefore we can ignore them.

Choosing the lattice terms invariant under the lattice symmetry operations ($\partial_\mu\to-\partial_\mu^+$, and $\pi/2$ rotations) the simplest transcription yields at this stage the one-loop improved lattice action

$$A^{1loop}_{imp} = a^{2+\varepsilon} \sum \Big\{ -\frac{1}{2}\vec{\phi}\cdot\tilde{K}\vec{\phi} - a^{-2-\varepsilon}(h\sigma + \tag{2.18}$$

$$+ a^{2-\varepsilon}\Big[\frac{1}{48} c_1(\varepsilon)\, g \sum_\mu \partial_\mu \partial_\mu^+ \vec{\phi}\cdot \partial_\mu \partial_\mu^+ \vec{\phi} -$$

$$- \frac{1}{12} c_2(\varepsilon)\, g^2 \sum_{\mu\nu} \Big(\frac{1}{4} (\partial_\mu + \partial_\mu^+)\vec{\phi}\cdot(\partial_\nu + \partial_\nu^+)\vec{\phi}\Big)^2 +$$

$$+ \frac{1}{48} c_2(\varepsilon)\, g^2 \, (\vec{\phi}\cdot K\vec{\phi})^2 -$$

$$- \frac{1}{16} c_3(\varepsilon)\, g^2 \sum_\mu (\vec{\phi}\cdot \partial_\mu \partial_\mu^+ \vec{\phi})^2 +$$

$$+ \frac{1}{4} c_4(\varepsilon)\, g\, \vec{\phi}\cdot(K\vec{\phi})\, \sigma^{-1} K\sigma -$$

$$- \frac{1}{4}(N-1)\, c_5(\varepsilon)\, (\sigma^{-1}K\sigma)^2 \Big]\Big\}.$$

Hereby, we wrote $\vec{\phi} \equiv (\vec{\pi}, g^{-1/2}\sigma)$ and used the fact that in the improvement parts in the square bracket, the unimproved lattice Laplacian in (2.2) of I could be used to sufficient accuracy such that in (2.18) at most next-nearest-neighbor couplings appear. Lattice symmetry invariance could also have been achieved in other ways than in (2.18), however, the difference to (2.18) would have consisted of $a^{4-\varepsilon}$ terms that would have an effect on improvement coefficients only in two-loop order when they themselves appear in loops.
The $c_i(\varepsilon)$ are meromorphic, and regular at $\varepsilon = 0$. The $c_i(0)$ are listed in (B.6).

Rather than calculating two-loop graphs with the one-loop-improved action (2.18), as would be the analog of the procedure in sect. 4 of I, we shall first transform (2.18) into manifestly O(N) invariant form by the tools of the following section, and take up the question of higher-loop orders only in sect. 4.3.

3. Operator identities and normal products

3.1 Not-manifestly-invariant operator identities

The action (2.3) amended by source terms $a^{2+\varepsilon}\sum_n (\vec{J}_n \cdot \vec{\pi}_n + g^{-1/2} h_n \sigma_n)$ leads to the field equation

$$-\tilde{K}\vec{\pi} + \sigma^{-1}\vec{\pi}\tilde{K}\sigma + \vec{J} - g^{1/2} h \sigma^{-1}\vec{\pi} + a^{-2-\varepsilon} g \sigma^{-2}\vec{\pi} = 0 \tag{3.1}$$

It is convenient to interpret $\vec{\pi}$ as $a^{-2-\varepsilon}\partial/\partial\vec{J}$ acting on the functional integral with sources, whereupon (3.1) becomes the Schwinger-Dyson equation. Multiplying (3.1) from the left by $g\vec{\pi}$ and rearranging yields

$$\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h) - g\,(\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma) + g\,(\vec{J}\cdot\vec{\pi} + g^{-1/2} h\sigma + a^{-2-\varepsilon} g \sigma^{-2} + (N-2)\, a^{-2-\varepsilon} g = 0 \tag{3.2}$$

which is the "counting identity" /9/ of the model.

From (2.3) with source terms included follows

$$\partial/(\partial g)(A_0 + \text{source terms}) = \frac{1}{2} g^{-2} a^{2+\varepsilon} \sum \big[\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h) + a^{-2-\varepsilon}\sigma^{-2} g \vec{\pi}^2\big]. \tag{3.3}$$

If we had included the h source term in the subtraction steps in sects. 2.3 and 2.4, $\sigma^{-1}\Delta\sigma$ in (2.12) would have been replaced, as expected from ref. /4/, by the first term in (3.2) remembering that, on the a^0 level, $\tilde{K}$ is the required transcription of $-\Delta$ onto the lattice. This confirms that $\vec{\pi}$

and $g^{-1/2}h$ would be renormalized to one-loop order, and by the argument of ref. /4/ to all orders, in the same way. (In this respect, $g^{-1/2}h$ is not analogous to $-\frac{1}{2}\Delta m_B{}^2\phi^2$ of ϕ^4 theory, since ϕ^2 there requires an independent renormalization factor as does $\vec{\pi}^2 - \langle\vec{\pi}^2\rangle$ in the present model /4,10/.) The two last terms in (3.2) and the last one in (3.3) are two-loop ones: Multiplying (3.2) by $a^{-\varepsilon}$, which is dimensionally its factor as integrand in (2.14), allows to write e.g. the last term as $a^{-2-\varepsilon}(ga^{-\varepsilon}) + a^{-2-\varepsilon}(ga^{-\varepsilon})(\sigma^{-2}-1)$, showing it to be a two-loop term in comparison with $-a^{-2-\varepsilon}\ln\sigma$ in (2.3) which is a one-loop one.

Multiplying (3.2) from the left by $g(\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma)$ yields

$$g(\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma)\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h) - g^2(\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma)^2 + g^2(\vec{J}\cdot\vec{\pi} + g^{-1/2}h\sigma)\cdot(\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma) + \text{2-loop terms} = 0. \tag{3.4}$$

Among the two-loop terms here is also the commutator

$$[g(\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma), g\vec{J}\cdot\vec{\pi}] = a^{-2-\varepsilon}g^2(\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma) - a^{-2-\varepsilon}g\sigma^{-1}\tilde{K}\sigma. \tag{3.5}$$

Namely, multiplying the r.h.s. by $a^{2-\varepsilon}$, dimensionally in (2.17) the coefficient of the first term in (3.4), yields

$$(ga^{-\varepsilon})^2[\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma] - (ga^{-\varepsilon})[a^{-\varepsilon}\sigma^{-1}\tilde{K}\sigma]$$

identifying these as a^0-level two-loop terms since the square brackets herein are zero- and one-loop terms, respectively.

Multiplying (3.2) from the left by the same expressions with all signs except the one of the first term reversed yields

$$[\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h)]^2 - g^2(\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma)^2 - g^2_{;}(\vec{J}\cdot\vec{\pi} + g^{-1/2}h\sigma)^2_{;} + 2g^2(\vec{J}\cdot\vec{\pi} + g^{-1/2}h\sigma)(\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma) + \text{2-loop terms} = 0. \tag{3.6}$$

Here the semicolon sign means ordering J to the left of π and σ. The difference relative to the ordinary square, with the factor $a^{2-\varepsilon}$ included, is

$$-a^{2-\varepsilon}g^2[(\vec{J}\cdot\vec{\pi} + g^{-1/2}h\sigma), \vec{J}\cdot]\vec{\pi} = -(ga^{-\varepsilon})^2[\vec{J}\cdot\vec{\pi} + g^{-1/2}h\sigma] + (ga^{-\varepsilon})^2[g^{-1/2}h\sigma^{-1}] \tag{3.7}$$

where the two last square brackets are a^0-level zero-loop terms. Also all other commutators arising in the step from (3.2) to (3.5) are two-loop terms as seen already from the "loop counting" factors $ga^{-\varepsilon}$. (3.4) and (3.6) will be used in sect. 4.1.

3.2 Manifestly invariant operator identities

Multiplying (3.1) from the left by J and rearranging yields

$$-(\vec{J}\cdot\tilde{K}\vec{\pi} + g^{-1/2}h\tilde{K}\sigma) + (\vec{J}^2 + h^2) + (\vec{J}\cdot\vec{\pi} + g^{-1/2}h\sigma)\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h) + a^{-2-\varepsilon}g(\vec{J}\cdot\vec{\pi} + g^{-1/2}h\sigma)\sigma^{-2} - a^{-2-\varepsilon}g^{1/2}h\sigma^{-1} = 0 \tag{3.8}$$

Multiplying (3.2) from the left by $\vec{J}\cdot\vec{\pi} + g^{-1/2}h\sigma$ and subtracting the result from (3.8) gives, with use of (3.7), in the notation introduced in (2.18)

$$-\vec{J}\cdot\tilde{K}\vec{\phi} + \vec{J}^{\,2} + g\,\vec{J}\cdot\vec{\phi}\;\vec{\phi}\cdot\tilde{K}\vec{\phi} - g\,;(\vec{J}\cdot\vec{\phi})^2_{\,;} - a^{-2-\varepsilon}(N-1)\,g\,\vec{J}\cdot\vec{\phi} = 0 \tag{3.9}$$

with $\vec{J} \equiv (\vec{J}, h)$ and semicolon sign as in (3.6). It will in sect. 4.3 turn out to be significant that in this invariant identity also the "higher-loop" term is so.

Multiplying (3.1) from the left by $\tilde{K}\vec{\pi}$ gives

$$-(\tilde{K}\vec{\pi})^2 + \vec{\pi}\cdot\tilde{K}\vec{\pi}\,\sigma^{-1}(\tilde{K}\sigma - g^{1/2}h + a^{-2-\varepsilon}g\sigma^{-1}) + \vec{J}\cdot\tilde{K}\vec{\pi} + \tfrac{5}{2}(2+\varepsilon)(N-1)a^{-4-\varepsilon} \tag{3.10}$$

where (2.4) has been used. Multiplying (3.2) from the left by $\vec{\pi}\cdot\tilde{K}\vec{\pi} + g^{-1}\sigma\tilde{K}\sigma$ and subtracting the result from (3.10) yields, using (3.5)

$$-(\tilde{K}\vec{\phi})^2 + g\,(\vec{\phi}\cdot\tilde{K}\vec{\phi})^2 - g\,(\vec{J}\cdot\vec{\phi})(\vec{\phi}\cdot\tilde{K}\vec{\phi}) + \vec{J}\cdot\tilde{K}\vec{\phi} - a^{-2-\varepsilon}(N-1)\,g\,\vec{\phi}\cdot\tilde{K}\vec{\phi} + \tfrac{5}{2}(2+\varepsilon)(N-1)a^{-4-\varepsilon} = 0 \tag{3.11}$$

again an entirely invariant identity. - If in (3.10) and (3.11) the unimproved Laplacian K instead of $\tilde{K}$ had been used, in the last terms the factor 5/2 had been replaced by 2.

3.3 Normal products

The terms in the one-loop improvement part (2.18) are designed to implement for $\mathrm{Re}\,\varepsilon > 2$ certain subtractions, and for $-2 < \mathrm{Re}\,\varepsilon < 2$ by analytic continuation corresponding substitutions, as described in sect. 2.1. The way the coefficients were determined in sect. 2.4 implies that those term in (2.18) should be interpreted as normal products in the sense of Zimmermann /8/. However, the difference between ordinary and normal products in (2.18) contributes only in two-loop order and thus can be disregarded in (2.18). Namely, in contractions two operators are replaced by const. $a^{-\varepsilon}$, which combines to a "loop-counting" factor $ga^{-\varepsilon}$ with the factor g that had to be present for dimensional balance relative to an operator product with two π operators less. Depending on the number of derivatives on the π operators, the contraction will be a two--loop order a^0 or a^2 part. The contractions of interest in (2.18) are worked out in Appendix C, and the formulae (C.6) and (C.8-11) show the described effect.

E.g., the difference on the l.h.s. of (C.6) would be relevant in the normal-ordering of (2.18) (at $\varepsilon = 0$) multiplied by g^2a^2, whereupon the first term on the r.h.s. of (C.6) becomes (comparing with (2.1)) a two-loop a^0 term and the second and third terms become (comparing with (2.18)) two-loop a^2 terms. One notes that on the lattice, contractions of "rotationally invariant" terms contain also "rotationally non-invariant" ones.

The only term which requires attention is the last one in (2.18) since, to lowest order:

$$\sigma^{-1}K\sigma = -g\,\vec{\pi}\cdot K\vec{\pi} + \tfrac{1}{2}g\,(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi} + \partial^+_\mu\vec{\pi}\cdot\partial^+_\mu\vec{\pi}).$$

The contraction of the square hereof contains $\langle \pi_i \pi_j \rangle$ of (C.4) with the factor $g^2 (K\pi_i)(K\pi_j)$. Such a term cannot be disregarded since it has no $\varepsilon \downarrow 0$ limit. However, in sect. 4.1 we shall replace the normal-ordered first term in (3.6) by the negative of the normal-ordered other terms, and the third term in (3.6) then contains a contraction $g^2 J_i J_j \langle \pi_i \pi_j \rangle$ that, to lowest order, balances the contraction of the first term of (3.6) to lowest order due to the lowest-order form $K\pi_i = J_i$ of (3.1). Thus, the IR problem is deferred and will be taken up in sect. 4.3.

4. Manifestly O(N) invariant improved action

4.1 Transformation of (2.18) into invariant form

Since the improvement terms in (2.18) are one-loop ones, the generating functional of one-loop improved GFs is

$$G_{imp}^{1loop}\{\vec{J}\} = \text{const} \int \mathcal{D}\pi \, (1 + A_{imp}^{1loop} - A_o) \cdot \exp\left[A_o + a^{2+\varepsilon} \sum \vec{J} \cdot \vec{\phi}\right]. \tag{4.1}$$

Here the identities (3.4) and (3.6) can be used to replace the two last terms of (2.18) by invariant ones, with the result to one-loop accuracy

$$G_{imp}^{1loop}\{\vec{J}\} = \text{const} \int \mathcal{D}\phi \, \Pi \delta(\vec{\phi}^{\,2} - g^{-1}) \cdot \exp\left\{ a^{2+\varepsilon} \sum \left(-\tfrac{1}{2} \vec{\phi} \cdot \tilde{K} \vec{\phi} + \vec{J} \cdot \vec{\phi}\right) + a^2 A_{1inv}^{1loop} \right\} \tag{4.2a}$$

with

$$\begin{aligned} A_{1inv}^{1loop} = {} & a^{2+\varepsilon} \sum a^{-\varepsilon} g \Big\{ \tfrac{1}{48} c_1(\varepsilon) \sum_\mu \partial_\mu \partial_\mu^+ \vec{\phi} \cdot \partial_\mu \partial_\mu^+ \vec{\phi} - \\ & - \tfrac{1}{12} c_2(\varepsilon) g \sum_{\mu\nu} \left[\tfrac{1}{4} (\partial_\mu + \partial_\mu^+) \vec{\phi} \cdot (\partial_\nu + \partial_\nu^+) \vec{\phi} \right]^2 + \\ & + \left[\tfrac{1}{48} c_2(\varepsilon) + \tfrac{1}{4} c_4(\varepsilon) - \tfrac{1}{4} (N-1) c_5(\varepsilon) \right] g (\vec{\phi} \cdot K \vec{\phi})^2 - \\ & - \tfrac{1}{16} c_3(\varepsilon) g \sum_\mu (\vec{\phi} \cdot \partial_\mu \partial_\mu^+ \vec{\phi})^2 + \\ & + \left[-\tfrac{1}{4} c_4(\varepsilon) + \tfrac{1}{2} (N-1) c_5(\varepsilon) \right] g \vec{J} \cdot \vec{\phi} \, \vec{\phi} \cdot K \vec{\phi} - \\ & - \tfrac{1}{4} (N-1) c_5(\varepsilon) g : (\vec{J} \cdot \vec{\phi})^2 : \Big\} \end{aligned} \tag{4.2b}$$

Hereby, according to sect. 3.3 and noting (C.11), all terms may be interpreted as either normal or ordinary products, with the exception of the last term which should be read as normal product, i.e. with its contraction subtracted to avoid perturbative IR difficulties. In (4.2b), K could be used instead of $\tilde{K}$ since the difference amounts to $O(a^4)$ terms that have $O(a^0)$ and $O(a^2)$ effects only in two-loop order.

The invariant identities (3.9) and (3.11) show that the one-loop-improved action is not unique, but that we can obtain a two-parameter family of such actions. This leads us to write down the Ansatz

$$A_{1\,\mathrm{inv}} = \tag{4.3a}$$

$$= a^{2+\varepsilon}\sum\Big\{\vec{J}\cdot\big[\bar{c}_1 g\,\vec{\phi}\,\vec{\phi}\cdot K\vec{\phi} + \bar{c}_2 K\vec{\phi}\big] + \bar{c}_3 g :(\vec{J}\cdot\vec{\phi})^2: + \bar{c}_4 \vec{J}^{\,2} + \bar{c}_5 (K\vec{\phi})^2 + \bar{c}_6 \sum_\mu (\partial_\mu\partial_\mu^+\vec{\phi})^2 + \bar{c}_7 g(\vec{\phi}\cdot K\vec{\phi})^2 + \bar{c}_8 g\sum_\mu(\vec{\phi}\cdot\partial_\mu\partial_\mu^+\vec{\phi})^2 + \bar{c}_9 g\sum_{\mu\nu}\Big[\tfrac{1}{4}(\partial_\mu+\partial_\mu^+)\vec{\phi}\cdot(\partial_\nu+\partial_\nu^+)\vec{\phi}\Big]^2\Big\}$$

with

$$\bar{c}_i \equiv \bar{c}_i(g a^{-\varepsilon}, N, \varepsilon) = \sum_{\ell=1}^{\infty} \bar{c}_{i\ell}(\varepsilon, N)(g a^{-\varepsilon})^{\ell}. \tag{4.3b}$$

Our result so far (4.2b) yields a two-parameter set of meromorphic coefficients $\bar{c}_{i1}(\varepsilon, N)$ regular at $\varepsilon = 0$. We shall discuss (4.3) after verifying the correctness of these coefficients directly.

4.2 Check of invariant Ansatz to one-loop order

We compute from (4.3a) upon functional differentiation w.r.t. J and h GFs and then VFs to one-loop order, write them in momentum space and expand them at fixed momenta w.r.t. small-a behaviour. There are a^0 and $a^{-\varepsilon}$ parts, which are "continuum" ones and ignored, their ε^{-1} singularities being related to renormalization. (There are for the one-loop VFs no genuine IR singularities at $\varepsilon = 0$; if they had occurred, they could have been identified by their dis-

appearance in a constant external field.) Due to the use of the improved Laplacian $\tilde{K}$ of (2.4) in A_0, there is no a^2 part, but an $a^{2-\varepsilon}$, a^4, $a^{4-\varepsilon}$ etc. part. Setting the $a^{2-\varepsilon}$ part zero gives constraints on the $\bar{c}_{i1}(\varepsilon, N)$, solvable by functions that are finite at $\varepsilon = 0$ (cp. end of sect. 4.2 of I).

The calculation leads to the same graphs figs. 1-3 that were discussed in sect. 2.4 and an additional one, and we list only the constraints obtained from the GFs (i.e. corresponding VFs) given:

$$\langle \pi_i(x)\pi_j(x)\rangle: \qquad \bar{c}_{61} - \tfrac{1}{48}c_1(\varepsilon) = 0, \qquad \bar{c}_{21} + \bar{c}_{41} + \bar{c}_{51} = 0 \tag{4.4a}$$

$$g^{-1}\langle \sigma(x)\sigma(y)\rangle: \qquad \bar{c}_{31} + \bar{c}_{41} + \tfrac{1}{4}(N-1)c_5(\varepsilon) = 0 \tag{4.4b}$$

$$g^{-1}\langle \pi_i(x)\pi_j(y)\sigma(z)\sigma(u)\rangle: \qquad \bar{c}_{11} + \bar{c}_{21} - \tfrac{1}{2}(N-1)c_5(\varepsilon) + \tfrac{1}{4}c_4(\varepsilon) = 0 \tag{4.4c}$$

$$\langle \pi_i(x)\pi_j(y)\pi_k(z)\pi_l(u)\rangle: \qquad \bar{c}_{81} + \tfrac{1}{16}c_3(\varepsilon) = 0, \qquad \bar{c}_{91} + \tfrac{1}{12}c_2(\varepsilon) = 0,$$
$$\bar{c}_{51} + \bar{c}_{71} - \tfrac{1}{48}c_2(\varepsilon) - \tfrac{1}{4}c_4(\varepsilon) + \tfrac{1}{4}(N-1)c_5(\varepsilon) = 0. \tag{4.4d}$$

Hereby, for (4.4c) the graph fig. 4 had to be evaluated at zero momentum, where it reduces to fig. 3a. In (4.c,d) only those constraints are listed that are not yet listed already. The $c_i(\varepsilon)$ are the ones in (B.1-5).

Further functions, e.g. $g^{-1/2}\langle \pi_i(x)\pi_j(y)\sigma(z)\rangle$, $\langle \pi_i\pi_j\pi_k\pi_l\sigma\rangle$, $\langle \pi\pi\pi\pi\pi\pi\rangle$ give no additional constraints, and functions $\langle\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\sigma\sigma\rangle$ etc. need no improvement. In fact, no further function gives additional constraints, for the following reason: Improvement is needed of the $(\text{momentum})^4$-term for graphs of type fig. 3c, of the $(\text{momentum})^2$-term for graphs of type fig. 3b, and of the $(\text{momentum})^0$-term for graphs of type fig. 3a, whereby the general type is obtained from the prototype fig. 3a–c by insertion of elements shown in fig. 5a, b, etc. (E.g., inserting fig. 5a into fig. 3a yields fig. 4.) Since infinitesimal momenta can enter into the fig. 3c, b, a type graphs at most at four, resp. two, resp. zero corners, all other corners carrying zero momenta, due to $RR^{-1} = 1$ no coefficient functions other than those calculated for figs. 3c, b, a themselves occur. A functional algorithm similar to the one used in sect. 2.3 then shows that no other constraints on the $\bar{c}_{i1}$ than those in (4.4) arise.

The identities (3.9) and (3.11) give the two invariances

$$\Delta\bar{c}_{11} = -\Delta\bar{c}_{21} = -\Delta\bar{c}_{31} = \Delta\bar{c}_{41} = x \qquad (4.5a)$$

and

$$\Delta\bar{c}_{11} = -\Delta\bar{c}_{21} = \Delta\bar{c}_{51} = -\Delta\bar{c}_{71} = y \qquad (4.5b)$$

with x and y arbitrary. These are indeed invariances of the equs. (4.4). Upon comparing (4.2b) with (4.3a) it is found that the coefficients in (4.2b) obey the constraints (4.4). (The improved action of ref. /11/ has at $\varepsilon = 0$ in (4.3a) $\bar{c}_6 = \frac{1}{48}gc_1(0)$; $\bar{c}_5 = -\frac{1}{4}gc_5(0)$, the other coefficients zero. Hereby the invariant one-loop two-point function is improved, not, however, e.g. the invariant one-loop four-point function.)

4.3 Generality of the Ansatz (4.3)

(4.3a) comprises in the curly bracket all (lattice-) local terms of (engineering) dimension $4+\varepsilon$ which can be formed polynomially from $\vec{\phi}$ and $\vec{\hat{J}}$, with factors g, in an O(N) invariant way. That (4.3a) with suitable coefficients of the form (4.3b) will improve all VFs to arbitrarily high loop order is shown by the following recursive argument: We have already shown improvement for $\mathcal{L} = 1$. VFs computed with the $\mathcal{L} = 1$-improved action of form (4.3a) to two-loop order have a^0, $a^{-\varepsilon}$, and $a^{-2\varepsilon}$ terms, the last ones in non-O(N)-invariant form as in sect. 2.3. As there, we can transform them by $\vec{\phi}$- and g-redefinition into O(N) invariant form. There remain, according to sect. 4 of I, in the small-a expansion local polynomial terms with factors $a^{2-2\varepsilon}$, the polynomials being the ones appearing in (2.17), including also the there accidentally missing term $(\Delta\vec{\pi})^2 + g^{-1}(\Delta\sigma)^2$, since these are all local solutions of the relevant Ward identities /4/. The corresponding lattice terms with opposite sign are then, with the help of the identities (3.4) and (3.6), brought into manifestly O(N) invariant form to yield the $\mathcal{L} = 2$ terms in (4.3b). Hereby the terms that would in addition appear in (3.4) and (3.6) due to use of action $A_0 + a^2 A_{1\,inv}^{1\,loop}$ rather than A_0 are $O(a^4)$ ones and negligible in the two-loop order under consideration, but they contribute to $O(a^0)$ and $O(a^2)$ terms in three-loop order. The $O(a^0)$ terms in that order are again transformed into O(N) invariant form, and the new $a^{2-3\varepsilon}$ terms treated as before, etc.

In each loop-order the freedom of choosing coefficients according to the invariances (4.5) arises anew such that in (4.3a) e.g. one pair $(\bar{c}_i, \bar{c}_j)$ from the following pairs of coefficient functions can be made zero to all orders: one member from $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$, $\bar{c}_4$, one member from $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_5$, $\bar{c}_7$, except for the pair $(\bar{c}_1, \bar{c}_2)$. That the identities (3.9) and (3.11) are invariant also in the "higher"-loop terms prevents inconsistencies.

Finally, we consider the normal product in the $\bar{c}_3$ term in (4.3a), which differs, by perturbative derivation in one-loop order as discussed in sect. 3.3, from the ordinary product by the perturbative contraction. In two dimensions in the absence of a symmetry-breaking external field (e.g. $\hat{\mathfrak{J}}_N \equiv h$ = const) only O(N) invariant linear combinations of GFs are perturbatively IR finite /2/, and this only if the action itself is O(N) invariant. This would be violated unless in the lattice action, to be used in two dimensions, all components of $\vec{\phi} \equiv (\vec{\pi}, g^{-1/2}\sigma)$ are treated (e.g., contracted) symmetrically. The perturbation theoretical contraction yields

$$(\vec{\mathfrak{J}}\cdot\vec{\phi})^2 - :(\vec{\mathfrak{J}}\cdot\vec{\phi})^2: = g^{-1}\hat{\mathfrak{J}}_N^{\,2} + \text{higher orders} \rightarrow (gN)^{-1}\vec{\mathfrak{J}}^2 + \text{higher orders}$$

upon symmetrization. The higher-order terms must again be local ones, and can thus be absorbed in the higher-$\mathcal{L}$ terms in (4.3) if they could affect GFs to the $O(a^2)$ accuracy of interest. Thus, the form (4.3a) is sufficiently general also without the double-dots in the $\bar{c}_3$ term, if in two dimensions we only consider O(N) invariant GFs. Outside of perturbation theory, there are for $N \geq 3$ no IR problems due to spontaneous mass gap. (See also the 1/N expansion analysis in the following paper /12/ of this series.) Perturbation theory is misleading for N = 2, and unavailable for N = 1.

5. Concluding remarks

5.1 Extension of procedure

The improvement detailed in sects. 2 and 4 can be extended to include $O(a^4)$

terms, $O(a^6)$ terms etc. as described in sect. 5.1 of I. Perturbation theory would yield (after transforming terms of lower loop-order and lower a^2-order into manifestly O(N) invariant form) as improvement terms higher-dimensional operators that correspond to local solutions of Ward identities /4/. The general form of their solutions was determined by Heidenreich and Kluberg-Stern /13/. The corresponding higher-dimensional operators are in general not manifestly O(N) invariant but can be replaced by invariant ones with the help of identities analogous to (3.4) and (3.6) derivable from (3.1). The result is that the improved action, analogous to (5.1) of I, can be chosen to have only manifestly O(N) invariant terms, with remaining arbitrariness due to the existence of O(N) invariant identities as discussed in sect. 4.3 for $O(a^2)$ order.

5.2 Nonperturbative determination of improvement coefficients

Improvement in the sense of sect. 1 of I requires to determine the coefficients in (4.3a) to all loop-orders "exactly". As for the ϕ^4 model (sect. 5.2 of I), this is possible only by Monte Carlo checks of improvement itself, of the simplest quantities on the finite lattice. These are, as in ϕ^4 theory, "generalized susceptibilities", i.e. Green functions with all momenta zero, or some momenta at the lowest discrete values available on the finite lattice with periodic boundary conditions. These would, if made normalization independent, have to obey equ. (1.1) of I with decreased r.h.s. That even one-loop-order improvement terms give improved scaling behaviour is shown in refs. /14/. (In not unlimited space-"time", perturbative calculations for vanishing IR regularizing field require care, since the limites $g \searrow 0$, i.e. weak-coupling expansion, and $H \searrow 0$ are there in general not interchangeable even for O(N) invariant quantities /15/. Available are then the "gauge choice" of David /16/ and an entirely O(N) invariant but only implicit method of Lüscher /17/.)

5.3 Other models, and limitations of improvement

In order to obtain a workable improved action for the model of this paper, it was essential to exploit the availability of alternative improved actions since an action of e.g. the direct perturbative form (2.18) is unsuitable for Monte Carlo simulation since σ can vanish. The present technique would be applicable to a large class of models with a global symmetry and a nonlinear constraint, e.g. to the CP^{N-1} model that has instantons for all N and "confinement" as nonperturbative effects. Such effects, leading to corrections to scaling, are expected also for the non-linear sigma model (e.g. refs. /18/). If such corrections, quantitatively not yet well understood, would not disappear faster than prop. a^2, they could not be removed by use of an action with local improvement terms only.

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Appendix A: Calculation of renormalization coefficients

The graph fig. 2a gives

$$\int_k N(k)^{-1} N(k+p) \equiv G(p) = a^{-2-\varepsilon} + \\ + a^{-\varepsilon} d_1(\varepsilon) p^2 + a^{2-\varepsilon} d_2(\varepsilon) \sum p_\mu^4 + O(p^6) \tag{A.1}$$

where

$$\int_k \equiv \prod_{\mu=1}^{2+\varepsilon} (2\pi)^{-1} \int_{-\pi/a}^{\pi/a} dk_\mu$$

and (see (2.4) and formula (4.1) of I)

$$N(k) \equiv \sum_{\mu=1}^{2+\varepsilon} a^{-2} \left[2(1-\cos k_\mu a) + \frac{1}{3}(1-\cos k_\mu a)^2 \right]$$

Thus,

$$d_1(\varepsilon) = \frac{1}{2} a^{\varepsilon} \int_k N(k)^{-1} N''(k_1) = \\ = (2\pi\varepsilon)^{-1} - .0971525 + O(\varepsilon) \tag{A.2}$$

The graph fig. 2b has the value

$$\int_k N(k)^{-1} \equiv a^{-\varepsilon} e(\varepsilon) \tag{A.3}$$

with

$$e(\varepsilon) = d_1(\varepsilon) + \frac{2}{3} a^{\varepsilon} \int_k N(k)^{-1} (1-\cos k_1 a)^2 = \\ = (2\pi\varepsilon)^{-1} + .0731455 + O(\varepsilon) \tag{A.4}$$

With unimproved propagator, we obtain instead

$$d_1(\varepsilon) \rightarrow d_{1stan}(\varepsilon) = (2\pi\varepsilon)^{-1} +$$

$$+ (4\pi)^{-1}\left(-\psi(1) + 3\ln 2 - \ln\pi\right) - \frac{1}{4} + O(\varepsilon) = \qquad \text{(A.5)}$$

$$= (2\pi\varepsilon)^{-1} - .129685 + O(\varepsilon)$$

and

$$e(\varepsilon) \rightarrow e_{stan}(\varepsilon) = d_{1stan}(\varepsilon) + \frac{1}{2}(2+\varepsilon)^{-1}. \qquad \text{(A.6)}$$

The "Λ-ratio" / /, which is up to a relative error O(g) the inverse ratio of the correlation lengths at the same value of g, is

$$\Lambda_{imp}/\Lambda_{stan} = \exp\left\{-2\pi(N-2)^{-1}\left[(N-1)\left(e(\varepsilon) - e_{stan}(\varepsilon)\right) - \right.\right.$$

$$\left.\left. - d_1(\varepsilon) + d_{1stan}(\varepsilon)\right]\right\}\Big|_{\varepsilon=0} = 2.21923 \text{ if } N = 3 \qquad \text{(A.7)}$$

Appendix B: Calculation of improvement coeffients

The graph fig. 2a gives, from (A.1), to fourth order

$$d_2(\varepsilon) = \frac{1}{24}\int_k N(k)^{-1} N''''(k_1) \equiv \frac{1}{24} c_1(\varepsilon) \qquad \text{(B.1)}$$

regular for Re$\varepsilon > -2$. The graph fig. 3a is, according to sect. 4.2 of I, to be computed for Re$\varepsilon > 2$ and analytically continued to $\varepsilon = 0$. In analogy to (4.5) of I we have, for $-2 <$ Re$\varepsilon < 2$

$$a^{-2+\varepsilon}\cdot(\text{fig. 3a}) = a^{-2+\varepsilon}\int_k \left[N(k)^{-2} - (k^2)^{-2}\right] -$$

$$- \left[\frac{1}{8}\pi^{-4}(\pi+2) + O(\varepsilon)\right] \equiv c_5(\varepsilon). \qquad \text{(B.2)}$$

The graph fig. 3b has the value $a^{-\varepsilon}e(\varepsilon)$ of (A.3) at p = 0 and at q = 0, such that

$$a^{-2+\varepsilon}\cdot(\text{fig. 3b}) = a^{-2}e(\varepsilon) + f(\varepsilon)pq + O(a^2)$$

with $f(\varepsilon)$ easiest by setting p = -q, such that for $-2 <$ Re$\varepsilon < 2$

$$f(\varepsilon) = -\frac{1}{2}a^{-2+\varepsilon}\int_k\left[N(k)^{-2}N''(k_1) - 2(k^2)^{-2}\right] +$$

$$+ \left[\frac{1}{8}\pi^{-4}(\pi+2) + O(\varepsilon)\right] \equiv -\frac{1}{2}c_4(\varepsilon). \qquad \text{(B.3)}$$

The graph fig. 3c reduces to fig. 2a upon setting p,q,r, or s = -p-q-r zero. Thus

$$(\text{fig. 3c}) = a^{-2-\varepsilon} G(p+r) + a^{2-\varepsilon}F(pqrs) + O(a^{4-\varepsilon})$$

with G(.) defined in (A.1) and F(pqrs) linear in all four momenta, such that it has the form

$$F(pqrs) = \alpha_1 p.qr.s + \alpha_2 p.rq.s + \alpha_3 p.sq.r + \alpha_0\sum_{\mu=1}^{2+\varepsilon} p_\mu q_\mu r_\mu s_\mu.$$

Setting p + q = 0 yields $\alpha_2 + \alpha_3 = 0$ and integrals for $\alpha_0 + \alpha_1$ and α_1. Setting p = t - q = -r, with $t_\mu \sim \delta_{\mu 1}$, $q_\mu \sim \delta_{\mu 2}$ allows to isolate easily $\alpha_2 = \frac{1}{3}\alpha_1$. Using this expansion of the graph fig. 3c in sect. 2.4 leads us to define, for $-2 <$ Re$\varepsilon < 2$

$$c_2(\varepsilon) \equiv a^{-2+\varepsilon}\int_k\left[N(k)^{-2}N''(k_1)N''(k_2) - 4(k^2)^2\right] - \left[\frac{1}{2}\pi^{-4}(\pi+2) + O(\varepsilon)\right] \qquad \text{(B.4)}$$

and for Re$\varepsilon > -2$

$$c_3(\varepsilon) \equiv a^{-2+\varepsilon} \int_k N(k)^{-2} \left[N''(k_1)^2 - N''(k_1) N''(k_2) \right]. \tag{B.5}$$

At $\varepsilon = 0$, the numerical values are

$$\begin{aligned} c_1(0) &= -\ .29702 & c_2(0) &= -\ .23324 \\ c_3(0) &= \ \ .13998 & c_4(0) &= -\ .069230 \\ & & c_5(0) &= 4.7976 \cdot 10^{-3} \end{aligned} \tag{B.6}$$

Appendix C: Evaluation of contractions

The contractions needed in sect. 3.3 are 2-point Green functions with coinciding arguments. In two dimensions, these are IR finite if at least one argument is (lattice-) differentiated. Since in sect. 3.3 it is argued that we can ignore contractions (with one exception, mentioned there), we will for simplicity take as Green function the free one with unimproved Laplacian:

$$G(a\vec{n}0) = 2a^{-2}(2\pi)^{-2-\varepsilon} \cdot \prod_{\mu=1}^{2+\varepsilon} \left(\int dk_\mu \right) \cdot \exp(-ia\vec{n}\cdot\vec{k}) \cdot \left[\sum_{\mu=1}^{2+\varepsilon} (1 - \cos k_\mu a) \right]^{-1}. \tag{C.1}$$

With use of

$$\partial_\mu - \partial_\mu^+ = a\, \partial_\mu \partial_\mu^+, \tag{C.2a}$$

$$\partial_\mu \partial_\mu^+ (AB) = A\, \partial_\mu \partial_\mu^+ B + B\, \partial_\mu \partial_\mu^+ A + \partial_\mu A\, \partial_\mu B + \partial_\mu^+ A\, \partial_\mu^+ B, \tag{C.2b}$$

$$A\, \partial_\mu B = -B\, \partial_\mu^+ A + \partial_\mu \left[B(1 - a\partial_\mu^+) A \right] \tag{C.2c}$$

(for notation see (2.2) of I, we do not use a summation convention) the following evaluations at coinciding arguments are straightforward (in (C.3b), $\mu \neq \nu$):

$$\begin{aligned} \langle \partial_\mu \pi_i\, \partial_\mu \pi_j \rangle &= \langle \partial_\mu^+ \pi_i\, \partial_\mu^+ \pi_j \rangle = \\ &= -\langle \pi_i\, \partial_\mu \partial_\mu^+ \pi_j \rangle = (2+\varepsilon)^{-1} \langle \pi_i\, K \pi_j \rangle = \\ &= (2+\varepsilon)^{-1} a^{-2-\varepsilon} \delta_{ij}, \end{aligned} \tag{C.3a}$$

$$\begin{aligned} \langle \partial_\mu \pi_i\, \partial_\nu \pi_j \rangle &= \langle \partial_\mu^+ \pi_i\, \partial_\nu^+ \pi_j \rangle = \\ &= -\langle \partial_\mu \pi_i\, \partial_\nu^+ \pi_j \rangle = -\langle \partial_\mu^+ \pi_i\, \partial_\nu \pi_j \rangle = \\ &= \tfrac{1}{2} a \langle \partial_\mu \pi_i\, \partial_\nu \partial_\nu^+ \pi_j \rangle = \\ &= -\tfrac{1}{2} a (1+\varepsilon)^{-1} \langle \partial_\mu \pi_i\, K \pi_j \rangle - \\ &\quad -\tfrac{1}{2} a (1+\varepsilon)^{-1} \langle \partial_\mu \pi_i\, \partial_\mu \partial_\mu^+ \pi_j \rangle = \\ &= \tfrac{1}{2} (1+\varepsilon)^{-1} a^{-2-\varepsilon} (1 - r(\varepsilon)) \delta_{ij}, \quad \mu \neq \nu, \end{aligned} \tag{C.3b}$$

$$\begin{aligned} \langle \partial_\mu \pi_i\, \partial_\mu^+ \pi_j \rangle &= \langle \partial_\mu \pi_i\, \partial_\mu \pi_j \rangle - \\ &\quad - a \langle \partial_\mu \pi_i\, \partial_\mu \partial_\mu^+ \pi_j \rangle = a^{-2-\varepsilon} \left[(2+\varepsilon)^{-1} - r(\varepsilon) \right] \delta_{ij}, \end{aligned} \tag{C.3c}$$

$$\begin{aligned} \langle \pi_i\, \partial_\mu \pi_j \rangle &= -\langle \pi_i\, \partial_\mu^+ \pi_j \rangle = \\ &= \tfrac{1}{2} a \langle \pi_i\, \partial_\mu \partial_\mu^+ \pi_j \rangle = \\ &= -\tfrac{1}{2} (2+\varepsilon)^{-1} a^{-1-\varepsilon} \delta_{ij} \end{aligned} \tag{C.3d}$$

where $r(\varepsilon) = 2\pi^{-1} + O(\varepsilon)$ from (C.1). The $\varepsilon \searrow 0$ singularity of the contraction (see (A.6))

$$\langle \pi_i \, \pi_j \rangle = \big[(2\pi\varepsilon)^{-1} + (4\pi)^{-1}(-\psi(1) + 3\ln 2 - \ln\pi) + O(\varepsilon) \big]\, \delta_{ij} \tag{C.4}$$

is the familiar perturbative IR one of non-O(N)-invariant expectations /2/ and would be avoided by use of an external field as intermediate IR regulator /4/.

From (C.2b) we obtain with $\sigma = (1-g\vec{\pi}^2)^{1/2}$ in lowest order

$$\vec{\pi}\cdot K\vec{\pi} + g^{-1}\sigma K\sigma = \frac{1}{2}\left(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi} + \partial_\mu^+\vec{\pi}\cdot\partial_\mu^+\vec{\pi}\right). \tag{C.5}$$

Using this and repeatedly (C.2a) and (C.3a-c) yields the contractions to lowest order, at $\varepsilon = 0$,

$$\begin{aligned} &(\vec{\pi}\cdot K\vec{\pi} + g^{-1}\sigma K\sigma)^2 - :(\vec{\pi}\cdot K\vec{\pi} + g^{-1}\sigma K\sigma)^2: = \\ &= \alpha^{-2}(N+1-4\pi^{-1})(\vec{\pi}\cdot K\vec{\pi} + g^{-1}\sigma K\sigma) + \\ &+ (-1+3\pi^{-1})\sum_\mu (\partial_\mu\partial_\mu^+\vec{\pi}\cdot\partial_\mu\partial_\mu^+\vec{\pi} + \\ &\qquad + g^{-1}\partial_\mu\partial_\mu^+\sigma\,\partial_\mu\partial_\mu^+\sigma) + \\ &+ \frac{1}{2}(1-2\pi^{-1})(K\vec{\pi}\cdot K\vec{\pi} + g^{-1}K\sigma K\sigma) \end{aligned} \tag{C.6}$$

up to a c-number. In (C.6), the double dots have their usual meaning with

respect to the π field as canonical one. - The notation introduced in (2.18) might suggest different "contractions":

$$\begin{aligned} &(\vec{\phi}\cdot K\vec{\phi})^2 - :(\vec{\phi}\cdot K\vec{\phi})^2: = \\ &= \sum_{ij}\big[\langle\phi_i\,\phi_j\rangle K\phi_i\,K\phi_j + \\ &+ 2\langle\phi_i K\phi_i\rangle\,\phi_j K\phi_j + \\ &+ 2\langle\phi_i K\phi_j\rangle\,\phi_j K\phi_i + \\ &+ \langle K\phi_i\,K\phi_j\rangle\,\phi_i\,\phi_j\big] + \text{c-number} = \\ &= (gN)^{-1}(K\vec{\phi})^2 + \\ &+ 2(1+N^{-1})\langle\vec{\phi}\cdot K\vec{\phi}\rangle\,\vec{\phi}\cdot K\vec{\phi} + \text{c-number} \end{aligned} \tag{C.7}$$

using that, due to symmetry restoration in two dimensions, $\langle\phi_i\,\phi_j\rangle = (gN)^{-1}\delta_{ij}$ and $\langle\phi_i K\phi_j\rangle \propto \delta_{ij}$. (C.7) differs from (C.6), the latter being the correct one in perturbation theory. (In ref. /14/, contractions of the type (C.7) instead of the, in restriction to one-loop order ignorable, type (C.6) ones were used, the difference in the coefficients is small, however, and without consequences for the conclusions of ref. /14/ since the improvement coefficients should really be optimized rather than taken from one-loop perturbation theory.)

Other lowest-order contractions at $\varepsilon = 0$, evaluated with (C.3a-d), are the following ones:

$$\frac{1}{4}\sum_{\mu\nu}\left[(\partial_\mu+\partial_\mu^+)\vec{\pi}\cdot(\partial_\nu+\partial_\nu^+)\vec{\pi}+ \right.$$
$$\left. + g^{-1}(\partial_\mu+\partial_\mu^+)\sigma\,(\partial_\nu+\partial_\nu^+)\sigma\right] \tag{C.8}$$

- : same expression : =

$$= (1-2\pi^{-1})(N+2)\Big[\alpha^{-2}(\vec{\pi}\cdot K\vec{\pi}+g^{-1}\sigma K\sigma)-$$
$$-\frac{1}{4}\sum_\mu(\partial_\mu\partial_\mu^+\vec{\pi}\cdot\partial_\mu\partial_\mu^+\vec{\pi}+g^{-1}\partial_\mu\partial_\mu^+\sigma\,\partial_\mu\partial_\mu^+\sigma)\Big],$$

$$\sum_\mu(\vec{\pi}\cdot\partial_\mu\partial_\mu^+\vec{\pi}+g^{-1}\sigma\,\partial_\mu\partial_\mu^+\sigma)^2- \tag{C.9}$$

- : same expression : =

$$(N+1-4\pi^{-1})\alpha^{-2}(\vec{\pi}\cdot K\vec{\pi}+g^{-1}\sigma K\sigma)+$$
$$+\sum_\mu(\partial_\mu\partial_\mu^+\vec{\pi}\cdot\partial_\mu\partial_\mu^+\vec{\pi}+g^{-1}\partial_\mu\partial_\mu^+\sigma\,\partial_\mu\partial_\mu^+\sigma),$$

and

$$(\vec{\pi}\cdot K\vec{\pi}+g^{-1}\sigma K\sigma)\,\sigma^{-1}K\sigma- \tag{C.10}$$

- : same expression : =

$$=(\tfrac{5}{2}-4\pi^{-1})\alpha^{-2}g\,(\vec{\pi}\cdot K\vec{\pi}+g^{-1}\sigma K\sigma)+$$
$$+g(-1+3\pi^{-1})\sum_\mu(\partial_\mu\partial_\mu^+\vec{\pi}\cdot\partial_\mu\partial_\mu^+\vec{\pi}+$$
$$+g^{-1}\partial_\mu\partial_\mu^+\sigma\,\partial_\mu\partial_\mu^+\sigma)+$$
$$+g(\tfrac{3}{4}-\pi^{-1})(K\vec{\pi}\cdot K\vec{\pi}+g^{-1}K\sigma K\sigma)-$$
$$-(2N-3)\,\alpha^{-2}\sigma^{-1}K\sigma$$

as well as

$$(\vec{J}\cdot\vec{\pi}+g^{-1/2}h\sigma)(\vec{\pi}\cdot K\vec{\pi}+g^{-1}\sigma K\sigma)- \tag{C.11}$$

- : same expression : =

$$=\frac{1}{4}(\vec{J}\cdot K\vec{\pi}+g^{-1/2}hK\sigma)+$$
$$+\frac{1}{4}g^{-1/2}h(\vec{\pi}\cdot K\vec{\pi}+g^{-1}\sigma K\sigma).$$

Figure Captions

Fig. 1. One-loop diagrams contributing to Z_{ij} and T_{ij} in (2.1)

Fig. 2. Graphs needed for renormalization coefficients

Fig. 3. Graphs needed for improvement coefficients (besides Fig. 2a)

Fig. 4. Graph needed in (4.4c)

Fig. 5. Insertions into graphs of Fig. 3.

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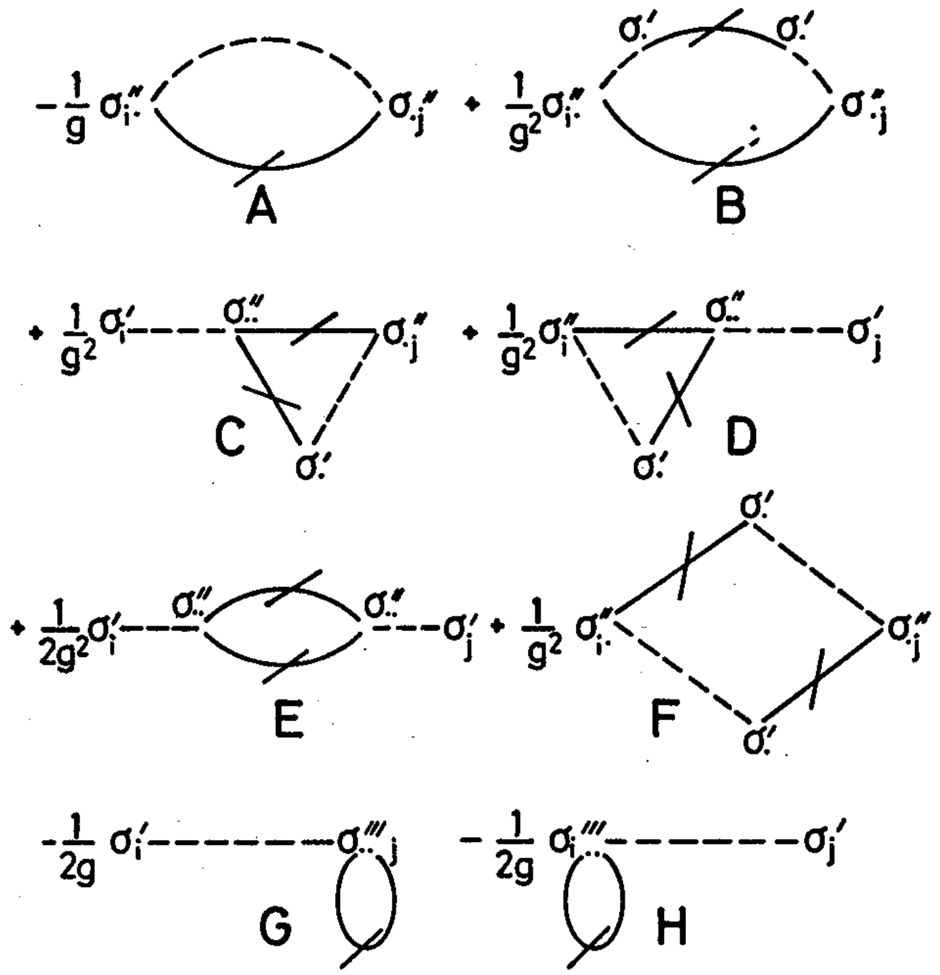

Fig.1

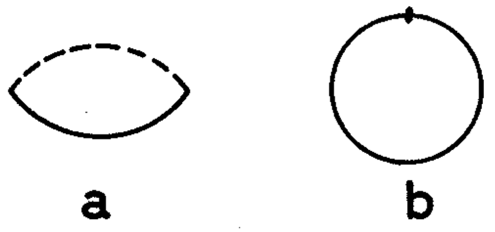

Fig.2

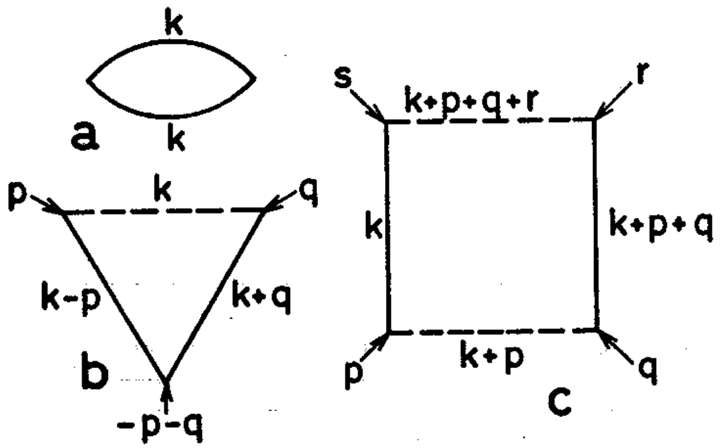

Fig.3

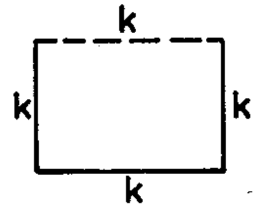

Fig.4

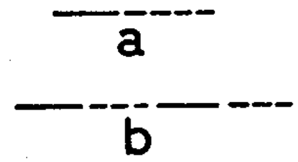

Fig.5