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MONOPOLES, VORTICES, AND CONFINEMENT

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Abstract An exact relation is established between an SO(3) lattice gauge theory model without monopoles, and a corresponding SU(2) model. Elimination of the monopoles (and their strings) leads to a substantial lowering of the entropy of thin vortices and a corresponding decrease of the string tension for low β. This is revealed by approximate calculations of the vortex free energy and is confirmed by Monte Carlo data. The value of the physical transition temperature to "hot gluon soup" is also lowered considerably.

1. Introduction and summary of Monte Carlo results

Four dimensional pure SU(2) lattice gauge theory can be interpreted as a Z_2 (gauge) theory with monopoles and fluctuating coupling constants [1,2]. Condensation of these monopoles and/or the associated Z_2 strings can lead to phase transitions [1,2-8]. In the present paper we investigate what happens if both the monopoles and Z_2 strings are eliminated from the model (by giving infinite energy to the strings). Such a modification does not affect the formal continuum limit.

We consider a 4-dimensional hypercubic lattice Λ made of sites x, links b, plaquettes p, cubes c, and hypercubes h. The boundary of a plaquette is sum of four links, etc. Let $\bar{U} = \{\bar{U}(b) \in SO(3)\}$ be an SO(3) lattice gauge field on Λ and choose representatives $U(b) \in SU(2)$ of the cosets $\bar{U}(b) \in SO(3) = SU(2)/Z_2$ in an arbitrary way. A gauge group SO(3) admits monopoles whose magnetic charge is added mod (2) because the fundamental group $\pi_1(SO(3)) = Z_2$. Translating the well known definition of monopoles in the continuum [9] to the lattice (see Appendix A) one defines [1]*

$$\rho_c(\bar{U}) = \prod_{p\in\partial c} \text{sign}\, \text{tr}\, U(\partial p) = \pm 1 \in Z_2 \tag{1.1a}$$

for every 3-dimensional cube c. The product runs over the six plaquettes in the boundary of c. $\rho_c(\bar{U})$ is independent of the choice of representatives U(b), and depends therefore only on $\bar{U}$, because it is invariant under the substitution $U(b) \to U(b)\gamma(b)$ with $\gamma(b) = \pm 1$. A conserved magnetic current $j_{\mu\nu\lambda}(x) \in \mathbb{F}_2 = \{0,1\}$ is now defined by

$$\rho_c = \exp i\pi j_{\mu\nu\lambda}(x) \tag{1.1b}$$

c = cube with corners $x, x+e_\mu, x+e_\mu+e_\nu, \dots, x+e_\lambda$.
At a given time the monopoles are in the spacelike cube c where $\rho_c(\bar{U}) = -1$. (The definitions generalize to $SU(N)/Z_N$ in the obvious way. Orientation of plaquettes has to be watched if $N \geq 3$.) The

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*Notation: If C is a path composed of links $b_1 \dots b_n$ then $U(C) = U(b_n) \dots U(b_1)$. In particular $U(\partial p) = U(b_4) \dots U(b_1)$ for a plaquette p with boundary $\partial p \equiv \dot{p}$ consisting of oriented links $b_1 \dots b_4$.

world lines of the monopoles are closed loops on the dual lattice. In an SU(2) theory the same definition (1.1) is used (i.e. the monopoles are the monopoles of the SO(3) gauge field that is obtained from the SU(2) gauge field by forming cosets.) The monopoles are now attached to Z_2 strings that carry energy [1,10]. They consist of plaquettes where

$$\text{sign tr } U(\partial p) = -1. \tag{1.2}$$

The world sheets of the Z_2 strings are 2-dimensional surfaces on the dual lattice. They are either closed or bordered by monopole loops.

In order to put our work into perspective we review briefly what is known about the above mentioned phase transitions.

The prototype of a phase transition associated with Z_2 string condensation occurs in Wegners pure Z_2 gauge theory [11,12]. Such a transition was proven to exist also in an SU(2) model in which the monopoles were eliminated by a constraint (MP model) [13]. Its transition point was determined by Monte Carlo computations by Brower, Kessler and Levine [5]. Our Monte Carlo data presented in figure 1 confirm that it is associated with condensation of Z_2 strings. The result for $1-\langle \text{sign tr } U(\partial p)\rangle$ behaves very much like the internal energy of Wegners Z_2 model.

Monte Carlo evidence for phase transitions associated with monopole condensation in SO(3) models was found by Halliday and Schwimmer and by Greensite and Lautrup [3,4]. Brower, Kessler and Levine [5] have shown that the monopole density in the standard SU(2) model with Wilson action rises rapidly in the vicinity of its "rapid transition" from weak to strong coupling, and they proposed that this transition is due to condensation of monopoles together with their strings.

One can suppress monopoles by adding a suitable second term L_2 to the standard SU(2) action of Wilson,

$$L_1 = \frac{\beta}{2} \sum_p \text{tr } U(\partial p) \tag{1.3}$$

One can either choose $L_2 = \sum_c \lambda \, \rho_c(\bar{U})$ as in ref. [5], or L_2 = action for a SO(3) theory [6-8]. The phase diagrams of such theories with two coupling parameters were studied with the Monte Carlo method by Brower, Kessler and Levine [5], Bhanot and Creutz [6], and Caneschi, Halliday and Schwimmer [7]. First order phase transition lines are found that are associated with either condensation of monopoles, or of Z_2 strings, or of both. Condensation of monopoles enhances the order parameters (monopole density ·2)

$$\tilde{M} = 1 - \langle \rho_c(\bar{U}) \rangle \tag{1.4}$$

while condensation of Z_2 strings in an SU(2) theory enhances the order parameter [7]

$$\bar{E} = 1 - \langle \varepsilon_b(U) \rangle \quad , \quad \varepsilon_b = \prod_{p \in \partial^* b} \text{sign tr } U(\partial p) \tag{1.5}$$

The product runs over the six plaquettes p that have b in their boundary. These order parameters were computed by Caneschi, Halliday and Schwimmer [7]. In this way they were able to elucidate the nature of the observed phase transitions.

One of the observed first order phase transition lines has a jump of both $\bar{E}$ and $\tilde{M}$ and projects towards the point in the two parameter plane which corresponds to the "rapid transition" in the standard SU(2) model with action L_1, $\beta \approx 2.2$, but it stops before reaching it [6,7]. A peak in the specific heat C_V of the standard SU(2) model at $\beta = 2.2$ was observed by Lautrup and Nauenberg [14]. Lang et al. [15] pointed out that it may represent a 3rd order phase transition (a change in slope of $\beta^{-2}C_V$). In figure 2 we present Monte Carlo data of our own for $\beta^{-2}C_V$. They are more accurate and from a larger lattice than data available before. Each data point is determined from the fluctuations in internal energy during 20 000 or more sweeps through the 6^4 lattice, using the heat bath method. The statistical errors were determined by dividing into M subsamples of about 400 sweeps each. They represent the mean square deviation of the averages over individual subsamples divided by $\sqrt{M}$. The result is consistent with a third order phase transition but it does not establish it since a smooth curve could also be drawn through the points. Moderate agreement is found with the fit of Nauenberg, Schalk and Brower [16] to their data for the internal energy. This fit obeys finite size scaling and reduces to a polynominal fit for infinite lattice size. High temperature series appear to favor absence of any phase transition in the standard model [17]. A tentative explanation of the absence of a (first order) phase transition is that small monopole loops become abundant before the monopoles are liberated, and their liberation (= condensation of Z_2 strings) becomes thereby thermodynamically insignificant. The new theory of first order phase transitions of Dobrushin, Shlosman and Kotecký [18] and the work of refs. [7,19] offer some promise that a better understanding will soon be reached.

In the formal continuum limit, the SU(2) lattice action L_1 becomes equal to the Yang Mills action for a gauge theory on continuous Euclidean space time [20] because $U(\partial p) \to 1$ as a random variable when $\beta \to \infty$ [1]. We see from this that the monopoles and Z_2 strings can be eliminated by a constraint sign $\mathrm{tr}\, U(\partial p) \equiv +1$ without affecting the formal continuum limit. We call the resulting model a "positive plaquette model".

Similarly, monopoles in the SO(3) theory with Wilson action can be eliminated by a constraint $\rho_c \equiv 1$ without affecting the formal continuum limit because $\rho_c(\bar{U}) = 1$ if $\bar{U}$ is locally close to a pure gauge. The formal continuum limit of the theories with gauge group SU(2) and SO(3) is the same.

Neither of these constraints violates gauge invariance of the models in question. On the basis of the above discussion of the nature of observed phase transitions we expect that elimination of the monopoles in the SO(3) model eliminates its phase transition, and the elimination of Z_2 strings and monopoles in the SU(2) models eliminates the "glitch" in its internal energy at the "rapid transition point" $\beta \approx 2.2$ [12]. This is indeed the case. In figure 3

we compare Monte Carlo data for the internal energy of an SO(3) model with and without monopoles. The monopole density $\bar{M}/2$ is also shown. Calculations for the SU(2) case were done by Brower, Kessler and Levine [5].

It is in fact not necessary to consider the constraint SU(2) and SO(3) models separately. In section 2 we establish an exact and explicit relation between such models. All observables, including in particular the expectation value of the Wilson loop [20] for fractionally charged static quarks can be transcribed from the positive plaquette SU(2) model to the corresponding monopoleless SO(3) model. The relation (2.14) for the Wilson loop is a simplified and more explicit version of a formula that was obtained by one of us in ref. [21]. It is very instructive and will elucidate both the topological origin of confinement and its basis in the peculiar locality properties of (classical) gauge field theories [33].

Chessboard estimates imply [1] that the probability

$$\mathrm{prob}\,(\mathrm{tr}\, U(\partial p) < 0) < \mathrm{const.}\, e^{-\beta/13} \tag{1.6}$$

in the standard SU(2) model, so that the density of Z_2 strings goes to zero exponentially in units of (lattice spacing)$^{-3}$, when $\beta \to \infty$. But this is insufficient to guarantee that they become unimportant dynamically, because also the string tension and mass squared in units of (lattice spacing)$^{-2}$ are expected to tend to zero exponentially. None of the existing proofs of confinement at strong coupling (small β) applies to the positive plaquette model. In fact we do not even have a proof for the special case $\beta = 0$. Moreover, if one eliminates the Villain monopoles [22] in a 3-dimensional U(1) lattice gauge theory by a constraint then linear confinement goes away (because the result is nothing but ordinary noncompact electrodynamics [23]). Therefore one might start to get worried whether elimination of monopoles and Z_2 strings might not destroy confinement. Theoretical arguments based on our present understanding of the mechanics of confinement for large β [24] imply that this should not happen. Nevertheless it seemed desirable to study the question by Monte Carlo computation.

The result of such computations is shown in figures 4. - 6. We consider lattices Λ of size $n_t \times n_s^3$ with periodic boundary conditions, and loops $C_{\underset{\sim}{x}}$ that wind through Λ as shown in figure 7. Let

$$L(\underset{\sim}{x}) = \mathrm{tr}\, U(C_{\underset{\sim}{x}}). \tag{1.7}$$

n_t^{-1} can be interpreted as a physical temperature in units of $(\text{lattice spacing})^{-1}$. A deconfining phase transition to "hot gluon soup" [25] with $L(\underset{\sim}{x}) \neq 0$ is expected to occur when n_t is lowered. Figure 4 shows that such a transition does indeed occur, but the transition temperature n_t^{-1} is lowered dramatically compared to the standard SU(2) model, for the range of values of the coupling parameter β where we have data. In the standard model the transition occurs at $n_t \approx 2$ when $\beta = 1.8$ [26]. In figure 5 we show the potential V(r) between two static quarks as a function of their distance r, for two values of β and with n_t^{-1} below the transition temperature. It is defined by [26]

$$V(r) = -n_t^{-1} \ln 4 < L(\underset{\sim}{x}) L(0) > \quad \text{for} \quad \underset{\sim}{x} = (00r) \tag{1.8}$$

It looks linear within the errors. This supports the belief that the confinement has not been destroyed. Assuming that V(r) can be fit by a linear function of r down to r = 1 one can determine the string tension α by

$$\alpha = V(2) - V(1) \tag{1.9}$$

The result is shown in figure 6. For comparison we show the old data of Creutz [27] for the string tension α in the standard SU(2) model. One sees that the string tension is considerably lower in the positive plaquette model, for $\beta \leq 1.5$. Calculation of α for larger β would have required lattices of impractical size, because of the low value of the physical transition temperature. (This problem cannot be avoided by considering Wilson loops, either.) Therefore, and to our great disappointment, we

were not able to determine the string tension for larger value of β as would have been necessary to see whether and how fast the string tension of the positive plaquette model approaches that of the standard model. We can only say that the approach is not as fast as one might have hoped, given the popular belief that one is close to the continuum limit of the standard SU(2) model as soon as one has passed the "rapid transition".

Now, we come to the theoretical arguments. Let us first straighten out the analogy with the U(1) theory. The analog of our monopoles (1.1) in a U(1) theory would be Glimm-Jaffe monopoles [28] rather than Villain monopoles [22]. The Villain monopoles are defined in terms of auxiliary variables that exist in some models. Eliminating them destroys the U(1) gauge invariance and leads to a gauge theory with gauge group $\mathbb{R}$ = universal covering of U(1). Villain monopoles can also be defined in the SO(3) model of Halliday and Schwimmer [3]. Eliminating them produces the standard SU(2) model which is known to have confinement at strong and intermediate coupling. In conclusion, analogy with U(1) theory produces no sound argument that elimination of monopoles should destroy confinement.

The effective Z_2 theory of quark confinement [24] explains confinement for large β as a consequence of condensation of vortices of thickness $d \geq d_c(\beta)$, with $d_c(\beta)/a \to \infty$ exponentially as $\beta \to \infty$ (a= lattice spacing), whereas thinner vortices freeze out. The Z_2 strings are a special kind of thin vortex of thickness 1 lattice spacing. Using the effective Z_2 theory, and the simplest of approximations to compute the free energy of thin vortices, the suppression of the string tension α at low β can be explained quantitatively. This will be shown in section 3. The calculations suggest that there are important correlations between vortices and monopoles, because elimination of monopoles cuts down the entropy associated with internal structure of the vortices.* It is tempting to speculate that the same kind of correlation exists between fat vortices and fat monopoles. The fat monopoles are defined by the same formula (1.1a) except that $U(\partial p)$ is replaced

*Samuel has argued before that monopoles are important for confinement, especially for gauge group SU(3) [43]. But his monopoles are defined differently and we are at present unable to say what the relation might be.

by the parallel transporter around the boundary of a square P of larger side length (= a plaquette of a block lattice) see [1].

In the last section of this paper we take the opportunity to present some new Monte Carlo data for the vortex free energy [29]. They add to those presented in our first paper [30]. Its conclusions are unchanged.

2. Exact relation between models with gauge group SO(3) and SU(2)

In this section we will establish an exact relation between monopoleless SO(3) models and corresponding "positive plaquette" SU(2) models.

Our SO(3) models without monopoles have Gibbs measure

$$d\bar{\mu}(\bar{U}) = \frac{1}{\bar{Z}} \prod_b d\bar{U}(b)\, e^{\Sigma_p \bar{\mathcal{L}}(\bar{U}(\partial p))} \prod_c \theta(\rho_c(\bar{U})) \tag{2.1a}$$

with

$$\bar{\mathcal{L}}(\bar{V}) = \frac{\beta}{2}\left[(\mathrm{tr}\,\bar{V}+1)^{1/2} - 2 \right] \quad (\text{pos. square root}) \tag{2.1b}$$

or

$$\bar{\mathcal{L}}(\bar{V}) = \frac{\beta}{3}\left[\mathrm{tr}\,\bar{V} - 3 \right] \quad , \qquad \text{for } \bar{V} \in SO(3). \tag{2.1c}$$

$d\bar{U}$ is normalized Haar measure on SO(3), and tr is here a trace of real orthogonal 3 × 3 matrices. Evidently the models are SO(3) gauge invariant. Expectation values of observables $\bar{F}(\bar{U})$ are defined by

$$\langle \bar{F} \rangle_{SO(3)} = \int d\bar{\mu}(\bar{U}) \bar{F}(\bar{U})$$

The corresponding SU(2) models involve variables which are 2 × 2 matrices $U(b) \in SU(2)$. Their Gibbs measure is given by

$$d\mu(U) = \frac{1}{Z} \prod_b dU(b)\, e^{\Sigma_p \mathcal{L}(U(\partial p))} \tag{2.2a}$$

with

$$\mathcal{L}(V) = \begin{cases} \frac{\beta}{2}[\mathrm{tr}\,V - 2] & \text{if } \mathrm{tr}\,V > 0 \\ -\infty & \text{otherwise} \end{cases} \tag{2.2b}$$

or

$$\mathcal{L}(V) = \begin{cases} \frac{\beta}{3}[(\mathrm{tr}\,V)^2 - 4] & \text{if } \mathrm{tr}\,V > 0 \\ -\infty & \text{otherwise} \end{cases} \tag{2.2c}$$

Expectation values of observables F(U) are defined by

$$\langle F \rangle_{SU(2)} = \int d\mu(U) F(U) \tag{2.2d}$$

Validity of the exact relation between these SU(2) models and the SO(3) models (2.1) will require matching boundary conditions. For the SU(2) model it is required that the boundary conditions are invariant under substitutions $U(b) \longrightarrow U(b)\gamma(b)$ with $\gamma(b) = \pm 1$, for $b \in \partial\Lambda$. Thus we may impose either free boundary conditions, or periodic boundary conditions for cosets $\bar{U}(b)$ (but not for $U(b)$ themselves), and the same boundary conditions for the SO(3) model.

Now we will transcribe expectation values of observables from this SU(2) model to the SO(3) model. The relation is simple for (gauge invariant local*) observables $F(U)=\bar{F}(\bar{U})$ which depend on U only through cosets $\bar{U} \in SO(3) = SU(2)/Z_2$, so that they can be regarded as observables of the SO(3) theory in a natural way:

$$\langle F \rangle_{SU(2)} = \langle \bar{F}(\bar{U}) \rangle_{SO(3)} \tag{2.3}$$

We proceed to the proof of this relation. From the Kronecker decomposition of the Kronecker product of two 2-dimensional representations of SU(2) it follows that

$$(\operatorname{tr} V)^2 = \operatorname{tr} \bar{V} + 1$$

Consequently

$$\bar{\mathcal{L}}(\bar{V}) = \mathcal{L}(V) \quad \text{if} \quad \operatorname{tr} V \geq 0 \tag{2.4}$$

We note also that the SU(2) models (2.2) have no monopoles. By definition, the monopoles are the monopoles of the SO(3) gauge field $\bar{U}$ that is obtained from U by forming cosets. Since $U(b)$ is a representative of $\bar{U}(b)$ it follows that

$$\rho_c(\bar{U}) = \prod_{p\in\partial c} \operatorname{sign} \operatorname{tr} U(\partial p) = 1 \tag{2.5}$$

because the measure (2.2) has in its support only gauge fields U with $\operatorname{tr} U(\partial p) > 0$.

*local means here that F should not depend on $U(b)$ attached to links b in the boundary $\partial\Lambda$ of the (finite) lattice Λ.

Because of relation (2.4) the expectation value $\langle F \rangle_{SU(2)}$ takes the form

$$\langle F \rangle_{SU(2)} = Z^{-1} \int \prod dU(b) e^{\Sigma_p \bar{\mathcal{L}}(\bar{U}(\partial p))} \prod_p \theta(\operatorname{tr} U(\partial p)) \bar{F}(\bar{U}) \tag{2.6}$$

We introduce normalized Haar measure on Z_2

$$\int d\gamma \,(\ldots) = \frac{1}{2} \sum_{\gamma=\pm 1} (\ldots) \tag{2.7}$$

The corresponding δ-function is $\delta(\gamma) = 1 + \gamma$. It satisfies $\int d\gamma \; \delta(\gamma) f(\gamma) = f(1)$. We will make use of identities of the form

$$\int_{SU(2)} dV f(V) = \int_{SO(3)} d\bar{V} \int_{Z_2} d\gamma \; f(V\gamma) \tag{2.8}$$

They are obtained by making a variable substitution $V \to V\gamma$, and averaging over γ. The relation (2.8) follows then from invariance of Haar measure; $\int d\gamma \; f(V\gamma)$ depends on V only through the coset $\bar{V}$. Our choice of boundary conditions admits a variable substitution $U(b) \to U(b)\gamma(b)$ in (2.6). Averaging over $\gamma(b)$ we obtain

$$\langle F \rangle_{SU(2)} = Z^{-1} \int \prod_b d\bar{U}(b) \, e^{\Sigma_p \bar{\mathcal{L}}(\bar{U}(\partial p))} \bar{F}(\bar{U}) \cdot 2^{-N_p} \int \prod d\gamma(b) \prod_b \delta(\gamma(\partial p) \operatorname{sign} \operatorname{tr} U(\partial p)).$$

N_p is the number of plaquettes in Λ. Integration over Z_2 gauge fields $\gamma(b)$ is equivalent to integration over Z_2 field strengths $\sigma(p)$, subject to the constraints imposed by the 2nd Maxwell equations. Thus

$$\langle F \rangle_{SU(2)} = 2^{-N_p} Z^{-1} \int \prod_b d\bar{U}(b) \, e^{\Sigma_p \bar{\mathcal{L}}(\bar{U}(\partial p))} \bar{F}(\bar{U}) \cdot \int \prod_p d\sigma(p) \, \delta(\sigma(p) \operatorname{sign} \operatorname{tr} U(\partial p)) \prod_c \delta(\prod_{p\in\partial c} \sigma(p))$$

$$= 2^{-N_p} Z^{-1} \int \prod d\bar{U}(b) e^{\Sigma_p \bar{\mathcal{L}}(\bar{U}(\partial p))} \bar{F}(\bar{U}) \prod_c \delta(\rho_c(\bar{U})) \tag{2.9}$$

$$= 2^{-N_p+N_c} (\bar{Z}/Z) \int d\bar{\mu}(\bar{U}) \bar{F}(\bar{U})$$

The partition functions are defined so that $<1> = 1$. Specializing to $F = \bar{F} = 1$ we conclude that $2^{-N_p+N_c}(\bar{Z}/Z) = 1$. Relation (2.9) reduces therefore to (2.3). Proof completed.

We wish to extend our relation to observables which are not of the simple form $F=\mathcal{F}(\bar{U})$. It will suffice to consider the Wilson loop expectation value for fractional charge.

Given a closed loop C which consists of links $b_1, \ldots, b_n$, the parallel transporter $U(C)$ around C is the ordered product $U(b_n) \ldots U(b_1)$. Let χ_ℓ be the character of the $2\ell+1$ dimensional representation of SU(2). Then $\chi_\ell(U(C))$ is the Wilson loop observable for static quarks of charge ℓ. It depends only on $\bar{U}$ if and only if ℓ is integer. It will be instructive to consider fractional and integer ℓ at the same time. Up to eq. (2.12) below our discussion will be general, without reference to particular models.

We will introduce new variables $\sigma(b) = \pm 1$ and $W(b) \in SU(2)$ with $\mathrm{tr}W(b) \geq 0$. They are functions of the original variables $U(b')$ with the following properties [21,31].

1. <u>Locality</u> $W(b)$ and $\sigma(b)$ depend only on $U(b')$ on links b' in a neighbourhood of one lattice spacing of b, and $W(b)$ depends in fact only on cosets $\bar{U}(b) \in SO(3) = SU(2)/Z_2$.

2. <u>Gauge invariance</u> $W(b)$ are gauge invariant, whereas a gauge transformation of the variables $U(b)$ induces a Z_2 gauge transformation $\sigma(b) \to \omega(x)\sigma(b)\omega(y)^{-1}$ for $b = (x,y)$, with $\omega(\cdot) = \pm 1$. Therefore the field strength $\sigma(\partial p) = \sigma(b_4) \ldots \sigma(b_1)$ (product over links on the boundary of a plaquette) is gauge invariant.

3. <u>Completeness</u> There exists a gauge transformation $S(\cdot)$ (dependent on U) such that

$$W(b)\sigma(b) = S(x)U(b)S(y)^{-1} \quad \text{for } b = (x,y), \tag{2.10}$$

and $\mathrm{tr}\,W(b) \geq 0$.

Properties 1. and 2. will be summarized by saying that $W(b)$ and $\sigma(\partial p)$ are <u>local gauge invariants</u>. One says that a <u>vortex soul</u> passes through the plaquette p if $\sigma(\partial p) = -1$. Vortex souls form closed 2-dimensional surfaces on the dual lattice, in 4-dimensions. For weak coupling β^{-1} these vortex souls have a topological interpretation analogous to the zero of the Higgs field in a Nielsen-Olesen vortex [32], see section 10 of ref. [21].

Properties 1 ... 3 above do not fix the variables W, σ uniquely. To get an explicit definition one must choose a local gauge [33] (a generalization of what is called a unitary gauge in Higgs models). An example will be given at the end of this section. It will therefore in general depend on the choice of local gauge where a vortex has its soul. (This explains the name 'soul': One does not know very precisely where it is, but vortices can be counted by counting souls.) The formulae below are however valid for any choice since they only depend on the above properties.

Consider a loop C which is boundary of a surface Ξ. From eq. (2.10) and the property of characters $\chi_\ell(\gamma V) = \gamma^{2\ell}\chi_\ell(V)$ for $\gamma = \pm 1$ it follows that

$$\chi_\ell(V) = \begin{cases} \chi_\ell(W(C)) \prod_{p\in\Xi} \sigma(\partial p) & \text{for } \ell = \frac{1}{2}, \frac{3}{2}, \ldots \\ \chi_\ell(W(C)) & \text{for } \ell = 0, 1, 2, \ldots \end{cases} \tag{2.11}$$

$\chi_\ell(W(C))$ is a sum of products of local gauge invariants that are localized near the path C. For instance,

$$\chi_{\frac{1}{2}}(W(C)) = \mathrm{tr}\, W(C) = \sum_{\alpha_1 \ldots \alpha_n} W_{\alpha_1\alpha_2}(b_n) \ldots W_{\alpha_2\alpha_1}(b_1) \tag{2.12}$$

if C is composed of links $b_1 \ldots b_n$. In the case of fractional charge there appears another factor which is not of this form. It counts the number of vortex souls that wind around C. It has been verified by Göpfert [34] to all orders of the high temperature expansion in the standard SU(2) model that $<\chi_{\frac{1}{2}}(W(C))>$ is perimeter law behaved, and that $<\chi_{\frac{1}{2}}(U(C))> \sim <\prod \sigma(\partial p)>$ up to a perimeter law behaved factor, so that the probability distribution of vortex souls determines the string tension. For those local gauges that were considered in [34], $\langle \prod \sigma(\partial p) \rangle$ depends on the choice of local gauge only through a (in some cases oscillatory) perimeter law behaved factor.

Now we return to our special models (2.2). The Gibbs measure (2.2) is supported on configurations U with sign tr $U(\partial p) \equiv 1$. Therefore it follows from eq. (2.10) that

$$\sigma(\partial p) = \text{sign tr } W(\partial p), \text{for models (2.2)}. \qquad (2.13)$$

As a result

$$\langle \chi_\ell (U(C)) \rangle_{SU(2)} = \langle \chi_\ell (W(C)) [\prod_{p \in \Xi} \text{sign tr } W(\partial p)]^{2\ell} \rangle_{SU(2)}$$

By property 1, W depends only on cosets $\bar{U}$. Therefore we may now apply eq. (2.3) to obtain our <u>final result</u>. If ℓ is integer, χ_ℓ is a character of an SO(3) representation and we may identify $\chi_\ell(V) = \chi_\ell(\bar{V})$. Thus finally

$$\langle \chi_\ell (U(C)) \rangle_{SU(2)} = \begin{cases} \langle \chi_\ell (W(C)) \prod_{p \in \Xi} \text{sign tr } W(\partial p) \rangle_{SO(3)} & \text{for } \ell = \frac{1}{2}, \frac{3}{2}, \ldots \\ \langle \chi_\ell (W(C)) \rangle_{SO(3)} = \langle \chi_\ell (\bar{U}(C)) \rangle_{SO(3)} & \text{for } \ell = 0, 1, 2, \ldots \end{cases} \qquad (2.14)$$

$\prod$ sign tr $W(\partial p)$ is independent of the choice of Ξ because of the constraints of the model which eliminate monopoles. This follows from eq. (2.13), since $\prod_{p \in \partial c} \sigma(\partial p) \equiv 1$ for $\sigma(\partial p) = \prod_{b \in \partial p} \sigma(b)$ (2nd Maxwell equation).

Finally we exhibit an example of a local gauge. To construct W, it suffies to specify the gauge transformation S(x) in (2.10). Since tr $W(b) \geqslant 0$ this fixes $\sigma(b) = \text{sign tr } S(x) U(b) S(y)^{-1}$. Let $p_{ij}(x)$ be the plaquette protruding from corner point x in positive ij direction. Then one may define the magnetic field matrix $\mathbb{B}(x) = (B^a{}_k(x))$ by

$$U(p_{ij}(x)) = 1 + i \sum_a \tau^a B^a{}_k(x) \qquad (ijk = 123 \text{ or cyclic}) \qquad (2.15)$$

τ^a are Pauli matrices. One can now define $\bar{S}(x) \in SO(3)$ by the decomposition

$$\mathbb{B}(x) = \bar{S}(x)\, \mathbb{P}(x) \qquad (2.16)$$

where the 3 × 3 matrix $\mathbb{P}$ is required to be either positive or negative semidefinite. S(x) is chosen as the representative of the coset $\bar{S}(x)$ with tr $S(x) \geqslant 0$. S(x) is uniquely defined for almost all gauge fields U, and W, σ are therefore well defined random variables (on a finite lattice, and also in general by an argument based on the Markov property, compare [35]). It is easy to verify that the properties 1 ... 3 above are satisfied.

3. Predictions of the effective Z_2-theory of confinement

The model (2.2a,b) differs from the SU(2) MP-model by the absence of Z_2-strings, and from the standard SU(2) model by the absence of monopoles as well as Z_2 strings. The world sheets of these Z_2 strings consist of plaquettes p in Λ where sign tr $U(\partial p) = -1$. They are closed surfaces on the dual lattice Λ^* in the MP-model, whereas in the standard model these surfaces may be either closed or have a boundary which is world line of a monopole. It consists of cubes c in Λ (= links in Λ^*) with $\rho_c = -1$.

The closed Z_2 strings are a special kind of "thin" vortices. They have a thickness of only one lattice spacing. At small β they condense in both models (MP- and standard ~), and this can be exploited to prove confinement. In the SU(2) model (2.2) the Z_2 strings are eliminated by a constraint. Therefore we expect a reduction of the string tension at small β. But we do not expect that confinement will be destroyed completely, because "fat" vortices can take over and confine static quarks in the same way as they do in the standard model at low β where the Z_2 strings are frozen (because they cost too much energy which cannot be compensated by their configurational entropy).

To obtain somewhat more quantitative information we appeal to the effective Z_2 theory of confinement which was described in ref. [24] In this theory, an effective Z_2-coupling constant $\beta_{eff}(d)$ is introduced which determines the chemical potential of a vortex of thickness d. More precisely it determines its energy, or internal free energy (which includes the entropy due to fluctuations in internal structure) whereas the configurational entropy is (approximately) the same as in a Z_2 theory on a lattice of lattice spacing d. Vortices of thickness d will condense if $\beta_{eff}(d) \leq \beta_c \approx 0.44$ (the transition point in Wegners Z_2 gauge theory model). This requires that $d \geq d_c(\beta)$. A distinction is made between a "high temperature" region where $\beta_{eff}(a) < \beta_c$ (a = lattice spacing) so that $d_c = a$ and vortices of thickness a condense, and a "low temperature" region where $d_c > a$. The high temperature region ends

at the value $\beta = \beta_1$ of the coupling parameter where $\beta_{eff}(a) = \beta_c$. In the low temperature region d_c is determined from the equation $\beta_{eff}(d_c) = \beta_c$, and the string tension is given by $\alpha = \alpha_0/d_c^2$, where α_0 is the string tension in Wegners Z_2 model just below its phase transition point. The value $\alpha_0 = 0.54$ is obtained by extrapolating the result of high temperature expansions to order 2k = 8,10,12,14 using a linear function of 1/k [36].

For the following calculations we need an approximate expression for $\beta_{eff}(d)$. It is given by 't Hoofts version of a vortex free energy $\nu(d)$ [29]. To obtain this approximation one imagines cutting a piece of area d'^2 out of a vortex sheet of thickness d and simulating the effect of its environment by imposing periodic boundary conditions [24]. This gives

$$\beta_{eff}(d) \approx \nu(d) = -\tfrac{1}{2}(d/d')^2 \ln\left[Z(\text{block}, \text{t.b.c.})/Z(\text{block}, \text{p.b.c.})\right] \quad (3.1)$$

and is supposed to be reasonably accurate for β not too large, depending on d, so that $d_c(\beta) \leq d$.
Z(...) are partition functions for a block of size $d \times d \times d' \times d'$ ($d' \geq d$) with periodic boundary conditions (p.b.c.), and twisted periodic boundary conditions (t.b.c.) with one twist in the 12-plane, respectively. The result should be independent of d' for $d' \gg d$.

The periodic (twisted) boundary conditions assure that there is an even (odd) number of vortex souls passing through the intersection Ξ of every plane x_3, x_4 = const. with the block. This is easily seen as follows[31] Periodic boundary conditions imply in particular periodicity of the Z_2-variables σ(b) that were introduced in the last section. It follows that the Z_2 parallel transporters satisfy

$$1 = \sigma(\partial\Xi) = \prod_{p\in\Xi} \sigma(\partial p) \qquad \text{for p.b.c.}$$

This says that an even number of vortex souls passes through Ξ. The singular gauge transformation on $\partial\Lambda$ (or a neighborhood of it) which changes periodic into twisted boundary configuration takes parallel transporters $\sigma(\partial\Xi) \to -\sigma(\partial\Xi)$. Therefore the number of vortex souls through Ξ is changed from even to odd.

We wish to obtain an estimate for the string tension of our positive plaquette model near $\beta = 0$. To get it we try to determine whether vortices of thickness 1 lattice spacing can condense at all in this model. The answer is not trivial, in spite of the constraint tr $U(\partial p) > 0$, if we count vortices by counting their souls as is suggested by the consideration of section 2. In ref. [30] we pointed out that twisted boundary conditions can be fulfilled by a pure gauge. This means that creation of vortex souls costs entropy but no energy. It follows that twisted b.c. can be fulfilled without making $\mathrm{tr}\, U(\partial p) \leq 0$ for any plaquette p, even in the case d = a. For instance $U(\partial p) = -U_1 U_2 U_1^{-1} U_2^{-2} = 1$ for $U_1 = i\sigma_1$, $U_2 = i\sigma_2$ (quaternions).

The effective Z_2 theory of confinement has so far given good results while using only the simplest of approximations. Encouraged by this we will be bold and use approximation (3.1) for d = a. Moreover we will drop the plaquettes in the boundary of the blocks from the action. (Consider them as part of the environment.) To check that these approximations are not unreasonable we will first try them out on the standard SU(2) model at low β. The partition function of that model is

$$Z = \int \prod dU(b) \exp \Sigma_p \tfrac{\beta}{2}\, \mathrm{tr}\, U(\partial p) \tag{3.2}$$

Therefore we obtain

$$Z(\mathrm{block}, \pm) = \left\{ \int dU dV \exp\left[\pm \tfrac{\beta}{2}\, \mathrm{tr}\, UVU^{-1}V^{-1} \right] \right\}^{d'^2/a^2}$$

+(-) stands for p.b.c. (t.b.c.)

We evaluate the integral by noting that $UVU^{-1}V^{-1} \in SU(2)$ for any U,V. Therefore there must exist a measure $\rho(W)dW$ on SU(2) such that

$$\int dU dV f(UVU^{-1}V^{-1}) = \int \rho(W) dW f(W) \tag{3.3}$$

Explicitly (see Appendix B)

$$\rho(W) = \frac{2\pi - \theta}{4 \sin \frac{1}{2}\theta} \quad \text{for} \quad W = S e^{i\sigma_3 \theta/2} S^{-1}, \ \theta = 0 \ldots 2\pi . \tag{3.4}$$

θ is the angle of rotation. For the purpose of integrating class functions we may use Weyls integration formula to substitute

$$dW = \frac{1}{\pi} \sin^2 \frac{\theta}{2}\, d\theta \qquad (\theta = 0 \ldots 2\pi) \tag{3.5}$$

As a result we obtain (I_0 is the modified Bessel function)

$$\int dU dV \exp\left[\tfrac{\beta}{2}\, \mathrm{tr}\, UVU^{-1}V^{-1}\right] = \beta^{-1}\left(e^{\beta} - I_0(\beta)\right) \tag{3.6}$$

Thus finally

$$\nu(a) = \frac{1}{2} \ln \left\{ (e^{\beta} - I_0(\beta)) / (I_0(\beta) - e^{-\beta}) \right\} = \frac{\beta}{4} + \ldots \tag{3.7}$$

for the standard model. We can use this to determine the end point β_1 of the high temperature region where vortices of thickness a cease to condense. Eq. (3.7) gives $\nu(a) = .44$ at $\beta = \beta_1 = 2.05$. This is in very good agreement with Monte Carlo data (s. section 4). Moreover, for $\beta < \beta_1$ the string tension α of the model should equal the string tension α_{Z_2} in Wegners Z_2 model at coupling parameter $\beta_{\mathrm{eff}}(a)$. Since $\alpha_{Z_2}(\beta) = -\ln\beta + \ldots$, we find, using eqs. (3.1) and (3.7)

$$\alpha = -\ln \frac{\beta}{4} + \ldots \tag{3.8}$$

This is the correct result, to order β^0.

Now that we have seen how well our simple approximation works for the standard model, we apply it to the positive plaquette model (2.2a,b). We obtain

$$Z(\mathrm{block}, \pm) = \left\{ \int dU dV\, \theta\left(\pm\, \mathrm{tr}\, UVU^{-1}V^{-1}\right) \exp\left[\pm \tfrac{\beta}{2}\, \mathrm{tr}\, UVU^{-1}V^{-1}\right] \right\}^{d'^2/a^2}$$

The integral can be evaluated in the same manner as before. It is expressible in terms of the modified Bessel- and Struve functions I_0 and $\mathbf{L}_0$ [37]. As a result

$$\begin{aligned} \nu(a) &= \frac{1}{2} \ln \left\{ \left(2e^{\beta} - 1 - I_0(\beta) - \mathbf{L}_0(\beta)\right) / \left(I_0(\beta) + \mathbf{L}_0(\beta) - 1\right) \right\} \\ &= \frac{1}{2} \ln(\pi - 1) + \frac{\pi(4-\pi)}{16(\pi - 1)} \beta + \ldots \\ &= 0.38 + 0.08\beta + \ldots \end{aligned} \tag{3.9}$$

This is close to and slightly below the critical value $\beta_c = 0.44$ for $\beta = 0$ and has a very small slope there. Therefore the string tension α in the model (2.2a,b) should be $(a=1)$

$$\alpha \gtrsim \alpha_0 = 0.54 \qquad \text{at} \quad \beta = 0 \tag{3.10}$$

(equal to the string tension of Wegners Z_2 model at $\beta = 0.38$) and it should change little with β near $\beta = 0$. Ignoring the errors inherent in our approximation, vortices of thickness a are found to just barely condense at $\beta = 0$, since $\nu(a) \leq 0.44$ for $\beta \leq 0.82$. In figure 6 we present some rough Monte Carlo data for the string tension of the model. They were obtained in the manner described in the introduction. We see that they are in agreement with the result of our theoretical calculations.

Finally we will now also consider the MP-model. It has Z_2 strings but no monopoles. It was proven in ref.[13] that this model has a phase transition that is associated with condensation of Z_2 strings. Partition functions are defined by the formula

$$Z = \int \prod dU(b) \exp\left[\frac{\beta}{2} \operatorname{tr} U(\partial p)\right] \prod_c \theta(\rho_c(\bar{U})) \qquad (3.11)$$

We compute the vortex free energy $\nu(a)$ using the same approximations as before. Because of the absence of monopoles and the periodic boundary conditions, we must either have $\operatorname{tr} U(\partial p) \geq 0$ for all plaquettes in the interior of the block, or $\operatorname{tr} U(\partial p) \leq 0$ for all of them. Z(block, $\pm$) are therefore sums of two contributions.

$$Z(\text{block},\pm) = \left[\int dU dV\, \theta(\pm \operatorname{tr} UVU^{-1}V^{-1})\, e^{\pm\frac{\beta}{2}\operatorname{tr} UVU^{-1}V^{-1}}\right]^{d'^2/a^2} + \left[\int dU dV\, \theta(\mp \operatorname{tr} UVU^{-1}V^{-1})\, e^{\pm\frac{\beta}{2}\operatorname{tr} UVU^{-1}V^{-1}}\right]^{d'^2/a^2}$$

The integrals can be evaluated in the same manner as before with the result

$$Z(\text{block},+) = \left[\frac{1}{2\beta}\left(2e^{\beta} - 1 - I_0(\beta) - L_0(\beta)\right)\right]^{d'^2/a^2} + \left[\frac{1}{2\beta}\left(1 - I_0(\beta) + L_0(\beta)\right)\right]^{d'^2/a^2}$$

$$Z(\text{block},-) = \left[\frac{1}{2\beta}\left(-1 + I_0(\beta) + L_0(\beta)\right)\right]^{d'^2/a^2} + \left[\frac{1}{2\beta}\left(-2e^{-\beta} + 1 + I_0(\beta) - L_0(\beta)\right)\right]^{d'^2/a^2}. \qquad (3.12)$$

We are interested in the behavior for $d'/a \gg 1$. In Z(block,+) the first term dominates in this limit for all β. In Z(block,-) the second term, which comes from the contributions of Z_2 strings, dominates for small β. Therefore

$$\nu(a) = \frac{3\pi}{8(\pi-1)}\beta + \dots \qquad \text{for small } \beta \qquad (3.13)$$

and the string tension behaves like

$$\alpha = -\ln \frac{3\pi}{8(\pi-1)}\beta + \dots \qquad \text{for small } \beta \qquad (3.14)$$

For large enough β the first term, which comes from vortices that are not Z_2 strings, dominates Z(block, -). The transition between the two possibilities occurs at the value of $\beta = \beta_{c,MP} = 0.82$ where both terms are equal. At this value of β the vortex free energy assumes the value $\nu(a) = 0.45$ which happens to be very nearly equal to the critical value $\beta_c = 0.44$ of the Z_2 coupling constant. From this we deduce that the string tension should behave as follows. For small β condensation of Z_2strings leads to a large value of the string tension as given by eq. (3.13). With increasing β the string tension falls. At $\beta = \beta_{c,MP} \approx 0.82$ it reaches a value around $\alpha_0 \approx 0.54$ which agrees with the prediction for the string tension for the positive plaquette model at that value of β. At $\beta > \beta_{c,MP}$ the Z_2 strings are no longer able to condense, and vortices which are not Z_2 strings take over. Their thickness has to grow with β.

We do not have Monte Carlo data for the string tension of the MP model. But there are indirect indications that the above predictions are essentially correct. First, the result (3.14) is consistent with the rigorous inequality of ref.[13] which implies that $\alpha \geq -\ln\beta$ for $\beta \to 0$. Second, the predicted value of the position $\beta_{c,MP}$ of the phase transition agrees well with the value $\beta_{c,MP} \approx 0.9$ that was obtained from Monte Carlo computations by Brower, Kessler and Levine [5], compare figure 1. Third, we have also Monte Carlo evidence that the probability of Z_2 strings in the MP-model becomes very small for β above $\beta_{c,MP}$. This follow from the results of figure 1 for $\langle \operatorname{sign} \operatorname{tr} U(\partial p)\rangle$.

Let us summarize our <u>conclusions</u>.

1. The effective Z_2 theory appears to work well with very simple approximations.

2. There is a strong correlation between thin vortices and monopoles. If the monopoles are suppressed by a constraint, the entropy of these vortices is lowered substantially. As a result, the string tension is also lowered substantially, for $\beta < \beta_1$ (the range of β where thin vortices condense in the standard model).

The simplicity of our approximations makes it very clear how the loss of entropy comes about because of a loss of possibilities in internal structure of the vortices.

Let us now turn to a discussion of what should happen at large β. Formally, the standard model, the MP-model, and the positive plaquette model (2.2) all have the same continuum limit. The example of the 3-dimensional U(1) lattice gauge theory [23] shows that arguments based on such formal properties need not be reliable. There is, however, a partial result. In ref. [1] it was shown that the monopoles in the standard SU(2) model are confined for large β. Moreover, it was predicted in ref. [24] on the basis of the effective Z_2 theory of confinement that the string tension for the standard model and for the MP-model should behave in the same way for large β. By the same argument the same should also hold for the positive plaquette model (2.2). One expects therefore that the correlation between monopoles and the fat vortices (which are needed to produce confinement for large β) will become less and less pronounced with increasing thickness d_c of these vortices. At the moment we can only hope that future Monte Carlo data will eventually confirm these predictions.

So far we have only considered monopoles of size 1 lattice spacing. In a sense such small monopoles in a SO(3) theory are the only ones which really deserve to be called monopoles (compare Appendix A). From the point of view of a block spin or renormalization group picture [38] it is however natural to consider also "fat monopoles" as have been introduced in ref. [1]. It would be interesting to know how they are correlated with fat vortices. An investigation in this direction was suggested in refs. [5,7]. A specific conjecture has been advanced by Iwasaki* [39]. He believes that the most important fat monopoles are those that are hidden inside instantons (in 4 dimensions). A stringent lower bound on the cost of energy of fat monopoles and their strings was established in [1].

*In Iwasaki's work only thin vortices are considered. The spreading mechanism, by which fat vortices can lower their free energy and confine static quarks for large β, is not considered there.

4. Monte Carlo data for the vortex free energy

In our first paper [30] we presented Monte Carlo data for the derivative $\partial\nu/\partial\beta$ of the vortex free energy ν as defined in eq. (3.1), for cubic lattices of side length $d' = d \leq 5a$, and we also compared them to predictions of the effective Z_2 theory of confinement. Since then we have collected more such data, and we have extended the computations to a lattice of side length $d' = d = 6a$. We take the opportunity to present these data in figure 8. The conclusions are the same as in ref. [30]. The fits represent the following predictions of the effective Z_2 theory for intermediate β [24]:

$$\nu(d) = [d/d_c(\beta)]^2 \exp[-\alpha d^2] \quad \text{for } d \gg d_c \tag{4.1}$$

where the string tension is given by

$$\alpha = \alpha_0 \exp[-\tfrac{6\pi^2}{11}(\beta-\beta_1)] \quad \text{for } \beta \geq \beta_1 , \tag{4.2}$$

and

$$(d_c/a)^2 = \begin{cases} \alpha_0/\alpha & \text{for } \beta \geq \beta_c & (4.3a) \\ 1 & \text{for } \beta \leq \beta_c & (4.3b) \end{cases}$$

a = lattice spacing, and α_0 and β_1 have the meaning described in section 3. In the fits we put $\alpha_0 = 0.54a^{-2}$ and $\beta_1 = 2.06$. All data are for $\beta > \beta_1$. (Below β_1, $\partial\nu/\partial\beta$ is too small to be computable on a reasonably big lattice.) Very good agreement is seen. Münster has shown [40] that the asymptotic behavior of ν is $\nu = (d^2/a^2)\exp[-\alpha d^2]$ to all orders of the high temperature expansion. If we were allowed to combine with eq. (4.2) for the string tension, which fits Creutz' Monte Carlo data [27] for $\beta > 2.15$ or so, we would get a ratio 16 : 25 : 36 of the height of the maxima of $\partial\nu/\partial\beta$ for lattice size 4a, 5a and 6a. In contrast, the predictions (4.1), (4.3a) of the effective Z_2 theory for $\beta > \beta_1$ differ from this expression by a factor (d^2/d_c^2) which depends exponentially on β. It predicts equal height of the maxima. The data decide in favor of this alternative.

Note added in manuscript. The asymptotic behavior of $\nu(d)$ for large β and fixed d/a can, in principle, be calculated by the saddle point method. In practise this is very difficult, but a calculation has now been performed for $\partial\nu/\partial\beta$ by Gonzalez-Arroyo, Groeneveld, Korthals Altes and Jurkiewicz [44]. They find that ν does not admit a series expansion in powers of $g^2 = 4/\beta$; instead there is an extra logarithmic term. It is also present in our result (3.7) for the special case d = a which gives

$$\nu(a) = \frac{1}{4} \ln 2\pi\beta + \dots \qquad \text{for large } \beta .$$

In ref. [30] we presented some Monte Carlo data for $\partial\nu/\partial\beta$ on a 3^4 lattice for large β (up to $\beta = 9$). They failed to reveal the presence of the logarithmic term in ν – the data could be fitted within errors without it.

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Appendix A SO(3)-monopoles in continuous space

We will adopt the fibre bundle point of view which has been advocated in particular by Wu and Yang [9].

Consider the 3-dimensional space $M = \mathbb{R}^3 - \{x_o\}$ which is obtained from Euclidean space by removing a single point x_o. x_o will be the monopoles site. Since the 2nd Maxwell equation (Bianchi identity) is not true at the site of a monopole, a gauge field (vector potential) will only exist away from x_o.

One imagines that a 3-dimensional real vector space V_x is attached to every point x of M. One may envisage introducing matter fields ϕ eventually which take their values $\phi(x) \in V_x$. To formulate differential equations for them one needs a topology (and differentiable structure) on the space E of pairs (x,v); $x \in M$, $v \in V_x$. It should satisfy certain requirements (the scalar product $\langle , \rangle$ should be continuous, and E should be a Euclidean space locally). When E is equipped with such a topology it is called a vector bundle with structure group SO(3).

The Naheinformationsprinzip [33] asserts that there is no a priori way of comparing directions in vector spaces V_x and V_y that are attached to different points x = y. Instead, a map

$$\mathcal{U}(C) : V_x \to V_y$$

is assumed to be given for every path C from x to y. It should depend smoothly on C, preserve the scalar product so that

$$\langle v_1, v_2 \rangle = \langle \mathcal{U}(C)v_1, \mathcal{U}(C)v_2 \rangle \quad \text{for } v_1, v_2 \in V_x$$

and obey the composition rule

$$\mathcal{U}(C_2 \circ C_1) = \mathcal{U}(C_2)\mathcal{U}(C_1) : V_x \to V_z$$

if $C_2 \circ C_1$ is obtained by juxtaposition of a path C_1 from x to y and a path C_2 from y to z. $\mathcal{U}(C)$ are called parallel transporters. They are said to specify a connection.

Our space M is topologically a sphere S^2. The classification of vector bundles over spheres has been described in Steenrods book [41].

The result is that they are specified by homotopy classes of maps from the structure group G of the vector bundle into the equator of the sphere. Here G = SO(3) and the equator of S^2 is a circle S^1. Since $\pi_1(SO(3)) = \mathbb{Z}_2$ there are two inequivalent bundles [42]. One is trivial, i.e. isomorphic to $M \times V_x$. This vector bundle can be extended to a vector bundle over $\mathbb{R}^3$, while the other one cannot. One says that there is a monopole at x_o if the vector bundle E is nontrivial.

To obtain a vector potential from $\mathcal{U}$ one needs to specify a moving frame first. A moving frame $\underline{e}$ specifies an orthonormal basis $\underline{e}(x) = (e_1(x), e_2(x), e_3(x))$ of vectors $e_i(x) \in V_x$ for every $x \in M$. One would like to have them depend smoothly on x; the problem caused by this requirement will be discussed below. A moving frame specifies a coordinate system on E: A point $(x,v) \in E$ can be specified by real numbers (x^μ, v^i), $v^i = \langle v, e_i(x) \rangle$.

Given a moving frame and a connection, i.e. parallel transporters $\mathcal{U}(C)$, one can define parallel transport matrices $\bar{U}(C) = (\bar{U}^i{}_j(C)) \in SO(3)$ by

$$\mathcal{U}(C) e_i(x) = e_i(y) \bar{U}^i{}_j(C)$$

x and y are initial and final point of C. The vector potential $A_\mu(x) = (A^i{}_{j\mu}(x))$ is a matrix in the Lie algebra $so(3)$ and is defined in terms of the parallel transport matrices by the familiar relation

$$U(C) = T \exp\left[- \int_C A_\mu \, dx^\mu \right]$$

One of the fundamental results in the theory of fibre bundles asserts that it is impossible to find an everywhere smooth moving frame in a nontrivial vector bundle (= a smooth section in the associated principal fibre bundle) [41] . It follows that the presence of a monopole at $x = x_o$ enforces the presence of singularities in any moving frame over $\mathbb{R}^3 - \{x_o\}$. In other words, a nonsingular

global coordinate system does not exist. Let C_D be any path from x_o to infinity. Since $\mathbb{R}^3 - C_D$ is topologically trivial, the fibre bundle E reduces to a trivial fibre bundle over $\mathbb{R}^3 - C_D$. Therefore the moving frame can be chosen so that it has singularities only on C_D. C_D is called a Dirac string. The singularities of the moving frame cause singularities of the vector potential on C_D as well. These are not singularities of the connection $\mathcal{U}$ but are merely due to the singularities of the coordinate system - the Dirac string is unphysical.

Given a vector potential $A_\mu(x)$, parallel transport matrices U(C) in SU(2) - rather than SO(3) - can also be defined by

$$U(C) = \exp\left[-\tfrac{i}{2} \int \varepsilon_{ikj} \frac{\tau^i}{2} A^j{}_{k\mu}(x) dx^\mu \right]$$

τ^i are Pauli matrices. The singularities of the vector potential on the Dirac string have the consequence that

$$\tfrac{1}{2} \operatorname{tr} U(C) \approx -1$$

if C is a closed path of infinitesimal length which winds around the Dirac string C_D, see figure 9. In contrast, $\frac{1}{2}$ tr U(C) ≈ 1 for any boundary of an infinitesimal area that is not crossed by C_D (and lies away from x_o).

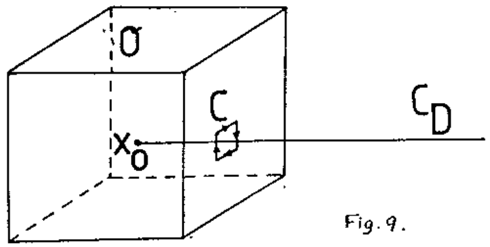

Fig. 9.

We will now consider a more general situation with several monopoles at some positions x_i. Given an open set σ, we may define the magnetic charge $Q = 0,1$ in σ to be equal to the number of monopoles in σ modulo 2. There is a formula for Q. Suppose that σ is a cube, and none of the monopoles is on its boundary $\partial\sigma$. We may superimpose a lattice of small lattice spacing a on the continuum $\mathbb{R}^3$ so that $\partial\sigma$ is a union of plaquettes p of this lattice. We can count monopole charge by counting Dirac strings. Therefore

$$e^{i\pi Q} = \prod_{p \in \partial\sigma} \operatorname{sign} \operatorname{tr} U(\partial p)$$

for sufficiently small lattice spacing a (depending on σ and U). This motivates the definition of monopoles in an SO(3) lattice gauge theory that is described in the introduction.

The world line of a monopole in space time $\mathbb{R}^4$ is a line without end points - either closed, or extending to infinity.

Let us emphasize once more that monopoles such as are discussed here can only exist when the gauge group G has a nontrivial first homotopy group $\pi_1(G)$, and that the Dirac string attached to such monopoles is unphysical and unobservable . If we had started with 2-dimensional complex vector spaces V_x in which (classical) fields or Schrödinger wave functions for quarks could take their values, and with a corresponding gauge group SU(2), then any vector bundle over $\mathbb{R}^3 - \{x_0\}$ would be trivial and there would be no monopoles. In an SU(2) lattice gauge theory one can interprete the gauge field as an SO(3) gauge field and use this to define monopoles as discussed in section 1, but the string attached to such monopoles is not unphysical - it costs energy [10].

Appendix B   Proof of formula (3.4)

We wish to determine the measure $\rho(W)dW$ on SU(2) which satisfies

$$\int dU dV\, f(UVU^{-1}V^{-1}) = \int \rho(W)dW\, f(W) \qquad \text{(B.1)}$$

dU is normalized Haar measure on SU(2). By making a substitution $U \to SUS^{-1}$, $V \to SVS^{-1}$ with $S \in SU(2)$ we see that ρ is a class function. Therefore it admits a character expansion

$$\rho(W) = \sum_{j=0,\frac{1}{2},1,\dots} a_j^{-1} \chi_j(W) \qquad \text{(B.2)}$$

χ_j is the character of the 2j + 1 dimensional irreducible representation of SU(2). It remains to determine the coefficients a_j^{-1}. We use orthogonality and the defining nonlinear integral equation of irreducible characters,

$$\int dU\, \chi_j(U)\chi_k(U^{-1}) = \delta_{jk} \quad ; \quad \int dU\, \chi_j(UV_1U^{-1}V_2) = \frac{1}{2j+1}\chi_j(V_1)\chi_j(V_2) . \qquad \text{(B.3)}$$

From eq. (B.1) we find that $a_j = 2j + 1$. Thus

$$\rho(W) = \sum_{j=0,\frac{1}{2},\dots} \frac{1}{2j+1} \frac{\sin(j+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} = \frac{2\pi - \theta}{4\sin\frac{1}{2}\theta} \quad \text{for } \theta = 0 \dots 2\pi \qquad \text{(B.4)}$$

The last equation is well known, especially to electrical engineers.

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Figure Captions

Figure 1. Monte Carlo data for $1-\langle \text{sign tr } U(\partial p)\rangle$ in the SU(2) MP-model (3.11). The calculations were done on a 3^4 lattice.

Figure 2. The specific heat C_V of the standard SU(2) model with Wilson action (1.3), computed on a 6^4 lattice. The line represents the finite size scaling polynomial fit of Nauenberg, Schalk and Brower to their data for the internal energy. For further explanation see text.

Figure 3. Internal energy (b) and monopole density (a) in the SO(3) model with Lagrangean $\frac{1}{3}\beta \text{ tr } \bar{U}(\partial p)$, compared with the internal energy of the corresponding monopoleless SO(3) model (2.1)(c).

Figure 4. The order parameter $\langle L(\underset{\sim}{Q})\rangle$ in the positive plaquette model (2.2a,b) for different sizes $n_t \times n_s^3$ of the lattice. The value of n_t^{-1} where $\langle L(\underset{\sim}{Q})\rangle$ reaches zero can be interpreted as a physical transition temperature to 'hot gluon soup' (in units of lattice spacing a^{-1}).

Figure 5. The potential V(r) between static quarks as defined by eq. (1.8) at two values of β, for n_t^{-1} below the transition temperature.

Figure 6. The string tension for the positive plaquette model (2.2a,b) compared with Creutz' Monte Carlo data for the string tension of the standard SU(2) model with Wilson action (1.3). The line represents the fit by eqs. (4.2), (4.3a), with the values of α_o, β_1 which give the best fit to the Monte Carlo data for $\partial \nu/\partial\beta$ (Fig. 8) in the standard model. Raising β_1 to 2.09 would give a perfect fit. We attribute this small discrepancy to systematic errors in the determination of α in [27].

Figure 7. The paths used in defining the order parameters and correlation functions $\langle L(\underset{\sim}{Q})\rangle$ and $\langle L(\underset{\sim}{x})L(\underset{\sim}{Q})\rangle$. See text before eq. (1.7).

Figure 8. Monte Carlo data for the derivative $\partial\nu/\partial\beta$ of the vortex free energy $\nu(d)$ of the standard SU(2) model with Wilson action (1.3). The lines represent a fit by eqs. (4.1) ... (4.3a), with $\alpha_o = 0.54\ a^{-2}$ and $\beta_1 = 2.06$.

Figure 9. A monopole at x_o and its Dirac string C_D. (Fig. on p.27)

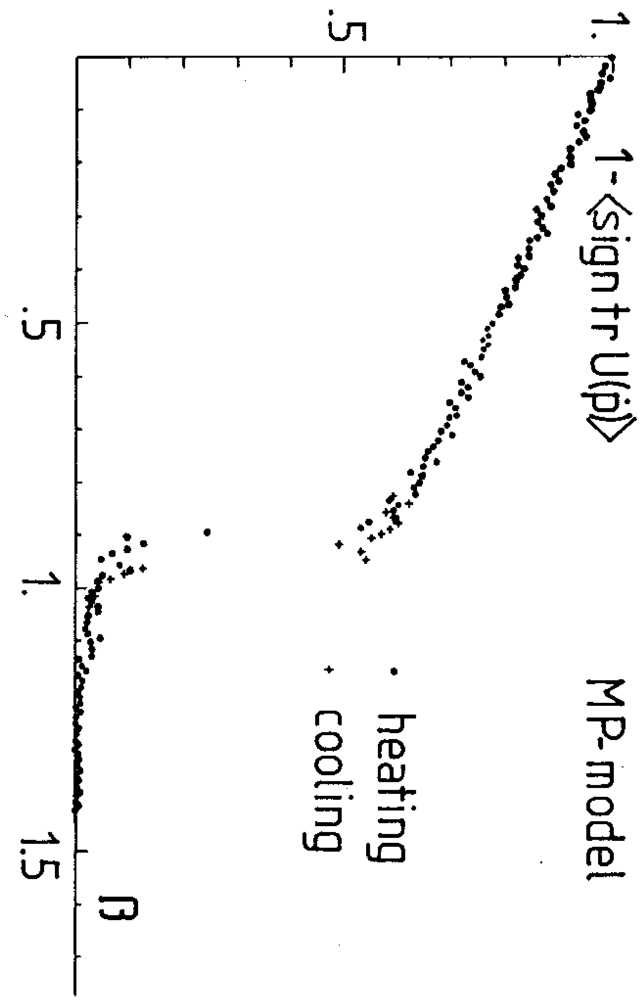

Figure 1

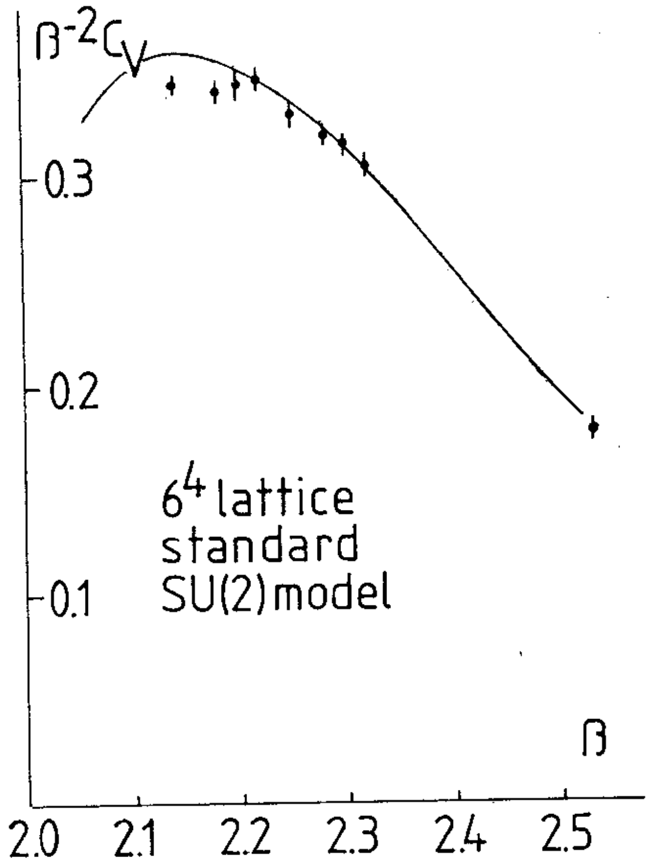

Figure 2

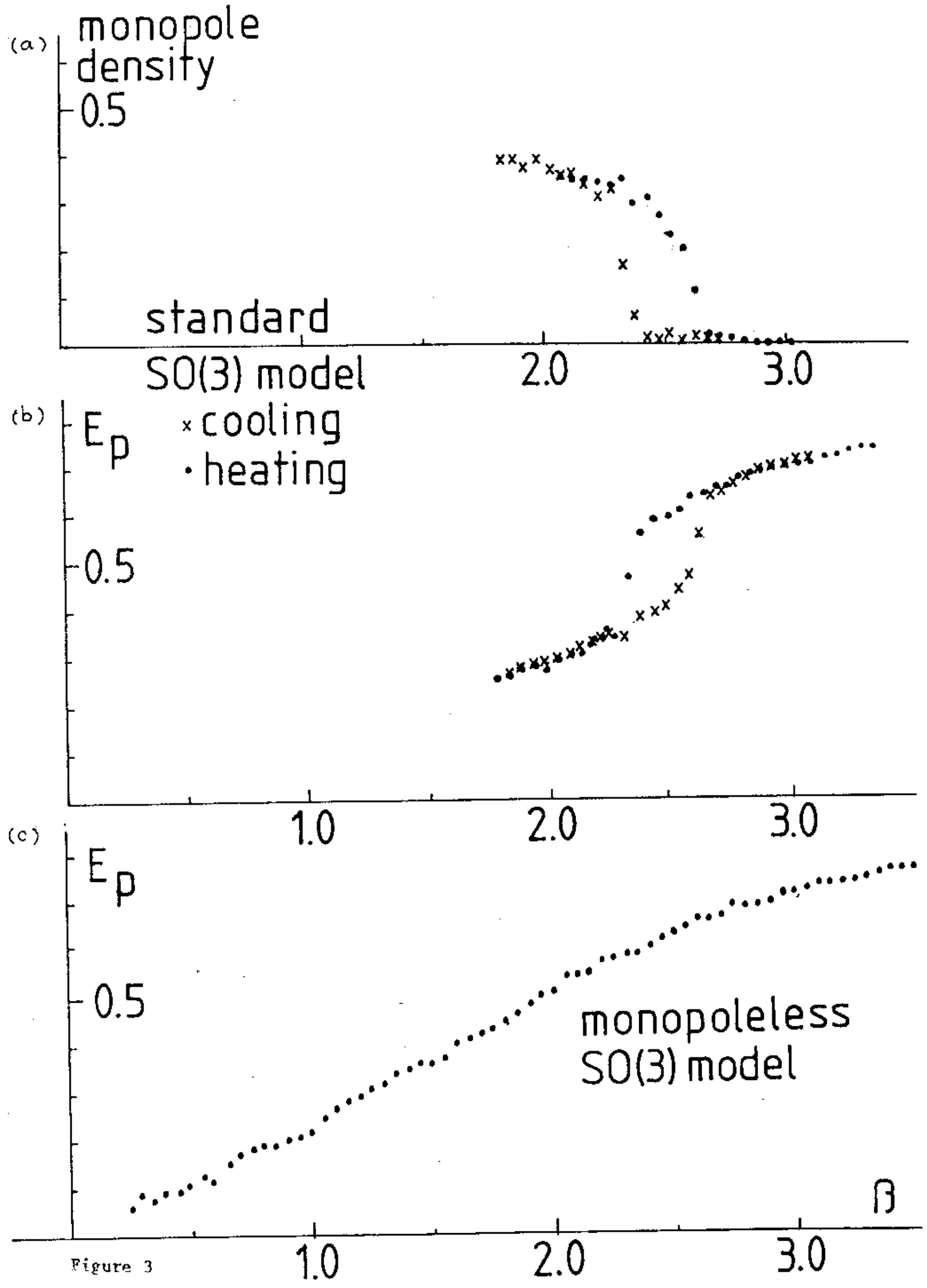

Figure 3

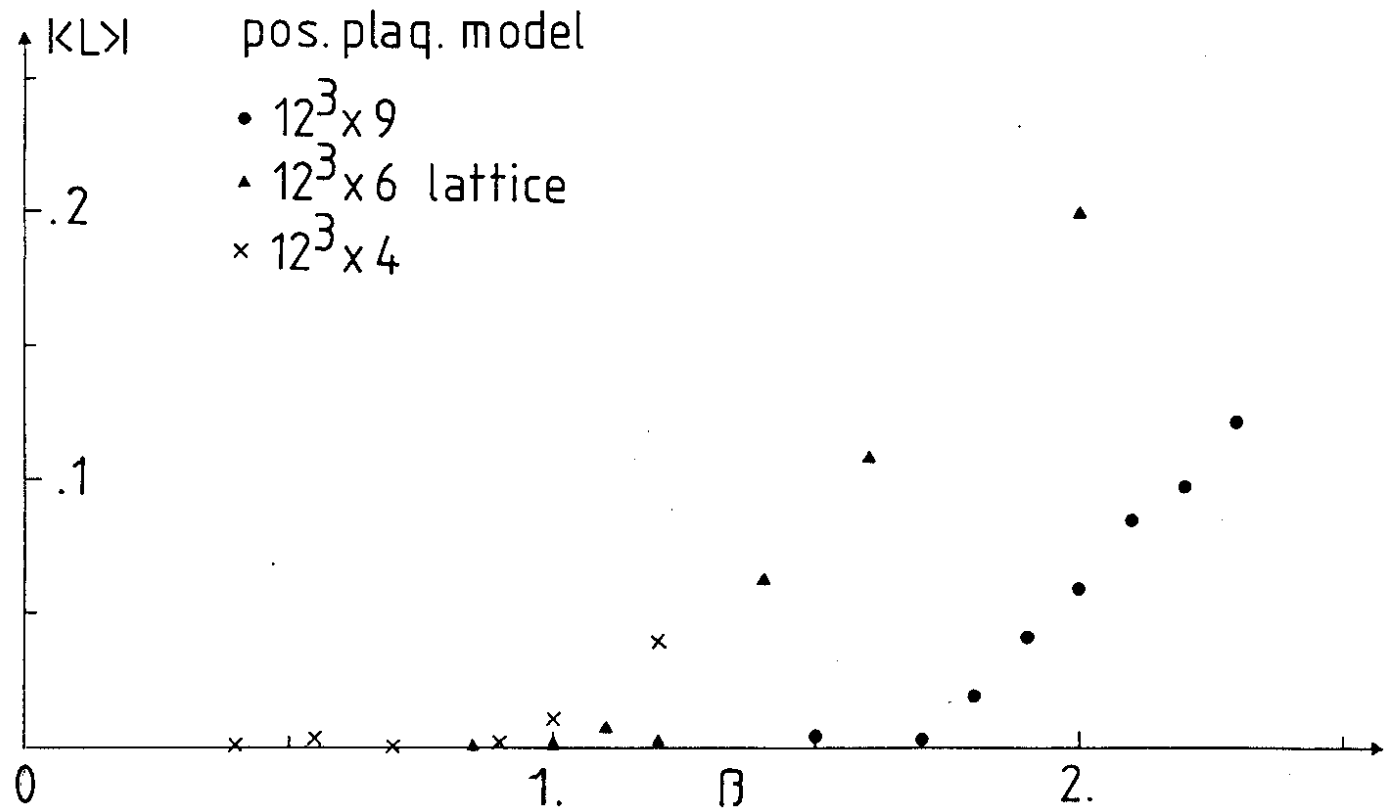

Figure 4

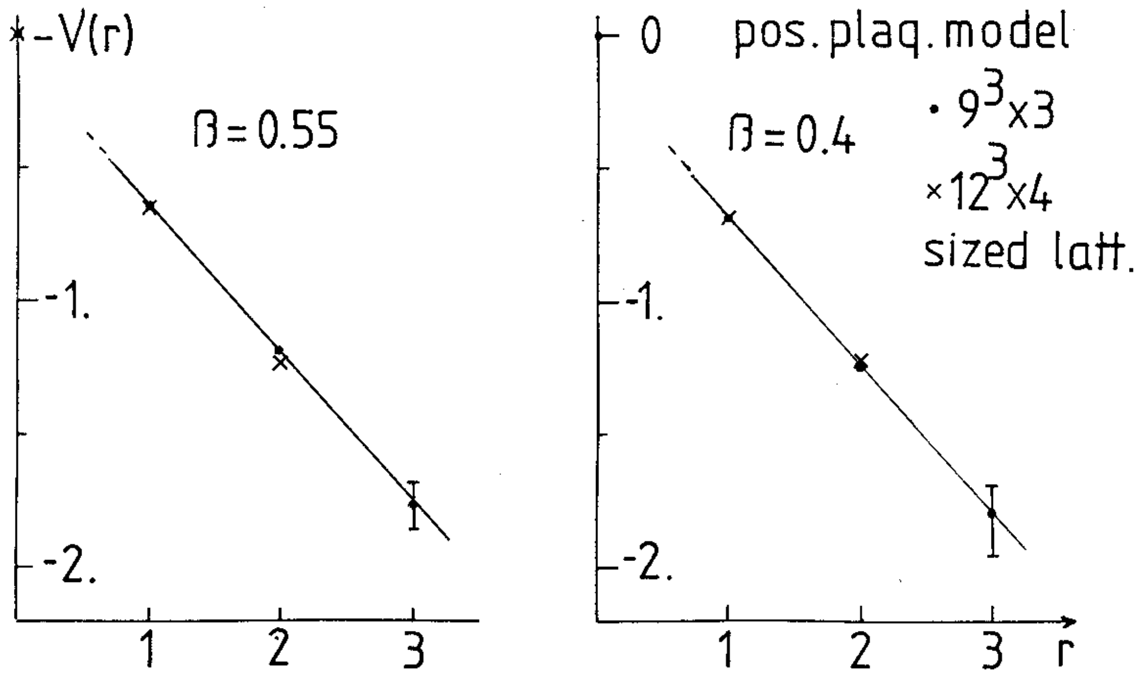

Figure 5

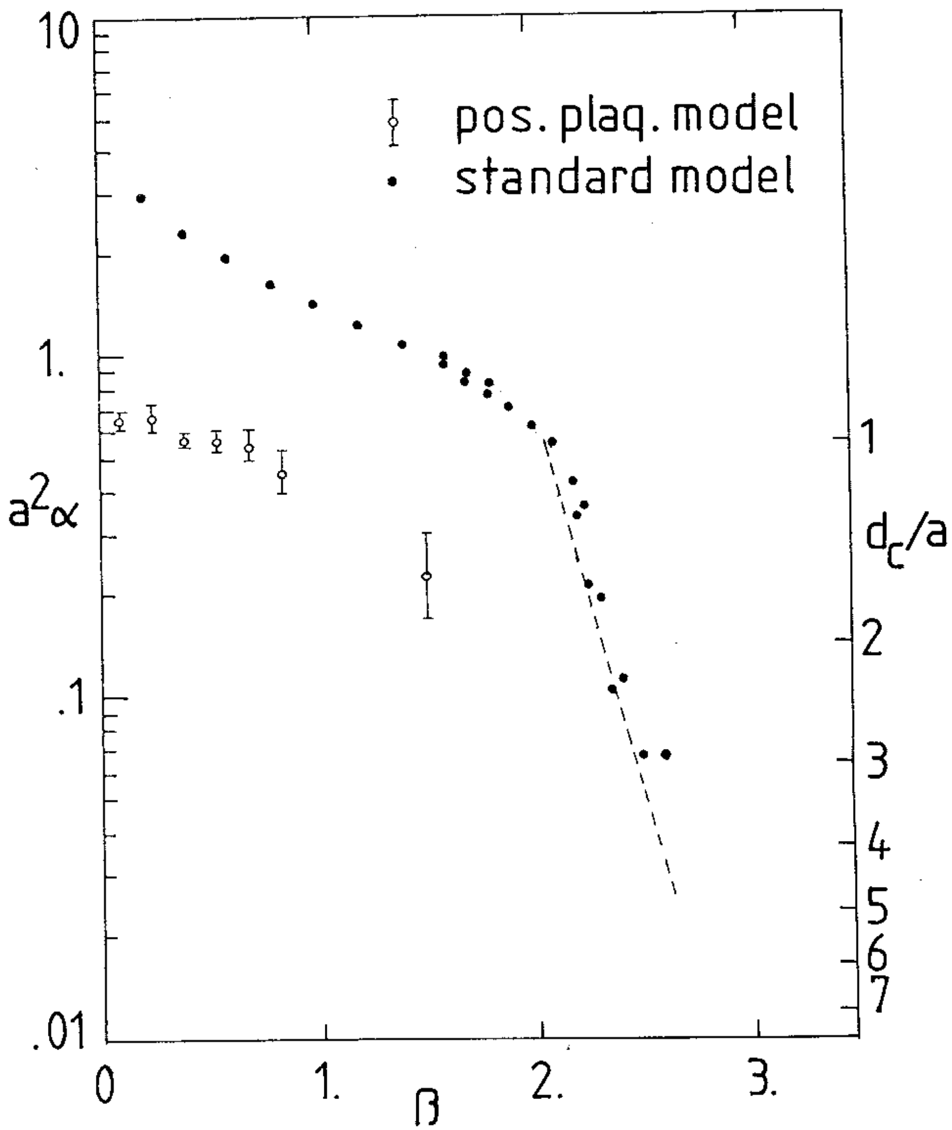

Figure 6

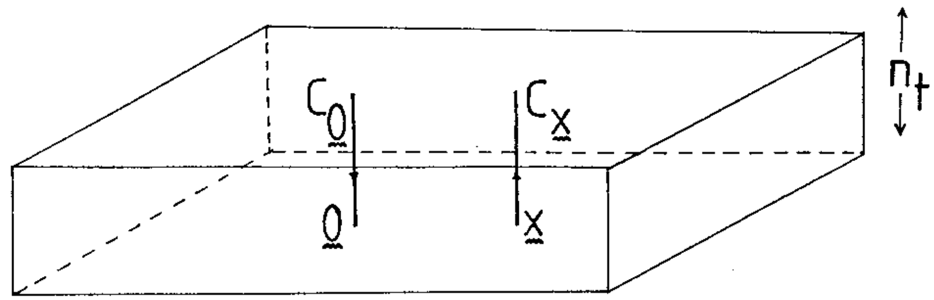

Figure 7

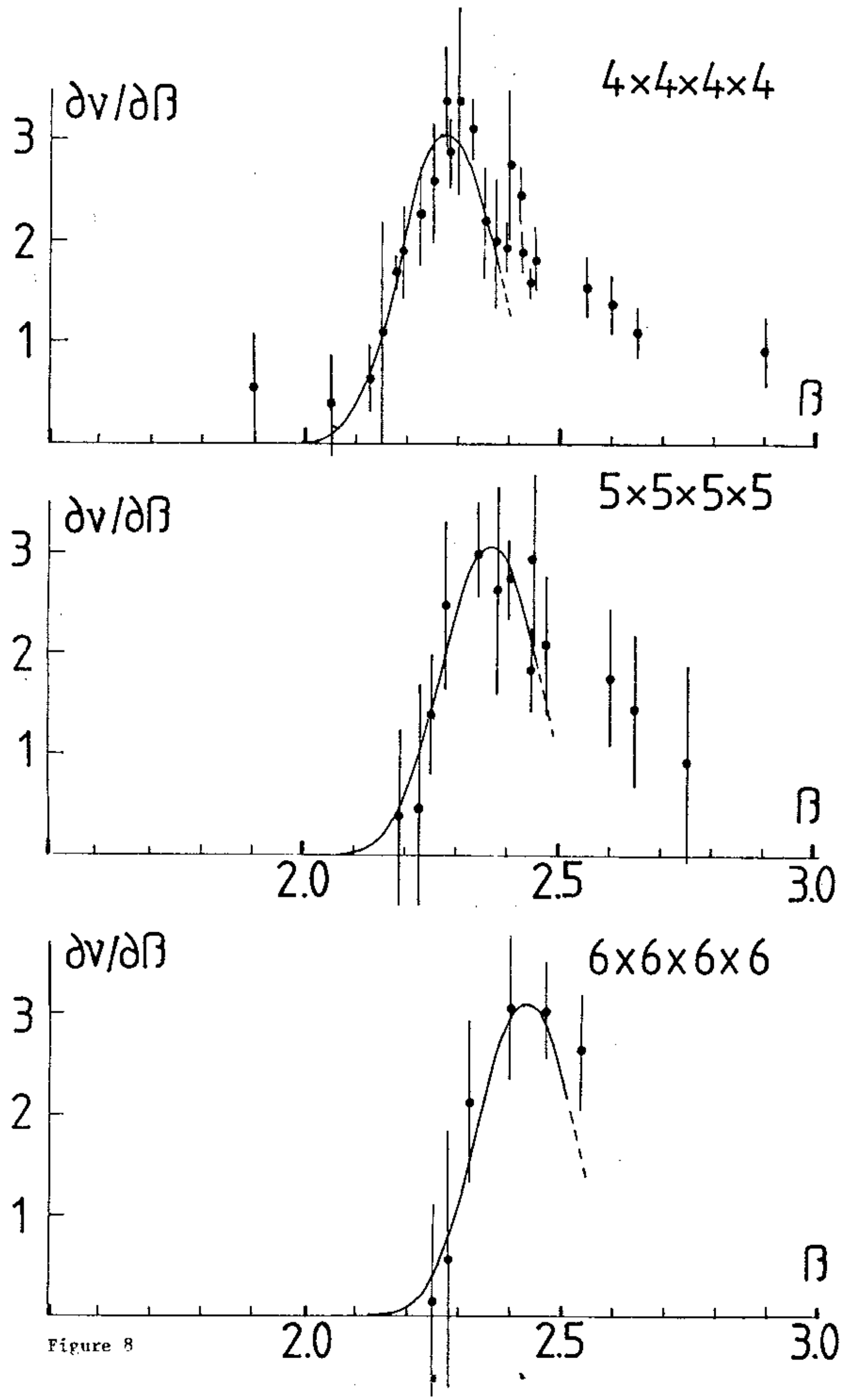

Figure 8