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THE CARDY-CARTAN MODULAR INVARIANT

Jürgen Fuchs [a], **Christoph Schweigert** [b], **Carl Stigner** [a]

[a] *Teoretisk fysik, Karlstads Universitet*
Universitetsgatan 21, S – 651 88 Karlstad

[b] *Organisationseinheit Mathematik, Universität Hamburg*
Bereich Algebra und Zahlentheorie
Bundesstraße 55, D – 20 146 Hamburg

Abstract

Using factorizable Hopf algebras, we construct modular invariant partition functions of charge conjugation, or Cardy, type as characters of coends in categories that share essential features with the ones appearing in logarithmic CFT. The coefficients of such a partition function are given by the Cartan matrix of the theory.

1 Introduction

Partition functions are among the most basic quantities of a quantum field theory. In large classes of two-dimensional rational conformal field theories, the (torus) partition function is quite explicitly accessible: via the principle of holomorphic factorization, it can be written as a bilinear combination

$$Z = \sum_{i,j} Z_{i,j}\, \chi_i^{\mathcal{V}} \otimes_{\mathbb{C}} \chi_j^{\mathcal{V}} \tag{1.1}$$

of the (finitely many) irreducible characters of the chiral symmetry algebra $\mathcal{V}$ of the CFT, with non-negative integer coefficients $Z_{i,j}$.

By the uniqueness of the vacuum one has $Z_{0,0} = 1$. A further necessary condition on $Z = Z(\tau)$ is invariance under the action of the mapping class group of the torus on the characters. This condition, briefly referred to as *modular invariance*, is rather restrictive. It has been the starting point of several classification programs. An important contribution by Max Kreuzer concerns a subclass of modular invariant partition functions: those, in which the vacuum representation with character $\chi_0^{\mathcal{V}}$ is only paired with irreducible $\mathcal{V}$-representations of a special type, so-called simple currents [SY], which are invertible objects in the representation category of $\mathcal{V}$. All modular invariants that come in infinite families turn out to be of this type; to classify such invariants is thus an evident problem.

It is characteristic for Max Kreuzer's approach to problems in mathematical physics that he did *not* address *this* problem. This is indeed most reasonable, because the mentioned constraints imposed on the coefficients $Z_{i,j}$ are just necessary conditions, but are far from sufficient. In fact, there are lots of examples of combinations (1.1) (with $Z_{0,0} = 1$) that are modular invariant, but are unphysical in the sense that they cannot be the torus partition function of any consistent CFT. One reason for the insufficiency of the usual constraints is that the space of bulk states, which are counted by Z, is not only a module over the tensor product $\mathcal{V} \otimes_{\mathbb{C}} \mathcal{V}$ of two copies of the chiral algebra, but has also more subtle properties, in particular it forms an algebra under the operator product, with an invariant bilinear form derived from the two-point functions on the sphere. To arrive at combinations (1.1) that can be expected to be compatible with such requirements as well, Max and Bert Schellekens took an approach inspired by orbifold techniques. The outcome [KS] is a general formula for Z that is both beautiful and of tremendous use in applications. Later on [FRS2] it could be proven rigorously that all modular invariants covered by their formula constitute physical partition functions, forming part of a consistent collection of correlators with any number of field insertions at all genera.

Today concise mathematical formulations of the various conditions on partition functions in rational CFT are available. The proper formalization of the Moore-Seiberg data associated to a chiral algebra $\mathcal{V}$ is the structure of a modular tensor category on the representation category $\mathcal{C}$ of $\mathcal{V}$. The bulk state space F is then required to be a commutative Frobenius algebra in the Deligne product $\overline{\mathcal{C}} \boxtimes \mathcal{C}$, which formalizes the physical idea of pairing left- and right-moving chiral degrees of freedom. If the category $\mathcal{C}$ is semisimple it is easy to give an example of such an algebra:

$$F = F_{\mathcal{C}} := \bigoplus_{i \in \mathcal{I}} S_i^{\vee} \boxtimes S_i\,, \tag{1.2}$$

with $(S_i)_{i \in \mathcal{I}}$ representatives for the finitely many isomorphism classes of simple objects of $\mathcal{C}$, i.e. of irreducible representations of the chiral algebra. The corresponding partition function $Z = Z_C$, called the charge conjugation modular invariant, has coefficients $Z_{i,j} = \delta_{i,\bar{j}}$. When compatible conformally invariant boundary conditions of the CFT are considered, this solution is also referred to as the 'Cardy case'.

What is relevant for capturing the Moore-Seiberg data is the category $\mathcal{C}$ as an abstract category (with much additional structure), not its concrete realization as the representation category of a chiral algebra $\mathcal{V}$. It is an old idea to rephrase these data by regarding the abstract category $\mathcal{C}$ as the category H-Mod of modules over a 'quantum group', i.e. over a Hopf algebra H with an R-matrix. In this case the Deligne product $\overline{\mathcal{C}}\boxtimes\mathcal{C}$ can be realized by the category H-Bimod of H-bimodules (alternatively, owing to modularity, as the equivalent category of Yetter-Drinfeld modules over H). The algebraic structure decribing the bulk algebra F is then the dual of the Hopf algebra H (with H seen as a bimodule over itself). In the present note the categories $\mathcal{C}$ and $\overline{\mathcal{C}}\boxtimes\mathcal{C}$ are treated in this spirit, i.e. are realized as categories of modules and bimodules over a quasitriangular Hopf algebra.

The novelty in our discussion is that we do *not* require the category $\mathcal{C}$ to be semisimple and are thereby transcending the Moore-Seiberg framework. Semisimplicity, meaning that any representation can be fully decomposed into a finite direct sum of irreducible representations, arises in quantum physics as a consequence of unitarity. Still our motivation is entirely physical: the categories we are working with are closely related to categories arising in logarithmic conformal field theories, with applications ranging from condensed matter physics to string theory (for a guide to the literature, see [FGST, Sect. 2]). We summarize our main findings:

- Even in the absence of semisimplicity there is a bulk algebra $F = F_C$ that generalizes (1.2). In particular it is modular invariant in the appropriate manner.

- The partition function for this bulk algebra F can still be expressed as a bilinear combination of the characters of simple objects of $\mathcal{C}$. Moreover, the matrix $Z_C = (Z_{i,j})$ turns out to be a natural quantity associated with the category $\mathcal{C}$: it is the *Cartan matrix* of $\mathcal{C}$, which describes [1] how projective objects decompose into simple objects.

  Put very briefly:
  The symbol C not only stands for <u>C</u>harge conjugation and <u>C</u>ardy, but also for <u>C</u>artan.

It is worth stressing that characters and partition functions, which count states, do not distinguish between direct sums and non-trivial extensions of representations. Thus when the underlying category is non-semisimple they carry much less physical information than in the semisimple case. It is thus crucial that we do not just obtain the partition function Z, but even the bulk algebra F that has Z as its character.

2 Summary of concepts and results

In this section we formulate, in Theorems 2 and 3 below, our main results and collect the relevant background information that is needed to appreciate them. The proofs, as well as a more detailed description of various pertinent concepts, will be given in Section 3.

[1] Concerning the notion of Cartan matrix of an associative algebra, or of its abelian category of modules, see e.g. [Be, Ch. 1.7] for a textbook reference, as well as [Lo].

2.1 Factorizable Hopf algebras

As already mentioned, we assume that we can realize the category $\mathcal{C}$, which for rational CFT is the modular tensor category that formalizes the Moore-Seiberg data, as the category H-Mod of left modules over a Hopf algebra H. More precisely, H comes endowed with additional structure, as stated in the following convention; for brevity we refer to such algebras as *factorizable Hopf algebras*:

Convention 1. *Throughout this note, a* factorizable Hopf algebra *is a finite-dimensional factorizable ribbon Hopf algebra over an algebraically closed field* $\Bbbk$ *of characteristic zero.*

In the CFT setting, $\Bbbk$ is the field of complex numbers. Let us summarize the meaning of the qualifications imposed on the $\Bbbk$-vector space H: That H is a *Hopf algebra* means that it is endowed with a product m, unit η, coproduct Δ, counit ε and antipode S, such that (H, m, η) is a unital associative algebra and (H, Δ, ε) is a counital coassociative coalgebra, with the coproduct being an algebra morphism from H to $H \otimes H$, and with the antipode satisfying $m \circ (id_H \otimes \mathrm{S}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{S} \otimes id_H) \circ \Delta$. A *quasitriangular* Hopf algebra is a Hopf algebra H endowed with an invertible element $R \in H \otimes_\Bbbk H$, called the *R-matrix*, that intertwines the coproduct Δ and the opposite coproduct $\Delta^{\mathrm{op}} = \tau_{H,H} \circ \Delta$ and satisfies [2]

$$(\Delta \otimes id_H) \circ R = R_{13} \cdot R_{23} \qquad \text{and} \qquad (id_H \otimes \Delta) \circ R = R_{13} \cdot R_{12}\,. \tag{2.1}$$

A *ribbon* Hopf algebra is a quasitriangular Hopf algebra H endowed with a central invertible element $v \in H$, called the *ribbon element*, that obeys

$$\mathrm{S} \circ v = v\,, \qquad \varepsilon \circ v = 1 \qquad \text{and} \qquad \Delta \circ v = (v \otimes v) \cdot Q^{-1}\,, \tag{2.2}$$

where $Q \in H \otimes_\Bbbk H$ is the *monodromy matrix* $Q = R_{21} \cdot R \equiv (\tau_{H,H} \circ R) \cdot R$. (In the CFT context, the R-matrix contains information about the braiding, while the eigenvalues of the action of the ribbon element on a module give the exponentiated conformal weights.)
A *factorizable* Hopf algebra is a quasitriangular Hopf algebra H whose monodromy matrix can be written as $Q = \sum_\ell h_\ell \otimes k_\ell$ with $\{h_\ell\}$ and $\{k_\ell\}$ two vector space bases of H.

The Hopf algebras that are presently thought to be of relevance for classes of logarithmic conformal field theories do not fully fit into our framework, but are very close. They do not have an R-matrix, but still a factorizable monodromy matrix (see e.g. [FGST, NT]) or live in a more general category than the one of finite-dimensional $\Bbbk$-vector spaces [ST].

2.2 Modules and bimodules over factorizable Hopf algebras

We denote by H-Mod the category of left H-modules and by H-Bimod the one of H-bimodules. Both of them are finite tensor categories in the sense of [EO], and they have a ribbon structure, i.e. there are families of duality, braiding and twist morphisms satisfying the usual axioms. In particular, the tensor product functor is exact in both arguments, and the set $\mathcal{I}$ of isomorphism classes of simple objects is finite. If H is semisimple, then H-Mod and H-Bimod are semisimple modular tensor categories, like the representation categories of chiral algebras in rational CFT. It is worth pointing out that the tensor product of H-Bimod is *not* the one over H, for which

[2] We identify H with the space $\mathrm{Hom}_\Bbbk(\Bbbk, H)$ of linear maps from $\Bbbk$ to H.

the vector space underlying a tensor product bimodule $X \otimes_H Y$ is a non-trivial quotient of the vector space tensor product $X \otimes_{\Bbbk} Y$ (and for which only the structure of H as an associative algebra is needed), but rather uses explicitly that H is a *bi*algebra: it is obtained by pulling back the natural $H \otimes H$-bimodule structure on $X \otimes Y$ along the coproduct to the structure of an H-bimodule (for more details see [FSS, Sect. 2.2]).

The *Deligne tensor product* of two locally finite $\Bbbk$-linear abelian categories $\mathcal{C}$ and $\mathcal{D}$ is a category $\mathcal{C} \boxtimes \mathcal{D}$ together with a bifunctor $\boxtimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ that is right exact and $\Bbbk$-linear in both variables and has a universal property by which, in short, bifunctors from $\mathcal{C} \times \mathcal{D}$ become functors from $\mathcal{C} \boxtimes \mathcal{D}$. If $\mathcal{C} \simeq A$-Mod and $\mathcal{D} \simeq B$-Mod are categories of left modules over associative algebras A and B, respectively, then their Deligne product $\mathcal{C} \boxtimes \mathcal{D}$ is equivalent to $(A \otimes B^{\mathrm{op}})$-Mod as a $\Bbbk$-linear abelian category. In our case, where $A = B = H$ is a factorizable Hopf algebra, upon an appropriate choice of braiding on the bimodule category this in fact extends to an equivalence

$$H\text{-Bimod} \simeq \overline{H\text{-Mod}} \boxtimes H\text{-Mod} \tag{2.3}$$

of ribbon categories, where $\overline{H\text{-Mod}}$ is H-Mod with opposite braiding and twist [FSS, App. A.3].

2.3 The bulk Frobenius algebra

It has been shown in [FSS] that for $\mathcal{C} = H$-Mod a natural candidate for the bulk state space is the *coregular bimodule* F, i.e. the dual space $\mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ of H endowed with the duals of the regular left and right actions of H. For semisimple H, this H-bimodule decomposes into simple bimodules as in (1.2).

As an object of $\overline{\mathcal{C}} \boxtimes \mathcal{C} = H$-Bimod, F has properties characteristic for the bulk state space of a CFT:

Theorem 2. (i) *The maps*

$$\begin{aligned} &m_F := \Delta^* , \qquad \eta_F := \varepsilon^* , \qquad \varepsilon_F := \Lambda^* \qquad \text{and} \\ &\Delta_F := [(id_H \otimes (\lambda \circ m)) \circ (id_H \otimes \mathrm{s} \otimes id_H) \circ (\Delta \otimes id_H)]^* \end{aligned} \tag{2.4}$$

(*with* Λ *and* λ *the integral and cointegral of* H*, respectively*) *endow the coregular bimodule* F *with the structure of a Frobenius algebra* $(F, m_F, \eta_F, \Delta_F, \varepsilon_F)$ *in the category* H-Bimod.
(ii) F *is commutative, cocommutative and symmetric and has trivial twist.*

For the proof, see Propositions 2.10 and 3.1, Theorem 4.4 and Remark 4.9 in [FSS]. We call F the *bulk Frobenius algebra*. If H is semisimple, F has the structure of a Lagrangian algebra in the sense of [DMNO, Def. 4.6].

Owing to factorizability, besides the equivalence (2.3) there is also an equivalence of ribbon categories between H-Bimod and the Drinfeld center of H-Mod, and thus between H-Bimod and the category of Yetter-Drinfeld modules over H. Hence instead of with H-bimodules we could equivalently work with Yetter-Drinfeld modules over H. In that setting, the algebra F arises as the so-called [FFRS, Da] *full center* of the tensor unit of H-Mod.

2.4 The partition function

Via the principle of holomorphic factorization, correlation functions in full CFT are, at least for rational CFTs, elements in spaces of conformal blocks of the associated chiral CFT. They must be invariant under actions of mapping class groups and obey sewing constraints. It is an obvious question whether solutions satisfying these conditions still exist when the theory is no longer rational so that the category $\mathcal{C}$ is non-semisimple. In fact, non-trivial solutions to locality and crossing symmetry constraints on the sphere with a modular invariant spectrum have been found in [GK] (compare also [GRW]).

In our setting, in which conformal blocks are specific morphism spaces $\mathrm{Hom}_H(-,-)$ in the category H-Mod, while correlation functions are elements of morphism spaces $\mathrm{Hom}_{H|H}(-,-)$ in H-Bimod, we are able to answer this question in the affirmative for any factorizable Hopf algebra H, for the particular case of the torus partition function, i.e. the zero-point correlator on the torus. The corresponding space of zero-point conformal blocks on the torus turns out to be $\mathrm{Hom}_H(L,\mathbf{1})$, where $\mathbf{1}$ is the tensor unit of H-Mod (given by the field $\Bbbk$ endowed with the trivial left H-action ε) and $L \in H$-Mod is a certain Hopf algebra internal to H-Mod; L, which is called the chiral handle Hopf algebra, will be described in detail in Section 3.3. Similarly, the torus partition function itself is the character

$$Z = \chi_F^K \ \in \mathrm{Hom}_{H|H}(K,\mathbf{1}) \tag{2.5}$$

of the bulk Frobenius algebra F with respect to a Hopf algebra K internal to H-Bimod. Here $\mathbf{1}$ is now the tensor unit of H-Bimod (again the field $\Bbbk$, now endowed with trivial left and right H-actions); the *bulk handle Hopf algebra* K will be described in detail in Section 3.5. It has been shown in [FSS] that the morphism χ_F^K is modular invariant, with respect to the natural action of the modular group that comes from the action [Ly2] of the modular group on the space $\mathrm{Hom}_H(L,\mathbf{1})$ of conformal blocks.

As we will see, K can be canonically identified with $L \otimes L$. Holomorphic factorization thus amounts to identifying χ_F^K with a bilinear expression of basis elements of $\mathrm{Hom}_H(L,\mathbf{1})$. We can show that this is indeed the case and, moreover, recognize the resulting coefficients as natural quantities for the category H-Mod:

Theorem 3. *The partition function* (2.5) *can be chirally decomposed as*

$$Z = \sum_{i,j\in\mathcal{I}} c_{\bar{\imath},j}\, \chi_i^L \otimes \chi_j^L\,, \tag{2.6}$$

*where $\{\chi_i^L \,|\, i \in \mathcal{I}\}$ are characters of L-modules, $\bar{\imath} \in \mathcal{I}$ is the label dual to i, and $C = \big(c_{i,j}\big)_{i,j\in\mathcal{I}}$ is the Cartan matrix of H-*Mod*.*

Remark 4. (i) The entries $c_{i,j}$ of the Cartan matrix are non-negative integers. In general, $c_{0,0}$ is larger than 1, but this is *not* in contradiction with the uniqueness of the vacuum.
(ii) Unless H is semisimple, the space $\mathrm{Hom}_H(L,\mathbf{1})$ is *not* spanned by characters alone. A complement is provided by so-called pseudo-characters (compare e.g. [Mi, GT, AN]). It is thus a non-trivial statement that a decomposition of the form (2.6) exists, irrespective of the precise values of the coefficients.
(iii) The result fits with predictions for the bulk state space of certain logarithmic CFTs, the $(1,p)$ triplet models and WZW models with supergroup target spaces [QS, GR].

The decomposition (2.6) is the main new result of this note. It would be difficult to establish this relation directly, as it is hard to describe the characters for the chiral and bulk handle

Hopf algebras sufficiently explicitly. Instead, our idea of proof is to relate these characters to characters for modules and bimodules over the underlying ordinary Hopf algebra H and then invoke classical results for the latter. In some more detail, we will proceed as follows.

1. Using general results about finite-dimensional associative algebras and their representations we deduce the formula (3.12) for the character of a self-injective algebra A as a bimodule over itself.
2. Using the fact that the bimodule structures of the regular and coregular bimodules H and F are intertwined by the Frobenius map, the character formula (3.11) is translated to the analogous formula (3.16) for the H-bimodule F.
3. We observe (Lemma 5) that the chiral handle Hopf algebra L acts via partial monodromy on any object of H-Mod. Based on this result we can show that L-characters are obtained from H-characters by composing them with the Drinfeld map, see formula (3.26).
4. We obtain a similar expression (3.31) of K-characters in terms of characters for the Hopf algebra $H \otimes H^{\mathrm{op}}$.
5. We show (Proposition 9) that K-characters can be written as bilinear combinations of L-characters. When applied to the character of F as a K-module, together with the previous results this yields the decomposition (2.6).

3 Details

3.1 Associative algebras and characters

For $A = (A, m, \eta)$ a finite-dimensional (unital, associative) algebra over the field $\Bbbk$, the *character* χ_M^A of a left A-module $M = (M, \rho)$ is, by definition, the partial trace of the representation morphism ρ. Here the trace is taken in the sense of linear maps, i.e. in the category of finite-dimensional $\Bbbk$-vector spaces. Thus

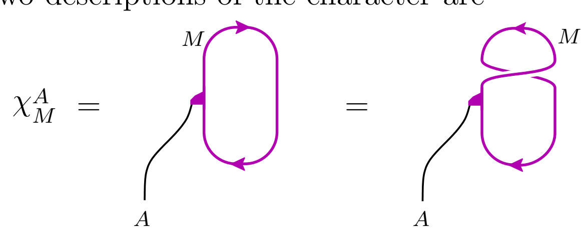
$$\chi_M^A := \mathrm{tr}_M(\rho) = \tilde{d}_M \circ (\rho \otimes id_{M^\vee}) \circ (id_A \otimes b_M) \ \in \mathrm{Hom}(A, \Bbbk)\,, \tag{3.1}$$

where $b_M \in \mathrm{Hom}_\Bbbk(\Bbbk, M \otimes_\Bbbk M^*)$ is the (right) coevaluation and $\tilde{d}_M \in \mathrm{Hom}_\Bbbk(M \otimes_\Bbbk M^*, \Bbbk)$ the (left) evaluation. Now for finite-dimensional $\Bbbk$-vector spaces the left and right dualities coincide, in the sense that $\tilde{d}_M$ can be expressed through the right evaluation $d_M \in \mathrm{Hom}_\Bbbk(M^* \otimes_\Bbbk M, \Bbbk)$ as

$$\tilde{d}_M = d_M \circ \tau_{M,M^*} \tag{3.2}$$

with τ the flip ma, and analogously for the two coevaluations.

In the sequel we will make use of the graphical calculus for strict [3] ribbon categories. In this pictorial notation, the two descriptions of the character are



(3.3)

[3] That the tensor product of the categories of our interest is strictly associative can – just like in many other situations in which associativity does not, a priori, hold on the nose – be assumed by invoking the Coherence Theorem.

Characters are class functions, i.e. satisfy $\chi^A_M \circ m = \chi^A_M \circ m^{\rm op}$; A is semisimple iff the space of class functions is already exhausted by linear combinations of characters of A-modules.

Characters are additive under short exact sequences, i.e. for any short exact sequence $0 \to U \to W \to V \to 0$ of A-modules one has [Lo, Sect. 1.5] $\chi^A_W = \chi^A_U + \chi^A_V$. It follows that

$$\chi^A_V = \sum_{i\in\mathcal{I}} [\,V : S_i\,]\, \chi^A_i\,, \tag{3.4}$$

where $\{S_i \,|\, i \in \mathcal{I}\}$ is a full set of representatives of the isomorphism classes of simple A-modules, $\chi^A_i \equiv \chi^A_{S_i}$ is the character of S_i and $[\,V : S_i\,]$ is the multiplicity of S_i in the Jordan-Hölder series of V. The simple A-modules S_i are given by $P_i/J(A)\,P_i$, where $J(A)$ is the Jacobson radical of A and P_i is the projective cover of S_i. The projective modules P_i, in turn, form a full set of representatives of the isomorphism classes of indecomposable projective A-modules and satisfy $P_i = A\,e_i$ with $\{e_i \in A \,|\, i \in \mathcal{I}\}$ a collection of primitive orthogonal idempotents. With the same idempotents e_i, $Q_i := e_i\,A$ are representatives for the isomorphism classes of indecomposable projective right A-modules.

The character of the projective module P_i decomposes as

$$\chi^A_{P_i} = \sum_{j\in\mathcal{I}} c_{i,j}\, \chi^A_j \tag{3.5}$$

with coefficients

$$c_{i,j} := [\,P_i : S_j\,] = \dim_{\Bbbk}(\mathrm{Hom}_A(P_i, P_j)) \tag{3.6}$$

for $i, j \in \mathcal{I}$. The matrix $C = \big(\, c_{i,j}\,\big)$, which only depends on the category A-Mod as an abelian category, is called the *Cartan matrix* of A, or of the category A-Mod.

As a left module over itself, A is projective and decomposes into indecomposable projective modules P_i according to [JS, Satz G.10]

$${}_AA \;\cong\; \bigoplus_{i\in\mathcal{I}} P_i \otimes_{\Bbbk} \Bbbk^{\dim(S_i)}. \tag{3.7}$$

An analogous decomposition is valid for A as a right module over itself. The structure of A as a bimodule over itself (with regular left and right actions) is, in general, much more complicated. Now the structure of an A-bimodule is equivalent to the one of a left $A\otimes A^{\rm op}$-module; accordingly, we define the character of A as a bimodule over itself as the character of A as an $A\otimes A^{\rm op}$-module.

For any two finite-dimensional $\Bbbk$-algebras A and B, the Jacobson radical of the tensor product algebra $A\otimes B$ (i.e. the vector space $A\otimes_{\Bbbk} B$, endowed with unit map $\eta_A \otimes \eta_B$ and product $(m_A \otimes m_B) \circ (id_A \otimes \tau_{A,B} \otimes id_B))$ satisfies $J(A\otimes B) = J(A) \otimes_{\Bbbk} B + A \otimes_{\Bbbk} J(B)$. Using that $\Bbbk$ is a field of characteristic zero, it follows that a complete set of primitive orthogonal idempotents of $A\otimes B$ is given by $\{e^A_i \otimes_{\Bbbk} e^B_j \,|\, i \in \mathcal{I}_A\,,\, j \in \mathcal{I}_B\}$, and complete sets of indecomposable projective and of simple $A\otimes B$-modules are given by $\{P^A_i \otimes_{\Bbbk} P^B_j \,|\, i \in \mathcal{I}_A\,,\, j \in \mathcal{I}_B\}$ and by $\{S^A_i \otimes_{\Bbbk} S^B_j \,|\, i \in \mathcal{I}_A\,,\, j \in \mathcal{I}_B\}$, respectively (see e.g. [CR, Thm. (10.38)]). Comparing with (3.4), it follows that the character of any $A\otimes B$- module X can be written as a bilinear combination

$$\chi^{A\otimes B}_X = \sum_{i\in\mathcal{I}_A, j\in\mathcal{I}_B} n_{i,j}\, \chi^A_i \otimes \chi^B_j\,, \tag{3.8}$$

where χ_k^A are the characters of simple left A-modules S_k, as above, and χ_l^B those of the simple left B-modules, and $n_{k,l}$ are non-negative integers.

Specializing to $B = A^{\mathrm{op}}$ and $X = A$ (with regular actions), we can use that

$$\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}\big((A\otimes A^{\mathrm{op}})\,(e_i\otimes e_j), A\big) \cong e_i\,A\,e_j \cong \mathrm{Hom}_A(Ae_i, Ae_j) = \mathrm{Hom}_A(P_i, P_j)\,, \tag{3.9}$$

which implies that the entries (3.6) of the Cartan matrix obey

$$c_{i,j} = \dim_{\Bbbk}\big(\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}((A{\otimes}A^{\mathrm{op}})(e_i{\otimes}e_j), A)\big) = [\,A : S_i \otimes_{\Bbbk} T_j\,] \tag{3.10}$$

with T_k the simple quotients of the projective right A-modules $e_k\,A$. Using also that $\chi_M^{A^{\mathrm{op}}} = \chi_{M^*}^A$, formula (3.4) yields

$$\begin{aligned} \chi_A^{A\otimes A^{\mathrm{op}}} &= \sum_{i,j\in\mathcal{I}} [\,A : S_i \otimes_{\Bbbk} T_j\,]\,\chi_{S_i\otimes_{\Bbbk} T_j}^{A\otimes A^{\mathrm{op}}} \\ &= \sum_{i,j\in\mathcal{I}} c_{i,j}\,\chi_{S_i\otimes_{\Bbbk} T_j}^{A\otimes A^{\mathrm{op}}} = \sum_{i,j\in\mathcal{I}} c_{i,j}\,\chi_i^A \otimes \chi_{T_j}^{A^{\mathrm{op}}} = \sum_{i,j\in\mathcal{I}} c_{i,j}\,\chi_i^A \otimes \chi_{S_j^*}^A \end{aligned} \tag{3.11}$$

as a linear map in $\mathrm{Hom}_{\Bbbk}(A\otimes A^{\mathrm{op}}, \Bbbk)$.

Moreover, if A is self-injective, then one has $T_k \cong S_k^*$ as right A-modules, so that (3.11) can be rewritten as

$$\chi_A^{A\otimes A^{\mathrm{op}}} = \sum_{i,j\in\mathcal{I}} c_{i,j}\,\chi_i^A \otimes \chi_j^A\,. \tag{3.12}$$

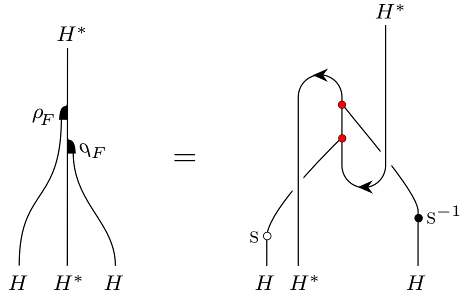
3.2 Factorizable Hopf algebras

Consider now the special case that $A = H$ is a *factorizable Hopf* algebra in the sense of Convention 1, with coproduct Δ, counit ε and antipode S. Then H is in particular self-injective. Thus by (3.12) the character of the *regular H-bimodule*, i.e. the vector space H together with the regular left and right actions, is given by

$$\chi_H^{H\otimes H^{\mathrm{op}}} = \sum_{i,j\in\mathcal{I}} c_{i,j}\,\chi_i^H \otimes \chi_j^H\,. \tag{3.13}$$

Since the Hopf algebra H is finite-dimensional, its antipode map S is invertible, and there are one-dimensional spaces of left integrals $\Lambda \in H$ and of right cointegrals $\lambda \in H^*$ [LS]. The composition $\lambda\circ\Lambda \in \Bbbk$ is invertible (unless λ or Λ is zero), and we can and will choose the integral and cointegral such that $\lambda\circ\Lambda = 1$. A factorizable Hopf algebra is unimodular [Ra3, Prop. 3(c)], meaning that the left integral Λ is also a right integral, and this implies that $\mathrm{S}\circ\Lambda = \Lambda$.

Next consider the *coregular H-bimodule F*, i.e. the vector space $H^* = \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ dual to H endowed with the dual of the regular left and right actions; graphically,



(3.14)

The bimodule F is isomorphic as a bimodule to the the regular bimodule. Indeed, for any linear map $\mu \in H^*$ the map

$$\Phi_\mu := ((\mu \circ m) \otimes id_{H^*}) \circ (\mathrm{S} \otimes b_H) \tag{3.15}$$

intertwines the regular and coregular left H-actions; if μ satisfies $\mu \circ m = \mu \circ m \circ \tau_{H,H} \circ (id_L \otimes \mathrm{S}^2)$ then Φ_μ intertwines the regular and coregular right H-actions as well and is thus of morphism of H-bimodules. In particular we can take $\mu = \lambda$ to be the cointegral, in which case $\Psi = \Phi_\lambda$ is the *Frobenius map* (see e.g. [CW2]); since the Frobenius map of a factorizable Hopf algebra is invertible (see e.g. [FS, App. A.2]), it follows that indeed H and F are isomorphic as H-bimodules.

It follows that the decomposition (3.13) applies to the coregular bimodule F as well, i.e.

$$\chi_F^{H\otimes H^{\mathrm{op}}} = \sum_{i,j\in\mathcal{I}} c_{i,j}\, \chi_i^H \otimes \chi_j^H \,. \tag{3.16}$$

3.3 The chiral handle Hopf algebra

To proceed, we introduce a certain Hopf algebra internal to the category H-Mod, the *chiral handle Hopf algebra L*. To this end we need a few notions from category theory. It is convenient to formulate them first for $\mathbb{k}$-linear abelian ribbon categories $\mathcal{C}$, and only later on specialize to the case that $\mathcal{C}$ is the category H-Mod of left H-modules, which for any factorizable Hopf algebra H belongs to this class of categories.

A *dinatural transformation* $F \Rightarrow B$ from a functor $F\colon\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ to an object $B \in \mathcal{D}$ is a family of morphisms $\varphi = \{\varphi_U\colon\ F(U,U) \to B\}_{U\in\mathcal{C}}$ such that the diagram

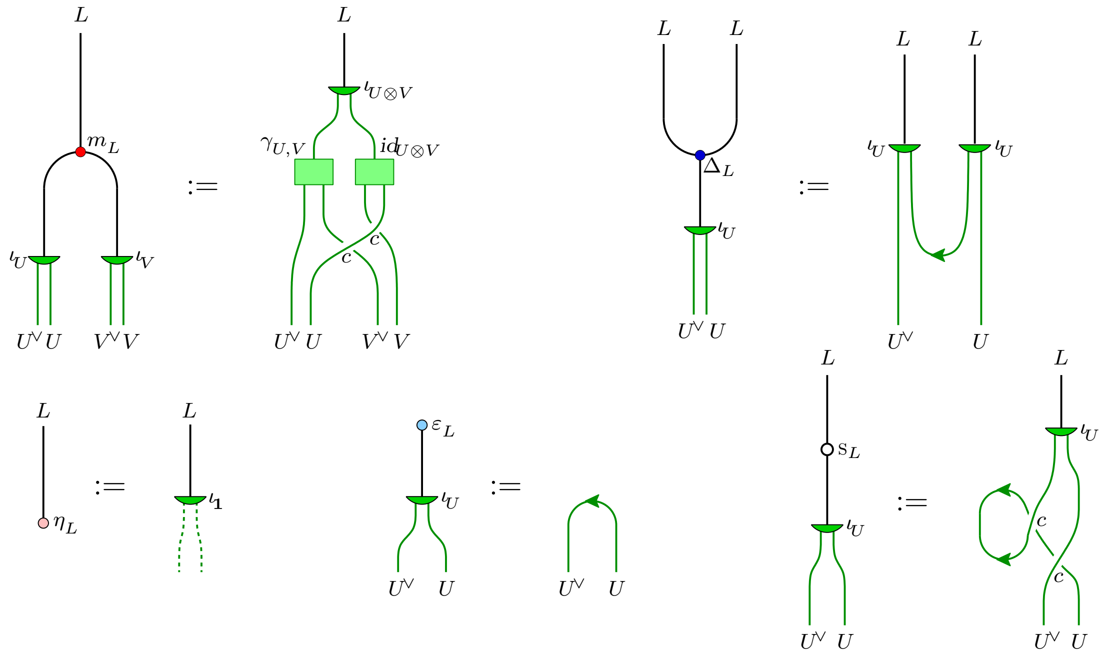
$$\begin{tikzcd}[column sep=small] & F(V,U) \arrow[dl, "F(f{,}id_U)"'] \arrow[dr, "F(id_V{,}f)"] & \\ F(U,U) \arrow[dr, "\varphi_U"] & & F(V,V) \arrow[dl, "\varphi_V"] \\ & B & \end{tikzcd} \tag{3.17}$$

commutes for all $f \in \mathrm{Hom}(U,V)$. A *coend* (C,ι) for a functor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object $C \in \mathcal{D}$ together with a dinatural transformation ι that has the universal property that for any dinatural transformation $\varphi\colon F \Rightarrow B$ there is a unique morphism $\kappa \in \mathrm{Hom}_{\mathcal{D}}(C,B)$ such that $\varphi_U = \kappa \circ \iota_U$ for all objects U of $\mathcal{C}$. Coends are unique up to unique isomorphism. The finiteness properties of the categories we are working with guarantee the existence of all coends we need.

Several different coends turn out to be of interest to us. The one relevant for us now is the coend $L := \int^U U^\vee \otimes U$ of the functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathcal{C}$ that acts on objects as $(U,V) \mapsto U^\vee \otimes V$. As shown in [Ma, Ly1], L has a natural structure of a Hopf algebra $(L, m_L, \eta_L, \Delta_L, \varepsilon_L, \mathrm{S}_L)$ internal to $\mathcal{C}$; its structural morphisms are given by

$$\begin{aligned} & m_L \circ (i_U \otimes i_V) := i_{V\otimes U} \circ (\gamma_{U,V} \otimes id_{V\otimes U}) \circ (id_{U^\vee} \otimes c_{U,V^\vee\otimes V})\,, \qquad \eta_L := i_{\mathbf{1}}\,, \\ & \Delta_L \circ i_U := (i_U \otimes i_U) \circ (id_{U^\vee} \otimes b_U \otimes id_U)\,, \qquad \varepsilon_L \circ i_U := d_U\,, \\ & \mathrm{S}_L \circ i_U := (d_U \otimes i_{U^\vee}) \circ (id_{U^\vee} \otimes c_{U^{\vee\vee},U} \otimes id_{U^\vee}) \circ (b_{U^\vee} \otimes c_{U^\vee,U})\,, \end{aligned} \tag{3.18}$$

where $\gamma_{U,V}$ is the canonical identification of $U^\vee \otimes V^\vee$ with $(V \otimes U)^\vee$. (Here it is used that a morphism f with domain the coend L is uniquely determined by the dinatural family $\{f \circ i_U\}$ of morphisms.) In graphical notation, [4]



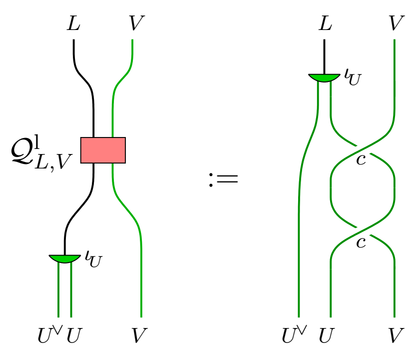
(3.19)

L also has a two-sided integral and a Hopf pairing. For semisimple modular $\mathcal{C}$, the coend L is given by $L = \bigoplus_{i\in\mathcal{I}} S_i^\vee \otimes S_i \in \mathcal{C}$.

Morphism spaces of the form $\mathrm{Hom}(L^g, V_1 \otimes \cdots \otimes V_n)$ carry [Ly2] natural representations of the mapping class group $\Gamma_{g,n}$ of Riemann surfaces of genus g with n marked points. We therefore call L the *chiral handle Hopf algebra*.

3.4 Modules over the chiral handle Hopf algebra

For any object V of $\mathcal{C}$, the family of morphisms from $U^\vee \otimes U \otimes V$ to $L \otimes V$ on the right hand side of



(3.20)

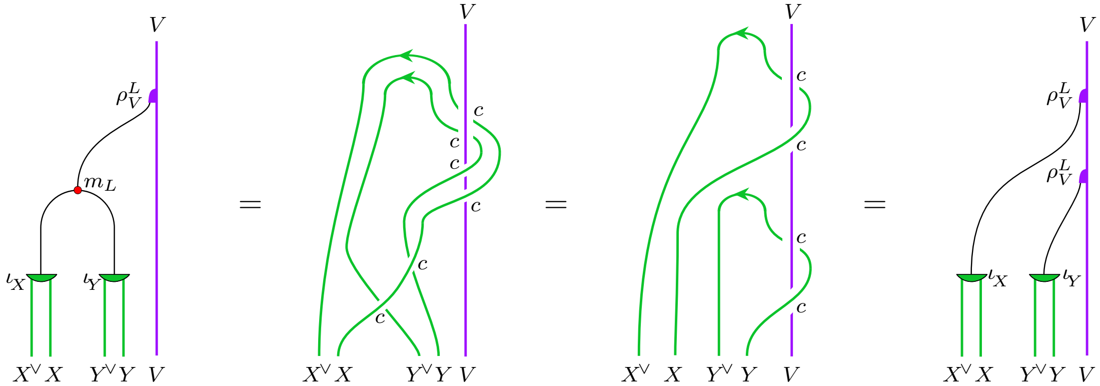
[4] The picture (3.19), as well as (3.20), (3.22) and (3.23) below, describe morphisms in the monoidal category $\mathcal{C}$, which (generically) is genuinely braided, i.e. over- and underbraiding are different morphisms. In the pictures this is indicated by labeling the braiding explicitly with the symbol c. In contrast, the other pictures we display refer to the category of finite-dimensional $\Bbbk$-vector spaces, for which the braiding is just the flip map τ, so that over- and underbraiding coincide. Note that τ is just a linear map, rather than a morphism of H-Mod or H-Bimod; nevertheless the maps described in those pictures *are* morphisms in the relevant categories.

is dinatural in the first two arguments and thus defines a morphism $\mathcal{Q}^{\mathrm{l}}_{L,V}$ in $\mathrm{End}_{\mathcal{C}}(L\otimes V)$, which we call the *partial monodromy* of V with respect to L. Composition with the counit of L supplies a morphism

$$\rho^L_V := (\varepsilon_L \otimes id_V) \circ \mathcal{Q}^{\mathrm{l}}_{L,V} \;\in\; \mathrm{Hom}_{\mathcal{C}}(L\otimes V, V)\,. \tag{3.21}$$

Lemma 5. *The morphism* (3.21) *endows the object V of $\mathcal{C}$ with the structure of an L-module internal to $\mathcal{C}$.*

Proof. Unitality follows directly from the definition of the unit of L. Compatibility with the product of L reduces to an application of the defining properties of the braiding:

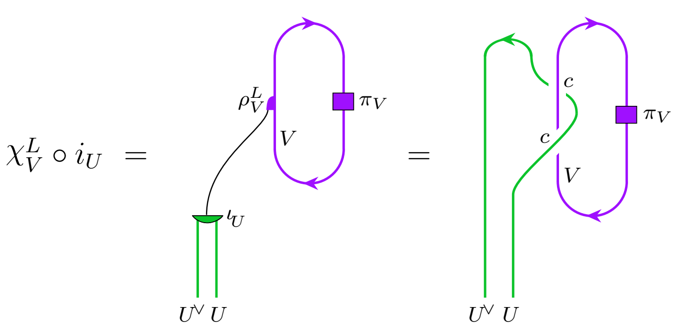


(3.22)

Here the first equality combines the definitions of ρ^L_V in (3.21) with those of m_L and ε_L in (3.18). ☐

The left action (3.21) of L on an object V of $\mathcal{C}$ should not be confused with the right *co*action of L on V that is obtained [Ly1] by combining the dinatural morphism i_U with the coevaluation for U. The latter only uses the duality of $\mathcal{C}$, whereas the former uses in addition the braiding, or rather, the monodromy of $\mathcal{C}$.

For algebras in monoidal categories one can set up their representation theory in a way very similar as for conventional $\Bbbk$-algebras. If the category is sovereign, one can in particular consider the character χ^L_V of the L-module (V,ρ^L_V); it is given by



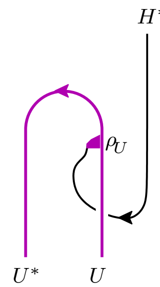
(3.23)

A new feature appearing here as compared to vector spaces is that the left and right dual of an object of $\mathcal{C}$ need not be equal, but only naturally isomorphic. This necessitates the insertion of an appropriate isomorphism $\pi_V \in \mathrm{Hom}_{\mathcal{C}}(V^\vee, {}^\vee V)$, forming part of a sovereign structure on $\mathcal{C}$.

Since L is a Hopf algebra, there is a natural notion of dual module. The character of the L-module $V^\vee$ dual to V turns out to be given by essentially the same morphism as in (3.23), except that the braidings are replaced by inverse braidings.

Now we specialize to the case $\mathcal{C} = H$-Mod for a factorizable Hopf algebra H. In this case one can describe the coend L explicitly [Ly2, Ke, Vi]: as an object of $\mathcal{C}$ it is the vector space H^* endowed with the coadjoint left H-action, and the morphisms of the dinatural family i are given by

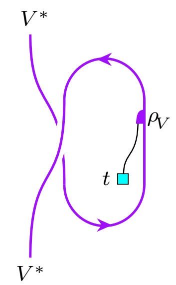
$$i_U \;=$$



(3.24)

Further, the monodromy appearing in (3.21) is now given by the action of the monodromy matrix Q of H on the tensor product H-module $U \otimes V$, and the sovereignty isomorphism is given by

$$\pi_V \;=$$



$$\in \mathrm{End}_{\Bbbk}(V^*) \qquad (3.25)$$

with $t = u\, v^{-1}$ the product of the Drinfeld element $u := m \circ (\mathrm{S} \otimes id_L) \circ R_{21} \in H$ and the inverse of the ribbon element of H.

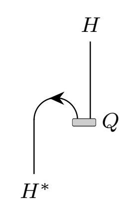
As a consequence we have the following description of L-characters:

Lemma 6. *The character of the L-module (M, ρ^L_M) obeys*

$$\chi^L_M = \chi^H_M \circ m \circ (t \otimes f_Q) = \chi^H_M \circ m \circ (f_Q \otimes t) \qquad (3.26)$$

with

$$f_Q \;:=\; (b_H \otimes id_H) \circ (id_{H^*} \otimes Q) \;=$$

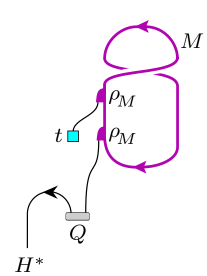


(3.27)

(*and with t regarded as an element of* $\mathrm{Hom}_{\Bbbk}(\Bbbk, H)$).

Proof. Inserting (3.24) and (3.25) into (3.23) and using that the monodromy in H-Mod is furnished by the action of the monodromy matrix Q, one obtains

$$\chi^L_M \;=$$



(3.28)

Using the representation property and comparing with (3.3) then yields the first of the equalities (3.26). The expression given by the second equality can be obtained in a similar way; that both expressions are valid is a consequence of the sphericity of the category of $\Bbbk$-vector spaces. $\square$

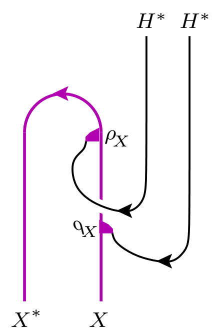
Remark 7. The mapping f_Q (3.27) is called the *Drinfeld map*. A priori f_Q is just a linear map in $\mathrm{Hom}_\Bbbk(H^*,H)$, but actually [CW1, Prop. 2.5(5)] it is a module morphism from H^* with the left coadjoint H-action to H with the left adjoint action on itself.

3.5 Modules over the bulk handle Hopf algebra

In CFT terms, what we have dealt with so far is a chiral half of the theory. We now proceed from the chiral to the full theory. In the present setting this means that we no longer work with the ribbon category H-Mod of left H-modules, but now with the ribbon category $H\text{-Bimod} \simeq \overline{H\text{-Mod}} \boxtimes H\text{-Mod}$ of H-bimodules. There are then *two* coends of interest to us. The first is the *bulk handle Hopf algebra* K. This is just the bimodule version of the coend L, i.e.

$$K = \int^{X\in H\text{-Bimod}} X^\vee \otimes X \,, \tag{3.29}$$

where the bifunctor $\otimes\colon\, H\text{-Bimod} \times H\text{-Bimod} \to H\text{-Bimod}$ is now the tensor product in H-Bimod. Explicitly [FSS, App. A.4], K is the *coadjoint bimodule*, i.e. the vector space $H^* \otimes_\Bbbk H^*$ endowed with the coadjoint left H-action on the first tensor factor and with the coadjoint right H-action on the second factor, and the dinaturality morphisms are given by [FSS, (A.30)]

$i_X^K \;=$ 

(3.30)

where ρ_X and $\text{\reflectbox{$\rho$}}_X$ are the left and right actions of H on the H-bimodule X (compare the chiral version (3.24)).

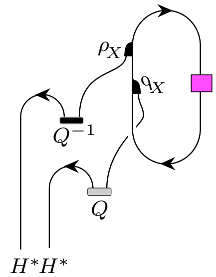
For the characters of modules over the bulk handle Hopf algebra the following analogue of Lemma 6 holds.

Lemma 8. *The K-character of a K-module (X,ρ_X^K) can be expressed through characters for the ordinary Hopf algebra $H\otimes H^{\mathrm{op}}$ as*

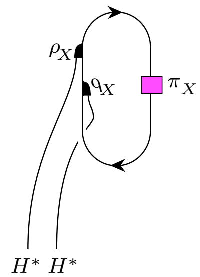
$$\chi_X^K = \chi_X^{H\otimes H^{\mathrm{op}}} \circ (m\otimes m) \otimes (t \otimes f_{Q^{-1}} \otimes f_Q \otimes t)\,. \tag{3.31}$$

Proof. Inserting (3.30) into the formula (3.23) for the character, as adapted to the present situation (i.e. in particular with L replaced by K and with the monodromy the one of H-Bimod), it follows that

$$\chi_X^K =$$



$$=$$



$$\circ\ (f_{Q^{-1}} \otimes f_Q)\,, \tag{3.32}$$

where f_Q is the Drinfeld map (3.27) and $f_{Q^{-1}}$ is the analogous morphism with Q replaced by Q^{-1}.
The sovereignty morphism π_X is given by the bimodule analogue of (3.25), i.e. with the left action by the element $t \in H$ complemented with a right action by t [FSS, (4.8)]. Each of the two occurrences of t can be manipulated in the same way as the single t in Lemma 6. Regarding the H-bimodule X as a left $H \otimes H^{\mathrm{op}}$-module, this yields (3.31). □

The result (3.31) is in fact not so surprising, because there is a ribbon equivalence between the categories of $H \otimes H$-modules and H-bimodules (the equivalence functor is given in formula (A.22) of [FSS]), and this equivalence maps the $H \otimes H$-module $L \otimes_{\Bbbk} L$ to the bimodule K. Furthermore, invoking also Lemma 6 and formula (3.8), we arrive at a *chiral decomposition* of K-characters:

Proposition 9. *The K-character of a K-module X can be expressed through characters for the chiral handle Hopf algebra in the form*

$$\chi_X^K = \sum_{i,j \in \mathcal{I}} n_{i,j}\, \chi_{\bar{i}}^L \otimes \chi_j^L\,, \tag{3.33}$$

where χ_i^L is the L-character that via (3.26) *corresponds to the irreducible H-character χ_i^H, $\chi_{\bar{i}}^L$ is the L-character of the corresponding dual L-module, and $n_{i,j}$ $(i, j \in \mathcal{I})$ are the non-negative integers that appear in formula* (3.8)*.*

Proof. Manipulating the sovereignty isomorphism in (3.32) in the same way as in the proof of Lemma 6 and invoking (3.8) for $A = H$ and $B = H^{\mathrm{op}}$ as well as the equivalence between H-bimodules and $H \otimes H^{\mathrm{op}}$-modules, we arrive at

$$\chi_X^K = \sum_{i,j \in \mathcal{I}} n_{i,j} \left[\chi_i^H \circ m \circ (t \otimes f_{Q^{-1}})\right] \otimes \left[\chi_j^H \circ m \circ (f_Q \otimes t)\right]. \tag{3.34}$$

By (3.26), the second tensor factor equals χ_j^L. For the first factor, the presence of $f_{Q^{-1}}$ instead of f_Q amounts to replacing the braiding by the opposite braiding in the representation morphism, and thus (compare the corresponding remark after (3.23)) (3.26) gives again an L-character, but now for the dual module. Together this yields (3.33). □

3.6 The character of the bulk Frobenius algebra

The second coend we need is the one for the functor from H-Mod$^{\mathrm{op}} \times H$-Mod to H-Bimod that on objects acts as $(U,V) \mapsto U^\vee \boxtimes V$; we denote it by

$$F = \int^{U\in H\text{-Mod}} U^\vee \boxtimes U \,. \tag{3.35}$$

As already suggested by the chosen notation, F is nothing but the coregular bimodule, i.e. the *bulk Frobenius algebra* featuring in Theorem 2, with dinatural family coinciding, as linear maps, with the one of the chiral handle Hopf algebra in (3.24) (for details see [FSS, App. A.2]).

We have now collected all ingredients for establishing Theorem 3.

Proof of Theorem 3.
According to (3.16), for $X = F$ the coefficients $n_{i,j}$ in (3.8), and thus those in (3.34), are given by the entries $c_{i,j}$ of the Cartan matrix of H-Mod. Using that $\bar{\bar{i}} = i$, we thus arrive at (2.6). □

4 Outlook

The quest for a classification of modular invariant partition functions has been an important activity in mathematical physics in the late 1980s and early 1990s. Nowadays it may be considered as superseded by approaches based on category-theoretic tools. In retrospect it is surprising how far one could get in this quest by imposing only a few convenient necessary conditions. The result by Max Kreuzer and Bert Schellekens [KS] is still the best available and, most probably, the best possible *systematic* result of this activity.

Th Kreuzer-Schellekens classification also played a central role in developments that led to the modern more mathematical approach to rational conformal field theory [FRS1, FjFRS]. This approach had in particular to reproduce their beautiful result, and indeed [FRS2] it does.

Today, one important activity is concerned with logarithmic conformal field theories, which amounts to dropping the condition of semisimplicity. For such theories, the only systematic information about torus partition functions seems to be the one about the Cardy-Cartan invariant discussed here (together with some automorphism-twisted versions [FSS, Sect. 6]. It is therefore encouraging that simple current symmetries, which are a crucial input for the Kreuzer-Schellekens result, appear to occur in logarithmic CFT [FHST, Rem. 5.3.2] as well. Moreover, module categories over non-semisimple tensor categories which only have invertible simple objects have been classified under certain assumptions [GM]. It is quite reasonable to expect that these results can be combined with the structural insight in the Kreuzer-Schellekens formula to give a handle on partition functions of logarithmic CFTs that are not of the Cardy-Cartan form. But this extension of Max' ideas still awaits its realization.

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