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FERMI PSEUDOPOTENTIAL IN HIGHER DIMENSIONS

by

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Abstract

The Fermi pseudopotential is generalized from three to five dimensions, and the case of an infinite, uniform, equidistant, linear chain of such pseudopotentials is studied in detail. Similar to the three-dimensional case, zero-width resonances are also present in five dimensions. While this generalization is natural and can be carried through formally when the strength is negative, there are basic changes in the underlying structure. These results in five dimensions also apply in four dimensions.

1. Introduction

The Fermi pseudopotential was first used nearly half a century ago [1]. The first applications were to problems of nuclear physics [2,3], and the later ones to many-body problems [4-7]. In three dimensions, it is

$$4\pi a\, \delta^3(\vec{r}) \frac{\partial}{\partial r} r \quad , \tag{1.1}$$

and is called a pseudopotential because it is an operator instead of a function. In the simplest case, a is a parameter with dimension length; more generally, a can be a function of the energy [4]. Here we are not going to deal with such energy dependence.

Recently, potentials of the form

$$\alpha\, \delta^3(\vec{r}) \tag{1.2}$$

were studied [8-10], where α is infinitesimal. It is now known that $-\nabla^2 + (1.1)$ and $-\nabla^2 + (1.2)$ are the same with α and a suitable functions of each other.

There is a general belief that the Fermi pseudopotential exists only in two and three dimensions, but not in higher dimensions [10]. The reason for this belief, in a nutshell, is as follows. Consider for $\kappa > 0$ the free-space Green's function in d dimensions

$$(-\nabla^2 + \kappa^2)\, G_o(r) = \delta^d(\vec{r}) \quad . \tag{1.3}$$

It is square integrable in two and three dimensions, but not for $d \geq 4$.

Intuitively, it seems that this condition of being square integrable is rather far removed from physics. One of the original ways of motivating the Fermi pseudopotential is to take a repulsive potential such as a hard sphere, and to continue analytically the s-wave part of the wave function outside the potential. Thus the wave function for the Fermi pseudopotential (1.1) is physically meaningful only for r away from the origin. With this motivation, square integrability in the vicinity of the origin has no physical interpretation.

It is the purpose of the present paper to initiate the study of Fermi pseudopotentials in higher dimensions by considering the case d = 5. This value of d has been chosen judiciously. On the one hand, because of the form of the free-space Green's function, the Fermi pseudopotential is simpler in form when d is odd than when d is even, as already evident by comparing the cases d = 2 and d = 3. On the other hand, the case d = 7 is much more complicated than the case d = 5, for reasons that are not yet understood.

In three dimensions, since the free-space Green's function has the singularity r^{-1}, the operator $\frac{\partial}{\partial r} r$ in (1.1) is chosen to annihilate this singularity. In an analogous way, since the free-space Green's function in five dimensions has the singularity r^{-3}, the Fermi pseudopotential is chosen to be

$$4\pi^2 a^3\, \delta^5(\vec{r}) \frac{\partial^3}{\partial r^3} r^3 \quad , \tag{1.4}$$

where the coefficient $4\pi^2$ is only for convenience. Here a has also the dimension of length. An alternative, entirely equivalent choice is

$$4\pi^2 a^3 \delta^5(\vec{r}) \frac{\partial}{\partial r} \frac{2}{r} \frac{\partial}{\partial r} r^3 \quad . \tag{1.5}$$

In Sec. 2, the Green's function in the presence of this Fermi pseudopotential (1.4) or (1.5) is obtained explicitly in a completely straightforward way. However, the result is interesting in that it makes sense only for

$$a \leq 0. \tag{1.6}$$

As in the three-dimensional case, $a = \infty$ is allowed. In Sec. 3, this Green's function is applied to the case of an infinite equispaced linear chain of Fermi pseudopotentials. It is found that the phenomenon of infinitely narrow resonances, previously known in three dimensions, also occurs in five dimensions. Sec. 4 is devoted to the simplest property of the Green's function of Sec. 2 as the resolvent operator, and the more mathematical questions of this Fermi pseudopotential in five dimensions are raised in Sec. 5, but by no means solved.

The failure of the present procedure in seven dimensions stems from the fact that there the condition (1.6) is replaced by $a \leq 0$ and $a \geq 0$, or more precisely $a = 0$ or ∞. Assuming that the mathematical problems discussed in Sec. 5 can be solved, then the present situation with Fermi pseudopotentials in higher dimensions is as follows. A Fermi pseudopotential can be defined in

all dimensions corresponding to $a = \infty$, meaning that the Green's function is defined and has the necessary analytic properties. In four and five dimensions, a one-parameter family of Fermi pseudopotentials can be defined, analogous to the known cases of two and three dimensions. The situation is as yet unclear in six dimensions.

2. Green's Function

A. Three Dimensions

Before obtaining the Green's function in five dimensions in the presence of the Fermi pseudopotential (1.4), we review the corresponding problem in three dimensions with (1.1). The Green's function is defined through the partial differential equation

$$\left[\nabla^2 + k^2 - 4\pi a\, \delta^3(\vec{r}) \frac{\partial}{\partial r} r \right] G(\vec{r}, \vec{r}_o; k) = - \delta^3(\vec{r} - \vec{r}_o) \tag{2.1}$$

together with the usual boundary conditions at infinity. Let

$$A = 4\pi a \frac{\partial}{\partial r} r\, G(\vec{r}, \vec{r}_o; k)\Big|_{r=0} \quad , \tag{2.2}$$

then

$$(\nabla^2 + k^2)\, G(\vec{r}, \vec{r}_o; k) = - \delta^3(\vec{r} - \vec{r}_o) + A\, \delta^3(\vec{r}) \quad , \tag{2.3}$$

and hence

$$G(\vec{r}, \vec{r}_o; k) = G_o(\vec{r}, \vec{r}_o; k) - A\, G_o(\vec{r}, 0; k) \quad , \tag{2.4}$$

where $G_o(\vec{r},\vec{r}_o;k)$ is the free-space Green's function given explicitly by

$$G_o(\vec{r}, \vec{r}_o; k) = \frac{e^{i k |\vec{r} - \vec{r}_o|}}{4\pi |\vec{r} - \vec{r}_o|} \quad . \tag{2.5}$$

The coefficient A is determined by substituting (2.4) into (2.2) and the result is

$$G(\vec{r}, \vec{r}_o; k) = G_o(\vec{r}, \vec{r}_o; k) - \frac{4\pi a}{1 + ika} G_o(\vec{r}, 0; k)\, G_o(\vec{r}_o, 0; k) \quad . \tag{2.6}$$

This same Green's function is also obtained from the "point interaction" (1.2). This shows that the Fermi pseudopotential and the point interaction are one and the same in three dimensions.

B. Five Dimensions

The derivation in five dimensions is step-by-step the same. We begin with the partial differential equation

$$\left[\nabla^2 + k^2 - 4\pi^2 a^3 \delta^5(\vec{r}) \frac{\partial^3}{\partial r^3} r^3\right] G(\vec{r}, \vec{r}_o; k) = -\delta^5(\vec{r} - \vec{r}_o) \tag{2.7}$$

in five dimensions. Let

$$A = 4\pi^2 a^3 \frac{\partial^3}{\partial r^3} r^3 G(\vec{r}, \vec{r}_o; k)\Big|_{r=0} \quad , \tag{2.8}$$

then

$$(\nabla^2 + k^2)\, G(\vec{r}, \vec{r}_o; k) = -\delta^5(\vec{r} - \vec{r}_o) + A\,\delta^5(\vec{r}) \quad , \tag{2.9}$$

and hence

$$G(\vec{r}, \vec{r}_o; k) = G_o(\vec{r}, \vec{r}_o; k) - A\, G_o(\vec{r}, 0; k) \quad , \tag{2.10}$$

where $G_o(r,r_o;k)$ is the free-space Green's function in five dimensions

$$G_o(\vec{r}, \vec{r}_o; k) = (8\pi^2)^{-1} |\vec{r} - \vec{r}_o|^{-3} e^{i k |\vec{r} - \vec{r}_o|} (1 - i k |\vec{r} - \vec{r}_o|) \quad . \tag{2.11}$$

Again, the coefficient A is determined by substituting (2.10) into (2.8), and the result is

$$G(\vec{r}, \vec{r}_o; k) = G_o(\vec{r}, \vec{r}_o; k) - B(k)\, G_o(\vec{r}, 0; k)\, G_o(\vec{r}_o, 0; k) \quad , \tag{2.12}$$

where

$$B(k) = \frac{24\pi^2 a^3}{1 + i k^3 a^3} \quad . \tag{2.13}$$

In spite of the close similarity of the results, B(k) has new and interesting features. If $a < 0$, there is a bound state with binding energy a^{-2} and bound-state wave function

$$r^{-3} e^{-r/|a|} (1 + r/|a|) \quad . \tag{2.14}$$

When $a > 0$, there are <u>complex</u> poles located at

$$k = a^{-1} e^{i\pi/6} \quad \text{and} \quad k = a^{-1} e^{5i\pi/6} \quad . \tag{2.15}$$

If such complex poles are not acceptable, in five dimensions a cannot be positive. More precisely, in five dimensions, a can be zero (which is trivial), negative, or infinity. In the last case, B(k) is simply

$$- i \, 24 \pi^2 k^{-3} \quad . \tag{2.16}$$

3. One-Dimensional Array

As an example of utilizing the Fermi pseudopotential (1.4) in five dimensions, we study in this section the case of the one-dimensional equispaced linear chain of infinite length. Before specializing to this case, let a finite or infinite number of Fermi pseudopotentials be located in general at $\vec{r}_j$. With $R_j = |\vec{r} - \vec{r}_j|$, the Hamiltonian is

$$H = -\nabla^2 + \sum_j \delta^5(\vec{r} - \vec{r}_j) \, 4\pi^2 a_j^3 \frac{\partial^3}{\partial R_j^3} R_j^3 \quad . \tag{3.1}$$

The solution of the Schrödinger equation

$$H \psi = k^2 \psi \tag{3.2}$$

follows closely the procedure of Sec. 2B. Eq. (3.2) is explicitly

$$(\nabla^2 + k^2) \psi = \sum_j A_j \, \delta^5(\vec{r} - \vec{r}_j) \quad , \tag{3.3}$$

where

$$A_j = 4\pi^2 a_j^3 \frac{\partial^3}{\partial R_j^3} R_j^3 \psi \Big|_{R_j = 0} \quad . \tag{3.4}$$

In the absence of an incident field, the solution of (3.3) is

$$\psi = - \sum_j A_j (8\pi^2 R_j^3)^{-1} e^{ikR_j} (1 - ikR_j) \quad . \tag{3.5}$$

Substitution into (3.4) then yields the following linear equations for the determination of A_j and hence ψ from (3.5):

$$(1 + ik^3 a_j^3) A_j + 3 a_j^3 \sum_{\ell \neq j} r_{j\ell}^{-3} e^{ikr_{j\ell}} (1 - ikr_{j\ell}) A = 0 \quad , \tag{3.6}$$

where $r_{j\ell} = |\vec{r}_j - \vec{r}_\ell|$ is the five-dimensional distance between the points $\vec{r}_j$ and $\vec{r}_\ell$. Given a_j and $\vec{r}_j$, (3.6) admits a non-trivial solution only for certain values of k: these are the resonances.

We now specialize to the case of an infinite uniform linear array of equal spacing b. Thus

$$r_{j\ell} = |j - \ell| \, b \, ,$$

$$a_j = a$$

independent of j, and

$$A_j = e^{ij\beta}, \tag{3.7}$$

where β specifies the reduced Hamiltonian. For this case, the fundamental equations (3.6) simplifies to one transcendental equation

$$1 + ik^3 a^3 + 3\,(a/b)^3 \sum_{j \neq 0} |j|^{-3} e^{ik|j|b} (1 - ik|j|b)\, e^{ij\beta} = 0 \ . \tag{3.8}$$

In three dimensions, the salient feature of such an infinite uniform linear chain is the presence of infinitely narrow resonances. The zero width of the resonance is intimately related to the infinite length of the array. If the array is suitably terminated or bent into a closed curve such as a circle, the width becomes non-zero but small [11]. Such narrow resonances are the quantum-mechanical analog of the corresponding electromagnetic phenomenon associated with the Yagi-Uda antenna array [12] often used with television sets.

This phenomenon also occurs in the present case of a linear chain in five dimensions. This means the existence of real k's as solutions to (3.8) provided that a,b, and β are in suitable ranges. First of all, the zero width of the resonance implies the absence of a radiation field, and hence

$$\beta > kb. \tag{3.9}$$

Let the condition (3.8) be rewritten in the form

$$\frac{1}{3}(b/a)^3 + \frac{1}{3} ik^3 b^3 = - \sum_{j \neq 0} |j|^{-3} e^{ik|j|b} (1 - ik|j|b)\, e^{ij\beta} \ , \tag{3.10}$$

then

$$\text{RHS of (3.10)} = \int_0^{e^{i(\beta + kb)}} z^{-1}\, dz[\ell n\,(1-z)](i\beta - \ell n\, z) + \beta \rightarrow -\beta \ . \tag{3.11}$$

Deform the path of integration in the z-plane so that it goes from 0 to 1 along the positive real axis and then from 1 to $e^{i(\pm\beta - \ell n\, z)}$ along the unit circle. As the first part of the contour gives a Riemann ζ-function $\zeta(3)$, (3.11) reduces to, by (3.9),

$$\text{RHS of (3.10)} = -2\,\zeta(3) + \int_0^{\beta + kb} d\theta\, (\theta - \beta)[\ell n(2\sin\tfrac{\theta}{2}) - \tfrac{1}{2}\, i\, (\pi - \theta)] + \int_0^{-\beta + kb} d\theta\, (\theta + \beta)[\ell n\,(-2\sin\tfrac{\theta}{2}) + \tfrac{1}{2}\, i\, (\pi + \theta)] \ . \tag{3.12}$$

Thus the imaginary part can be explicitly calculated as

$$\text{Imaginary part of RHS of (3.10)} = \frac{1}{3} k^3 b^3, \tag{3.13}$$

and consequently (3.10) has the real form

$$\frac{1}{3}(b/a)^3 + 2\,\zeta(3) - \left[\int_0^{\beta + kb} + \int_0^{\beta - kb}\right] d\theta\, (\theta - \beta)\, \ell n\, (2\sin\tfrac{\theta}{2}) = 0 \ . \tag{3.14}$$

Remember that $a < 0$ while $b > 0$. The solutions of (3.14) give the locations of infinitely narrow resonances.

4. Resolvent Equation

The example of last section shows that the Fermi pseudopotential in five dimensions is of use. We now return to the case of a single Fermi pseudopotential already treated in Sec. 2B to study some simple properties of the Green's function (2.12). Let the Hamiltonian be

$$H = -\nabla^2 + 4\pi^2 a^3 \delta^5(\vec{r}) \frac{\partial^3}{\partial r^3} r^3 \quad , \tag{4.1}$$

then $G(\vec{r},\vec{r}_o;k)$ is the coordinate representation of $(H-k^2)^{-1}$. Since H commutes with itself, this operator satisfies the resolvent equation

$$(k^2-k'^2)(H-k^2)^{-1}(H-k'^2)^{-1} = (H-k^2)^{-1}-(H-k'^2)^{-1}. \tag{4.2}$$

Expressed in terms of $G(\vec{r},\vec{r}_o;k)$, (4.2) is

$$(k^2-k'^2)\int d^5\vec{r}''\, G(\vec{r},\vec{r}'';k)G(\vec{r}',\vec{r}'';k')$$
$$= G(\vec{r},\vec{r}';k) - G(\vec{r},\vec{r}';k'). \tag{4.3}$$

In what sense is this formula correct?

The difficulty stems from the fact that in five dimensions the free-space Green's function is not square integrable, as discussed in the introduction. Thus, in the usual sense of integration, the integral on the left-hand side of (4.3) does not exist.

Fortunately, since the divergent integral is of the form $\int r^{-6} d^5\vec{r}$, formulas familiar in dimensional regularization [13,14] can be invoked. The relevant

one is

$$\int r^{-6} d^5\vec{r} = 0. \tag{4.4}$$

Eq. (4.4) gives a meaning to the left-hand side of (4.3), because the G_o of (2.11) is of the form

$$G_o(\vec{r}, 0; k) = (8\pi^2)^{-1} r^{-3} [1 + O(r^2)] \tag{4.5}$$

for small r. The rest of the verification for (4.3) is straightforward.

5. Discussions

We have seen that the Fermi pseudopotential (1.1) in three dimensions can be naturally generalized to (1.4) or equivalently (1.5) in five dimensions. With this natural generalization, the formal manipulations are only slightly changed. The underlying mathematical structure, however, is altered in a profound manner. While $-\nabla^2 +$ (1.1) gives rise to a self-adjoint operator in the space of square integrable functions, it is not clear whether (4.1) is self-adjoint in any sense. A first question that needs to be answered is: Does there exist a Hilbert space of functions such that the resolvent as explicitly given by (2.12) is bounded and satisfies the resolvent equation (4.2)? We do not know the answer to this and numerous related questions. By explicit examples, we do know, however, that the Fermi pseudopotential (1.4) in five dimensions is useful. It is a challenge to mathematical physicists to construct a consistent theory of this new Fermi pseudopotential.

There is one other fundamental difference between (1.1) and (1.4). In three dimensions, (1.1) can be approximated by a short-range square-well attractive potential of suitable strength. This is the basis for the successful application of non-standard analysis [8]. On the contrary, in five dimensions, such an approximation does not seem to exist for the Fermi pseudopotential (1.4).

We conclude with a list of Fermi pseudopotentials in two to six dimensions, where the overall constant is omitted:

Two dimensions $\delta^2(\vec{r})\ r\ (\ln r)^2 \frac{\partial}{\partial r} (\ln r)^{-1}$;

Three dimensions $\delta^3(\vec{r}) \frac{\partial}{\partial r} r$;

Four dimensions $\delta^4(\vec{r})\ r\ (\ln r)^2 \frac{\partial}{\partial r} (\ln r)^{-1} r^{-1} \frac{\partial}{\partial r} r^2$;

Five dimensions $\delta^5(\vec{r}) \frac{\partial^3}{\partial r^3} r^3$;

Six dimensions $\delta^6(r)\ r\ (\ln r)^2 \frac{\partial}{\partial r} (\ln r)^{-1} r^{-1} \frac{\partial}{\partial r} r^{-1} \frac{\partial}{\partial r} r^4$.

There may be some problems with the last one.

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