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INFRARED DYNAMICS OF QUANTUM ELECTRODYNAMICS AND THE ASYMPTOTIC BEHAVIOR OF THE ELECTRON FORM FACTOR

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H. D. Dahmen
Fachbereich Physik, Universität Siegen

B. Scholz
Deutsches Elektronen-Synchrotron DESY, Hamburg

F. Steiner
II. Institut für Theoretische Physik der Universität Hamburg

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*) On leave of absence from Fachbereich Physik, Universität Siegen

Abstract

A general method is presented for the study of the infrared dynamics in quantum electrodynamics. As an illustrative example the method is applied to the electron form factor. The proof of the famous exponentiation is very simple and needs no order by order investigation of perturbation theory. The correct renormalization of the form factor is automatically obtained. Its asymptotic behavior at large momentum transfer shows the well-known Sudakov suppression in addition to the explicit infrared divergences. The Sudakov behavior is intimately related to the infrared divergent Coulomb phase via an Omnès representation.

I. Introduction

In this work we present a new method for a consistent treatment of the infrared behavior in QED. It is based on a Hamiltonian which contains the full infrared dynamics of QED. This Hamiltonian is derived from the exact QED Hamiltonian. In contrast to an earlier treatment of the infrared problem by Kulish and Faddeev [1] we use a fictitious photon mass λ instead of the time t as an infrared regulator. This allows us to perform explicitly the limit $t \longrightarrow \infty$ and thereby we obtain a well defined unitary S-operator in Fock space. Due to the simple structure of the infrared Hamiltonian we obtain a closed expression for the S-operator which does no more involve the complicated time ordering of the full S-operator. Our S-operator has a nice physical interpretation since it creates coherent states which describe a cloud of an infinite number of soft photons.

We demonstrate the power of our method by applying it to the calculation of the renormalized Dirac form factor of the electron. The famous exponentiation [2-11] of the form factor in the infrared region follows immediately from the exponential form of the S-operator. We thereby avoid a tedious study of an infinite number of Feynman diagrams. A diagrammatic interpretation of our result can be given in terms of three graphs only. These graphs include already the ultraviolet renormalization of the exponentiated form factor.

Since we work in the time-like region we also get the complex phase of the form factor which is analoguous to the well-known Coulomb phase in electron scattering. It turns out that the Coulomb phase of the form factor is identical to the infrared contribution to the imaginary part of the exact order e^2

form factor. This allows to cast the exponentiated result into the form of an Omnès representation. In the high energy limit we obtain for the form factor a double logarithmic suppression in agreement with the well-known leading logarithmic result [2-11]. This behavior can be seen to be a direct consequence of the analytic structure of the form factor as expressed by the Omnès representation.

The method described in this paper can also be applied to Yang-Mills theories. A first step in this direction has been undertaken in ref. [12,13]. We hope that our treatment of the infrared behavior provides also in Quantum Chromo-dynamics a systematic method to attack the soft gluon problem.

II. The Infrared Dynamics of QED

Our starting point of the study of infrared properties in QED is the observation made by Kulish and Faddeev [1] that the infrared structure is completely determined by the large time behavior of the Hamiltonian *). In the interaction picture the interaction Hamiltonian - in terms of free fields - is given by

$$H_I(t) = \int d^3x \; j_\mu(x) \, A^\mu(x) \tag{1}$$

with

$$j_\mu(x) = -e :\bar{\psi}(x)\gamma_\mu \psi(x):$$

We now decompose the above Hamiltonian into

$$H_I(t) = \widetilde{H}_I(t) + H_I'(t) \tag{2}$$

*) This is easily seen from the fact that the U-operator at finite times shows no infrared singularities.

where the infrared Hamiltonian $\tilde{H}_I(t)$ is defined as the dominant term of $H_I(t)$ in the limit $t = x^0$ going to infinity. In the Kulish-Faddeev approach where massless photons are used the infrared singularities appear as singularities in the time t. This means that t plays the rôle of an infrared regulator. Since the evaluation of the S-operator involves an infinite time limit one obtains an S-operator which is no more a unitary operator in the usual Fock space but acts instead in the larger space of coherent states.

If one introduces a fictitious photon mass as done in most of the calculations one can carry out the time limit and obtains a well-behaved S-operator in Hilbert space. This leads us to the modification of the Kulish-Faddeev approach where the decomposition, eq. (2), is performed by giving the photon a fictitious mass. A study of the large time limit of $H_I(t)$ with massive photons leads to the following form of the spin non-flip part of the infrared Hamiltonian $\tilde{H}_I$ (see Appendix A)

$$\tilde{H}_I(t) = \int d^3\vec{x}\, \tilde{j}_\mu(x)\, A^\mu(x) \tag{3}$$

where the asymptotic current is found to be

$$\tilde{j}_\mu(x) = \int d^3\vec{p}\, \rho(\vec{p})\, \frac{p_\mu}{\omega}\, \delta^{(3)}\!\left(\vec{x} - \frac{\vec{p}}{\omega}t\right) \tag{3a}$$

with the charge density operator

$$\rho(\vec{p}) = -e\sum_r \left[b^+(\vec{p},r)\, b(\vec{p},r) - d^+(\vec{p},r)\, d(\vec{p},r) \right] \tag{3b}$$

Here $p^\mu = (\omega, \vec{p})$ denotes the electron/positron 4-momentum, $\omega = \sqrt{\vec{p}^2 + m^2}$, m = electron mass *). (For details of the notation see App. A). Eq. (3) has the same form as in the Kulish-Faddeev approach apart from the fact that $A^\mu(x)$ stands now for the massive photon field. The infrared Hamiltonian $\tilde{H}_I$ describes the interaction of the photon with a quasiclassical electron-positron current. Since the charge operator commutes with the fermion number operator the current $\tilde{j}_\mu$ conserves separately the numbers of electrons and positrons. Thus the infrared Hamiltonian, eq. (3), describes the absorption and emission of photons and does not contain the annihilation and creation of electron-positron pairs. This was to be expected since it is well-known that fermion loops do not lead to infrared singularties. A further important property of the asymptotic current $\tilde{j}_\mu(x)$ consists in the fact that it commutes with itself, i.e.:

$$\left[\tilde{j}_\mu(x), \tilde{j}_\nu(x') \right] = 0 \qquad \text{for all } x, x' \tag{4}$$

The infrared structure of QED is completely described by the time evolution operator U(t) which satisfies the equation

$$\frac{d\,U(t)}{dt} = -i\, \tilde{H}_I(t)\, U(t) \tag{5}$$

The solution of this equation can be given in a closed form according to a theorem by W. Magnus [17]

*) The mass m and the charge e are the renormalized physical quantities all throughout. This follows from the classical nature of the current, eq. (3a).

$$\mathcal{U}(t) = \exp\left\{-i\int_{t_0}^{t}\tilde{H}_I(t_1)\,dt_1 + \frac{(-i)^2}{2}\int_{t_0}^{t}dt_2\int_{t_0}^{t_2}dt_1\left[\tilde{H}_I(t_2),\tilde{H}_I(t_1)\right]\right\} \qquad (6)$$

with $U(t_o) = 1$.

This closed nonperturbative expression for the infrared dynamics is equivalent to the commonly used Dyson expression for U(t). It has the above simple structure due to the fact that

$$\left[\left[\tilde{H}_I(t_1),\tilde{H}_I(t_2)\right],\tilde{H}_I(t_3)\right] = 0 \qquad (7)$$

as follows from eq. (4).

The transformation of the Fock space states $|\zeta\rangle$ of the interaction picture by virtue of U(t) creates a photon cloud around the electrons/positrons present in the state $|\zeta\rangle$. The new state $|\tilde{\zeta}\rangle$ is given by

$$|\tilde{\zeta}\rangle = \mathcal{U}^{+}|\zeta\rangle \qquad (8)$$

and is a coherent state [14]. It still lies in the original Fock space as long as the photon mass is kept unequal zero.

III. The On Shell Electron Form Factor

We now apply the formalism outlined in section II to the calculation of the

on shell electron form factor.

The form factor F(s) is defined by means of the matrix element of the full electromagnetic current $J_\mu(x) = -e:\bar{\Psi}(x)\gamma_\mu\Psi(x):$ between the Fock vacuum and an outgoing e^-e^+ coherent state $|\widetilde{e^-e^+}\rangle$ *):

$$\begin{aligned}
&\lim_{t\to\infty}\langle \widetilde{e^-(p_1)\,e^+(p_2)}\,|\,J_\mu(0)\,|\,0\rangle \\
&= \lim_{t\to\infty}\langle e^-(p_1)\,e^+(p_2)\,|\,\mathcal{U}(t)\,J_\mu(0)\,|\,0\rangle \qquad (9)\\
&= \lim_{t\to\infty}\langle e^-(p_1)\,e^+(p_2)\,|\,\mathcal{U}(t)\,|\,e^-(p_1)\,e^+(p_2)\rangle\langle e^-(p_1)\,e^+(p_2)\,|\,J_\mu(0)\,|\,0\rangle \\
&\equiv F(s)\,\langle e^-(p_1)\,e^+(p_2)\,|\,J_\mu(0)\,|\,0\rangle
\end{aligned}$$

$s = (p_1 + p_2)^2$, $p_1^{\,2} = p_2^{\,2} = m^2$.

Here we have used the diagonality of U(t) with respect to electron/positron number and their momenta.

If we now project the operator U(t) onto the subspace of one electron and one positron we obtain the following expression for the form factor:

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*) As can be seen from eq. (9) the on shell vertex function in the infrared region has only a γ_μ-term, since the Hamiltonian $\tilde{H}_I$, eq. (3), does not contain any spin dependence. This implies that the form factor F(s) is the Dirac form factor $F_1(s)$ in the usual notation.

$$F(s) = \lim_{t\to\infty} \langle 0 | \mathcal{U}_{e^-e^+}(t;p_1,p_2) | 0 \rangle \tag{10}$$

where $U_{e^-e^+}(t;p_1,p_2)$ has the same form as in eq. (6) but with $\tilde{H}_I(t)$ replaced by its projection onto the one electron/one positron subspace:

$$\tilde{H}_{e^-e^+}(t;p_1,p_2) = -e\,\frac{p_1\cdot A(t,\frac{\vec{p}_1}{\omega_1}t)}{\omega_1} + e\,\frac{p_2\cdot A(t,\frac{\vec{p}_2}{\omega_2}t)}{\omega_2} \tag{11}$$

$$\equiv -e\,G(\xi) + e\,G(\eta)$$

with $\xi^\mu = (t, \frac{\vec{p}_1}{\omega_1}t) \quad ; \quad \eta^\mu = (t, \frac{\vec{p}_2}{\omega_2}t)$

It turns out that the second term in the exponent of $U_{e^-e^+}(t;p_1,p_2)$ is a pure c-number

$$\frac{(-i)^2}{2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \left[\tilde{H}_{e^-e^+}(t_2;p_1,p_2), \tilde{H}_{e^-e^+}(t_1;p_1,p_2)\right]$$

$$= \frac{e^2}{2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \Big\{ [G(\xi_2),G(\eta_1)] + [G(\eta_2),G(\xi_1)] - [G(\xi_2),G(\xi_1)] - [G(\eta_2),G(\eta_1)] \Big\} \tag{12}$$

Defining the commutator function $d(\xi-\eta)$ in Feynman gauge as

$$d(\xi-\eta) \equiv [G(\xi),G(\eta)] \tag{13}$$

$$= \frac{p_1^\mu p_2^\nu}{\omega_1\omega_2}[A_\mu(\xi), A_\nu(\eta)] = i\,\frac{p_1\cdot p_2}{\omega_1\omega_2} D(\xi-\eta)$$

we obtain for the expression in eq. (12)

$$\frac{e^2}{2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \Big\{ d(\xi_2-\eta_1) + d(\eta_2-\xi_1) - d(\xi_2-\xi_1) - d(\eta_2-\eta_1) \Big\} \tag{14}$$

The vacuum matrix element of the first term in $U_{e^-e^+}$ is evaluated by decomposing the exponent in its positive and negative frequency parts and applying the Baker-Campbell-Hausdorff formula $\left(G^{(+)}|0\rangle = \langle 0|G^{(-)} = 0\right)$:

$$\langle 0 | \exp\Big\{ie\int_{t_0}^{t} dt_1 \big(G(\xi_1) - G(\eta_1)\big)\Big\} | 0\rangle$$

$$= \langle 0 | \exp\Big\{ie\int_{t_0}^{t}\big(G^{(-)}(\xi_1) - G^{(-)}(\eta_1) + G^{(+)}(\xi_1) - G^{(+)}(\eta_1)\big)dt_1\Big\}|0\rangle \tag{15}$$

$$= \exp\Big\{-\frac{1}{2}(ie)^2\int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1 \Big[\big(G^{(-)}(\xi_2) - G^{(-)}(\eta_2)\big), \big(G^{(+)}(\xi_1) - G^{(+)}(\eta_1)\big)\Big]\Big\}$$

Defining another commutator function $d^{(+)}(\xi-\eta)$ in Feynman gauge as

$$d^{(+)}(\xi-\eta) \equiv [G^{(-)}(\xi), G^{(+)}(\eta)] \tag{16}$$

$$= \frac{p_1^\mu p_2^\nu}{\omega_1\omega_2}[A_\mu^{(-)}(\xi), A_\nu^{(+)}(\eta)] = i\,\frac{p_1\cdot p_2}{\omega_1\omega_2} D^{(+)}(\xi-\eta)$$

we obtain for the form factor the following result

$$F(s) = e^{I_1 - I_2} \tag{17a}$$

where

$$I_1(p_1,p_2) = \frac{e^2}{2}\int_0^\infty dt_2 \int_0^\infty dt_1 \left\{\theta(t_2-t_1)\, d(\xi_2-\eta_1) - d^{(+)}(\xi_2-\eta_1) + (\xi \leftrightarrow \eta)\right\}$$

$$I_2(p_1,p_2) = \frac{e^2}{2}\int_0^\infty dt_2 \int_0^\infty dt_1 \left\{\theta(t_2-t_1)\, d(\xi_2-\xi_1) - d^{(+)}(\xi_2-\xi_1) + (\xi \leftrightarrow \eta)\right\} \tag{17b}$$

Here we have chosen the time $t_o = 0$ in order to get the correct in- and outgoing fermion currents. This is explained in detail in Appendix B. An explicit evaluation of the integral I_1 in eq. (17b) shows that the terms containing $d^{(+)}$ are purely real for $-\infty < S < \infty$ and those containing d are purely imaginary and vanish below threshold, $s = 4m^2$.
The form factor can be written in a more compact form using the following relation

$$\theta(t_2-t_1)\, d(\xi_2-\eta_1) - d^{(+)}(\xi_2-\eta_1) = i\, \frac{p_1\cdot p_2}{\omega_1\omega_2}\, D_F(\xi_2-\eta_1)$$

$$= -\frac{1}{(2\pi)^4}\, \frac{p_1\cdot p_2}{\omega_1\omega_2} \int d^4k\, \frac{e^{ik\cdot(\xi_2-\eta_1)}}{k^2-\lambda^2+i\varepsilon} \tag{18}$$

where D_F is the Feynman propagator of the photon in Feynman gauge. We obtain

$$I_1 = ie^2\, \frac{p_1\cdot p_2}{\omega_1\omega_2}\int_0^\infty dt_2\int_0^\infty dt_1\, D_F(\xi_2-\eta_1) = -\frac{ie^2}{(2\pi)^4}\int d^4k\, \frac{p_1\cdot p_2}{(k^2-\lambda^2+i\varepsilon)(k\cdot p_1+i\varepsilon)(k\cdot p_2-i\varepsilon)} \tag{19a}$$

$$I_2 = \frac{ie^2}{2}\int_0^\infty dt_2\int_0^\infty dt_1 \left\{\frac{m^2}{\omega_1^2}\, D_F(\xi_2-\xi_1) + \frac{m^2}{\omega_2^2}\, D_F(\eta_2-\eta_1)\right\} \tag{19b}$$

$$= -\frac{ie^2}{(2\pi)^4}\,\frac{1}{2}\int d^4k\, \frac{m^2}{k^2-\lambda^2+i\varepsilon}\left\{\frac{1}{(k\cdot p_1+i\varepsilon)(k\cdot p_1-i\varepsilon)} + \frac{1}{(k\cdot p_2+i\varepsilon)(k\cdot p_2-i\varepsilon)}\right\}$$

The result for I_1 coincides with that usually obtained by a rather involved investigation of complicated Feynman diagrams and their final summation in the leading logarithmic approximation [2-9] . From the eq. (19a/b) follows

$$I_1(p_1,-p_1) = I_2(p_1,-p_1) \tag{20}$$

which implies

$$F(0) = 1 \tag{21}$$

The function F(s) is therefore the renormalized Dirac form factor

$$F(s) = F_R(s) = Z_1\, F_{UR}(s) \tag{22}$$

For the vertex renormalization constant Z_1 we get from eqs. (17) and (22)

$$Z_1 = e^{-I_2} \tag{23}$$

After carrying out the k^0-integration in eq. (19a) we obtain for the unrenormalized form factor

$$F_{UR}(s) = e^{I_1} = e^{I_{1.1} + I_{1.2}} \tag{24}$$

where

$$I_{1.1} = -\frac{e^2}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\sqrt{\vec{k}^2+\lambda^2}}\, \frac{p_1\cdot p_2}{k\cdot p_1\; k\cdot p_2}$$

$$I_{1.2} = \frac{e^2}{(2\pi)^3}\int \frac{d^3\vec{k}}{\vec{\beta}\cdot\vec{k}-i\varepsilon}\, \frac{p_1\cdot p_2}{k\cdot p_1\; k\cdot p_2} \tag{25}$$

with $k^2 = \lambda^2$ and $\vec{\beta} = \frac{\vec{p}_1}{\omega_1} = -\frac{\vec{p}_2}{\omega_2}$ in the center of mass system of the electron and positron.

In expression (25) the photon is on shell; therefore energy conservation restricts the integration over photon momenta $\vec{k}$ to the range $0 \le |\vec{k}| \le \sqrt{s-4m^2}$.

The integral $I_{1.1}$ in eq. (25) is identical to the $d^{(+)}$-contribution in eq. (17) and is evaluated as follows

$$I_{1.1} = -\frac{\alpha}{\pi}\frac{1+\beta^2}{2\beta}\int_0^{\sqrt{s-4m^2}}\frac{dk\, k}{k^2+\lambda^2}\ln\frac{\sqrt{k^2+\lambda^2}+\beta k}{\sqrt{k^2+\lambda^2}-\beta k} \tag{26}$$

$$= -\frac{\alpha}{2\pi}\frac{1+\beta^2}{2\beta}\int_0^{\left(1+\frac{\lambda^2}{s-4m^2}\right)^{-1/2}} du \left\{\frac{\ln(1+\beta u)}{1-u} - \frac{\ln(1+\beta u)}{1+u} - \frac{\ln(1-\beta u)}{1-u} + \frac{\ln(1-\beta u)}{1+u}\right\}$$

$$k = |\vec{k}|\ ;\ \beta \equiv |\vec{\beta}| = \sqrt{1-4\frac{m^2}{s}}\ ;\ \alpha = \frac{e^2}{4\pi}\ .$$

Here we have introduced the photon velocity $u = \frac{k}{\sqrt{k^2+\lambda^2}}$ as integration variable. We finally obtain after the use of several functional relations of $\mathcal{L}_2(x)$

$$I_{1.1} = \frac{\alpha}{\pi}\frac{1+\beta^2}{2\beta}\left\{\ln\frac{1+\beta}{1-\beta}\ln\frac{\lambda}{2m} - \frac{1}{4}\ln^2\frac{1+\beta}{1-\beta} - \mathcal{L}_2\left(\frac{1-\beta}{1+\beta}\right) + \frac{\pi^2}{6}\right\} \tag{27}$$

with

$$\mathcal{L}_2(x) = -\int_0^x \frac{\ln(1-z)}{z}\,dz$$

In the limit $s \gg 4m^2$ we get

$$I_{1.1} = -\frac{\alpha}{4\pi}\left\{\ln^2\frac{s}{m^2} - 4\ln\frac{s}{m^2}\ln\frac{\lambda}{2m}\right\} \tag{28}$$

For the integral $I_{1.2}$ in eq. (25) we obtain in the CMS

$$I_{1.2} = \frac{\alpha}{\pi}\frac{1+\beta^2}{2}\int_0^{\sqrt{s-4m^2}} dk\, k^2 \int_{-1}^{1} d\cos\vartheta \left[P\frac{1}{\beta k\cos\vartheta} + i\pi\,\delta(\beta k\cos\vartheta)\right] \cdot \frac{1}{\sqrt{k^2+\lambda^2}-\beta k\cos\vartheta}\,\frac{1}{\sqrt{k^2+\lambda^2}+\beta k\cos\vartheta} \tag{29}$$

$$= i\alpha\frac{1+\beta^2}{2\beta}\int_0^{\sqrt{s-4m^2}} dk\,\frac{k}{k^2+\lambda^2}$$

$$= i\,\frac{\alpha}{2}\,\frac{s-2m^2}{\sqrt{s(s-4m^2)}}\ln\frac{s-4m^2}{\lambda^2}$$

$$\longrightarrow i\,\frac{\alpha}{2}\cdot\ln\frac{s}{\lambda^2} \quad \text{for} \quad s \gg 4m^2 .$$

Putting the results of eq. (28) and (29) into eq. (24) we get for the large momentum behavior of the unrenormalized on shell electron form factor the following result

$$F_{UR}(s) = \exp\left\{-\frac{\alpha}{4\pi}\left[\ln^2\frac{s}{m^2} - 4\ln\frac{s}{m^2}\ln\frac{\lambda}{2m}\right] + i\frac{\alpha}{2}\ln\frac{s}{\lambda^2}\right\} \; ; \quad s \gg 4m^2 \tag{30}$$

For the integral I_2 in eq. (19b) we get

$$I_2 = -\frac{e^2}{(2\pi)^3}\frac{m^2}{2}\int_0^{\sqrt{s-4m^2}}\frac{dk\,k^2}{2\sqrt{k^2+\lambda^2}}\int d\Omega\left[\frac{1}{(k\cdot p_1)^2} + \frac{1}{(k\cdot p_2)^2}\right]$$

$$= \frac{\alpha}{\pi}\left\{\ln\frac{\lambda}{2m} + \frac{1-\beta}{2\beta}\ln\frac{1+\beta}{1-\beta} - \ln\frac{2\beta}{1+\beta}\right\} \tag{31}$$

$$\longrightarrow \frac{\alpha}{\pi}\ln\frac{\lambda}{2m} \qquad \text{for} \quad s \gg 4m^2.$$

The eqs. (17), (19a/b), (27), (28), (29), (31) give for the renormalized Dirac form factor the final result

$$F(s) = \exp\left\{\frac{ie^2}{2(2\pi)^4}\int\frac{d^4k}{k^2-\lambda^2+i\varepsilon}\left(\frac{p_1}{k\cdot p_1} - \frac{p_2}{k\cdot p_2}\right)^2\right\} \tag{32a}$$

$$F(s) = \exp\left\{\frac{\alpha}{\pi}\left[B(s)\ln\frac{\lambda}{2m} + C(s)\right]\right\} \tag{32b}$$

$$\xrightarrow[s\gg 4m^2]{} \exp\left\{\frac{\alpha}{\pi}\left[\left(\ln\frac{s}{m^2} - 1\right)\ln\frac{\lambda}{2m} - \frac{1}{4}\ln^2\frac{s}{m^2}\right] + i\frac{\alpha}{2}\ln\frac{s}{\lambda^2}\right\} \tag{32c}$$

where

$$B(s) = \frac{1+\beta^2}{2\beta}\left[\ln\frac{1+\beta}{1-\beta} - i\pi\right] - 1 \tag{33a}$$

$$C(s) = -\frac{1+\beta^2}{2\beta}\left[\frac{1}{4}\ln^2\frac{1+\beta}{1-\beta} + \mathcal{L}_2\left(\frac{1-\beta}{1+\beta}\right) + \frac{1-\beta}{1+\beta^2}\ln\frac{1+\beta}{1-\beta} - \frac{2\beta}{1+\beta^2}\ln\frac{2\beta}{1+\beta} - \frac{\pi^2}{6} - i\frac{\pi}{2}\ln\frac{\beta^2}{1-\beta^2}\right] \tag{33b}$$

$$\beta = \sqrt{1 - 4\frac{m^2}{s}} \;, \qquad s > 4m^2 .$$

IV. Discussion and Conclusions

In section III we have calculated the renormalized on shell Dirac electron form factor in the infrared region. Our result for the unrenormalized form factor, eq. (24), agrees in the space-like region with the famous exponentiation obtained by previous authors [2-11]. The proofs given by these authors are rather complicated since they required a detailed study of perturbation theory order by order. The simplicity of the result suggested an easier derivation which has been given in the present work. Our simple proof was based on a modification of the Kulish-Faddeev approach. The complicated order by order investigation in perturbation theory has been replaced by the closed exponential form of the U-operator, eq. (6). Due to this exponential form of the U-operator we also obtained an exponentiated result for the vertex renormalization constant Z_1, eq. (23), and we therefore got automatically the multiplicative vertex renormalization. This led us to the exponentiation of the complete renormalized form factor

$$F(s) = \exp\left\{ \frac{ie^2}{2(2\pi)^4} \int \frac{d^4k}{k^2-\lambda^2+i\varepsilon} \left(\frac{p_1}{k\cdot p_1} - \frac{p_2}{k\cdot p_2} \right)^2 \right\} \tag{32a}$$

which is manifestly gauge invariant. The real part of the expression in the exponent is the well-known order e^2 result for radiative corrections to electron scattering [15].

The evaluation of the integral in eq. (32a) gave the result

$$F(s) = \exp\left\{ \frac{\alpha}{\pi} \left[B(s) \ln\frac{\lambda}{2m} + C(s) \right] \right\} \tag{32b}$$

where the infrared finite functions B(s) and C(s) have been given in eqs. (33a/b). The infrared divergent part in the exponent, $\frac{\alpha}{\pi} B(s) \ln\frac{\lambda}{2m}$,

has no imaginary part for $s < 0$ and has already appeared in Schwinger's calculation of the radiative corrections [15].

For large s in the time-like region the renormalized form factor behaves as

$$F(s) \longrightarrow \exp\left\{ \frac{\alpha}{\pi} \left[\left(\ln\frac{s}{m^2} - 1 \right) \ln\frac{\lambda}{2m} - \frac{1}{4} \ln^2\frac{s}{m^2} \right] + i\frac{\alpha}{2} \ln\frac{s}{\lambda^2} \right\} \tag{32c}$$

The double logarithmic terms in eq. (32c) have been found in the leading logarithmic approximation under the conditions $\alpha \ln^2\frac{s}{m^2} \lesssim 1$ and $\alpha \ln\frac{s}{m^2} \ll 1$. In our derivation of the form factor no such restrictions are needed. At first sight it would seem that the double logarithmic term $\ln^2\frac{s}{m^2}$ is not correlated to the infrared behavior. That this term is indeed entirely of infrared origin can be seen from the imaginary part in the exponent, eq. (32c), which necessarily leads via analyticity to the above double logarithm, see eq. (33). This intimate connection between the double logarithm and the infrared behavior could not directly be seen in previous investigations since these were carried out in the space-like region where the imaginary part vanishes. It is interesting to notice that the imaginary part is nothing else than the well-known divergent Coulomb phase the physical interpretation of which has always been somewhat unclear. A relation between the corresponding Coulomb phase in quark-antiquark scattering and the quark-antiquark potential has been established in ref. [13].

It is remarkable that our form factor result has an Omnès representation

$$F(s) = \exp\left\{\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\,\frac{\mathrm{Im}\,I_1(s')}{s'-s-i\varepsilon}\right\} \tag{33}$$

where the imaginary part $i\,\mathrm{Im}\,I_1 = I_{1,2}$ has been calculated in eq. (29).

From eq. (32b) it follows immediately that our renormalized form factor obeys the infrared differential equation

$$\lambda\,\frac{\partial F(s)}{\partial\lambda} = \frac{\alpha}{\pi}\,B(s)\,F(s) \tag{34}$$

which has been shown to hold in all orders of perturbation in ref. [9,16]. Moreover, these authors established a Callan-Symanzik type equation for the infrared finite form factor $\mathcal{F}(s)$:

$$\left(-\frac{\partial}{\partial t} + \beta(\alpha)\alpha\frac{\partial}{\partial\alpha} - \Gamma(t,\alpha)\right)\mathcal{F} = 0 \tag{35}$$

where

$$\mathcal{F}(s) \equiv e^{-\frac{\alpha}{\pi}B(s)\ln\frac{\lambda}{2m}}\,F(s) = e^{\frac{\alpha}{\pi}C(s)}$$

$$t = \frac{1}{2}\ln\frac{s}{m^2}$$

$$\beta(\alpha) \equiv \frac{m}{\alpha}\frac{d\alpha}{dm} = \frac{2}{3}\frac{\alpha}{\pi} + O(\alpha^2)$$

The quantity Γ plays the rôle of a t dependent "anomalous dimension". With our form factor result, eq. (32c), the eq. (35) is fulfilled and we obtain

$$\Gamma(t,\alpha) = 2\,\frac{\alpha}{\pi}\,t + O\left(\left(\frac{\alpha}{\pi}\right)^2\right) \tag{36}$$

The leading term in eq. (36) agrees with ref. [9].

Furthermore, we would like to point out that the terms in the exponent of the form factor, eqs. (17) and (19), have a simple diagrammatic interpretation. The integrals of eq. (19) can be obtained by the following infrared graph rules in momentum space. A fermion propagator is represented by 1, since the fermions are considered to be classical particles. The photon propagator is identical to the usual Feynman propagator $-i g_{\mu\nu}\,(k^2-\lambda^2+i\varepsilon)^{-1}$. The electron-(positron)-photon-vertex is given by $\mp ie\,p_\mu\,(2k\cdot p)^{-1}$ where k is the photon and p the fermion momentum. The integration over internal photon momenta has to be carried out with the measure $(2\pi)^{-4}d^4k$. There is a factor 2 for each infrared virtual photon line corresponding to the two directions that each line might be thought to flow. In addition another factor 2 is needed for each virtual photon line connecting two different fermions. This weight factor can be inferred from the apparent symmetry in eq. (17b). We want to emphasize that these rules are by no means Feynman rules in the sense of perturbation theory since they do not generate the S matrix but rather the logarithm of the S matrix. Due to the simple structure of the infrared Hamiltonian only diagrams of order e^2

have to be taken into account. For the case of the form factor considered in this work we are left with three diagrams only, depicted in the figure.

Our starting point of the modification of the Kulish-Faddeev approach (as explained in section II) was the following result derived in this approach for the unrenormalized electron form factor

$$F_{UR}(s) = \exp\left\{ \frac{\alpha}{\pi} \left[-\ln\frac{s}{m^2} \ln\frac{t}{t_0} + \frac{1}{2} \ln^2 \frac{s}{m^2} \right] + i\alpha \ln\frac{t}{t_0} \right\} \tag{37}$$

Here the time t plays the rôle of an infrared regulator; the time t_o is arbitrary. With the relation $t/t_o = \sqrt{s}/\lambda$ we realize that the imaginary part in the exponent of eq. (37) is identical to the one in eq. (30) obtained in our approach. This identification implies for the real part in the exponent of eq. (37)

$$\frac{\alpha}{\pi} \ln\frac{s}{m^2} \ln\frac{\lambda}{m}$$

which is identical to the corresponding infrared part in eq. (30). However, it does not contain the double logarithmic term $-\frac{\alpha}{4\pi} \ln^2 \frac{s}{m^2}$. We observe that the original Kulish-Faddeev approach produces correctly the infrared divergent terms but does not lead to the desired double logarithm.

In this paper we have presented a new treatment of the infrared problem in QED based on an infrared Hamiltonian with fictitious photon mass λ, eq. (3). This Hamiltonian leads to a closed expression for the S-operator which is a

well-defined unitary operator in Fock space due to the nonzero photon mass. This S-operator dresses the electron with a photon cloud in form of coherent states. This simple S-operator allows a rather elegant and consistent treatment of the infrared behavior. This has been illustrated by applying our method to the on shell electron form factor. We thereby immediately obtained the famous exponentiation of both, the renormalized and unrenormalized form factor. There was no need to carry out an explicit renormalization procedure. This method constitutes an important improvement compared to the conventional treatment which requires an order by order study in perturbation theory. The infinite number of Feynman diagrams are replaced by the three diagrams shown in the figure. The success of our method encourages us to attack the crucially important soft gluon problems in Quantum Chromodynamics, see ref. [12,13].

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Appendix A: Derivation of the infrared Hamiltonian

We are studying the large time behavior of the interaction Hamiltonian in the interaction picture

$$H_I(t) = \int d^3\vec{x}\; j_\mu(x) A^\mu(x) \tag{A1}$$

where

$$j_\mu(x) = -e : \bar{\Psi}(x)\gamma_\mu \Psi(x):$$

This study requires the use of explicit representations of the free field operators:

$$\Psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega}} \sum_{r=1,2} \left\{ b(\vec{p},r) u(\vec{p},r) e^{-ip\cdot x} + d^\dagger(\vec{p},r) v(\vec{p},r) e^{ip\cdot x} \right\} \tag{A2a}$$

$$\equiv \Psi^{(+)}(x) + \Psi^{(-)}(x)$$

where m is the electron mass, ω its energy and $\Psi^{(+)}$, $\Psi^{(-)}$ denote the positive and negative frequency parts of Ψ; $\bar{\Psi} = \Psi^\dagger \gamma^0$.

In Feynman gauge we have

$$A_\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2k^0} \sum_{\sigma=0}^{3} \left\{ a(k,\sigma)\varepsilon_\mu(k,\sigma) e^{-ik\cdot x} + a^\dagger(k,\sigma)\varepsilon_\mu^*(k,\sigma) e^{ik\cdot x} \right\} \tag{A2b}$$

where $k^0 = \sqrt{\vec{k}^2 + \lambda^2}$ and λ denotes the fictitious photon mass.

Rewriting the interaction Hamiltonian by means of the positive and negative frequency parts of the electron field

$$H_I(t) = -e \int d^3\vec{x} \left\{ \bar{\Psi}^{(+)}(x)\gamma_\mu \Psi^{(+)}(x) + \bar{\Psi}^{(+)}(x)\gamma_\mu \Psi^{(-)}(x) + \bar{\Psi}^{(-)}(x)\gamma \Psi^{(+)}(x) + \bar{\Psi}^{(-)}(x)\gamma_\mu \Psi^{(-)}(x) \right\} A^\mu(x) \tag{A3}$$

we obtain for the first term of eq. (A3)

$$\int d^3\vec{x}\; \bar{\Psi}^{(+)}(x)\gamma_\mu \Psi^{(+)}(x) A^\mu(x)$$

$$= \sum_{r,r',\sigma} \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{d^3\vec{p}\,'}{(2\pi)^{3/2}} \frac{d^3\vec{k}}{2k^0} \frac{m}{\sqrt{\omega'\omega}}\; b^\dagger(\vec{p}\,',r') b(\vec{p},r) \bar{u}(\vec{p}\,',r')\gamma_\mu u(\vec{p},r)\; \bullet \tag{A4}$$

$$\bullet \left\{ a(k,\sigma)\varepsilon^\mu(k,\sigma) \delta^{(3)}(\vec{p}\,'-\vec{p}-\vec{k}) e^{i(\omega'-\omega-k^0)t} + a^\dagger(k,\sigma)\varepsilon^{\mu *}(k,\sigma)\delta^{(3)}(\vec{p}\,'-\vec{p}+\vec{k}) e^{i(\omega'-\omega+k^0)t} \right\}$$

We can perform the $\vec{p}\,'$-integration. This yields us for the energy factors in the time exponentials for small photon momenta $\vec{k}$:

$$\omega'-\omega-k^0 = \sqrt{(\vec{p}+\vec{k})^2+m^2} - \sqrt{\vec{p}^2+m^2} - k^0 = -\frac{k\cdot p}{\omega} + O(\vec{k}^2)$$

$$\omega'-\omega+k^0 = \sqrt{(\vec{p}-\vec{k})^2+m^2} - \sqrt{\vec{p}^2+m^2} + k^0 = \frac{k\cdot p}{\omega} + O(\vec{k}^2) \tag{A5}$$

Since in the limit $t \longrightarrow \infty$ the dominant contribution to the integral (A4) comes from $\vec{k} = 0$ one gets for the spin non-flip part

$$\int d^3\vec{x}\; \bar{\Psi}^{(+)}(x)\gamma_\mu \Psi^{(+)}(x) A^\mu(x)$$

$$\xrightarrow[|t|\to\infty]{} \sum_{r,\sigma} \int d^3\vec{p}\; b^\dagger(\vec{p},r) b(\vec{p},r) \frac{p_\mu}{\omega} \int \frac{d^3\vec{k}}{(2\pi)^3 2k^0} \left\{ a(k,\sigma)\varepsilon^\mu(k,\sigma) e^{-i\frac{k\cdot p}{\omega}t} + a^\dagger(k,\sigma)\varepsilon^{\mu *}(k,\sigma) e^{i\frac{k\cdot p}{\omega}t} \right\} \tag{A6}$$

Carrying out the same steps for the second term of eq. (A3) we get

$$\int d^3x\, \bar{\psi}^{(+)}(x)\gamma_\mu \psi^{(-)}(x)\, A^\mu(x)$$

$$\xrightarrow[|t|\to\infty]{} \sum_{r,r',\sigma} \int d^3p\, b^\dagger(-\vec{p},r')\, d^\dagger(\vec{p},r) \frac{m}{\omega}\, \bar{u}(-\vec{p},r')\gamma_\mu v(\vec{p},r)\, \cdot \tag{A7}$$

$$\cdot \int \frac{d^3k}{(2\pi)^3 2k^0} \left\{ a(-\vec{k},\sigma)\varepsilon^\mu(-\vec{k},\sigma) e^{i(2\omega-\frac{k\cdot p}{\omega})t} + a^\dagger(-\vec{k},\sigma)\varepsilon^{\mu *}(-\vec{k},\sigma) e^{i(2\omega+\frac{k\cdot p}{\omega})t} \right\}$$

In the limit $|t| \to \infty$ the time exponentials in eqs. (A6) and (A7) become $\mp \frac{k\cdot p}{\omega} t \longrightarrow \mp \lambda t$ and $(2\omega \mp \frac{k\cdot p}{\omega})t \longrightarrow (2\omega \mp \lambda)t$. It is evident that for $\lambda \to 0$ eq. (A6) gives the dominant contribution since ω can never vanish for massive fermions. If the same procedure is performed for the third and fourth term of eq. (A3) it is found that only the fourth term is leading. We therefore obtain for the spin non-flip part

$$H_I(t) \xrightarrow[|t|\to\infty]{} \tilde{H}_I(t) = \sum_\sigma \int d^3p\, \rho(\vec{p}) \frac{p_\mu}{\omega} \int \frac{d^3k}{(2\pi)^3 2k^0} \left\{ a(k,\sigma)\varepsilon^\mu(k,\sigma) e^{-i\frac{k\cdot p}{\omega}t} + a^\dagger(k,\sigma)\varepsilon^{\mu *}(k,\sigma) e^{i\frac{k\cdot p}{\omega}t} \right\} \tag{A8}$$

$$= \int d^3x\, \tilde{j}_\mu(x)\, A^\mu(x)$$

where

$$\tilde{j}_\mu(x) = \int d^3p\, \rho(\vec{p}) \frac{p_\mu}{\omega} \delta^{(3)}(\vec{x} - \frac{\vec{p}}{\omega}t)$$
$$\rho(\vec{p}) = -e\sum_r \left[b^\dagger(\vec{p},r)\, b(\vec{p},r) - d^\dagger(\vec{p},r)\, d(\vec{p},r) \right] \tag{A9}$$

Appendix B: The asymptotic fermion currents in momentum space

The eq. (A9) yields for the asymptotic one-electron current the classical electron current

$$\tilde{j}^{e^-}_\mu(x) = \frac{\langle e^-(\vec{p}) \,|\, \tilde{j}_\mu(x) \,|\, e^-(\vec{p})\rangle}{\langle e^-(\vec{p}) \,|\, e^-(\vec{p})\rangle} = -e \frac{p_\mu}{\omega} \delta^{(3)}(\vec{x} - \frac{\vec{p}}{\omega}t)$$
$$= -e \frac{p_\mu}{\omega} \theta(t-t_0)\, \delta^{(3)}(\vec{x} - \frac{\vec{p}}{\omega}t) - e\frac{p_\mu}{\omega}\theta(-(t-t_0))\, \delta^{(3)}(\vec{x} - \frac{\vec{p}}{\omega}t) \tag{B1}$$
$$\equiv \tilde{j}^{e^-}_\mu(x;\mathrm{OUT}) + \tilde{j}^{e^-}_\mu(x;\mathrm{IN})$$

which we have decomposed into out- and ingoing currents with respect to the time t_o. In momentum space we obtain only for the choice $t_o = 0$ the correct classical expression

$$\int d^4x\, e^{ik\cdot x}\, \tilde{j}^{e^-}_\mu(x;{}^{\mathrm{OUT}}_{\mathrm{IN}}) = \frac{\mp i e p_\mu}{k\cdot p \pm i\varepsilon} \tag{B2}$$

This shows that $t_o = 0$ is the correct initial time for solving the time evolution equation, eq. (5).

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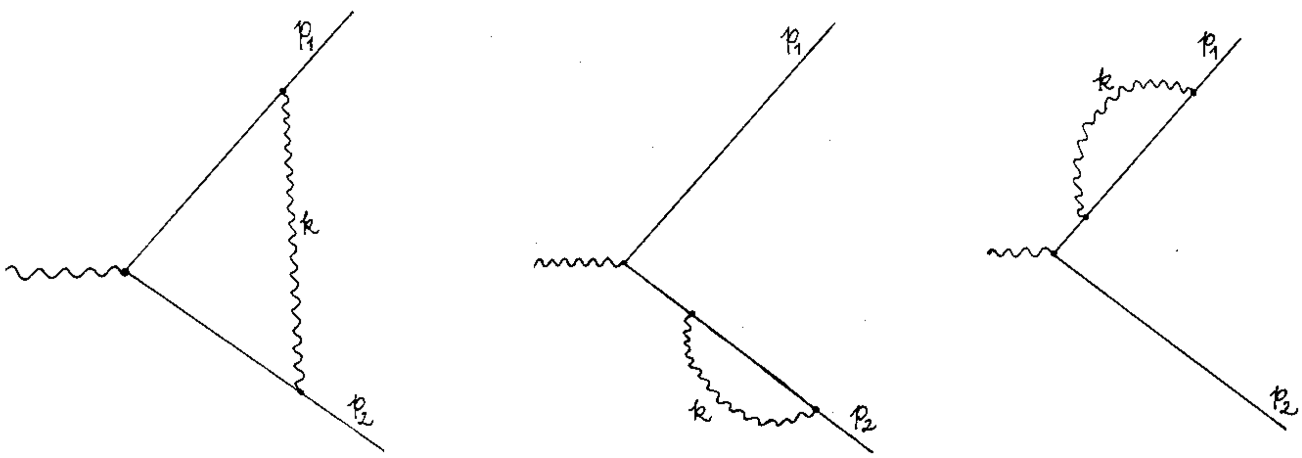

Figure

Diagrammatic interpretation of the renormalized electron form factor, $F(s) = e^{I_1 - I_2}$, eq. (17a). The first diagram represents the integral I_1, eq. (19a), and determines the unrenormalized form factor $F_{UR}(s) = e^{I_1}$. The second and third diagram represent the integral I_2, eq. (19b), and determine the vertex renormalization constant $Z_1 = e^{-I_2}$.