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NONPERTURBATIVE METHODS

by

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0. INTRODUCTION

During the last ten years, quantum field theory and classical statistical mechanics have merged into a single subject and the same methods are used in both fields. Accordingly, speaking of nonperturbative methods in quantum field theory one usually means methods of classical statistical mechanics (other than standard perturbation theory). This includes the Monte Carlo method. Since this method is covered by other lectures at the school, I will concentrate on analytical methods, chiefly expansion methods. For illustration, applications to some models - ferromagnets and pure Yang Mills theory on a lattice - will be discussed. Presentation of this material covered the first 3 lectures at the school. For further reading I recommend E. Seilers book [1]. The 4-th lecture dealt with the effective Z(N) theory (vortex condensation theory) of quark confinement. It is omitted in these notes, see refs. 2a. Tests of this theory by Monte Carlo computations were performed by Pietarinen and the author [2b].

In classical statistical mechanics and Euclidean quantum field theory one wants to compute partition functions (free energies) and correlation functions. This involves computation of n-dimensional integrals (in the n-volume limit). Expansion methods to achieve that fall into two categories

a) simple. By a suitable expansion, the whole problem is reduced to the computation of finite dimensional integrals.

b) sophisticated: The integration variables are divided into groups. (Often this step is preceded by variable transformations, and sometimes by use of integral representations for some of the factors in the integrand, such as a Kramers Wannier duality transformation). Then the groups of variables are treated individually (one after the other) by suitable expansion methods of type a) (high temperature expansions, low temperature expansions, Mayer expansions, cluster expansions of constructive field theory [3], to name the most important ones). Renormalization group calculations [4] fall into this category, they involve many identical steps of type a). More generally, after some of the

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integrations are done, the integrand $\mathcal{Z}$ still depends on the remaining groups of variables. $\ln \mathcal{Z}$ is then called an effective action. The procedure amounts to compute a sequence of effective actions by suitable expansion methods.

Quite complicated systems have already been analyzed by this method, and significant progress is still being made. Here are some examples. Glimm, Jaffe and Spencer [5] have developed a method to deal with field theories with spontaneously broken discrete symmetry, it uses a combination of low temperature - and cluster expansions. Brydges and Federbush have established Debye screening in very dilute 3-dimensional Coulomb gases [6]. Fröhlich and Spencer were able to analyze the Kosterlitz Thouless phase of the two-dimensionale plane rotator model [7]. Göpfert and the author have proven confinement of static quarks in 3-dimensional U(1) lattice gauge theory for all values of the coupling constant [8]. (The results of this work will be described in lecture 3.) Finally, Gawedzki and Kupiainen have announced a rigorous renormalization group treatment of the dipole gas and anharmonic crystal [9]. This was a particularly difficult problem because it requires an infinite number of renormalization group steps to determine the long distance behavior of correlation functions. All of this was achieved by combination of standard expansion techniques of classical statistical mechanics.

1. EXPANSION METHODS

The derivation of any expansion for a free energy or correlation functions may be divided into two steps: i) transformation of the model into a 'polymer system', and ii) application of expansion formulae for polymer systems [10]. Different expansions (for instance high and low temperature expansions) are based on different transformations into polymer systems, while the second step ii) is always essentially the same. I will illustrate the method first at the example of high temperature expansions for pure Yang Mills theory on a lattice. In the course of this discussion I will also review the proof of confinement of static quarks for strong coupling [11].

1.1. High temperature expansions,

and reasons to hope that their intrinsic limitations are not more stringent than those of the Monte Carlo Method if suitable "partially summed" expansions are used.

Sites, links, plaquettes, cubes of the lattice will be denoted by x, ℓ, p, c respectively. In pure Yang Mills theory on a lattice, the basic variables U_ℓ are attached to the links ℓ of the lattice, they are unitary matrices in the gauge group .

Let C be a closed loop which consists of links $\ell_1 \dots \ell_n$. Then the parallel transporter around C is defined by $U_C = U_{\ell_1} \dots U_{\ell_n}$. In particular, the boundary ∂p of a plaquette consists of four links $\ell_1 \dots \ell_4$ and $U_{\partial p} = U_{\ell_1} \dots U_{\ell_4}$.

Let D^k be some representation of the gauge group, and $\chi_k = \mathrm{tr}\, D^k$ the corresponding character. According to Wilson, static quarks which transform according to representation D^k of the gauge group will be confined by a linearly rising potential αr if the Wilson loop expectation value $\langle \chi_k(U_C) \rangle$ obeys an area law [12],

$$\langle \chi_k(U_C) \rangle \sim \exp[-\alpha A]$$

where A is the minimal area of a surface whose boundary is C. α is called the string tension.

It will be usefull to consider partition functions $Z(X)$ that are associated with subsets X of the lattice Λ. The action for such a subset X in a typical pure lattice gauge theory model is

$$L_X(U) = \sum_{p \in X} \mathcal{L}_p(U) \tag{1.1}$$

Summation extends only over those plaquettes p in Λ whose corners are all in X.

$$\mathcal{L}_p(U) = \frac{\beta}{2} \mathrm{tr}\, U_{\partial p} + \mathrm{const.} \qquad \text{for the SU(2) Wilson action} \tag{1.2}$$

The partition function $Z(X)$ for an arbitrary sublattice $X \subset \Lambda$ is defined by

$$Z(X) = \int \prod_{\ell \in X} dU_\ell \; e^{L_X(U)} \tag{1.3}$$

The Wilson loop expectation values on Λ read in this notation

$$\langle \chi_k(U_C) \rangle = \frac{1}{Z(\Lambda)} \int \prod_{\ell \in \Lambda} dU_\ell \; \chi_\ell(U_C)\, e^{L_\Lambda(U)} \tag{1.4}$$

Reformulation as a polymer system.

One writes

$$e^{\mathcal{L}_p(U)} = 1 + f_p(U) , \tag{1.5}$$

considers f_p as "small", and expands in products of f_p's

$$e^{L_X(U)} = \prod_{p \in X} [1 + f_p(U)] = 1 + \sum_{\mathcal{B} \subseteq X} \prod_{p \in \mathcal{B}} f_p(U) \tag{1.6}$$

Summation is over all nonempty sets $\mathcal{B}$ of plaquettes p on X; $\mathcal{B} \subset X$ is a somewhat imprecise short hand notation to keep track of this restriction. The partition functions become

$$Z(X) = 1 + \sum_{\mathcal{B} \subseteq X} \int \prod_{\ell \in X} dU_\ell \prod_{p \in X} f_p(U) \tag{1.7}$$

Next one decomposes $\mathcal{B}$ into connected pieces P, they will be called polymers. A set $\mathcal{B}$ is said to decompose into two disjoint subsets $\mathcal{B}_1$ and $\mathcal{B}_2$ if no plaquette p in $\mathcal{B}_1$ shares a link in its boundary with a plaquette p' in $\mathcal{B}_2$. A nonempty set P of plaquettes is called connected, or a polymer, if it is not a union of two nonempty disjoint subsets $\mathcal{B}_1$ and $\mathcal{B}_2$. Examples are shown in figure 1.

I will use the symbol Σ for union of disjoint subsets.

For polymers P one defines "activities" A(P) by

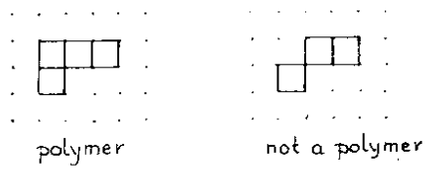


Figure 1

$$A(P) = \int \prod_{\ell \in P} dU_\ell \prod_{p \in P} f_p(U) \qquad (1.8)$$

"$\ell \in P$" is shorthand for "ℓ is a link in the boundary of a plaquette p in P".

If B decompoes into disjoint pieces P_i, then the integral in eq. (1.7) factorizes. (If ℓ is not in the boundary of any plaquette $p \in B$ then the integration over U_ℓ is trivial and gives a factor 1 because the Haar measure dU_ℓ is normalized.) Therefore

$$Z(X) = 1 + \sum_{n \geq 1} \sum_{\substack{(P_1,\dots,P_n) \\ \sum P_i \subseteq X}} \prod_{i=1}^{n} A(P_i) \qquad (1.9)$$

Let us temporarily write $X_\square$ for the set of all plaquettes on X. Summation in eq. (1.9) extends over all partitions of $X_\square$ into disjoint polymers and empty plaquettes. The reader is invited to think of a chessboard as $X_\square$ and of an ample supply of polymers that are cut out of card board. Each polymer can cover a certain number of plaquettes (squares) on the chessboard. Any union of squares that can be cut out of card board without falling apart is a polymer. A certain weight $\mu(P)$ can be given to every such polymer by glueing pieces of lead on top of them; polymers of the same shape are not distinguished and should have the same weight. Now the chessboard can be covered or partly covered by polymers. The rule is that no squares may be covered by more than one polymer, and no two polymers may touch along a line. (They are allowed to touch at corners.) Every such covering adds a contribution exp(total weight of all polymers on the chessboard) to the partition function. The activities are $A(P) = e^{\mu(P)}$ in this example.

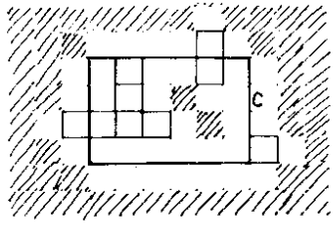
An expression of the form (1.9) is called <u>partition function of a polymer system</u>. It is sometimes useful to admit activities which are not necessarily positive, but it is always required that Z(X) is positive for all X. In our applications to lattice gauge theory this requirement is fulfilled by definition (1.3) of Z(X).

<u>Extension to Wilson loops.</u>

Suppose the loop C consists of links $\ell_1 \dots \ell_m$ so that $U_C = U_{\ell_1} U_{\ell_2} \dots U_{\ell_m}$. The same steps that lead to eq. (1.7) give

$$Z(\Lambda)\langle \chi_k(U_C)\rangle = \int \prod_{\ell \in \Lambda} dU_\ell \, \chi_k(U_{\ell_1} \dots U_{\ell_m}) \Big\{ 1 + \sum_{B \subseteq \Lambda} \prod_{p \in B} f_p(U) \Big\} \qquad (1.10)$$

We want to exploit the factorization properties of the integral again, to do so we introduce a suitable new definition of polymer. Every set B of plaquettes on Λ specifies a set of $n+1 \geq 1$ polymers. We decompose B into connected pieces as before. Polymers P_0 consists of all (possibly none) those connected pieces of B that touch the Wilson loop C along a link. An example is shown in figure 2. It is convenient to consider the links ℓ in the loop C as part of P_0 also. (It can happen that P_0 consists only of C.)



P_0 : loop C and displayed plaquettes

$\Lambda - \bar{P}_0$: shaded plaquettes

figure 2

$P_1, \dots, P_n$ are the other connected pieces of B. They do not touch the loop C along any link. An activity $K_C(P_0)$ is defined by

$$K_C(P_0) = \int \prod_{\ell \in P_0} dU_\ell \, \chi_k(U_C) \prod_{p \in P_0} f_p(U) \qquad (1.11)$$

The activities of the other polymers which do not touch C are defined by eq. (1.8) as before.

Proceeding as in the derivation of eq. (1.9) one obtains

$$Z(\Lambda)\langle \chi_k(U_C)\rangle = \sum_{n \geq 0} \sum_{(P_0,\dots,P_n)} K_C(P_0) \prod_{i=1}^{n} A(P_i) \qquad (1.12)$$

An empty product, which arises when $n = 0$, is read as 1. Summation is over partitions of Λ into disjoint polymers $P_0, \dots, P_n$. The loop C is considered as part of P_0, and disjoint means that different polymers may not overlap or touch along a link.

The next step is a partial resummation [11] : One sums over all those partitions $(P_0, P_1, \dots, P_n)$ with a fixed P_0. They are in one to one correspondence with partitions $(P_1, \dots, P_n)$ of $\Lambda - \bar{P}_0$. $\Lambda - \bar{P}_0$ consists of all those plaquettes on Λ which do not touch P_0 along a link

(see figure 2). Making use of eq. (1.9) the result of this partial resummation takes the form

$$\langle \chi_k(U_C)\rangle = \sum_{P_o} \frac{Z(\Lambda - \tilde{P}_o)}{Z(\Lambda)} K_C(P_o) \tag{1.13}$$

Summation is over polymers P_o as described above; they contain the path C.

Proof of confinement of static quarks (with nontrivial transformation law under the center of the gauge group) for small β (strong coupling) [11].

To be specific, let us consider a theory with gauge group SU(2) and Wilson action. I choose the additive constant in expression (1.2) for $\mathcal{L}_p$ so that

$$\mathcal{L}_p(U) = \frac{\beta}{2}\left[\tfrac{1}{2} U_{\partial p} + 2\right] \geq 0 \tag{1.14}$$

Then

$$0 \leq f_p(U) \leq \kappa\beta \tag{1.15}$$

for a constant $\kappa \geq 1$ and sufficiently small β. Since integrations over U_ℓ for $\ell \notin X$ are trivial, expression (1.3) for Z(X) is equivalent to

$$Z(X) = \int \prod_{\ell\in\Lambda} dU_\ell \prod_{p\in\Lambda} \left[1 + \tilde{f}_p(U)\right]$$

with

$$\tilde{f}_p(U) = \begin{cases} f_p & \text{if } p \in X \\ 0 & \text{otherwise} \end{cases}$$

Since $f_p \geq 0$ it follows that

$$0 < Z(X) \leq Z(\Lambda) \quad \text{for } X \subseteq \Lambda$$

Consequently, eq. (1.13) tells us that

$$|\langle \chi_k(U_C)\rangle| \leq \sum_{P_o} |K_C(P_o)| \tag{1.16}$$

The crucial question is now: For which P_o is $K_C(P_o) \neq 0$? Here the center of the gauge group comes in.

The center $\mathbb{Z}_2$ of SU(2) consists of the two matrices ± 1, and the

character χ_k of the (2k+1)-dimensional representation of SU(2) obeys

$$\chi_k(-U_C) = (-1)^{2k} \chi_k(U_C) \tag{1.17}$$

The static quarks transform nontrivially under the center of the gauge group SU(2) if they have fractional colour-isospin $k = \frac{1}{2}, \frac{3}{2}, \ldots$. It follows from eq. (1.17) that

$$K_C(P_o) = 0 \tag{1.18}$$
unless P_o contains all plaquettes in a surface with boundary C.

Idea of the proof of assertion (1.18) [13]. Suppose P_o leaves a hole. Then I can find a coclosed set **T** of plaquettes which winds around C and shares no plaquette with P_o. "Coclosed" means that every 3-dimensional cube in Λ has an even number of plaquettes in P_o in its boundary. An example is shown in fig. 3.

Because **T** is coclosed, there exists a variable transformation

$$U_\ell \to U'_\ell = U_\ell \gamma_\ell \quad , \gamma_\ell \in \mathbb{Z}_2 \tag{1.19}$$

with the property that

$$U_{\partial p} \to \begin{cases} -U_{\partial p} & \text{if } p \in \mathbf{T} \\ U_{\partial p} & \text{otherwise} \end{cases} \tag{1.20}$$

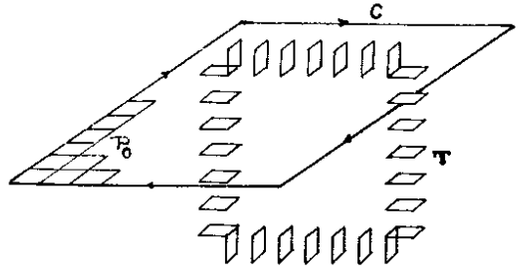


figure 3

For instance, if **T** is as in figure 4, one chooses $\gamma_\ell = -1$ for $\ell \in S$, and $\gamma_\ell = +1$ otherwise. Moreover, since **T** winds around C, the transformation takes

$$U_C \to U'_C = -U_C \tag{1.21}$$

Finally, because of invariance of Haar measure

$$dU_\ell = dU'_\ell \tag{1.22}$$

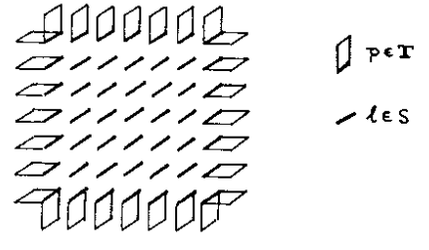
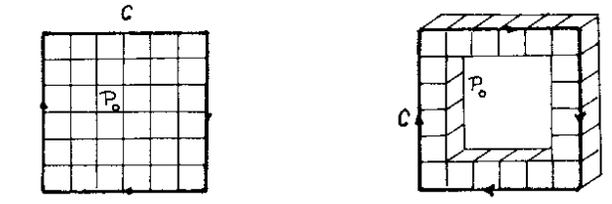


figure 4

Since **T** does not intersect P_o, $f_p(U) = f_p(U')$ for $p \in P_o$.

On the other hand, because of eq. (1.17), $\chi_k(U'_C) = -\chi_k(U_C)$ if $k = \frac{1}{2}, \frac{3}{2}, \ldots$. Therefore the integrand of expression (1.11) for $K_C(P_o)$ changes sign under the above variable transformation. In conclusion, $K_C(P_o) = 0$ if P_o leaves a hole, q.e.d.

If the static quarks carry integral isospin $k = 1, 2, \ldots$ then assertion (1.17) does not hold. In this case $K_C(P_o) \neq 0$ for P_o as shown in figure 5b.



a) $k = \frac{1}{2}$ b) $k = 1$

fig. 5 leading contributions to (1.13)

Now we return to inequality (1.16). Suppose that C bounds a rectangle if size L · T. Then a surface with boundary C contains at last L · T plaquettes. Therefore (1.18) implies that

$$|\langle \chi_k(U_C)\rangle| \leq \sum_{\substack{P_o \\ |P_o| \geq L\cdot T}} |K_C(P_o)|$$

if $k = \frac{1}{2}, \frac{3}{2}, \ldots$. I write $|P_o|$ for the number of plaquettes in P_o.

Since $\int dU_{\cdot} = 1$ and $0 \leq f_p \leq \varkappa\beta$, if follows from definition (1.11) that

$$|K_C(P_o)| \leq \chi_k(1)\,(\varkappa\beta)^{|P_o|} \tag{1.23}$$

Finally we need a combinatorial estimate. It is a corollary of Eulers solution of the Königsberg bridge problem [14] and says that the number of polymers P_o with a given number $n = |P_o|$ of plaquettes is bounded by $\varkappa_1^{|C|+n}$, where $|C|$ is the length of C (= 2(L+T)) and $\varkappa_1$ is some constant. It follows that

$$\begin{aligned} |\langle \chi_k(U_C)\rangle| &\leq \sum_{n \geq L\cdot T} \varkappa_1^{|C|}\,(\varkappa\varkappa_1\beta)^n\,\chi_k(1) \\ &= \varkappa_1^{|C|}\,(\varkappa_1\varkappa\beta)^{L\cdot T}\,(1-\varkappa\varkappa_1\beta)^{-1}\,\chi_k(1) \\ &\leq \text{const}\cdot \varkappa_1^{|C|}\,e^{-\alpha_o L\cdot T} \end{aligned} \tag{1.24}$$

if β is small, and $k = \frac{1}{2}, \frac{3}{2}, \ldots$, with $\alpha_o = -\ln \varkappa\varkappa_1\beta > 0$. This is the promised area law. If k is integer, the contribution of the polymer P_o of figure 5b produces a perimeter law.

Möbius inversion [15]

So far we have obtained only an upper bound on the Wilson loop expectation value. If one wants to actually calculate $\langle \chi_k(U_C)\rangle$ one needs $Z(X)/Z(\Lambda)$ for subsets $X \subset \Lambda$ since

$$\langle \chi_k(U_C)\rangle = \sum_{P_o} \frac{Z(\Lambda - \bar{P}_o)}{Z(\Lambda)}\, K_C(P_o)$$

by eq. (1.13). This leads us to ask for expansions for free energies $-\ln Z(X)$, since

$$Z(X)/Z(\Lambda) = \exp\left[\ln Z(X) - \ln Z(\Lambda)\right] \tag{1.25}$$

$Z(X)$ has been exhibited as partition function of a polymer system in eq. (1.9), for $X \subset \Lambda$. From now on we will regard X and Λ as sets of plaquettes. We look for a "Lagrangean" $\mathbf{L}(Y)$ of our polymer system which is defined for all subsets $Y \subset \Lambda$ and has the property that

$$\ln Z(X) = \sum_{Y \subseteq X} \mathbf{L}(Y) \tag{1.26}$$

$\mathbf{L}(Y)$ is uniquely determined by this equation, since it can be inductively computed from it. The explicit inversion formula is given in eq. (1.27) below. Eq. (1.26) provides an inductive specification of localization of free energy . $-L(Y)$ is the part of the free energy of a system in Y that is spread out throughout Y, i.e. equal to the free energy in Y minus whatever part of it is already localized in some proper subsets of Y.

The inversion formula for $\mathbf{L}(Y)$ reads

$$\mathbf{L}(Y) = \sum_{W \subseteq Y} \vartheta_{YW} \ln Z(W) \quad \text{with} \quad \vartheta_{YW} = (-1)^{|Y|-|W|} \tag{1.27}$$

$|Y|$ = number of plaquettes in Y. $L(Y)$ has the important property that

$$\mathbf{L}(Y) = 0 \text{ if } Y \text{ is not polymer-connected.} \tag{1.28}$$

Y is "not polymer connected" if it is union of two nonempty subsets Y_1, Y_2 such that there exists no polymer which is a subset of Y and intersects

both Y_1 and Y_2. We say that Y_1, Y_2 are "polymer disjoint" in this case.

The Möbius inversion formulae (1.26), (1.27) have been known for a long time, but they appear to be not as generally known as they ought to be. We pause to give the proof of (1.27) and (1.28).

The proof of eq. (1.27) proceeds in two steps. First one notes that

$$\sum_{\substack{Y \\ X \subseteq Y \subseteq X'}} \vartheta_{YX} = \begin{cases} 1 & \text{if } X = X' \\ 0 & \text{otherwise} \end{cases} \tag{1.29}$$

Then one uses this to show that expression (1.27) statisfies (1.26). By uniqueness of $\mathbf{L}(Y)$ it must therefore be true. Inserting (1.27) into (1.26) we obtain

$$\text{r.h.s. of (1.26)} = \sum_{W \subseteq Y \subseteq X} \sum_{W} \vartheta_{YW} \ln Z(W) = \ln Z(X)$$

by eq. (1.29). The proof of eq. (1.29) follows from the binomial theorem. Let $s = |X|$, $t = |X'|$, $n = |Y|$ be the number of plaquettes in X, X', Y, respectively. Then $\vartheta_{YX} = (-1)^{n-s}$. Y is fixed by selecting n-s elements of X'-X. This can be done in $\binom{t-s}{n-s}$ ways. So

$$\text{l.h.s. of (1.29)} = \sum_{\substack{n \\ s \le n \le t}} (-1)^{n-s} \binom{t-s}{n-s} = \sum_{k=0}^{t-s} (-1)^k \binom{t-s}{k}$$

$$= (1-1)^{t-s} = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{otherwise} \end{cases}$$

Since $X \subseteq X'$, $t = s$ implies $X = X'$. □

Next, I give the proof of (1.28). Suppose that Y is not polymer connected, so that it is union of two polymer disjoint subsets Y_1, Y_2. Let X be any subset of Y. Then $X_1 = Y_1 \cap X$ and $X_2 = Y_2 \cap X$ are also polymer disjoint if they are both nonempty. It follows from formula (1.9) for the partition function that in this case

$$Z(X) = Z(X_1) Z(X_2)$$

If X_1 or X_2 is empty, the same is also true because $Z(\emptyset) = 1$. From its definition it follows that ϑ_{YX} also factors: $\vartheta_{YX} = \vartheta_{Y_1 X_1} \vartheta_{Y_2 X_2}$. Therefore

$$\mathbf{L}(Y) = \sum_{X_1 \subseteq Y_1} \sum_{X_2 \subseteq Y_2} \vartheta_{Y_1 X_1} \vartheta_{Y_2 X_2} \left[\ln Z(X_1) + \ln Z(X_2) \right] \tag{1.30}$$

Y_1 and Y_2 are nonempty by hypothesis. It follows from eq. (1.29) with X = empty set that both terms in (1.30) are zero. Thus $\mathbf{L}(Y) = 0$. □

Use as an expansion formula

I propose to use eq. (1.26) as an expansion formula for the free energy

$$\ln Z(X) = \sum_{\substack{Y \\ (Y \subseteq X)}} \mathbf{L}(Y) \tag{1.26}$$

To obtain a useful approximation for the free energy of an arbitrarily large system X, one should truncate the sum over Y. For instance, one may consider omitting all subsets Y with a diameter bigger than some d. To use a truncated expansion (1.26) for computational purposes, one has to compute $\mathbf{L}(Y)$ for small sets Y from eq. (1.27) combined with eq. (1.9) for Z(W), see eq. (1.32) below. Both equations involve finite sums.

Expansion (1.26) has the essential property that it is a finite sum for a finite lattice X. It can therefore be expected to converge fast for an infinite system if the polymer system has no long range correlations. This is in sharp contrast with the standard high temperature expansions (s. below). They involve infinite sums on finite lattices X already, and they may diverge on a finite lattice. Their divergence is therefore not in general indicative of any kind of long range correlations in the polymer systems. This is important for instance in the SU(2) lattice gauge theory model (1.2). This model is thought (or hoped) not to have infinite range correlations for any finite value of the coupling parameter β. Nevertheless the standard high temperature expansions appear to diverge outside the strong coupling regime $\beta < 2.2$. It is hoped that use of the expansion (1.26) offers a way to overcome this limitation. As we shall see in the next subsection, (1.26) may be regarded as a partially summed form of the standard high temperature expansions.

In the expansions (1.26) we expect "divergence" (or very slow convergence) only if the polymer system has long range correlations, for instance

$$\frac{Z(\Lambda - A - B)}{Z(\Lambda)} - \frac{Z(\Lambda - A)}{Z(\Lambda)} \frac{Z(\Lambda - B)}{Z(\Lambda)} \xrightarrow{\text{fast}} 0 \tag{1.31a}$$

when distance $(A,B) \to \infty$

that is

$$\ln \frac{Z(\Lambda - A - B) Z(\Lambda)}{Z(\Lambda - A) Z(\Lambda - B)} = - \sum_{\substack{Y \\ A \cup B \subset Y \subseteq \Lambda}} \mathbf{L}(Y) \xrightarrow{\text{fast}} 0 \tag{1.31b}$$

The r.h.s. of eq. (1.31b) involves the sum over (large) polymer-connected sets Y that contain both the widely separated subsets A and B. Expressions of the form on the r.h.s. of (1.31a) are called reduced correlation functions in the theory of polymer systems [10] .

The relation between convergence and long range correlations in expansions (1.26) for infinite systems deserves further study.

It is amusing to see how the expansions react when the additive constant in the Lagrangean $\mathcal{L}_p$ of our lattice gauge theory model is changed: $\mathcal{L}_p \to \mathcal{L}_p + c$. If $f_p = e^{\mathcal{L}_p} - 1$ is small for some choice, it is not small for another choice. As a result, the convergence of the standard high temperature expansions is affected by such a change. Not so for the expansions (1.26). The change $\mathcal{L}_p \to \mathcal{L}_p + c$ changes $Z(X) \to Z(X) e^{c|X|}$. As a result $\mathbf{L}(Y) \to \mathbf{L}(Y) + c$ if Y is a single plaquette, while $\mathbf{L}(Y)$ for all other sets Y are unchanged. (To see this it suffices to note that eq. (1.26) remains valid for any finite X after these substitutions, if it was valid before.) Therefore, convergence properties of (1.26) are unaffected.

The standard high temperature expansions

They can be obtained from eq. (1.26) by further expansion. One starts from expansion (1.26) with eqs. (1.27), (1.9) for $\mathbf{L}(Y)$ inserted, viz.

$$\ln Z(\Lambda) = \sum_Y \mathbf{L}(Y) \tag{1.26}$$

with

$$\mathbf{L}(Y) = \sum_{X \subseteq Y} \vartheta_{YX} \ln \Big(1 + \sum_{\substack{(P_1,\dots,P_n) \\ \sum P_i \subseteq X}} \prod_i A(P_i) \Big) \tag{1.32}$$

Now one expands $\mathbf{L}(Y)$ in powers of the activities, using $\ln(1+x) = x - \frac{x^2}{2} + \dots$ and the multinominal theorem. As a result one obtains an expression of the form [10]

$$\mathbf{L}(Y) = \sum_Q a(Q) A(Q) \tag{1.33}$$

Summation is over sets Q of not necessarily distinct polymers, polymer P_i may occur with multiplicity n_i. Every plaquette in Y must be contained in at last one polymer P_i in Q. a(Q) are combinatorial coefficients independent of the activities, and

$$A(Q) \equiv A(P_1)^{n_1} \dots A(P_k)^{n_k} \quad \text{if } Q = (P_1^{n_1}, \dots, P_k^{n_k})$$

The set Y of plaquettes which are contained in at least one polymer in Q is called the shadow of Q.

Inserting (1.33) into (1.26) one obtains the final result

$$\ln Z(\Lambda) = \sum_Q a(Q) A(Q) \tag{1.34}$$

Summation is over all Q whose shadow is polymer connected and contained in Λ.

Expansion (1.34) may diverge on a finite lattice.

Example: Λ = a single plaquette, a single polymer P = Λ with activity A.

$$\ln Z(\Lambda) = \ln(1+A) = \sum \frac{(-1)^{n+1}}{n} A^n \quad \text{diverges for } |A| > 1 .$$

Sufficient conditions for convergence of expansions (1.34) are known. (They are derived by use of Kirkwood Salsburg equations, cp ref. 10.) Convergence is assured if there exists a number $\xi > 1$ such that

$$\frac{1}{\xi} \Big[1 + \sup_P \sum_{\bar{P}: P \in \bar{P}} \xi^{|\bar{P}|} A(\bar{P}) \Big] < 1 \tag{1.35}$$

Summation is over all polymers which contain a given plaquette (in case the polymers are made of plaquettes).

The problem of the roughening transition

Let us now return to the expansion for Wilson loop expectation values, eq. (1.13). In the high temperature regime (β small enough) only polymers P_o (with boundary C = Wilson loop) which differ from the minimal surface with boundary C by small deformations need be taken into account. An example is given in figure 6.

One can make use of this to obtain an expansion for $\ln \langle \chi_k(U_C) \rangle$ in the limit of large loops, that is for the string tension α. This is done by performing another high temperature expansion in which the deformations in P_o (and the clusters which come from the expansion of $\ln Z(\Lambda - \bar{P}_o)/Z(\Lambda)$) are treated as the polymers. Series expansions for the string tension α were computed in this way by Münster [16].

This series for the string tension α itself must be expected to diverge when $\beta > \beta_R$, where β_R is the roughening transition point [17]. A roughening transition is expected to take place on the basis of general

arguments [18]. It means the following. One may look at fluctuations of the surface P_o, i.e. the average deflection d squared, see figure 7. It is determined by the relative size of the contribution from surfaces P_o in expansion (1.13). For $\beta < \beta_R$, d is bounded

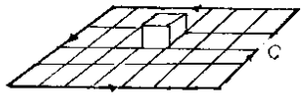


figure 6

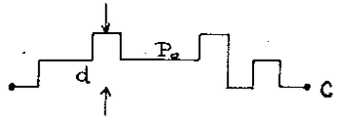


figure 7

independent of the side length T of the Wilson loop. In contrast, for $\beta > \beta_R$ is grows logarithmically with T: $d \propto \ell nT$. (*) This means that increasingly larger deformations in P_o become important for increasing T and therefore the series expansion for α, which is an expansion in increasing size of the deformation for infinitely large loops C, cannot converge for $\beta > \beta_R$. In SU(2) lattice gauge theory will Wilson action, $\beta_R \approx 2$ according to ref. 17. On the other hand, the roughening transition is a change in asymptotic behavior for large Wilson loop, it is not expected to correspond to nonanalytic behavior of $\langle \chi_k(U(C)) \rangle$ for fixed finite C.

Conclusion: If one wants to use series expansions beyond $\beta = \beta_R$, one must be content with expansions for finite Wilson loops, and the "partially summed" high temperature expansions which I have described earlier in this lecture should be used. The larger the loop, the more terms in the expansion will be needed.

Let us note that the Monte Carlo Method is also limited to finite Wilson loops (3 x 3 or so).

1.2 Other expansions

I will very briefly mention two other expansions.

1) Low temeprature expansions, e.g. for lattice gauge theories with discrete gauge group G [19]

example: $G = Z_2$, $U_\ell = \pm 1$

$$Z = \sum_U \exp \sum_P \beta [U_{\partial p} - 1]$$

so that

(*) The roughening transition can also be defined without recourse to expansions in terms of the behavior of expectaion values $\langle tr\, U_C\ tr\, U_{\partial p} \rangle$.

$$Z = 1 + \sum_{\mathcal{B}} e^{-2\beta|\mathcal{B}|} \tag{1.36}$$

B is the set of plaquettes where $U_{\partial p} = -1$. It is coclosed, i.e. every 3-dimensional cube c in A contains an even number of plaquettes in B in its boundary. $|B|$ is the number of plaquettes in B. B can be decomposed into connected components P. In this way the partition function is found to equal the partition function of a polymer system. The polymers P are connected coclosed sets of plaquettes. An example is shown in figure 8. Their activities are

$$A(P) = e^{-2\beta|P|} \tag{1.37}$$

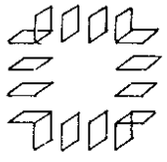


figure 8

2. Mayer expansion for dilute gases [20]

example: A gas of particles with "charges" $q_j = \pm 1$ which interact through a potential $q_i q_j v(x_i - x_j)$ of finite range.

The canoncial partition function of N particles is

$$Z_N = \sum_{\{q_j\}} \int dx_1 \ldots dx_n \exp\Big[-\beta \sum_{(ij)} q_i q_j v(x_i - x_j)\Big] \tag{1.38}$$

Polymers P are subsets of the N particles {1 ... N}, their activities will only depend on the number of particles that are selected. We may label them by 1 ... n = |P|

$$A(P) = \sum_{\mathcal{G}} \int dx_1 \ldots dx_n \sum_{\{q_j\}} \prod_{(ij)\in\mathcal{G}} f_{ij} \tag{1.39}$$

with

$$f_{ij} = -1 + \exp\left[-\beta q_i q_j v(x_i - x_j)\right] \tag{1.40}$$

A graph $\mathcal{G}$ on n vertices {1 ... n} is specified by prescribing the pairs (ij) of vertices which are joined by a line. These pairs are considered as the elements $(ij) \in \mathcal{G}$ of $\mathcal{G}$. Summation in (1.39) is over all connected graphs on vertices {1 ... n}. There exists a useful formula (the "tree formula") with which the sum over graphs can be reexpressed, see ref. 21.

A simplification occurs when one considers the grand canoncial

partition function with fugacity $z = e^{\beta\mu}$ (μ = chem. potential)

$$Z = \sum \frac{z^N}{N!} Z_N$$

It is expressed in terms of the activities by

$$\ln Z = \sum_P z^{|P|} A(P)$$

provided the sum is absolutely convergent.

2. THE NOTION OF AN EFFECTIVE ACTION (WITH ILLUSTRATION USING A SIMPLE APPROXIMATION)

I will consider lattice models (ferromagnets) with n-component spin variables $\varphi(x) = (\varphi^1(x), \ldots \varphi^n(x))$, and with partition function of the form

$$Z = \int \prod_x d^n\varphi(x)\, e^{L_0(\varphi)} \prod_x f(\varphi(x)) \tag{2.1a}$$

with $f(\varphi(x)) \geq 0$ and

$$L_0(\varphi) = -\frac{n}{2f_0} \sum_x \nabla_\mu \varphi^i(x) \nabla_\mu \varphi^i(x) + \text{const.} \tag{2.1b}$$

Example 1. O(n)-symmetric Heisenberg ferromagnet:

$$f(\varphi(x)) = \delta\left(|\varphi(x)|^2 - 1\right) \tag{2.2a}$$

Example 2. Discrete Gaussian Model = Z-ferromagnet: n = 1, and

$$f(\varphi(x)) = \sum_{k=0,\pm1,\pm2,\ldots} \delta(\varphi(x) - 2\pi k) \tag{2.2b}$$

Let us write $\tilde{\varphi}(k)$ for the Fourier transform of $\varphi(x)$. Instead over $\varphi(x)$ we may integrate $\tilde{\varphi}(k)$. Suppose that the integrations over $\varphi(k)$ with $k \geq M$ have been done. Let $\Phi(x)$ be the sum of the Fourier components $\varphi(k)e^{ikx}$ with $|k| < M$, i.e. equal to $\varphi(x)$ minus its high frequency components. Then the result of the integrations will be of the form

$$Z = \int \mathcal{D}\Phi\ e^{L_{eff}(\Phi)} \tag{2.3}$$

This defines the effective action L_{eff}. It is convenient to take out a kinetic term from $L_{eff}(\Phi)$ and to write

$$Z = \int d\mu_C(\Phi)\, e^{-V_{eff}(\Phi)} \tag{2.4}$$

$d\mu_C$ is a free field measure with UV-cutoff M. (This means that it contains δ-functions which constrain the Fourier components $\tilde{\varphi}(k)$ with $k \geq M$ to zero.)

It is of importance to study the localization properties of L_{eff}. (Because of the integrations that have been performed, L_{eff} will include nonlocal terms.) [4]. By using the technique of Möbius inversion of

section 1 one can exhibit L_{eff} as a sum of terms which are localized in regions Y of space (Euclidean spacetime if we consider QFT). As an example, the complete effective action for the Z-ferromagnet with UV-cutoff M is exhibited in eq. (3.2). (For technical reasons, a Pauli Villars UV-cutoff is used there instead of a sharp momentum cutoff $|k| < M$, and Φ is rescaled by a factor $\beta^{\frac{1}{2}}$.)

In this section I will use a simple approximation to illustrate how (approximate) computations of effective actions can be used to determine the phase structure of a model.

2.1 Normal ordering.

A free field theory is uniquely determined by its propagator

$$C = \beta(-\Delta + m^2)^{-1} \quad \text{without UV-cutoff, or on a lattice} \tag{2.5a}$$

or

$$C = \beta[(-\Delta+m^2)^{-1} - (-\Delta+M^2)^{-1}] \quad \text{with a Pauli-Villars cutoff M} \tag{2.5b}$$

etc. . $-\Delta$ should be read as k^2 (continuum) or $-2\sum_\mu[\cos k_\mu a - 1]$ (lattice) to obtain the propagator in momentum space. The factor β could be absorbed by rescaling the field.

The corresponding Euclidean free field measure is

$$d\mu_C(\varphi) = \frac{1}{Z_0} \prod_x d^n\varphi(x) \exp\left[-\tfrac{1}{2}(\varphi, C^{-1}\varphi)\right] \tag{2.6}$$

C is also called the covariance of this Gaussian measure.

Normal ordering with respect to a Gaussian measure with covariance C is defined by

$$e^{ik\cdot\varphi(x)} = :e^{ik\cdot\varphi(x)}: \int d\mu_C(\varphi') e^{ik\cdot\varphi'(x)}$$

$$= e^{-k^2C(0)/2} :e^{ik\cdot\varphi(x)}: \tag{2.7}$$

C(0) = propagator at distance 0.

To write an arbitrary function $f(\varphi(x))$ in normal ordered form, one expands it in a Fourier series or - integral and uses (2.7). It follows that, if f is any tempered distribution then

$$f(\varphi(x)) = :F(\varphi(x)): \tag{2.8a}$$

with

$$F(\xi) = (2\pi C(0))^{-\frac{n}{2}} \int d^n\eta\, f(\eta)\, e^{-\frac{1}{2C(0)}(\xi-\eta)^2} \tag{2.8b}$$

and F is an entire analytic function of ξ, hence posesses an everywhere convergent power series (for any finite number n of components of φ - in the $n \to \infty$ limit it need not be so [30]). Note in particular that also δ-functions can be written in normal ordered form, and then expanded in normal ordered products.

Generalization to functions which depend on the field at several points is obvious.

example: $\varphi(x)^2 = :\varphi(x)^2: + nC(0)$

2.2 A simple approximation

To carry out the integrations of high frequency components of the fields, we will use the approximation (for C a propagator which propagates the high frequency components)

$$\int d\mu_C(\varphi) \prod_x f_x(\varphi(x)) \equiv \int d\mu_C(\varphi) \prod_x :F_x(\varphi(x)): \approx \prod_x F_x(0) \tag{2.9}$$

In this formula, :: is normal ordering will respect to Gaussian measure with covariance C; it depends on C because C(0) enters into eq. (2.8b).

Motivation for the approximation

Write the integral as a sum of Feynman vacuum-diagrams by use of Wicks theorem. Neglect all nontrivial diagrams (with > 1 vertex. Diagrams ○ + ◎ + ◎◎ +...are taken into account by the above normal ordering). For instance, let n=1. Expand

$$F_x(\varphi(x)) = F_x(0) + F_x'(0)\varphi(x) + \cdots$$

The leading term in $\prod_x :F_x(\varphi(x)):$ is $\prod_x F_x(0)$. Other terms include for instance something proportional to

$$:\varphi^2(x_1)::\varphi^2(x_2): \overset{\text{Wick}}{=} :\varphi^2(x_1)\varphi^2(x_2): + C(x_1-x_2):\varphi(x_1)\varphi(x_2):$$

$$+ 4C(x_1-x_2)^2$$

The last term is graphically represented by a graph ⟲ with two vertices, which we neglect. The first two terms integrate to zero.

2.3 Application

I use this approximation to carry out the integration of the high

frequency parts of the field $\varphi(x)$, viz. $\tilde{\varphi}(k)$ with $|k| > M$. The aim is to obtain in this way an effective action with UV-cutoff M, with M of the order of the ultimate physical mass. The (approximate) true vacuum etc. is then obtained by a semiclassical treatment of this effective action. This reveals then the phase structure of the model.

NB: In very nonlinear theories (with δ-functions etc.), semiclassical approximations are only trustworthy once the cutoff M has been brought down to O(phys.mass). To obtain a rigorous justification of a semiclassical approximation when M is low enough one relies on Glimm-Jaffe-Spencer expansions (compare introduction) if n = 1.

Let us split the original propagator ($\beta = f_0/n$)

$$C = \beta(v+u) \qquad , \quad \beta\tilde{u}(k) = \tilde{C}(k)\,\theta(\pi M - |k|)$$

Then

$$d\mu_C(\varphi) = d\mu_{\beta u}(\phi)\, d\mu_{\beta v}(\chi) \qquad (2.10)$$

$$\varphi = \phi + \chi$$

Φ has only (propagating) components $\Phi(k)$ with $|k| < \pi M$, the other components are called χ. (Note that $d\mu_{u=0}(\Phi)$ = Dirac measure concentrated at $\Phi = 0$.) We obtain

$$Z = \int d\mu_C(\varphi) \prod_x f(\varphi(x)) = \int d\mu_{\beta u}(\phi)\, d\mu_{\beta v}(\chi) \prod_x f(\varphi(x)+\chi(x))$$

$$= \int d\mu_{\beta u}(\phi)\, d\mu_{\beta v}(\chi) \prod_x :F(\varphi(x)+\chi(x)):$$

where :: is normal ordering of the function $F_x(\chi(x)) = F(\Phi(x) + \chi(x))$ with respect to covariance βv. Using our simple approximation, the above expression becomes

$$\approx \int d\mu_{\beta u}(\phi) \prod_x F(\phi(x)) \equiv \int d\mu_{\beta u}(\phi)\, e^{-V_{eff}(\phi)}$$

with

$$V_{eff}(\phi) = -\sum_x \ln F(\phi(x))$$

The effective action consists of $-V_{eff}$ plus the "kinetic term" from the Gaussian measure.

Thus we obtain the following Rules for an approximate determination of L_{eff}:

Suppose the partition function to be evaluated is of the form

$$Z = \int d\mu_C(\varphi) \prod_x f(\varphi(x))$$

Then proceed as follows. Normal order f with respect to a covariance βv that equals C except for the presence of an infrared cutoff at M, viz. $f(\chi(x)) = :F(\chi(x)):$. Drop the dots ::, modify the kinetic term by an UV-cutoff M, set

$$L_{eff}(\phi) = -\tfrac{1}{2\beta}(\phi, u^{-1}\phi) - V_{eff}(\phi) \quad ; \quad V_{eff}(\phi) = -\sum_x \ln F(\phi(x)). \qquad (2.11)$$

Application to the Z-ferromagnet (example 1, eq. (2.2b)).

In 3 dimensions this model is the (Kramers Wannier) dual transform of the U(1) lattice gauge theory with Villain action [22].

In 2 dimensions it is the dual transform of the plane rotator (XY-model) with Villain action.

The partition function is ($\beta = 1/f_0$, $v_{Cb} = (-\Delta)^{-1}$)

$$Z = \int d\mu_{\beta v_{Cb}}(\varphi) \sum_k \prod_x \delta(\varphi(x) - 2\pi k(x))$$

We split

$$v_{Cb} = v + u$$

Instead of a sharp momentum cutoff one could also use a Pauli Villars-cutoff. In this case

$$v = (-\Delta + M^2)^{-1} \qquad , \quad u = (-\Delta)^{-1} - (-\Delta + M^2)^{-1}$$

Alternatively, one may use a lattice cutoff $M = L^{-1}$. In this case the propagator u(x,y) is obtained by averaging $v_{Cb}(x,y)$ over blocks of side length L in each argument. We want M of the order of the ultimate physical mass m. According to our rules, we have to write the periodized δ-function in normal ordered form. The result is immediate from eq. (2.8b): it is equal to a normal ordered periodized Gaussian. As a result of the application of our rules,

$$Z \approx \int d\mu_{\beta u}(\phi) \sum_k \prod_x e^{-\frac{1}{2\beta v(0)}(\phi(x) - 2\pi k(x))^2} \qquad (2.12)$$

For $\beta v(0)$ large this becomes

$$\approx \int d\mu_{\beta u}(\phi) \exp\left[z \sum_x \cos\phi(x) + \text{const} \right]$$

with

$$z = \exp\left[-\beta v(0)/2\right] \qquad (2.13)$$

The effective action in the presence of a cutoff M is thus given by

$$V_{eff}(\phi) = -z \sum_x \cos \phi(x) \tag{2.14}$$

We now proceed to the further analysis for the 3-dimensional case (U(1) lattice gauge theory).

Suppose that $\beta \to \infty$ in units of lattice spacing, and $M = \lambda m$, m = phys.mass. For large β, m will be small and $v(0) \approx v_{Cb}(0)$. Assume for simplicity that we use a lattice cutoff $M = L^{-1}$. Then the field $\Phi(x)$ is constant on blocks of lattice spacing L, and the effective action becomes

$$V_{eff}(\phi) = -z L^3 \sum_{\text{blocks}} \cos \phi(x) \tag{2.15}$$

We look at this as an effective potential for a theory an a block lattice of lattice spacing L. Because of the large factor zL^3 (s. below), V_{eff} has minima which are separated by very high maxima. Expanding the cos around $\Phi = 0$ we obtain the mass m,

$$m^2 = 2\beta z e^{-\beta v_{Cb}(0)/2} \tag{2.16}$$

Therefore $zL^3 = \lambda^{-3}(m\beta)^{-1} \to \infty$ exponentially as $\beta \to \infty$. Here $\lambda = (M/m) = 0(1)$. An approximate expression for the surface tension (see figure 10) can be obtained by a (semi)classical approximation also. The result is

$$\alpha = 8m\beta^{-1} \tag{2.17}$$

Duality transformation shows that the surface tension in the Z-ferromagnet equals the string tension in the U(1) lattice gauge theory.

Let me briefly discuss the 2-dimensional case for contrast. In this case the factor zL^3 in eq. (2.15) is replaced by zL^2. This makes a profound difference - we are now lacking the one factor $L = 0(m^{-1})$ which made $zL^3 \to \infty$ as $\beta \to \infty$.

Consider the situation with small or zero physical mass $0 \leq m \ll 1$ in units of (lattice spacing)$^{-1}$. The mass m is determined by the curvature of V_{eff} at its minima, in the limit of low enough cutoff.

$$m^2 = 2\beta z \quad \text{with} \quad z = \exp\left[-\beta v(0)/2\right] \tag{2.18}$$

z depends on M, and therefore indirectly on m because M is constraint to values $\geq 0(m)$. For small M

$$v(0) \approx -\frac{1}{2\pi} \ln M a' \qquad \text{with } a' = 0 \text{ (lattice spacing)} \tag{2.19}$$

Suppose that $\beta > \beta_c = 8\pi$. Then z is exceedingly small for moderately small M, and $m^2 = 2\beta z$ tends to 0 faster than M^2. Therefore $m \to 0$ as I lower M, maintaining $M \geq 0(m)$. Thus

$$m = 0 \qquad \text{for } \beta > \beta_c \approx 8\pi \tag{2.20}$$

This is the Kosterlitz-Thouless phase.

Application to the nonlinear σ-model (Heisenberg ferromagnet)

This is example 1, eq. (2.2a). According to our result, we must find the normal ordered form (with respect to covariance βv) of the δ-function that is concentrated on the unit sphere S_{n-1}.

According to eq. (2.8b)

$$\delta(\varphi(x)^2 - 1) = :F(\varphi(x)):$$

with ($\beta = f_0/n$, $\xi = (\xi^1, \ldots, \xi^n)$)

$$F(\xi) = \left[2\pi\beta v(0)\right]^{-\frac{n}{2}} \int d^n\eta \; \delta(\eta^2 - 1) e^{-\frac{1}{2\beta v(0)}(\xi - \eta)^2} \tag{2.21}$$

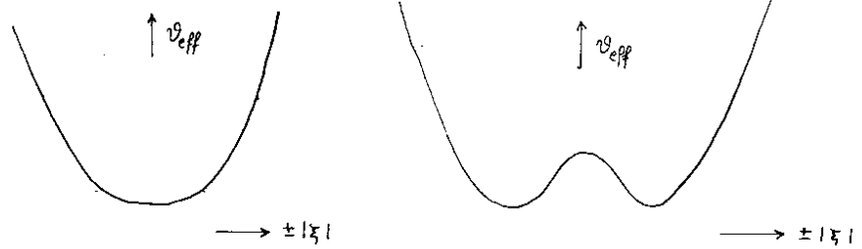
We split the propagator as before

$$C = \beta(v + u)$$

Application of our rules gives

$$V_{eff}(\Phi) = \sum_x \vartheta_{eff}(\Phi(x))$$
$$\vartheta_{eff}(\xi) = -\ln F(\xi) \tag{2.22}$$

To find the behavior V_{eff} at $\xi = 0$ (maximum or minimum), one computes V_{eff}'' from eqs. (2.21), (2.22). Qualitatively, V_{eff} looks as follows as a function of $|\xi|$



a) $n\beta v(0) > 1$ figure 9 b) $n\beta v(0) < 1$

We are interested in the behavior when the cutoff M is small compared to the (lattice spacing)$^{-1}$. $M \to 0$ gives $v(0) \to (-\Delta)^{-1}(0)$.

In 2 dimensions, $v(0) \to \infty$ as $(-\Delta+m^2)^{-1}(0) \to \infty$ when $m \to 0$ in 2 dimensions. As a result, V_{eff} alway looks as in figure 9a) for sufficiently low cutoff M. Therefore there will be a finite mass and no spontaneous symmetry-breakdown.

In more than 2 dimensions, there will be spontaneous symmetry-breaking if $n\beta v_{cb}(0) < 1$, i.e. if $f_o^{-1} < (-\Delta)^{-1}(0)$. The potential V_{eff} looks like figure 9b) for this range of f_o and low cutoff M.

These results agree with the results of the 1/n expansion.

3. RIGOROUS RESULTS FOR THE 3-DIMENSIONAL U(1) LATTICE GAUGE THEORY

In this lecture I will report results of a study by M. Göpfert and myself [8]. It dealt with lattice gauge theory in 3 space time dimensions without matter fields and with gauge group U(1). It is known from the work of A. Guth [23], that such a model shows a deconfining phase transition in 4 dimensions, so that its weak coupling continuum limit will not show confinement (if it exists at all). In contrast, in 3 dimensions, the work of Polyakov [24], Banks, Myerson and Kogut [22], Drell et al. [25] and DeGrand and Toussaint [26] lead to the belief that confinement will be true for all values of the coupling constant. We proved that this is indeed the case, for the model with Villain action. It turned out, however, that the ratio α/m_D^2 of the string tension to asymptotic physical mass squared becomes infinite in the weak coupling limit (continuum limit).

The model lives on a 3-dimensional cubic lattice of lattice spacing a. It is a classical statistical mechanical system whose random variables are attached to the links b = (x,y) of the lattice

$$U_\ell = e^{i\theta_\mu(x)} \quad \text{for} \quad \ell = (x, x+e_\mu) \quad , \; -\frac{\pi}{a} \le \theta_\mu \le \frac{\pi}{a} \tag{3.1a}$$

(e_μ = lattice vector in μ direction). The action is of the form

$$L(U) = \sum_p \mathcal{L}(U_{\partial p}) \tag{3.1b}$$

with $U_{\partial p} = U_{\ell_1} \dots U_{\ell_4}$ if p is the plaquette whose boundary consists of links $\ell_1 \dots \ell_4$, and

$$\mathcal{L}(e^{i\varphi}) = \ln \sum_{m=0,\pm1,\pm2,\dots} \exp\left[-\frac{1}{2ag^2}(\varphi - 2\pi m)^2\right] \tag{3.1c}$$

g^2 is the unrenormalized electric charge squared. It has dimension of a mass in 3 dimensions. The Boltzmann factor is exp L(U).

Since the gauge group U(1) of this model is abelian, the model can be subject to a Kramers-Wannier duality transformation. As a result one obtains a ferromagnet with a global symmetry group $\mathbb{Z}$. Its random variables n(x) are attached to the sites of a 3-dimensional cubic lattice Λ (the dual of the origianl one) and assume values which are integer multiples of 2π. The new action is

$$\hat{L}(n) = -\frac{1}{2\beta}\int_x [\nabla_\mu n(x)]^2 \qquad \text{with } \beta = 4\pi^2/g^2$$

We use the standard notations

$$\int_x = a^3 \sum_x \qquad , \qquad \nabla_\mu f(x) = a^{-1}[f(x+e_\mu) - f(x)]$$

a is the lattice spacing, I prefer not to set a = 1 in this section. Expectation values are computed with the help of the Boltzmann factor $\exp \hat{L}(n)$. This model is known as the "discrete Gaussian model", for obvious reasons. We call it the "Z-ferromagnet" in order to emphasize its symmetry properties. I have already considered this model as an example in section 2.

The global Z-symmetry of this model is always spontaneously broken (if $\langle n(x)\rangle$ exists at all) since the equation

$$\langle n(x)\rangle = \langle n(x)\rangle + 2\pi I \qquad , \; I \in \mathbb{Z}$$

has no solution if $I \neq 0$. The surface tension α of the model is defined as the cost of free energy per unit area of a domain-wall which separates two domains whose spontaneous magnetization $\langle n(x)\rangle$ differs by 2π, see figure 10. The duality transformation shows that α equals the string tension of the U(1) gauge model.

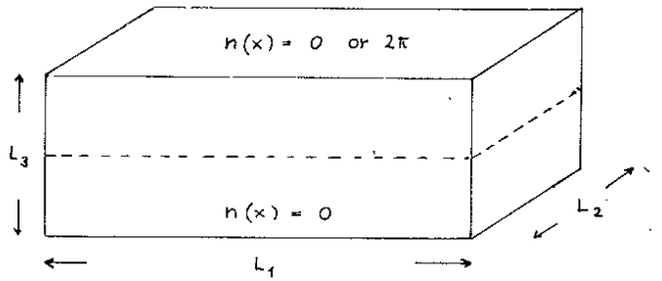


figure 10: Definition of the surface tension α. One considers the free energy of a box for the two choices of boundary conditions i) $n(x) = 0$ everywhere and ii) $n(x) = 2\pi$ above the dashed line and $n(x) = 0$ below. α equals the difference of the free energy for the two boundary conditions, divided by L_1L_2, in the limit $L_i \to \infty$ $(i = 1,2)$, followed by $L_3 \to \infty$.

It is convenient to introduce the following quantity with the dimension of a mass squared

$$m_D^2 = (2\beta/a^3)\exp[-\beta v_{Cb}(0)/2]$$

where v_{Cb} is the lattice Coulomb potential. As was shown by Banks et al.,

the model can also be transformed into a (special) Coulomb system. m_D^{-1} is the prediction of a Debye Hückel approximation for the sceening length of that system, for large β/a.

Our main results are as follows.

Theorem 1 There is a dimensionless constant $C > 0$ such that

$$\alpha \geq C \cdot m_D \beta^{-1} \qquad \text{for sufficiently large } \beta/a.$$

Since αa^2 is a monotone decreasing function of β/a by Guth's inequality [23], it follows that $\alpha > 0$ for all values of the coupling constant $g^2 > 0$. We believe that the r.h.s. of the inequality of theorem 1 represents the true asymptotic behavior of α. It is amusing to compare with the leading term of the high temperature expansion, which is valid when β/a is small. It reads $\alpha = 2\pi^2 a^{-1}\beta^{-1} + \ldots$.

The meaning of m_D as an asymptotic mass is clarified by the second result. The effective action L_{eff} mentioned below depends on a real field $\Phi(x)$. It is obtained by integrating out the high frequency components of $\beta^{-\frac{1}{2}} n(x)$. Symbolically we may write

$$\exp -L_{eff}(\Phi) = \int \exp -\hat{L}(n) \prod_x \delta(\Phi(x) - \beta^{-1/2} n(x)) \qquad \text{with Pauli-Villars cutoff } M.$$

Theorem 2 Consider the correlation functions $\langle \Phi(x_1) \ldots \Phi(x_n)\rangle$ for fixed distances $m_D|x_i - x_j|$ in units of m_D. They tend to the correlation functions of a massive free field theory with mass m_D as $\beta/a \to \infty$ and $M/m_D \to \infty$ (proportional $(\beta/a)^{1/12}$, for instance).

These results were obtained by a rigorous block spin calculation, and are therefore perfectly consistent with the general renormalization group theory [4]. However, they contradict what would be obtained by making simple but popular approximations.

Suppose that one could set up a renormalization group procedure (block spin calculation) for the U(1) gauge theory (3.1) in such a way that the effective action at each step of this interative procedure is still (approximately) of the same form (3.1), except for the replacement of g^2 by a running coupling constant $g^2_{eff}(a')$ and a new value $a' > a$ of the lattice spacing. Suppose moreover that $a' g^2_{eff}(a')$ reaches values in the domain of validity of high temperature expansions after sufficiently many iteration steps (depending on ag^2), no matter how small the bare coupling constant $g^2 a$ is. Then it would follow immediately that the string tension α should be proportional to the physical mass (= mass gap) squared.

In contrast, theorem 1 tells us that

$$\alpha / m_D^2 \geq C (\beta m_D)^{-1} = C (2\beta^3/a^3)^{-1/2} \exp \tfrac{1}{4}\beta v_{Cb}(0) \to \infty$$

as $\beta/a = 4\pi^2/g^2 a \to \infty$. (The numerical values of $v_{Cb}(0)$ is found in the literature to be $0.2527\ a^{-1}$).

I will now briefly describe the main steps of the analysis. The first step is to integrate out the high frequency components of the field $\varphi(x) = \beta^{-\frac{1}{2}} n(x)$. This produces an effective action $L_{eff}(\Phi)$ for a real field $\Phi(x)$ (on the original lattice) with Pauli-Villars cutoff M. $L_{eff}(\Phi)$ is obtained in the form of an infinite series of the following form, with real coefficients $\rho_s(\ldots)$.

$$L_{eff}(\Phi) = -\tfrac{1}{2} \int_x \Phi(x) \left\{ -\Delta \left(1 - \frac{\Delta}{M^2}\right) \right\} \Phi(x) + \qquad (3.2)$$
$$+ \sum_{s \geq 1} \frac{1}{s!} \sum_{m_1 \ldots m_s} \int_{x_1 \ldots x_s} \rho_s(m_1, x_1, \ldots, m_s, x_s) \left[e^{i m_1 \beta^{1/2} \Phi(x_1)} - 1 \right] \ldots \left[e^{i m_s \beta^{1/2} \Phi(x_s)} - 1 \right]$$

m-summations are over $m_j = \pm 1, \pm 2, \ldots$. The first term is the usual kinetic term for a real field with a Pauli-Villars cutoff M [27].

The main problem was to establish convergence of the expansion, for $M > \lambda m_D$ (λ independent of β) and large β/a, and bounds on the individual terms. In particular, it was shown that the coefficients ρ_s for $s > 2$ decay exponentially with distances $x_i - x_j$ so that they become negligibly small for distances $|x_i - x_j| \quad M^{-1}$. This means that no interactions of range much larger than the cutoff length M^{-1} have been generated in the process of integrating out the high frequency components of the field $\varphi(x)$. This is a basic requirement in a block spin calculation.

Moreover, the bounds show also that the dominant terms are the terms with $s = 1$ and $m_1 = \pm 1$, for large β/a, and that $\rho_1(m,x) \approx \frac{1}{2}\beta^{-1} m_D$. Thus

$$L_{eff}(\Phi) = \text{kinetic term} - m_D^2 \beta^{-1} \int_x \left[1 - \cos \beta^{1/2} \Phi(x) \right] + \ldots \qquad (3.3)$$

This reproduces the result (2.13), (2.14) of the simple approximation that I used in section 2. The rigorous treatment of the model justifies this approximation, for large β/a, by producing bounds on the correction terms ... in the effective action. If the resulting theory with effective action (3.3)(without the correction terms ...)is treated by a classical approximation, one obtains the result

$$\alpha = 8 m_D \beta^{-1}$$

It is possible that this is exact for $\beta/a \to \infty$, but our bounds are not sharp enough to prove it. There is, however, an upper bound on α due

to Ito which complements the lower bound of theorem 1 and is very close to it[28]

The second step is the analysis of a theory with action (3.2). Such an analysis was already performed by Brydges and Federbush [6] and we could use their result. The Glimm-Jaffe-Spencer expansion of constructive field theory [5] (see introduction) in the basis tool in this step.

Finally I would like to explain briefly how the effective action (3.2) is obtained. It is known from the work of Banks, Myerson and Kogut [22] that the model can be transformed into a Coulomb system with partition function

$$Z = \sum_{m \in \mathbb{Z}^\Lambda} e^{-\beta (m, v_{Cb} m)/2}$$

Here we set the lattice spacing equal to one, $m(x) = 0, \pm 1, \pm 2, \ldots$ is the charge at site x of Λ, $v_{Cb} = (-\Delta)$ is the lattice Coulomb potential, and

$$(m, v_{Cb} m) = \sum_x \sum_y m(x) v_{Cb}(x-y) m(y)$$

A self-interaction term $x = y$ is included. Following Fröhlich [29] one splits the Coulomb potential into a Pauli-Villars cutoff Coulomb potential $u = (-\Delta)^{-1} - (-\Delta + M^2)^{-1}$, and a Yukawa potential $v = (-\Delta + M^2)^{-1}$ of range M^{-1}.

$$v_{Cb} = v + u$$

One inserts this and uses the formula for the characteristic function of a Gaussian measure with covariance u,

$$\int d\mu_u(\Phi)\, e^{i(f,\Phi)} = e^{-\frac{1}{2}(f, uf)}$$

to rewrite the partition function as

$$Z = \int d\mu_u(\Phi)\, \mathcal{Z}(\Phi)$$

with

$$\mathcal{Z}(\Phi) = \sum_{m \in \mathbb{Z}^\Lambda} e^{i\beta^{1/2}(m,\Phi)}\, e^{-\beta(m, vm)/2}$$

$\mathcal{Z}(\Phi)$ is the partition function of a Yukawa gas with complex space dependent fugacity $z = \exp i\beta^{\frac{1}{2}} m(x)\Phi(x)$. Its logarithm is the desired effective action

$$L_{eff}(\Phi) = \text{kinetic term} + \ln \mathcal{Z}(\Phi)$$

It is natural to try to use a Mayer expansion to compute $\ln \Xi$ (compare section 1). The leading term in such a Mayer expansion comes from clusters with only one particle and gives (3.3). Unfortunately, known methods to prove convergence of such a Mayer expansion were not nearly good enough to cover the values of parameters (fugacity z, inverse temperature β) of interest here. We have therefore developed a refined version of such Mayer expansions. It is based on splitting the Yukawa potential v into a sum of interactions of decreasing strength and increasing range, and then treating one after the other of these by Mayer expansions as usual. Recursive bounds are established, and these combine to prove convergence of the complete expansion and produce bounds on the individual terms.

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