Making Lifting Obstructions Explicit

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Abstract

If $P \to X$ is a topological principal K-bundle and $\widehat{K}$ a central extension of K by Z, then there is a natural obstruction class $\delta_1(P) \in \check{H}^2(X, \underline{Z})$ in sheaf cohomology whose vanishing is equivalent to the existence of a $\widehat{K}$-bundle $\widehat{P}$ over X with $P \cong \widehat{P}/Z$. In this paper we establish a link between homotopy theoretic data and the obstruction class $\delta_1(P)$ which in many cases can be used to calculate this class in explicit terms. Writing $\partial_d^P \colon \pi_d(X) \to \pi_{d-1}(K)$ for the connecting maps in the long exact homotopy sequence, two of our main results can be formulated as follows. If Z is a quotient of a contractible group by the discrete group Γ, then the homomorphism $\pi_3(X) \to \Gamma$ induced by $\delta_1(P) \in \check{H}^2(X, \underline{Z}) \cong H^3_{\mathrm{sing}}(X, \Gamma)$ coincides with $\partial_2^{\widehat{K}} \circ \partial_3^P$ and if Z is discrete, then $\delta_1(P) \in \check{H}^2(X, \underline{Z})$ induces the homomorphism $-\partial_1^{\widehat{K}} \circ \partial_2^P \colon \pi_2(X) \to Z$. We also obtain some information on obstruction classes defining trivial homomorphisms on homotopy groups.

Contents

1 Introduction

Let K be a topological group and $q\colon P \to X$ be a K-principal bundle on the paracompact Hausdorff space X. Suppose further, that $\widehat{K}$ is a topological central extension of K by Z (which is always assumed to be a locally trivial Z-bundle over K). We say that P *lifts to a $\widehat{K}$-bundle* if there exists a $\widehat{K}$-principal bundle $\widehat{P}$ over X with $\widehat{P}/Z \sim P$ (as K-bundles). By local trivializations and the corresponding transition functions, we obtain a bijection between the set $\mathrm{Bun}(X,K)$ of equivalence classes of topological principal K-bundles over the paracompact space X and the sheaf cohomology set $\check{H}^1(X,\underline{K})$, where $\underline{K}$ is the sheaf of germs of continuous K-valued functions on X. Now the short exact sequence $Z \hookrightarrow \widehat{K} \to K$ of topological groups induces a short exact sequence of sheaves of groups

$$\mathbf{1} \to \underline{Z} \to \underline{\widehat{K}} \to \underline{K} \to \mathbf{1},$$

which in turn leads to an exact sequence in sheaf cohomology

$$\begin{aligned} C(X,K) \cong \check{H}^0(X,K) \xrightarrow{\delta_0} \mathrm{Bun}(X,Z) \cong \check{H}^1(X,\underline{Z}) \\ \to \mathrm{Bun}(X,\widehat{K}) \cong \check{H}^1(X,\underline{\widehat{K}}) \to \mathrm{Bun}(X,K) \cong \check{H}^1(X,\underline{K}) \xrightarrow{\delta_1} \check{H}^2(X,\underline{Z}) \end{aligned} \tag{1}$$

(cf. [Gr58]). For the cohomology class $[P] \in \check{H}^1(X,\underline{K})$ representing the K-bundle P, the class $\delta_1([P]) \in \check{H}^2(X,\underline{Z})$ obtained from the connecting map can therefore be interpreted as an *obstruction class*. It vanishes if and only if P lifts to a $\widehat{K}$-bundle. In the present paper we address the problem to make this obstruction class as explicit as possible in terms of characteristic data attached to the central extension $\widehat{K}$ and the bundle P. This is motivated in particular by the occurence of such problems in Quantum Field Theory and the representation theory of infinite dimensional Lie groups (cf. [CM00, CM02]).

To this end, we assume that the identity component Z_0 of Z is open and a $K(\Gamma,1)$-*group*, i.e., locally contractible, $\pi_1(Z)$ is isomorphic to the discrete abelian group Γ, and its simply connected covering group $\widetilde{Z}_0$ is contractible and divisible (cf. Definition 2.7). [1] This implies in particular that Z_0 is divisible, so that Z is a direct product $Z \cong Z_0 \times D$, where $D \cong \pi_0(Z)$ is a discrete abelian group. From the short exact sequence $\Gamma \to \widetilde{Z}_0 \to Z_0$ and the corresponding short exact sequence

$$0 \to \underline{\Gamma} \to \underline{\widetilde{Z}_0} \to \underline{Z_0} \to 0$$

[1] Typical examples of $K(\Gamma,1)$-groups are quotients of a topological vector space $\mathfrak{z}$ by a discrete subgroup Γ.

of abelian group sheaves we obtain the corresponding long exact sequence in sheaf cohomology. For each paracompact space X, the vanishing of the cohomology $\check{H}^n(X, \underline{Z_0})$ for $n > 0$ ([Hu61, Prop. 4]) therefore yields natural isomorphisms

$$\iota_n \colon \check{H}^n(X, \underline{Z_0}) \to \check{H}^{n+1}(X, \underline{\Gamma}) \cong \check{H}^{n+1}(X, \Gamma), \tag{2}$$

where the latter isomorphism follows from the discreteness of Γ ([Hu61, Prop. 4], resp. [Go58, Ex. 5.1.10, p. 230]). Therefore our problem boils down to describe a cohomology class in

$$\check{H}^2(X, \underline{Z}) \cong \check{H}^2(X, \underline{Z_0}) \oplus \check{H}^2(X, \underline{D}) \cong \check{H}^3(X, \Gamma) \oplus \check{H}^2(X, D).$$

In particular, we can study the case of a contractible group $Z = Z_0$ and a discrete group $Z = D$ independently.

Throughout we shall assume that X is paracompact (needed for the homotopy theory of bundles) and that X is locally contractible, which leads to a natural isomorphism $\check{H}^k(X, \Gamma) \to H^k_{\rm sing}(X, \Gamma)$ between Čech cohomology and singular cohomology. Since we can evaluate singular cohomology classes on homology classes, we obtain for each $n \in \mathbb{N}_0$ homomorphisms

$$\alpha_n \colon \check{H}^n(X, \Gamma) \to \mathrm{Hom}(\pi_n(X), \Gamma) \tag{3}$$

of abelian groups (see Definition 4.2 for details). Therefore our first step in analyzing the obstruction classes consists in identifying their image under α_2, resp., α_3. To formulate our results, we write

$$\partial_j^P \colon \pi_j(X) \to \pi_{j-1}(K) \tag{4}$$

for the connecting map in the long exact homotopy sequence of the bundle $P \to X$. Then our main result can be stated as follows:

Theorem 1.1. *Let X be a paracompact locally contractible Hausdorff space, K a connected locally contractible topological group, $P \to X$ a principal K-bundle, and $\widehat{K}$ be a central extension of K by Z. Then the following assertions hold:*

(a) *If Z is a $K(\Gamma, 1)$-group, then $\alpha_3(\iota_2(\delta_1([P]))) = \partial_2^{\widehat{K}} \circ \partial_3^P$.*

(b) *If $Z = D$ is discrete, then $\alpha_2(\delta_1([P])) = -\partial_1^{\widehat{K}} \circ \partial_2^P$.*

Since $\pi_2(K)$ vanishes for every finite dimensional Lie group K ([Ca36]), part (a) of the preceding theorem shows that for these groups the obstruction class defines the trivial homomorphism $\pi_3(X) \to \pi_1(Z)$. If X is 2-connected, then $H^3(X, \Gamma) \cong \mathrm{Hom}(\pi_3(X), \Gamma)$, so that (a) completely determines the obstruction class. If X is 1-connected, then $H^2(X, \Gamma) \cong \mathrm{Hom}(\pi_2(X), \Gamma)$ and (b) likewise determines the obstruction class for $Z = D$.

For the proof of these results we proceed as follows. We start in Section 2 with some generalities on obstruction classes. Since we want to express these obstruction classes in terms of easily accessible data attached to the central extension $\widehat{K}$ and the bundle P, it is important to understand which kind of data is relevant for our purpose. Since we are mainly interested in the case where K is a Lie group and Z_0 is a quotient of a topological vector space, we show in Section 3 that, for a connected locally contractible topological group K, the relevant data related to $\widehat{K}$ consists of the homomorphisms

$$\partial_1^{\widehat{K}} : \pi_1(K) \to \pi_0(Z) \quad \text{and} \quad \partial_2^{\widehat{K}} : \pi_2(K) \to \pi_1(Z),$$

and, if $\partial_2^{\widehat{K}}$ vanishes, the abelian extension

$$[\pi_1(\widehat{K})] \in \operatorname{Ext}(\pi_1(K), \pi_1(Z)).$$

The obstruction class $\delta_1([P])$ depends only on this data. In the context of Lie group extensions, $\partial_2^{\widehat{K}}$ can be written as the period homomorphism of a left invariant 2-form ([Ne02, Prop. 5.11]) which is a crucial key for its explicit determination and hence for the calculation of obstruction classes.

In Section 4 we collect various results on spheres. In particular, we describe an explicit way to calculate Čech cohomology classes with values in discrete abelian groups, introduce the suspension homomorphism and recall how K-bundles on $\mathbb{S}^n$ are classified by elements of $\pi_{n-1}(K)$. Then we turn in Section 5 to the case where $Z = D$ is discrete and prove Theorem 1.1(b). As a byproduct, we obtain an analog of Theorem 1.1 for $\delta_0 : C(X, K) \to \check{H}^1(X, \underline{Z})$ (Proposition 5.7).

In Section 6 we reduce the proof of Theorem 1.1(a) first to the case $X = \mathbb{S}^3$ by pulling back geometric data by continuous maps $\mathbb{S}^3 \to X$ and then we complete the proof by verifying the theorem for $X = \mathbb{S}^3$ by direct computation. In the course of the proof, the following diagram plays a central role:

$$\begin{array}{ccc} \check{H}^0(\mathbb{S}^2, \underline{K}) \cong C(\mathbb{S}^2, K) & \xrightarrow{\delta_0} & \check{H}^1(\mathbb{S}^2, \underline{Z}) \\ \downarrow & & \downarrow{\scriptstyle -s} \\ \check{H}^1(\mathbb{S}^3, \underline{K}) \cong \operatorname{Bun}(\mathbb{S}^3, K) & \xrightarrow[\delta_1]{} & \check{H}^2(\mathbb{S}^3, \underline{Z}) \end{array} \qquad (5)$$

Here the vertical maps are suspension maps, so that the commutativity of the diagram expresses a compatibility of connecting maps with suspensions.

In Section 7, we discuss situations where the obstruction classes lie in the subgroup $\Lambda^3(X, \Gamma) := \ker \alpha_3 \subseteq \check{H}^3(X, \Gamma)$ of *aspherical cohomology classes* inducing the zero homomorphism $\pi_3(X) \to \Gamma$. If X is 1-connected, then $\Lambda^3(X, \Gamma) \cong \operatorname{Ext}(\pi_2(X), \Gamma)$. Here our main result is Theorem 7.12 which provides complete information on the obstruction class for $\partial_2^P = 0$ and a flat extension of K defined by a homomorphism $\gamma : \pi_1(K) \to Z$, which implies that $\delta_1([P]) \in \operatorname{Ext}(H_2(X), \Gamma) \subseteq \Lambda^3(X, \Gamma)$. We also explain how to construct other types of aspherical obstruction classes.

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Conventions

Unless stated otherwise X and Y are paracompact locally contractible topological Hausdorff spaces, Γ is a discrete abelian group, Z is a locally contractible abelian topological group whose identity component Z_0 is a $K(\Gamma,1)$-group. Furthermore, $\mathfrak{z}$ always is a topological vector space and when writing $\mathfrak{z}/\Gamma$ we always assume that $\Gamma \subset \mathfrak{z}$ is a discrete *subgroup*. We assume that we are given a central extension

$$Z \hookrightarrow \widehat{K} \to K, \tag{6}$$

where K is a locally contractible topological group and (6) is required to be a locally trivial principal bundle. The set of equivalence classes of such extensions form a group K is denoted by $\mathrm{Ext}_c(K,Z)$. If, in addition, K is a Lie group we write $\mathrm{Ext}_s(K,Z)$ for the group of smooth central extensions, i.e., central extensions for which the group $\widehat{K}$ in (6) is a Lie group and a smooth principal Z-bundle. If K and Z are discrete, then $H^n_{\mathrm{grp}}(K,Z) \cong H^n(BK,Z)$ denotes the ordinary group cohomology. We will also sometimes use the locally continuous cohomology groups $H^n_c(K,Z)$ and the locally smooth cohomology groups $H^n_s(K,Z)$ in case that K is a Lie group (cf. [WW]). Whilst we are using different kinds of cohomology groups we will throughout only use singular homology of spaces, which we denote by $H_n(X)$. We also recall that the Čech cohomology groups $\check{H}^n(X,\Gamma)$ and the singular cohomology groups $H^n(X,\Gamma) := H^n_{\mathrm{sing}}(X,\Gamma)$ are isomorphic because X was assumed to be locally contractible ([Go58, Thm. 5.10.1, p. 228 and Ex. 3.9.1, p. 159]).

We recall that, for $k=0,1$ (and for $k \in \mathbb{N}$ if $\widehat{K}$ is abelian), the connecting homomorphisms

$$\delta_k \colon \check{H}^k(X,\underline{K}) \to \check{H}^{k+1}(X,\underline{Z})$$

can be constructed by representing a cohomology class c by a cocycle $(g_{i_0,\ldots,i_k})$ on an open cover $\mathcal{U} = (U_i)_{i\in I}$ of X for which all functions $g_{i_0,\ldots,i_k}$ have continuous lifts leading to a $\widehat{K}$-valued cochain $\widehat{g}_{i_0,\ldots,i_k}$. Then taking $\delta_k(c)$ is the cohomology class represented by the $\underline{Z}$-valued cocycle $\delta(\widehat{g})_{i_0,\ldots,i_{k+1}}$.

2 Generalities on the obstruction class

In this section we start with some general remarks on the obstruction class $\delta_1([P])$. We write

$$\mathrm{obs}^{\widehat{K}}([P]) := \mathrm{obs}_P([\widehat{K}]) := \delta_1([P]) \in \check{H}^2(X,\underline{Z})$$

for the corresponding obstruction class and recall that the group structure on $\mathrm{Ext}_c(K,Z)$ is defined by the Baer sum

$$[\widehat{K}_1] \oplus [\widehat{K}_2] := [\widehat{K}_3] \quad \text{with} \quad \widehat{K}_3 := (\widehat{K}_1 \times_K \widehat{K}_2)/\{(z,z^{-1})\colon z\in Z\}.$$

Lemma 2.1. (a) *The map*

$$\mathrm{obs}_P\colon \mathrm{Ext}_c(K,Z) \to \check{H}^2(X,\underline{Z})$$

is a group homomorphism.

(b) *If K is discrete, then obs_P factors through a homomorphism*

$$\mathrm{obs}_P^d\colon \mathrm{Ext}_c(K,Z) \to \check{H}^2(X,Z_d),$$

where Z_d denotes Z, endowed with the discrete topology.

Proof. (a) Let $q_j\colon \widehat{K}_j \to K$ be two Z-extensions of K, $(g_{k\ell}) \in \check{Z}^1(X,\underline{K})$ be a 1-cocycle and $(h_{k\ell}) \in \check{C}^1(X,\underline{\widehat{K}_1})$, $(h'_{k\ell}) \in \check{C}^1(X,\underline{\widehat{K}_2})$ be lifts of $(g_{k\ell})$. Then $h''_{k\ell} := (h_{k\ell}, h'_{k\ell})$ is a lift of $(g_{k\ell})$ with values in the $Z \times Z$-extension of K given by the fiber product $\widehat{K}_1 \times_K \widehat{K}_2$ and $\delta h'' = (\delta h, \delta h') \in \check{Z}^2(X,\underline{Z^2})$ represents $(\mathrm{obs}_P([K_1]), \mathrm{obs}_P([K_2]))$.

The central extension $[\widehat{K}_1] \oplus [\widehat{K}_2] \in \mathrm{Ext}_c(K,Z)$ is defined as the quotient $\widehat{K}_3$ of $\widehat{K}_1 \times_K \widehat{K}_2$ by the central subgroup $\{(z,z^{-1})\colon z \in Z\}$, which is the kernel of the multiplication homomorphism $\mu_Z\colon Z^2 \to Z$. From that it follows that $\mathrm{obs}_P([\widehat{K}_3]) = \mathrm{obs}_P([\widehat{K}_1]) + \mathrm{obs}_P([\widehat{K}_2])$.

(b) This follows immediately from the bijection $\mathrm{Ext}_c(K,Z_d) \to \mathrm{Ext}_c(K,Z)$ induced from the continuous bijection $Z_d \to Z$. □

From the naturality of the connecting maps in (1), we immediately obtain the naturality of the obstruction classes:

Lemma 2.2. (a) *If $f\colon Y \to X$ is a continuous map, $P \to X$ a K-bundle and $\widehat{K}$ a Z-extension of K, then*

$$\delta_1([f^*P]) = f^*\delta_1([P]) \in \check{H}^2(Y,\underline{Z}).$$

(b) *If $\phi\colon K_1 \to K_2$ is a continuous morphism of topological groups,*

$$\phi^*\colon \mathrm{Ext}_c(K_2,Z) \to \mathrm{Ext}_c(K_1,Z), \quad [\widehat{K}_2] \mapsto [\phi^*\widehat{K}_2],$$

$P \to X$ a K_1-bundle and $\phi_*P := P \times_\phi K_2$ the associated K_2-bundle, then

$$\mathrm{obs}_P([\phi^*\widehat{K}_2]) = \mathrm{obs}_{\phi_*P}([\widehat{K}_2]).$$

(c) *If $\phi\colon Z_1 \to Z_2$ is a continuous morphism of abelian groups and*

$$\phi_*\colon \mathrm{Ext}_c(K,Z_1) \to \mathrm{Ext}_c(K,Z_2), \quad [\widehat{K}] \mapsto [\phi_*\widehat{K}], \quad \phi_*\widehat{K} = \widehat{K} \times_\phi Z_2,$$

then

$$\phi_*\,\mathrm{obs}_P([\widehat{K}]) = \mathrm{obs}_P([\phi_*\widehat{K}]).$$

Definition 2.3. For a principal K-bundle $q_P\colon P \to X$ over the topological space X, we write

$$\partial_n^P\colon \pi_n(X) \to \pi_{n-1}(K)$$

for the *connecting homomorphisms* in the long exact homotopy sequence of P. Since we shall need this information below, we recall how these maps are constructed. First we choose base points $p_0 \in P$ and $x_0 \in X$ with $q_P(p_0) = x_0$ and observe that the map $\iota\colon K \to P_{x_0}, k \mapsto p_0.k$ is a homeomorphism. Let $I := [0,1]$ be the unit interval. For a continuous based map $\sigma\colon \mathbb{S}^n \cong I^n/\partial I^n \to X$ we choose a continuous lift $\widetilde{\sigma}\colon I^n \to P$. Then $\widetilde{\sigma}(\partial I^n) \subseteq P_{x_0}$, and we put

$$\partial_n^P([\sigma]) = [\iota^{-1} \circ \widetilde{\sigma}\mid_{\partial I^n}] \in \pi_{n-1}(K).$$

As explained in the following remark, the connecting map ∂_1^P is the obstruction for the reduction of the structure group to the identity component K_0. As Theorem 1.1 shows, the higher connecting maps ∂_2^P, resp., ∂_3^P are closely related to the obstruction classes for discrete, resp., connected groups Z.

Remark 2.4. (Geometric interpretation of ∂_1^P) Let $K_0 \subseteq K$ be the identity component. Then $BK_0 := EK/K_0$ is a classifying space for K_0. As K_0 is connected, BK_0 is simply connected. Moreover, the natural map $BK_0 \to BK \cong EK/K$ is a covering because K_0 is open in K. Therefore BK_0 is the universal covering space of BK.

Since a continuous map $f\colon X \to BK$ can be lifted to $\widetilde{BK}$ if and only if $\pi_1(f) = \partial_1^{f^*EK}\colon \pi_1(X) \to \pi_1(BK) \cong \pi_0(K)$ vanishes, it follows that a K-bundle $P \to X$ has a reduction to a K_0-bundle if and only if the connecting homomorphism $\partial_1^P\colon \pi_1(X) \to \pi_0(K)$ is trivial.

In Remark 5.4(c) below we shall see that the vanishing of the connecting map $\partial_2^P\colon \pi_2(X) \to \pi_1(K)$ also has a simple geometric interpretation because it is equivalent to the existence of a $\widetilde{K}$-lift of the pullback of the K-bundle P to the universal covering space of X. A geometric interpretation of the vanishing of ∂_3^P can be found in Remark 7.19(b).

Lemma 2.5. *If $P \to X$ is a K-bundle and $f\colon (Y, y_0) \to (X, x_0)$ is a continuous based map, then*

$$\partial_n^P \circ \pi_n(f, y_0) = \partial_n^{f^*P}\colon \pi_n(Y, y_0) \to \pi_{n-1}(K).$$

Proof. Pick a base point $p_0 \in P$ over x_0. For a continuous based map $\sigma\colon \mathbb{S}^n \cong I^n/\partial I^n \to Y$ we have

$$\partial_n^{f^*P}([\sigma]) = [\iota^{-1} \circ \widetilde{\sigma}\mid_{\partial I^n}],$$

where $\widetilde{\sigma}\colon I^n \to f^*P$ is a continuous base point preserving lift of σ. Since the projection $\mathrm{pr}_P\colon f^*P \to P$ preserves base points, $\mathrm{pr}_P \circ \widetilde{\sigma}$ is a base point preserving lift of $f \circ \sigma$, so that

$$\partial_n^P([f \circ \sigma]) = [\iota^{-1} \circ \mathrm{pr}_P \circ \widetilde{\sigma}\mid_{\partial I^n}] = \partial^{f^*P}([\sigma]).$$

This proves our assertion. □

Remark 2.6. Since Z_0 is open in Z and divisible, $Z \cong Z_0 \times D$ is a direct product of the discrete group $D := Z/Z_0 =: \pi_0(Z)$ and the connected group Z_0. This product decomposition leads to

$$\mathrm{Ext}_c(K, Z) \cong \mathrm{Ext}_c(K, Z_0) \oplus \mathrm{Ext}_c(K, D)$$

and

$$\check{H}^2(X, \underline{Z}) \cong \check{H}^2(X, \underline{Z_0}) \oplus \check{H}^2(X, \underline{D}) \cong \check{H}^3(X, \Gamma) \oplus \check{H}^2(X, D).$$

This splits the problem to determine the obstruction class into two cases, where Z is either discrete or a quotient $\mathfrak{z}/\Gamma$.

Definition 2.7. We call a connected abelian topological group Z a *$K(\Gamma, 1)$-group* if it is locally contractible, $\pi_1(Z) \cong \Gamma$, and $\widetilde{Z}$ is contractible and divisible.[2] Typical examples arise for $Z = \mathfrak{z}/\Gamma$, where $\mathfrak{z}$ is a topological vector space and $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup.

The following examples show how obstruction classes are connected to various well known constructions in topology and group theory.

Example 2.8. (a) (Chern classes) Let Z be a $K(\Gamma, 1)$-group. Then $\widetilde{Z}$ is a central extension of Z by Γ, and the obstruction class defines a homomorphism

$$\mathrm{obs}^{\widetilde{Z}} = \iota_1 \colon \check{H}^1(X, \underline{Z}) \to \check{H}^2(X, \underline{\Gamma}) \cong \check{H}^2(X, \Gamma) \cong H^2_{\mathrm{sing}}(X, \Gamma),$$

which is a group isomorphism assigning to a Z-bundle $P \to X$ its *Chern class* $\mathrm{Ch}(P) \in H^2(X, \Gamma)$.

(b) If K is discrete, then BK can be realized as a CW complex ([Ro94, Thm. 5.1.15]), hence in particular as a locally contractible space. Now $\check{H}^3(BK, \Gamma) \cong H^3(BK, \Gamma) \cong H^3_{\mathrm{grp}}(K, \Gamma)$ is the group cohomology of K with values in the trivial K-module Γ ([EML45]). For the universal K-bundle $q \colon EK \to BK$, we obtain for a $K(\Gamma, 1)$-group Z the obstruction map

$$\iota_3 \circ \mathrm{obs}_{EK} \colon \mathrm{Ext}_c(K, Z) \cong H^2_{\mathrm{grp}}(K, Z) \to \check{H}^3(BK, \Gamma) \cong H^3_{\mathrm{grp}}(K, \Gamma).$$

It is easy to see that this coincides with the natural connecting map

$$\delta_2 \colon H^2_{\mathrm{grp}}(K, Z) \to H^3_{\mathrm{grp}}(K, \Gamma)$$

in the long exact sequence in group cohomology induced by the short exact sequence $\Gamma \hookrightarrow \widetilde{Z} \twoheadrightarrow Z$.

[2] $K(\Gamma, 1)$ groups exist for every abelian group Γ. Let $G(\Gamma)$ be the group of all measurable functions $[0, 1] \to \Gamma$ with finitely many values, modulo functions supported in a zero set and endowed with the metric $d(f, g) := |\{f \neq g\}|$, then $G(\Gamma)$ is contractible, locally contractible and contains the subgroup Γ of constant functions as a discrete subgroup. If Γ is divisible, then $G(\Gamma)$ inherits this property. Since every abelian group Γ can be embedded in a divisible group D, the group $G(D)/\Gamma$ is a $K(\Gamma, 1)$-group.

(c) Let $q_X\colon \widetilde{X} \to X$ be a simply connected covering of X, consider the group $K = \pi_1(X)$ of deck transformations as a discrete group, and $\widetilde{X}$ as a principal K-bundle. Then the obstruction class leads to a homomorphism

$$\mathrm{obs}_{\widetilde{X}}\colon H^2_{\mathrm{grp}}(\pi_1(X), \Gamma) \to \check{H}^2(X, \Gamma) \cong H^2(X, \Gamma)$$

for every discrete abelian group Γ. Theorem 7.9 below implies that $\mathrm{obs}_{\widetilde{X}}$ is injective and that its range is the group $\Lambda^2(X, \Gamma) := \ker \alpha_2$ of aspherical cohomology classes. This is an obstruction theoretic interpretation of Hopf's Theorem asserting that $\Lambda^2(X, D) \cong H^2_{\mathrm{grp}}(\pi_1(X), D)$ ([EML45]).

3 Topological data of central group extensions

To understand the homomorphism obs_P for connected K, it is important to know which topological data coming along with a central Z-extension $\widehat{K}$ of K can be used to express obs_P more directly. Clearly, the homomorphisms $\partial_2^{\widehat{K}}$ and $\partial_1^{\widehat{K}}$ are two such pieces of data appearing in Theorem 1.1. To see what else we have, it is instructive to take a look at those extensions for which $\partial_2^{\widehat{K}}$ vanishes.

First we note that, for connected K and every abelian topological group Z, we have a natural homomorphism

$$E^*\colon \operatorname{Hom}(\pi_1(K), Z) \to \operatorname{Ext}_c(K, Z), \quad \gamma \mapsto [\gamma_* \widetilde{K}],$$

where

$$\gamma_* \widetilde{K} := (\widetilde{K} \times Z)/\{(d, -\gamma(d))\colon d \in \pi_1(K)\}$$

is the central extension of K by Z associated to the universal covering $q_K\colon \widetilde{K} \to K$ with kernel $\pi_1(K)$ by the homomorphism γ.

Remark 3.1. Suppose that K is connected.

(a) If D is a discrete abelian group, then [Ne02, Prop. 2.6] implies that the homomorphism

$$E^*\colon \operatorname{Hom}(\pi_1(K), D) \to \operatorname{Ext}_c(K, D)$$

is an isomorphism of abelian groups. For the so obtained central extensions $\widehat{K} = \gamma_* \widetilde{K}$ we have $\partial_2^{\widehat{K}} = 0$ and $\partial_1^{\widehat{K}} = \gamma$, so that all information is encoded in $\partial_1^{\widehat{K}}$.

(b) If Z is a $K(\Gamma, 1)$-group and $\widehat{K}$ a central extension of K by Z, then the Chern class $\mathrm{Ch}([\widehat{K}])$ is an element of $H^2_{\mathrm{sing}}(K, \Gamma)$. From the Universal Coefficient Theorem we obtain the short exact sequence

$$0 \to \operatorname{Ext}(H_1(K), \Gamma) \cong \operatorname{Ext}(\pi_1(K), \Gamma) \to H^2(K, \Gamma) \to \operatorname{Hom}(H_2(K), \Gamma) \to 0,$$

where we have used that $\pi_1(K)$ is abelian, so that $H_1(K) \cong \pi_1(K)$ by the Hurewicz Theorem. The Z-bundles over K corresponding to extensions of the form $\gamma_* \widetilde{K}$ for $\gamma \in \operatorname{Hom}(\pi_1(K), Z)$ define the trivial homomorphism $H_2(K) \to \Gamma$ which can be interpreted as the "curvature" of $\widehat{K}$. Since the Z-bundle $\gamma_* \widetilde{K}$ is topologically trivial if and only if the homomorphism $\gamma\colon \pi_1(K) \to Z$ lifts to a homomorphism $\widetilde{\gamma}\colon \pi_1(K) \to \widetilde{Z}$, we see that $\mathrm{im}(E^*) \cong \mathrm{Ext}(\pi_1(K), \Gamma)$, and we conclude that the Chern class provides a surjection

$$\mathrm{Ch} : \mathrm{Ext}_c(K, Z)_{\mathrm{flat}} := \mathrm{im}(E^*) \to \mathrm{Ext}(\pi_1(K), \Gamma),$$

showing that all flat Z-bundles over K can be realized by flat central Z-extensions.

When specialized to Lie groups and $Z = \mathfrak{z}/\Gamma$, the following proposition corrects a wrong claim in [Ne02, Cor. 7.15].

Proposition 3.2. *Let Z be a $K(\Gamma, 1)$-group and K be a connected locally contractible paracompact topological group. For the subgroup*

$$\mathrm{Ext}_c(K, Z)_0 := \{[\widehat{K}] \in \mathrm{Ext}_c(K, Z) \colon \partial_2^{\widehat{K}} = 0\},$$

we have an exact sequence

$$\mathrm{Ext}_c(K, \widetilde{Z}) \xrightarrow{(q_Z)_*} \mathrm{Ext}_c(K, Z)_0 \xrightarrow{\zeta} \mathrm{Ext}(\pi_1(K), \Gamma), \quad \zeta([\widehat{K}]) = [\pi_1(\widehat{K})].$$

Note that $\pi_1(\widehat{K})$ is abelian because $\widehat{K}$ is a topological group.

Proof. If $K^\sharp$ is a central $\widetilde{Z}$-extension of K, then $\widehat{K} := (q_Z)_* K^\sharp = K^\sharp/\Gamma$ is a central extension for which $\partial_2^{\widehat{K}} = \pi_1(q_Z) \circ \partial_2^{K^\sharp} = 0$ follows from $\pi_1(q_Z) = 0$. Moreover, the isomorphism $\pi_1(K^\sharp) \to \pi_1(K)$ and the natural homomorphism $\pi_1(K^\sharp) \to \pi_1(\widehat{K})$ lead to a splitting of the extension $\pi_1(\widehat{K})$ of $\pi_1(K)$ by Γ, so that $[\widehat{K}] \in \ker \zeta$.

If, conversely, $[\widehat{K}] \in \ker \zeta$, then Proposition 7.11(ii) below[3] implies that $\widehat{K}$ is a trivial Z-bundle, hence of the form $\widehat{K} = Z \times K$ with a multiplication

$$(z, k)(z', k') = (zz' f(k, k'), kk'),$$

where $f\colon K \times K \to Z$ is a continuous 2-cocycle. The splitting homomorphism $\sigma\colon \pi_1(K) \to \pi_1(\widehat{K}) \subseteq (\widehat{K})\widetilde{\ }$ maps onto a discrete central subgroup of $(\widehat{K})\widetilde{\ }$. The latter group is of the form $\widetilde{Z} \times \widetilde{K}$ with a multiplication of the form

$$(z, k)(z', k') = (zz' \widetilde{f}(k, k'), kk'),$$

where $\widetilde{f}\colon \widetilde{K} \times \widetilde{K} \to \widetilde{Z}$ is a continuous 2-cocycle satisfying $q_Z \circ \widetilde{f} = f \circ (q_K \times q_K)$. We conclude that $K^\sharp := (\widehat{K})\widetilde{\ }/\mathrm{im}(\sigma)$ is a central $\widetilde{Z}$-extension of K with $K \cong K^\sharp/\Gamma = (q_Z)_* K^\sharp$. This completes the proof. ∎

[3] This reference requires the paracompactness of the topological group which actually follows from its local paracompactness ([AU06]).

For central Lie group extension $\widehat{K}$ by $Z = \mathfrak{z}/\Gamma$, one can argue differently, thus bypassing the requirement of paracompactness: If $[\widehat{K}] \in \ker \zeta$, [Ne02, Thm. 7.12] implies the existence of a central $\mathfrak{z}$-extension $K^\sharp$ of K with the same Lie algebra cocycle. Then $[K^\sharp/\Gamma] - [\widehat{K}] \in \mathrm{Ext}_s(K, Z)$ lies in the image of $\mathrm{Hom}(\pi_1(K), Z)$ by [Ne02, Thm. 7.12]. To show that $[\widehat{K}] \in \mathrm{im}((q_Z)_*)$, we may therefore assume that $\widehat{K} = \gamma_* \widetilde{K}$ for some homomorphism $\gamma\colon \pi_1(K) \to Z$. Then $\pi_1(\widehat{K}) \cong \gamma^*\mathfrak{z}$ as a Γ-extension of $\pi_1(K)$ (cf. Remark 7.14 below). Since $\zeta([\widehat{K}]) = [\pi_1(\widehat{K})]$ vanishes, γ lifts to a homomorphism $\widetilde{\gamma}\colon \pi_1(K) \to \mathfrak{z}$. Therefore $\widehat{K} \cong (\widetilde{\gamma}_* \widetilde{K})/\Gamma$ is contained in the image of $(q_Z)_*$.

Proposition 3.3. *Suppose that $\widehat{K}$ is an extension of the connected paracompact group K by the $K(\Gamma,1)$-group Z. If*

$$\partial_2^{\widehat{K}} \colon \pi_2(K) \to \Gamma \quad \textit{and} \quad [\pi_1(\widehat{K})] \in \mathrm{Ext}(\pi_1(K), \Gamma)$$

vanish, then $\mathrm{obs}^{\widehat{K}}([P])$ *vanishes for every K-bundle P.*

Proof. Proposition 3.2 implies the existence of a central extension $K^\sharp$ of K by $\widetilde{Z}$ with $K^\sharp/\Gamma \cong \widehat{K}$. Therefore the obstruction class $\mathrm{obs}^{\widehat{K}}([P]) \in \check{H}^2(X, \underline{Z})$ is the image of the corresponding obstruction class $\mathrm{obs}^{K^\sharp}([P]) \in \check{H}^2(X, \underline{\widetilde{Z}}) = \{0\}$ (Lemma 2.2(c)), hence trivial. □

Remark 3.4. Since $\widetilde{Z}$-extensions of K define trivial obstruction classes, the preceding proof shows that the information on $[\widehat{K}]$ that is relevant for $\mathrm{obs}_P([\widehat{K}])$ only depends on $[\widehat{K}]$ modulo the image of $(q_Z)_*$. Two such classes coincide if and only if

$$\partial_2^{\widehat{K}_1} = \partial_2^{\widehat{K}_2} \colon \pi_2(K) \to \Gamma \quad \text{and} \quad 0 = \zeta([\widehat{K}_1] - [\widehat{K}_2]) \in \mathrm{Ext}(\pi_1(K), \Gamma).$$

Remark 3.5. In view of Proposition 3.2 and Remark 3.1(b),

$$\mathrm{Ext}_c(K, Z)_0 = \mathrm{im}((q_Z)_*) + \mathrm{im}(E^*). \tag{7}$$

Flat central extensions are precisely those whose Chern class defines the trivial homomorphisms $H_2(K) \to \Gamma$ and $\mathrm{Ext}_c(K, Z)_0$ consists of those central extensions for which this homomorphism vanishes on the image of $\pi_2(K)$ under the Hurewicz homomorphism $h_2 \colon \pi_2(K) \to H_2(K)$.

Let $[\widehat{K}] \in \mathrm{Ext}_c(K, Z)_0$. In view of (7), $[\widehat{K}]$ is a sum of a flat extension and an extension of the form $(q_Z)_* K^\sharp$, where $K^\sharp$ is a central extension of K by $\widetilde{Z}$. Since the Z-bundle $(q_Z)_* K^\sharp$ over K has a trivial Chern class, it follows that, whenever $\partial_2^{\widehat{K}}$ vanishes, then also the homomorphism $H_2(K) \to \Gamma$ defined by the Chern class of the Z-bundle $\widehat{K}$ vanishes. Therefore no additional information is gained by considering $H_2(K)$ instead of $\pi_2(K)$.

4 Čech cohomology on spheres

We introduce a good cover of $\mathbb{S}^n$ with respect to which we specify Čech cohomology classes, the isomorphisms we need in the following, and the suspension homomorphism in Čech cohomology. We also recall the gluing construction of K-bundles over $\mathbb{S}^n$ from elements of $\pi_{n-1}(K)$.

4.1 Calculating Čech cohomology classes on spheres

Suppose that Z is connected. For the spheres $\mathbb{S}^n$, $n > 1$, we obtain from (2) isomorphisms

$$\iota_{n-1} \colon \check{H}^{n-1}(\mathbb{S}^n, \underline{Z}) \to \check{H}^n(\mathbb{S}^n, \Gamma) \cong \Gamma.$$

Below we have to deal with the suspension map $\text{sp}_1 \colon \check{H}^1(\mathbb{S}^2, \underline{Z}) \to \check{H}^2(\mathbb{S}^3, \underline{Z})$. To make this map more explicit, we shall identify both sides with Γ, and to do that, we need explicit isomorphisms $S_n \colon \check{H}^n(\mathbb{S}^n, \Gamma) \to \Gamma$.

To this end, we realize $\mathbb{S}^n$ as the (relative) boundary of the standard simplex

$$\Delta^{n+1} = [e_0, \ldots, e_{n+1}] \subseteq \mathbb{R}^{n+2},$$

where $e_0, \ldots, e_{n+1}$ are the canonical basis vectors in $\mathbb{R}^{n+2}$. Then the jth face

$$F_j := [e_0, \ldots, \widehat{e_j}, \ldots, e_{n+1}], \qquad j = 0, \ldots, n+1$$

is a closed subset of $\mathbb{S}^n = \partial\Delta^{n+1}$ and we write $\Lambda_j := F_j^c$ for its open complement, which is the intersection of $\partial\Delta^{n+1}$ with a convex subset of Δ^{n+1}. The latter property is inherited by all finite intersections of the Λ_j, which immediately implies that $\mathcal{U} := (\Lambda_0, \ldots, \Lambda_{n+1})$ is a *good cover* of $\mathbb{S}^n$, i.e., all intersections of covering sets are contractible. Hence we can calculate Čech cohomology directly with the cover $\mathcal{U}$ via[4]

$$\check{H}^k(\mathbb{S}^n, \underline{Z}) \cong \check{H}^k_{\mathcal{U}}(\mathbb{S}^n, \underline{Z}) \quad \text{and} \quad \check{H}^k(\mathbb{S}^n, \Gamma) \cong \check{H}^k_{\mathcal{U}}(\mathbb{S}^n, \Gamma).$$

We also note that $\bigcup_{j=0}^{n+1} F_j = \mathbb{S}^n$ implies that $\bigcap_{j=0}^{n+1} \Lambda_j = \emptyset$. In particular, the space $\check{C}^{n+1}_{\mathcal{U}}(\mathbb{S}^n, \Gamma)$ of $(n+1)$-cochains is trivial, which implies that

$$\check{H}^n_{\mathcal{U}}(\mathbb{S}^n, \Gamma) \cong C^n_{\mathcal{U}}(\mathbb{S}^n, \Gamma)/\delta(C^{n-1}_{\mathcal{U}}(\mathbb{S}^n, \Gamma)).$$

Elements of $\check{C}^n_{\mathcal{U}}(\mathbb{S}^n, \Gamma)$ assign to $(n+1)$-fold intersections

$$\Lambda_{i_1, \ldots, i_{n+1}} := \bigcap_{j=1}^{n+1} \Lambda_{i_j}$$

elements of Γ. Since there are only $n+2$ such intersections:

$$\Lambda_{(j)} := \Lambda_{0,1,\ldots,\widehat{j},\ldots,n+1} = \bigcap_{k \neq j} \Lambda_k = \text{int}(F_j)$$

we have $C^n_{\mathcal{U}}(\mathbb{S}^n, \Gamma) \cong \Gamma^{n+2}$.

Lemma 4.1. *The map*

$$\widetilde{S}_n \colon C^n_{\mathcal{U}}(\mathbb{S}^n, \Gamma) \cong \Gamma^{n+2} \to \Gamma, \quad \widetilde{S}_n(\gamma_0, \ldots, \gamma_{n+1}) := \sum_{j=0}^{n+1} (-1)^j \gamma_j, \tag{8}$$

factors through an isomorphism $S_n \colon \check{H}^n(\mathbb{S}^n, \Gamma) \to \Gamma$.

[4] From the isomorphisms $\check{H}^n(X, \underline{Z}) \cong \check{H}^{n+1}(X, \Gamma)$, it follows that these cohomology groups vanish if X is contractible. Hence this follows from [Go58, Cor.,p. 213].

Proof. For $\beta \in C^{n-1}_{\mathcal{U}}(\mathbb{S}^n, \Gamma)$, the coboundary is given by

$$\delta(\beta)_{(j)} = \beta_{1,\ldots,\widehat{j},\ldots,n+1} - \beta_{0,2,\ldots,\widehat{j},\ldots,n+1} \pm \cdots \\ + (-1)^{j-1}\beta_{0,1,\ldots,j-2,\widehat{j-1},\widehat{j},\ldots,n+1} + (-1)^j \beta_{0,1,\ldots,\widehat{j},\widehat{j+1},\ldots,n+1} \pm \cdots .$$

For the cochains β^k, $k = 0, \ldots, n$, assigning the value γ to $\Lambda_{0,1,\ldots,\widehat{k},\widehat{k+1},\ldots,n+1}$ and 0 to all other $(n-1)$-fold intersections, we obtain

$$\delta(\beta^k)_{(j)} = (-1)^{j-1}\delta_{j-1,k}\gamma + (-1)^j \delta_{j,k}\gamma = (-1)^k(\delta_{j-1,k} + \delta_{j,k})\gamma.$$

This implies that $\delta(C^{n-1}_{\mathcal{U}}(\mathbb{S}^n, \Gamma))$ coincides with the kernel of $\widetilde{S}_n$, and this proves the lemma. □

Definition 4.2. Since Γ is discrete, Lemma 4.1 provides for each $n > 0$ a natural isomorphism $S_n \colon \check{H}^n(\mathbb{S}^n, \Gamma) \to \Gamma$. For a paracompact locally contractible space X, we thus obtain natural homomorphisms

$$\alpha_n \colon \check{H}^n(X, \Gamma) \to \mathrm{Hom}(\pi_n(X), \Gamma), \quad \alpha_n(c)([\sigma]) = S_n(\sigma^* c) \tag{9}$$

whose kernel $\Lambda^3(X, \Gamma) \subseteq$ we call the *aspherical classes*.

In fact, for the natural isomorphism $\check{H}^n(X, \Gamma) \to H^n(X, \Gamma)$ ([Br97b, p. 184]), the evaluation map

$$\beta_n \colon H^n(X, \Gamma) \to \mathrm{Hom}(H_n(X), \Gamma)$$

coming from the Universal Coefficient Theorem (cf. [Br97a]) and the Hurewicz homomorphism

$$h_n \colon \pi_n(X) \to H_n(X),$$

we have

$$\alpha_n(c) = \beta_n(c) \circ h_n. \tag{10}$$

Since α_n, β_n and h_n are functorial in X, it suffices to verify this relation for $X = \mathbb{S}^n$, where it is an immediate consequence of the compatibility of both sides with the suspension map (cf. Proposition 4.3 below and the Mayer–Vietoris Sequence in singular (co)homology [Br97a, §IV.15]).

4.2 The suspension map in Čech cohomology

For each abelian topological group A, there is a suspension homomorphism

$$\mathrm{sp}_k : \check{H}^k(\mathbb{S}^n, \underline{A}) \to \check{H}^{k+1}(\mathbb{S}^{n+1}, \underline{A}).$$

To describe this map, observe that the good open cover we used for $\mathbb{S}^{n+1}$ reduces to the one defined for $\mathbb{S}^n$ by restriction to the equator $\partial F_{n+2} \cong \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. Moreover, there exists a retraction

$$p \colon \mathbb{S}^{n+1} \backslash (\mathrm{int}(F_{n+2}) \cup \{e_{n+2}\}) \to \partial F_{n+2} \cong \mathbb{S}^n, \tag{11}$$

restricting for $0 \leq j \leq n+1$ to a retraction $F_j \setminus \{e_{n+2}\} \to F_j \cap F_{n+2}$. It satisfies $p(\Lambda_j \setminus \operatorname{int}(F_{n+2})) \subseteq \Lambda_j \cap \mathbb{R}^{n+1}$ and thus maps the open subset Λ_j of $\mathbb{S}^{n+1}$ to the corresponding open subset $\Lambda_j \cap \mathbb{S}^n$.

The suspension map sp_k is defined by sending a cochain $(g_{i_0,\ldots,i_k})$ on $\mathbb{S}^n$ to the cochain $(h_{i_0,\ldots,i_{k+1}})$ on $\mathbb{S}^{n+1}$ defined by

$$h_{i_0,\ldots,i_{k+1}} := \begin{cases} g_{i_0,\ldots,i_k} \circ p & \text{for } i_0 < \ldots < i_{k+1} = n+2 \\ 0 & \text{for } n+2 \notin \{i_0,\ldots,i_{k+1}\}. \end{cases}$$

We work in this article always with totally antisymmetric Čech cocycles ([Ha77, Rem. 4.01, p. 218]), so if i_{n+1} appears in the list of indices, it may always be moved to the last position. The so defined cochain $(h_{i_0,\ldots,i_{k+1}})$ then satisfies

$$(\delta h)_{i_0,\ldots,i_{k+2}} = \begin{cases} (\delta g \circ p)_{i_0,\ldots,i_{k+1}} & \text{for } i_{k+2} = n+2 \\ 0 & \text{otherwise.} \end{cases}$$

This shows that, if g is a cocycle, then so is h, and we likewise see that, if g is a coboundary, then h is a coboundary.

Proposition 4.3. *If $A = \Gamma$ is discrete, then the diagram*

$$\begin{array}{ccc} \check{H}^n(\mathbb{S}^n, \Gamma) & \xrightarrow{\mathrm{sp}_n} & \check{H}^{n+1}(\mathbb{S}^{n+1}, \Gamma) \\ & {\scriptstyle S_n} \searrow \quad \swarrow {\scriptstyle S_{n+1}} & \\ & \Gamma & \end{array}$$

commutes and sp_n is an isomorphism. If Γ is discrete in the topological vector space $\mathfrak{z}$ and $Z = \mathfrak{z}/\Gamma$, then the diagram

$$\begin{array}{ccc} \check{H}^k(\mathbb{S}^n, \underline{Z}) & \xrightarrow{\mathrm{sp}_k} & \check{H}^{k+1}(\mathbb{S}^{n+1}, \underline{Z}) \\ \downarrow{\scriptstyle \iota_k} & & \downarrow{\scriptstyle \iota_{k+1}} \\ \check{H}^{k+1}(\mathbb{S}^n, \Gamma) & \xrightarrow{\mathrm{sp}_{k+1}} & \check{H}^{k+2}(\mathbb{S}^{n+1}, \Gamma) \end{array}$$

also commutes.

Proof. From the construction of S_n, S_{n+1} and sp_n one immediately deduces $S_{n+1} \circ \mathrm{sp}_n = S_n$. This also implies that sp_n is an isomorphism since S_n and S_{n+1} are so.

The isomorphism ι_k is defined by sending the cocycle $(g_{i_0,\ldots,i_k})$ to the cocycle $\delta(\widetilde{g})_{i_0,\ldots,i_{k+1}}$, where $\widetilde{g}_{i_0,\ldots,i_k}$ are arbitrarily chosen lifts of $g_{i_0,\ldots,i_k}$. Since $(\widetilde{g}_{i_0,\ldots,i_k} \circ p)$ lifts the image $(g_{i_0,\ldots,i_k} \circ p)$ of $g_{i_0,\ldots,i_k}$ under sp_k, the commutativity of the second diagram follows. $\square$

The above description also carries over to the case of a non-abelian topological group K. In this case it defines a map

$$\mathrm{sp}_0 \colon \check{H}^0(\mathbb{S}^n, \underline{K}) \to \check{H}^1(\mathbb{S}^{n+1}, \underline{K}), \quad \mathrm{sp}_0(g)_{ij} = \begin{cases} g_i \circ p & \text{for } j = n+2 \\ \mathbf{1} & \text{for } j < n+2. \end{cases}$$

4.3 Bundles over spheres

Definition 4.4. For connected K, elements of $\mathrm{Bun}(\mathbb{S}^n, K)$ are classified by homotopy classes $[f] \in [\mathbb{S}^n, BK] \cong \pi_n(BK)$. Since

$$\partial_n^{EK} \colon \pi_n(BK) \to \pi_{n-1}(K)$$

is an isomorphism for $n \geq 1$, the characteristic class of $P \cong f^*EK$ is given by

$$h([P]) := \partial_n^{EK}([f]) = \partial_n^P([\mathrm{id}_{\mathbb{S}^n}]) \in \pi_{n-1}(K), \tag{12}$$

where the second equality follows from Lemma 2.5.

Remark 4.5. (a) Applying Lemma 2.5 to $Y = \mathbb{S}^n$, we obtain for a K-bundle $P \to X$ the following interpretation of ∂_n^P in terms of characteristic classes of bundles over $\mathbb{S}^n$:

$$\partial_n^P([f]) = \partial_n^{f^*P}([\mathrm{id}_{\mathbb{S}^n}]) = h(f^*P) \in \pi_{n-1}(K). \tag{13}$$

(b) For $n = 2$ and $Z = \mathfrak{z}/\Gamma$, we obtain in particular for any Z-bundle Q over $\mathbb{S}^2$

$$h(Q) = \partial_2^Q([\mathrm{id}_{\mathbb{S}^2}]) \in \pi_1(Z) \cong \Gamma. \tag{14}$$

Remark 4.6. Let $n \geq 2$ and $f \colon \mathbb{S}^{n-1} \to K$ be a continuous map. Below we recall the clutching construction of a principal K-bundle $P_f \to \mathbb{S}^n$ satisfying $\partial_n^{P_f}([\mathrm{id}]) = [f]$ and $[P_f] = \mathrm{sp}_0(f)$, where $\partial_n^{P_f}$ is the connecting map from (4) and sp_0 is the suspension map.

To this end, let $\mathbb{B}^n$ be the closed unit ball in $\mathbb{R}^n$ and $\eta_j \colon \mathbb{B}^n \to \mathbb{S}^n$, $j = 0, 1$, be topological embeddings with $\eta_0(\mathbb{S}^{n-1}) = \eta_1(\mathbb{S}^{n-1})$ and for which the closed subsets $A_j := \mathrm{im}(\eta_j)$ satisfy

$$\mathbb{S}^n = A_0 \cup A_1 \quad \text{ and } \quad A_0 \cap A_1 = \eta_j(\mathbb{S}^{n-1}) \cong \mathbb{S}^{n-1}.$$

We now consider the quotient

$$P_f := (\mathbb{B}^n \times \{0, 1\} \times K)/\sim,$$

where the equivalence relation is given by $(x, j, k) \sim (x', j', k')$ if and only if $x = x'$, $j = j'$, and $k = k'$, or $x \in \mathbb{S}^{n-1}$, $\eta_0(x) = \eta_1(x)$ and $k = f(x)k'$. We write $[(x, j, k)]$ for the equivalence class of (x, j, k) in P_f. Then it is easy to see that K acts from the right on P_f by $[(x, j, k)].k' := [(x, j, kk')]$ and that $q \colon P_f \to \mathbb{S}^n, [(x, j, k)] \mapsto \eta_j(x)$ is a continuous map whose fibers are the K-orbits. Clearly, $\sigma_j(\eta_j(x)) := [(x, j, \mathbf{1})]$, $j = 1, 2$, define continuous sections on the closed subsets A_j, and on $A_0 \cap A_1$ these sections satisfy

$$\sigma_1(\eta_1(x)) = \sigma_0(\eta_0(x)).f(x) \quad \text{ for } \quad x \in \partial\mathbb{S}^{n-1}.$$

Extending the map $\eta_0(x) \mapsto f(x)$ to a neighborhood of $A_0 \cap A_1$ in A_0, the assignment $x \mapsto \sigma_0(\eta_0(x)).f(x)$ defines a continuous section on an open neighborhood of A_1 extending σ_1. This implies that P_f is locally trivial.

To verify the relation

$$\partial_n^{P_f}([\mathrm{id}_{\mathbb{S}^n}]) = [f] \in \pi_{n-1}(K), \tag{15}$$

we consider the quotient map

$$p\colon \widehat{A}_0 := \mathbb{S}^{n-1} \times I \to A_0, \quad (x,t) \mapsto \eta_0(tx)$$

collapsing $\mathbb{S}^{n-1} \times \{0\}$ to a single point. Gluing $\widehat{A}_0$ in the canonical way to A_1, we obtain a space $\widehat{\mathbb{S}}^n := \widehat{A}_0 \cup A_1$ homeomorphic to $\mathbb{B}^n$ and a map $\xi\colon \widehat{\mathbb{S}}^n \to \mathbb{S}^n$ inducing a homeomorphism $\overline{\xi}\colon \widehat{\mathbb{S}}^n/(\mathbb{S}^{n-1} \times \{0\}) \to \mathbb{S}^n$. Now

$$\widetilde{\xi}\colon \widehat{\mathbb{S}}^n \to P_f, \quad \widetilde{\xi}(x,t) \mapsto \sigma_0(\eta_0(tx)).f(x)$$

defines a continuous lift of $\overline{\xi}$. Its restriction to $\mathbb{S}^{n-1} \times \{0\}$ is given by $(x,0) \mapsto \sigma_0(\eta_0(0)).f(x)$, and this implies that $\partial_n^{P_f}([\mathrm{id}]) = [f]$.

In order to verify the relation $[P_f] = \mathrm{sp}_0(f)$, we observe that we may identify $\mathbb{S}^n$ with $\partial\Delta^{n+1}$ in such a way that F_{n+2} gets identified with A_2 and the closure of Λ_{n+1} with A_1. Then we can use the map $f \circ p$ to extend the section σ_2 to $A_1 \backslash \{e_{n+2}\}$ and the transition functions of this system of sections (restricted to the Λ_i's) has precisely $\mathrm{sp}_0(f)$ as associated Čech cocycle.

We end this section with showing that suspension and clutching are compatible in the following sense. This will be the basis for our upcoming arguments.

Proposition 4.7. *For an arbitrary abelian topological group Z, $[\widehat{K}] \in \mathrm{Ext}_c(K,Z)$ and $n \in \mathbb{N}$ the diagram*

$$\begin{array}{ccc}
\check{H}^0(\mathbb{S}^n, \underline{K}) \cong C(\mathbb{S}^n, K) & \xrightarrow[f \mapsto [P_f]]{\mathrm{sp}_0} & \check{H}^1(\mathbb{S}^{n+1}, \underline{K}) \\
\big\downarrow \delta_0^{\mathbb{S}^n} & & \big\downarrow \delta_1^{\mathbb{S}^{n+1}} \\
\check{H}^1(\mathbb{S}^n, \underline{Z}) & \xrightarrow{-\,\mathrm{sp}_1} & \check{H}^2(\mathbb{S}^{n+1}, \underline{Z})
\end{array} \tag{16}$$

commutes. Moreover, we have $\delta_0^{\mathbb{S}^n}(f) = [f^*\widehat{K}]$.

Proof. We first perform some basic constructions. Recall the covering $\Lambda_0, \ldots, \Lambda_{n+2}$ of $\mathbb{S}^{n+1} \cong \partial\Delta^{n+2}$ from Section 4.1.

1. For $\widetilde{f}\colon \mathbb{S}^n \cong \partial F_{n+2} \to K$ we set

$$f := \widetilde{f} \circ p\colon \mathbb{S}^{n+1} \backslash (\mathrm{int}(F_{n+2}) \cup \{e_{n+2}\}) \to K,$$

where p is the projection map from (11). With this we define

$$f_i := f|_{\Lambda_i \backslash \mathrm{int}(F_{n+2})} \to K.$$

2. Define $g_{ij}\colon \Lambda_{ij} \to K$ for $j = n+2$ by $g_{i\,n+2} := f_i$ and $g_{ij} = \mathbf{1}$ if $n+2 \notin \{i,j\}$. By the previous remark, these are transition functions for the bundle P_f on the cover $(\Lambda_i)_{i=0,\ldots,n+2}$.

3. Since $\Lambda_i \cap \partial F_{n+2}$ is contractible for $i < n+2$, there exist lifts $\widetilde{f_i}\colon \Lambda_i \cap \partial F_{n+2} \to \widehat{K}$. Then $\widehat{f_i} := \widetilde{f_i} \circ p$ is a lift of f_i satisfying $\widehat{f_i} = \widetilde{f_i}\Big|_{\Lambda_i \cap \partial F_{n+2}} \circ p$.

4. The maps $\widehat{g}_{i\, n+2} := \widehat{f_i}$ if $i < n+2$ and $\widehat{g}_{ij} = 1$ if $n+2 \notin \{i,j\}$ define a lifting cochain of the cocycle g_{ij}. Thus we have

$$\begin{aligned} \delta_1^{\mathbb{S}^{n+1}}(g)_{ij\, n+2} &= \widehat{g}_{ij} \cdot \widehat{g}_{j\, n+2} \cdot \widehat{g}_{i,n+2}^{-1} = \widehat{f_j} \cdot \widehat{f_i}^{-1} && \text{if } i,j < n+2 \\ \delta_1^{\mathbb{S}^{n+1}}(g)_{ijk} &= 0 && \text{if } n+2 \notin \{i,j,k\}. \end{aligned}$$

Since the cocycle (g_{ij}) represents $[P_f]$ and $\widehat{g}_{ij}$ lifts g_{ij}, the above cocycle represents $\delta_1^{\mathbb{S}^{n+1}}([P_f])$. On the other hand, $\widehat{f_i} \cdot \widehat{f_j}^{-1} \mid_{\Lambda_{ij} \cap \partial F_{n+2}}$ is a cocycle representing $\delta_0^{\mathbb{S}^n}(f)$. Thus we have

$$\begin{aligned} \big(\operatorname{sp}_1(\delta_0^{\mathbb{S}^n}(f))\big)_{ij\, n+2} &= (\delta_0^{\mathbb{S}^n}(f))_{ij} \circ p \\ &= \left(\widehat{f_i} \mid_{\Lambda_{ij} \cap \partial F_{n+2}} \circ p\right) \cdot \left(\widehat{f_j}^{-1} \mid_{\Lambda_{ij} \cap \partial F_{n+2}} \circ p\right) = \widehat{f_i} \cdot \widehat{f_j}^{-1}, \end{aligned}$$

showing our first claim. Since $\widehat{f_i}$ can be used to construct sections of $f^*\widehat{K}$ on the cover $(\Lambda_i)_{i=0,\ldots,n+2}$ it also follows that $\delta_0^{\mathbb{S}^n}(f) = [f^*\widehat{K}]$. □

Remark 4.8. The commutativity of the diagram in the previous proposition can also be understood in the language of (smooth) bundle gerbes (cf. [Mu96], [Go03], [SW10]). First we note that the diagram (16) may be rephrased in the smooth setting by considering sheaves of germs of continuous K- and Z-valued functions. The proof of the commutativity of this diagram then carries over literally to the smooth setting.

Bundle gerbes are defined analogously to the local definition of principal bundles, except that the cohomological dimension is raised by one. The role of an open trivializing cover of a smooth manifold M will be taken by an arbitrary surjective submersion[5] $Y \to M$, giving rise to the k-fold fiber products $Y^{[k]} := Y \times_M \cdots \times_M Y$ (replacing the k-fold intersections of the open cover). A principal Z-bundle over M is then a Z-valued function $z\colon Y^{[2]} \to Z$ satisfying the cocycle condition

$$\pi_{12}^* z \cdot \pi_{23}^* z = \pi_{13}^* z$$

on $Y^{[3]}$. A bundle gerbe thus consists of a principal Z-bundle $Q \to Y^{[2]}$ and an isomorphism

$$\mu\colon \pi_{12}^* Q \otimes \pi_{23}^* Q \to \pi_{13}^* Q$$

[5] In the case of a (trivializing) cover $(U_i)_{i \in I}$ of M, the surjective submersion would be $\coprod U_i \to M$.

of principal Z-bundles over $Y^{[3]}$ such that the diagram

$$\begin{array}{ccc} \pi_{12}^*Q \otimes \pi_{23}^*Q \otimes \pi_{34}^*Q & \xrightarrow{\pi_{123}^*\mu\otimes \mathrm{id}} & \pi_{13}^*Q \otimes \pi_{34}^*Q \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi_{234}^*\mu} & & \Big\downarrow{\scriptstyle \pi_{134}^*\mu} \\ \pi_{12}^*Q \otimes \pi_{24}^*Q & \xrightarrow{\pi_{124}^*\mu} & \pi_{14}^*Q \end{array} \tag{17}$$

of isomorphisms of principal Z-bundles over $Y^{[4]}$ commutes.

Bundle gerbes have characteristic classes in $\check{H}^2(M,\underline{Z})$, which can be constructed as follows. One chooses an open covering $(U_i)_{i\in I}$ such that there exist sections $\sigma_i\colon U_i \to Y$ of the submersions $Y \to M$, yielding $\sigma_{ij}\colon U_i\cap U_j \to Y^{[2]}$. Let s_{ij} be a section of $Q_{ij} := \sigma_{ij}^* Q$. On $U_i\cap U_j\cap U_k$ we thus obtain a section $\mu_*(s_{ij}, s_{jk})$ of Q_{ik} and thus a unique smooth function $\gamma_{ijk}\colon U_i\cap U_j\cap U_k \to Z$ satisfying $\mu_*(s_{ij}, s_{jk}) = s_{ik}\cdot \gamma_{ijk}$. Using (17) one easily checks that (γ_{ijk}) indeed comprises a Čech 2-cocycle whose cohomology class is independent of all choices.

Starting with a smooth function $f\colon \mathbb{S}^n \to Z$, we can now construct two bundle gerbes of $\mathbb{S}^{n+1}$. The first one is the *lifting bundle gerbe*, obtained by taking as surjective submersion the bundle $P := P_f \to \mathbb{S}^{n+1}$ from Remark 4.6 (which actually can be realized as a smooth principal K-bundle). Then $P^{[2]} \cong P\times K$ and we take $Q := P\times \widehat{K} \to P\times K$ as principal Z-bundle. With respect to the identifications $P^{[3]}\cong P\times K\times K$ we then have

$$\begin{aligned} &\pi_{12}^*(Q) = P\times \widehat{K}\times K,\\ &\pi_{23}^*(Q) = P\times K\times \widehat{K} \text{ and}\\ &\pi_{13}^*(Q) = \{(p,k,k',\widehat{k})\in P\times K\times K\times \widehat{K} : kk' = q(\widehat{k})\} \end{aligned}$$

with the corresponding natural maps to $P\times K\times K$ (projection to the second and third entry in the last case). Then $\pi_{12}^*(Q)\otimes \pi_{12}^*(Q) \cong P\times (\widehat{K}\times\widehat{K})/Z$ (for the anti-diagonal action of Z) and μ is given by

$$\mu(p,\widehat{k},\widehat{k'}) = (p, q(\widehat{k}), q(\widehat{k'}), \widehat{k}\widehat{k'})$$

The diagram (17) commutes for this choice of μ since the multiplication on $\widehat{K}$ is associative.

The second bundle gerbe is the *suspension gerbe* constructed by taking as surjective submersion the one obtained from an open cover of $\mathbb{S}^{n+1}$ by two open hemispheres U_1, U_2 intersecting in $]-\varepsilon,\varepsilon[\times \mathbb{S}^n$ (where $\mathbb{S}^n$ is identified with the equator in $\mathbb{S}^{n+1}$). Then $f^*\widehat{K}$ is a principal Z-bundle over $U_1\cap U_2$ and we obtain a Z-bundle over

$$Y^{[2]} := U_1 \sqcup (U_1\cap U_2)\sqcup (U_2\cap U_1)\sqcup U_2$$

by taking $-f^*\widehat{K}$ on $U_2\cap U_1$ and the trivial Z-bundle over U_1 and U_2. Since $Y^{[3]} = Y^{[2]}$ and since tensoring an arbitrary bundle with the trivial bundle yields the same bundle (up to a canonical isomorphism) we have that $\pi_{12}^*P \otimes \pi_{23}^*P$ and π_{13}^*P are canonically isomorphic. If we take this isomorphism as μ it clearly makes (17) commute and thus completes the specification of the second bundle gerbe.

From its construction described above, it follows that the characteristic class of the lifting bundle gerbe is given by $\delta_1^{\mathbb{S}^{n+1}}([P_f])$ and the characteristic class of the suspension bundle gerbe is given by $\mathrm{sp}_1([f^*\widehat{K}])$. Thus the commutativity of (16) implies that these two classes coincide (up to sign).

5 Central extensions by discrete groups

In Remark 2.6, we have broken up the problem to describe obstruction classes into the two cases where Z is connected or discrete. In this section we take a closer look at the discrete case $Z = D$, i.e. where $\widehat{K}$ is a covering of K. Then we have to analyze the homomorphism

$$\mathrm{obs}_P \colon \operatorname{Ext}_c(K, D) \to \check{H}^2(X, D).$$

We first take a closer look at bundles over $\mathbb{S}^2$.

Proposition 5.1. *Let K be a connected locally contractible topological group, P be a K-bundle over $\mathbb{S}^2$ and $\widehat{K}$ be a central extension of K by a discrete group D. Then*

$$S_2(\delta_1([P])) = -\partial_1^{\widehat{K}}(h(P)) \in D.$$

Proof. Let $f \colon \mathbb{S}^1 \cong \partial\Delta^2 \to K$ be a classifying map (cf. Remark 4.6), so that $h([P]) = [f]$. This means that $P \sim P_f$, and Proposition 4.7 implies that

$$\mathrm{sp}_1([f^*\widehat{K}]) = -\delta_1([P_f]),$$

so that Proposition 4.3 leads to

$$S_2(\delta_1([P_f])) = -S_1([f^*\widehat{K}]).$$

So it suffices to verify $S_1([f^*\widehat{K}]) = \partial_1^{\widehat{K}}([f])$. A cocycle representative of $[f^*\widehat{K}] = \delta_0(f)$ is obtained from lifting $f|_{\Lambda_i}$ to $\widehat{f}_i \colon \Lambda_i \to \widehat{K}$ for $i = 0, 1, 2$, where we use the open cover $(\Lambda_i)_{i=0,\ldots,3}$ of $\partial\Delta^2$ from Section 4. We may assume that $\widehat{f}_1$ agrees with $\widehat{f}_2$ on the open line segment $\Lambda_{12} =]e_1, e_2[$ and that $\widehat{f}_2$ agrees with $\widehat{f}_0$ on $\Lambda_{02} =]e_0, e_2[$. Then

$$S_1([f^*\widehat{K}]) = \widehat{f}_1 \cdot \widehat{f}_2^{-1} - \widehat{f}_0 \cdot \widehat{f}_2^{-1} + f_0 \cdot \widehat{f}_1^{-1} = \widehat{f}_0 \cdot \widehat{f}_1^{-1}$$

Note that $\widehat{f}_i \cdot \widehat{f}_j^{-1}$ is in D and thus constant. Fixing a base point in $]e_0, e_1[= \Lambda_{01}$, the lifts $\widehat{f}_1, \widehat{f}_2$ and $\widehat{f}_0$ on

$$\Lambda_1 =]e_0, e_1] \cup [e_1, e_2[, \quad \Lambda_2 =]e_1, e_2] \cup [e_2, e_0[, \quad \Lambda_0 =]e_2, e_0] \cup [e_0, e_2[$$

combine to a lift of a parametrization $[0,1] \to \partial\Delta^2 \cong \mathbb{S}^1$ starting in the base point to a curve in $\widehat{K}$. This implies that

$$\partial_1^{\widehat{K}}([f]) = (\widehat{f}_1^{-1}\widehat{f}_0) \mid_{]e_0,e_1[} = (\widehat{f}_0\widehat{f}_1^{-1}) \mid_{]e_0,e_1[} = S_1([f^*\widehat{K}]),$$

where the next to last equality follows from the fact that D is central in $\widehat{K}$. This completes the proof. ☐

Corollary 5.2. *If Z is a $K(\Gamma,1)$-group and $Q \to \mathbb{S}^2$ a Z-bundle, then the Chern class $\delta_1([Q]) \in \check{H}^2(\mathbb{S}^2,\Gamma)$ satisfies*

$$S_2(\delta_1([Q])) = -h([Q]) \in \pi_1(Z) \cong \Gamma.$$

Proof. Applying the preceding proposition with $K = Z$ and $\widehat{K} = \widetilde{Z} \cong E\Gamma$ shows the claim since ∂_1^Z implements the isomorphism $\pi_1(Z) \cong \Gamma$. ☐

The following corollary now implies Theorem 1.1(b).

Corollary 5.3. *Assume that K is connected and that $\widehat{K}$ is a central extension of K by the discrete group D. For a K-bundle $P \to X$ we then have*

$$\alpha_2(\delta_1([P])) = -\partial_1^{\widehat{K}} \circ \partial_2^P \colon \pi_2(X) \to D.$$

Proof. With Proposition 5.1, the naturality of δ_1 and Remark 4.6 we obtain

$$\begin{aligned}\alpha_2(\delta_1([P]))([\sigma]) &= S_2(\sigma^*\delta_1([P])) = S_2(\delta_1([\sigma^*P])) \\ &= -\partial_1^{\widehat{K}} h(\sigma^*P) = -\partial_1^{\widehat{K}} \circ \partial_2^P([\sigma]).\end{aligned}$$ ☐

Remark 5.4. (a) If X is 1-connected, then $H^2(X,D) \cong \mathrm{Hom}(\pi_2(X),D)$ by the Universal Coefficient Theorem and the Hurewicz Isomorphism $h_2 \colon \pi_2(X) \to H_2(X)$. Therefore Corollary 5.3 determines in this case the obstruction class completely.

(b) If X is not 1-connected and $q_X \colon \widetilde{X} \to X$ is a simply connected covering, then $\delta_1([q_X^*P]) = q_X^*\delta_1([P])$ by Lemma 2.2, and Corollary 5.3 thus leads with Lemma 2.5 to

$$\alpha_2(\mathrm{obs}_{q_X^*P}([\widehat{K}])) = -\partial_1^{\widehat{K}} \circ \partial_2^P \circ \pi_2(q_X).$$

As $\pi_2(q_X) \colon \pi_2(\widetilde{X}) \to \pi_2(X)$ is an isomorphism, (a) shows that Corollary 5.3 determines the obstruction class of the K-bundle q_X^*P over $\widetilde{X}$.

(c) (Geometric interpretation of $\partial_2^P = 0$) From (b) we derive in particular that $\mathrm{obs}_{q_X^*P}([\widehat{K}])$ vanishes if and only if $\partial_1^{\widehat{K}} \circ \partial_2^P$ vanishes. For $\widehat{K} = \widetilde{K}$ the homomorphism $\partial_1^{\widehat{K}}$ also is an isomorphism, so that q_X^*P lifts to a $\widetilde{K}$-bundle if and only if ∂_2^P vanishes.

Proposition 5.5. *If Z is $K(\Gamma,1)$-group, then the Chern class*

$$\mathrm{Ch}\colon \mathrm{Bun}(X,Z) \to \check{H}^2(X,\Gamma)$$

defines an isomorphism of abelian groups mapping

$$\mathrm{Bun}(X,Z)_0 := \{[P] \in \mathrm{Bun}(X,Z) \colon \partial_2^P = 0\}$$

onto the subgroup $\Lambda^2(X,\Gamma)$*. If* $q_X \colon \widetilde{X} \to X$ *is the universal covering map, then* $\partial_2^P = 0$ *is equivalent to the triviality of the bundle* $q_X^* P$ *on* $\widetilde{X}$*.*

Proof. We know already that Ch is an isomorphism of abelian groups (Example 2.8(a)). From $\mathrm{Ch}(P) = \mathrm{obs}^{\widetilde{Z}}([P])$ and Remark 5.4(b) we further derive that $\alpha_2(\mathrm{Ch}(P))$ vanishes if and only if $\partial_2^P = 0$. Finally, Remark 5.4(c) shows that this in turn is equivalent to the triviality of the bundle $q_X^* P$ on $\widetilde{X}$. □

Remark 5.6. Recall that for each topological space X the group $H_2(X)$ is generated by the classes obtained from continuous maps $\Sigma \to X$, where Σ is an orientable surface. This comes from the fact that the cone over each connected compact 1-manifold is a disc.

Let $q \colon \Sigma \to \mathbb{S}^2$ be a map inducing an isomorphism $H_2(q) \colon H_2(\Sigma) \to H_2(\mathbb{S}^2)$. If Σ is of genus g and obtained from gluing the sides of a $4g$-gon, then such a map can be obtained by collapsing the whole boundary of this polygon to a point, which leads to a 2-sphere. Let $\widetilde{S}_2 \colon \check{H}^2(\Sigma,\Gamma) \to \Gamma$ be the unique isomorphism with $\widetilde{S}_2 \circ q^* = S_2$, so that we can evaluate $c \in \check{H}^2(X,\Gamma)$ on the homology class defined by a continuous map $\sigma \colon \Sigma \to X$ by $\widetilde{S}_2(\sigma^* c) \in \Gamma$.

For a connected Lie group K, the classifying space BK is 1-connected, so that [Br97a, Cor. 13.16] implies that

$$\mathrm{Bun}(\Sigma, K) \cong [\Sigma, BK] \to H^2(\Sigma, \pi_2(BK)) \cong H^2(\Sigma, \pi_1(K)), \quad f \mapsto f^*[u]$$

is bijective, where

$$u \in H^2(BK, \pi_2(BK)) \cong \mathrm{End}(\pi_2(BK)) \cong \mathrm{End}(\pi_1(K))$$

is the element corresponding to $\mathrm{id}_{\pi_1(K)}$.[6] Since $H_1(\Sigma)$ is free and $H_2(\Sigma) \cong \mathbb{Z}$, the Universal Coefficient Theorem implies that

$$\mathrm{Bun}(\Sigma, K) \cong H^2(\Sigma, \pi_1(K)) \cong \mathrm{Hom}(H_2(\Sigma), \pi_1(K)) \cong \pi_1(K).$$

The preceding discussion now implies that the pullback map

$$q^* \colon \mathrm{Bun}(\mathbb{S}^2, K) \to \mathrm{Bun}(\Sigma, K)$$

is a bijection. If $h(P) \in \pi_1(K)$ is the characteristic class of the K-bundle P over Σ and $P \cong q^* Q$ for a K-bundle Q on $\mathbb{S}^2$, then

$$\partial_1^{\widehat{K}} h(P) = \partial_1^{\widehat{K}} h(Q) = -S_2(\delta_1([Q])) = -\widetilde{S}_2(\delta_1([P])).$$

[6] Here we assume that BK is homotopy equivalent to a a CW complex. This is the case for each metrizable Lie group, cf. [NSW11, Lemma 4.4]

For a general K-bundle $P \to X$ and the corresponding classifying map $f\colon X \to BK$, we thus obtain for a continuous map $\sigma\colon \Sigma \to X$:

$$\begin{aligned}\widetilde{S}_2(\sigma^*\delta_1([P])) &= \widetilde{S}_2(\delta_1([\sigma^*P])) = -\partial_1^{\widehat{K}} h(\sigma^*P) \\ &= -\partial_1^{\widehat{K}} \circ \partial_2^{EK}[f \circ \sigma] = -\partial_1^{\widehat{K}} \circ H_2(f)[\sigma].\end{aligned}$$

This means that the homomorphism $H_2(X) \to \pi_1(K)$ defined by the obstruction class $\delta_1([P]) \in H^2(X, \Gamma)$ coincides for $P \sim f^*EK$ with $-\partial_1^{\widehat{K}} \circ H_2(f)$.

A degree zero analog

As a byproduct of Proposition 5.1, we obtain the following analog of Theorem 1.1, resp., (19), for the connecting map δ_0. For a based map $f \in C(X, K) = \check{H}^0(X, \underline{K})$, we think of the maps

$$\pi_n(f)\colon \pi_n(X) \to \pi_n(K)$$

as analogs of the connecting maps $\partial_n^P\colon \pi_n(X) \to \pi_{n-1}(K)$ associated to $[P] \in \check{H}^1(X, \underline{K})$ and recall that $\delta_0([f]) \in \check{H}^1(X, \underline{Z}) \cong \check{H}^2(X, \Gamma)$ corresponds to the Z-bundle $f^*\widehat{K}$ over X.

Proposition 5.7. *For $f \in C(X, K)$ and the central extension $\widehat{K}$ of K by the $K(\Gamma, 1)$-group Z, we have*

$$\partial_2^{\widehat{K}} \circ \pi_2(f) = -\alpha_2(\iota_1(\delta_0(f)))\colon \pi_2(X) \to \Gamma. \tag{18}$$

Proof. Corollary 5.2 implies that the negative of

$$\alpha_2(\iota_1(\delta_0(f)))([\sigma]) = \alpha_2(\iota_1([f^*\widehat{K}]))([\sigma]) = S_2(\iota_1([\sigma^* f^*\widehat{K}])) = S_2(\delta_1([\sigma^* f^*\widehat{K}]))$$

(note that $\iota_1 = \delta_1$ in this case) equals

$$h([\sigma^* f^*\widehat{K}]) = \partial_2^{\widehat{K}}([f \circ \sigma]) = \partial_2^{\widehat{K}}(\pi_2(f)([\sigma])).$$

This proves (18). ∎

6 Proof of Theorem 1.1

The first part in the proof of Theorem 1.1 is to reduce the problem to the case $X = \mathbb{S}^3$. Our goal is to show the following equality of two group homomorphisms

$$\partial_2^{\widehat{K}} \circ \partial_3^P = \alpha_3(\iota_2(\delta_1([P])))\colon \pi_3(X) \to \Gamma. \tag{19}$$

Lemma 6.1. *If (19) holds for $X = \mathbb{S}^3$, then it holds for arbitrary X.*

Proof. For a continuous map $\sigma\colon \mathbb{S}^3 \to X$, we obtain with the naturality of δ_1 and ι_2:

$$\begin{aligned}\alpha_3(\iota_2^X(\delta_1^X([P])))([\sigma]) &= S_3(\sigma^*\iota_2^X(\delta_1^X([P]))) \\ &= S_3\big(\iota_2^{\mathbb{S}^3}(\delta_1^{\mathbb{S}^3}([\sigma^*P]))\big) = \alpha_3\big(\iota_2^{\mathbb{S}^3}(\delta_1^{\mathbb{S}^3}([\sigma^*P]))\big)([\mathrm{id}_{\mathbb{S}^3}]).\end{aligned}$$

If (19) holds for $\mathbb{S}^3$, then

$$\partial_2^{\widehat{K}} \circ \partial_3^{\sigma^*P} = \alpha_3(\iota_2^{\mathbb{S}^3}(\delta_1^{\mathbb{S}^3}([\sigma^*P])))\colon \pi_3(\mathbb{S}^3) \cong \mathbb{Z} \to \Gamma,$$

and we obtain with Lemma 2.5

$$\begin{aligned}\alpha_3(\iota_2^X(\delta_1^X([P])))([\sigma]) &= \alpha_3\big(\iota_2^{\mathbb{S}^3}(\delta_1^{\mathbb{S}^3}([\sigma^*P]))\big)([\mathrm{id}_{\mathbb{S}^3}]) \\ &= \partial_2^{\widehat{K}} \circ \partial_3^{\sigma^*P}([\mathrm{id}_{\mathbb{S}^3}]) = \partial_2^{\widehat{K}} \circ \partial_3^{P}([\sigma]),\end{aligned}$$

which implies (19). □

In view of Lemma 6.1, it remains to show (19) for $X = \mathbb{S}^3$. Since $\pi_3(\mathbb{S}^3)$ is cyclic generated by the homotopy class of $\mathrm{id}_{\mathbb{S}^3}$, we have to verify for every K-bundle P over $\mathbb{S}^3$ the relation

$$\partial_2^{\widehat{K}} \circ \partial_3^{P}([\mathrm{id}_{\mathbb{S}^3}]) = \partial_2^{\widehat{K}}(h(P)) \overset{!}{=} S_3(\iota_2(\delta_1([P]))) = \alpha_3(\iota_2(\delta_1([P])))([\mathrm{id}_{\mathbb{S}^3}]).$$

We address this problem by verifying it for the bundles P_f associated to a continuous map $f\colon \mathbb{S}^2 \to K$ in such a way that $h([P_f]) = [f]$. Since each K-bundle on $\mathbb{S}^3$ arises from this construction, this will prove our theorem. In view of

$$\partial_2^{\widehat{K}}([f]) = \partial_2^{f^*\widehat{K}}([\mathrm{id}_{\mathbb{S}^2}]) = h([f^*\widehat{K}]) = -S_2(\delta_1([f^*\widehat{K}])) = -S_2(\iota_1([f^*\widehat{K}])),$$

(Corollary 5.2) and $[f^*\widehat{K}] = \delta_0^{\mathbb{S}^2}(f)$ (Proposition 4.7) we have to verify that

$$-S_2(\iota_1(\delta_0^{\mathbb{S}^2}(f))) = S_3(\iota_2(\delta_1^{\mathbb{S}^3}([P_f]))). \tag{20}$$

From Propositions 4.3 we know that

$$S_3 \circ \mathrm{sp}_2 = S_2, \quad \mathrm{sp}_2 \circ \iota_1 = \iota_2 \circ \mathrm{sp}_1 \quad \text{and} \quad \mathrm{sp}_0(f) = [P_f].$$

With Proposition 4.7, we thus obtain

$$\iota_2\delta_1^{\mathbb{S}^3}([P_f]) = \iota_2\delta_1^{\mathbb{S}^3}\,\mathrm{sp}_0(f) = -\iota_2(\mathrm{sp}_1(\delta_0^{\mathbb{S}^2}(f)) = -\,\mathrm{sp}_2(\iota_1(\delta_0^{\mathbb{S}^2}(f))).$$

Applying S_3 now completes the proof of Theorem 1.1.

7 Aspherical obstruction classes

In this section we study those obstruction classes which cannot be detected by the corresponding homomorphisms on $\pi_2(X)$, resp., $\pi_3(X)$. These are the classes contained in the subgroups $\Lambda^k(X,\Gamma) = \ker \alpha_k \subseteq \check{H}^k(X,\Gamma)$ of those cohomology classes inducing the trivial homomorphism $\pi_k(X) \to \Gamma$ $(k = 2, 3)$. The Universal Coefficient Theorem (cf. [Br97a]) leads to the short exact sequence

$$0 \to \mathrm{Ext}(H_{k-1}(X), \Gamma) \to H^k(X, \Gamma) \xrightarrow{\beta_k} \mathrm{Hom}(H_k(X), \Gamma) \to 0$$

and the map β_k satisfies

$$\alpha_k(c) = \beta_k(c) \circ h_k \colon \pi_k(X) \to \Gamma \quad \text{for} \quad c \in H^k(X, \Gamma).$$

The image $\Sigma_k(X) := h_k(\pi_k(X))$ is called the *subgroup of spherical cycles* in $H_k(X)$. Note that cohomology classes coming from elements of $\mathrm{Ext}(H_{k-1}(X), \Gamma)$ are in particular contained in $\Lambda^k(X,\Gamma)$ and that this inclusion may be proper. For $k = 2$ Hopf's Theorem asserts that $\Lambda^2(X,\Gamma) \cong H^2_{\mathrm{grp}}(\pi_1(X), \Gamma)$. For an interpretation of this result in terms of obstruction classes see Theorem 7.9 below.

7.1 Central extensions by discrete groups

If K is connected, then ∂_1^P is always trivial. Therefore we first take a closer look at the set

$$\mathrm{Bun}(X, K)_0 := \{[P] \in \mathrm{Bun}(X, K) \colon \partial_2^P = 0\}$$

because the vanishing of ∂_1^P and ∂_2^P implies that the corresponding obstruction classes are aspherical (cf. Theorem 1.1). It turns out that, for every $[P] \in \mathrm{Bun}(X, K)_0$, the fundamental group $\pi_1(P)$ is a central extension of $\pi_1(X)$ by $\pi_1(K)$, and that the class $[\pi_1(P)] \in H^2_{\mathrm{grp}}(\pi_1(X), \pi_1(K))$ coincides, up to sign, with the lifting obstruction $\mathrm{obs}^{\widetilde{K}}([P])$. This observation complements Theorem 1.1 and leads to a description of all obstruction classes associated to flat central extensions of K by $K(\Gamma, 1)$-groups as elements of $\mathrm{Ext}(H_2(X), \Gamma) \subseteq H^3(X, \Gamma)$ (Theorem 7.12).

Suppose that X is connected and let $q_X : \widetilde{X} \to X$ denote a universal covering. We identify $\pi_1(X)$ with the group of deck transformations of X and view $\widetilde{X}$ as a $\pi_1(X)$-principal bundle. In this section K denotes a connected locally contractible topological group.

Remark 7.1. (a) If $\pi_2(X)$ or $\pi_1(K)$ vanishes, then $\mathrm{Bun}(X, K) = \mathrm{Bun}(X, K)_0$. This is in particular the case if X is *aspherical*, i.e., if $\pi_k(X)$ vanishes for $k \geq 2$.

Typical examples of aspherical spaces are: connected surfaces of positive genus, tori, solvmanifolds (the covering is a simply connected solvable Lie group) and hyperbolic manifolds.

(b) Suppose that $q_X^* P$ is trivial. Since the homomorphisms $\pi_n(q_X) \colon \pi_n(\widetilde{X}) \to \pi_n(X)$ are isomorphisms for $n > 1$, all the connecting maps $\partial_n^P \colon \pi_n(X) \to$

$\pi_{n-1}(K)$ vanish. For $n = 1$ we have a homomorphism $\partial_1^P \colon \pi_1(X) \to \pi_0(K)$ which is also trivial if K is connected. For a connected group K, we thus obtain K-bundles P for which all connecting maps ∂_d^P vanish. In view of Theorem 1.1, the corresponding obstruction classes are aspherical.

The following lemmas show that the obstruction classes defined by central extensions of $\pi_1(X)$ are aspherical.

Lemma 7.2. *If K is discrete and $P \to X$ a principal K-bundle, then*

$$\mathrm{obs}_P(\mathrm{Ext}_c(K, Z)) \subseteq \Lambda^2(X, Z)$$

for every abelian group Z. In particular,

$$\mathrm{obs}_{\widetilde{X}}(H^2_{\mathrm{grp}}(\pi_1(X), Z)) \subseteq \Lambda^2(X, Z).$$

Proof. We have to show that $\sigma^* \mathrm{obs}_P([\widehat{K}])$ vanishes for every continuous map $\sigma \colon \mathbb{S}^2 \to X$. In view of the naturality of the obstruction class,

$$\sigma^* \mathrm{obs}_P([\widehat{K}]) = \mathrm{obs}_{\sigma^* P}([\widehat{K}]),$$

and since the K-bundle $\sigma^* P \to \mathbb{S}^2$ is trivial because K is discrete and $\mathbb{S}^2$ is simply connected, all corresponding obstruction classes vanish. □

We now turn the universal covering space of a K-principal bundle into a principal bundle for a suitable extension of K by the discrete group $\pi_1(P)$.

Lemma 7.3. *Let $q \colon P \to X$ be a connected K-bundle and $\alpha \colon \pi_1(K) \to \pi_1(P)$ be the homomorphism defined by the choice of a base point in P. Then*

$$K^\sharp := (\widetilde{K} \times \pi_1(P))/\{(d, \alpha(d)^{-1}) \colon d \in \pi_1(K)\}$$

is an extension of K by the discrete group $\pi_1(P)$ with $\pi_0(K^\sharp) \cong \pi_1(P)/\mathrm{im}(\alpha) \cong \pi_1(X)$ and the universal covering space $\widetilde{P}$ of P carries a natural $K^\sharp$-bundle structure such that $P \sim \widetilde{P}/\pi_1(P)$ is an equivalence of K-bundles.

Proof. From basic covering theory we know that the K-action on P lifts to a $\widetilde{K}$-action on the universal covering space $\widetilde{P}$. Here the action of $\pi_1(K) \cong \ker q_K$ is given by the homomorphism $\alpha \colon \pi_1(K) \to \pi_1(P)$ obtained from the orbit map of a base point $p_0 \in P$, where we identify $\pi_1(P) \subseteq \mathrm{Homeo}(\widetilde{P})$ with the group of deck transformations of the covering $q_P \colon \widetilde{P} \to P$. In particular, $D_0 := \ker \alpha$ acts trivially on $\widetilde{P}$ and the group $\widetilde{K}/D_0$ acts faithfully on $\widetilde{P}$.

Since $\widetilde{K}$ is connected and normalizes the subgroup $\pi_1(P) \subseteq \mathrm{Homeo}(\widetilde{P})$ whose orbits are discrete, it actually centralizes $\pi_1(P)$. We thus obtain an action of the group $K^\sharp$ because the discrete subgroup $\{(d, \alpha(d)^{-1}) \colon d \in \pi_1(K)\}$ acts trivially.

If $U \subseteq X$ is an open 1-connected subset for which we have a continuous section $\sigma \colon U \to P$, then σ also lifts to a continuous map $\widetilde{\sigma} \colon U \to \widetilde{P}$. Now the action map leads to a local homeomorphism

$$U \times K^\sharp \to q_P^{-1}(U) \subseteq \widetilde{P}, \quad (x, k) \mapsto \widetilde{\sigma}(x)k,$$

and since $\pi_1(P)$ acts freely on $\widetilde{P}$, this actually is a $K^\sharp$-equivariant homeomorphism. Therefore $\widetilde{P}$ is a $K^\sharp$-principal bundle and $q_P \colon \widetilde{P} \to P$ induces an equivalence $\widetilde{P}/\pi_1(P) \to P$ of K-bundles. □

Remark 7.4. (a) From the long exact homotopy sequence of the K-bundle P we obtain the exact sequence

$$\pi_2(X) \xrightarrow{\partial_2^P} \pi_1(K) \xrightarrow{\alpha} \pi_1(P) \to \pi_1(X) \to \pi_0(K) = \{\mathbf{1}\}, \tag{21}$$

so that $\partial_2^P = 0$ is equivalent to α being injective, resp., to $\widetilde{P}$ carrying the structure of a $\widetilde{K}$-principal bundle.

(b) Since the $\widetilde{K}$-action on $\widetilde{P}$ commutes with $\pi_1(P)$, the subgroup $\alpha(\pi_1(K)) \subseteq \pi_1(P)$ is central in $\pi_1(P)$. In particular, $\pi_1(P)$ is a central extension of $\pi_1(X)$ by $\operatorname{im}(\alpha)$, and if ∂_2^P vanishes, then $[\pi_1(P)] \in H^2_{\mathrm{grp}}(\pi_1(X), \pi_1(K))$.

(c) Let $K^+ := (K^\sharp)_0 \cong \widetilde{K}/\operatorname{im}(\partial_2^P)$ be the identity component of $K^\sharp$. Since K^+ is connected and $\widetilde{P}$ is simply connected, the long exact homotopy sequence of the K^+-bundle $\widetilde{P} \to \widetilde{P}/K^+$ implies that this quotient is simply connected. It also is a $\pi_1(X)$-bundle and a covering of X, hence a model for the universal covering space $\widetilde{X}$ of X.

(d) If the homomorphism $\pi_2(q) \colon \pi_2(P) \to \pi_2(X)$ vanishes, then (21) defines an exact four term sequence

$$\{0\} \to \pi_2(X) \xrightarrow{\partial_2^P} \pi_1(K) \xrightarrow{\alpha} \pi_1(P) \to \pi_1(X) \to \{\mathbf{1}\}. \tag{22}$$

This is the four terms sequence defined by the crossed module obtained from α and the trivial action of $\pi_1(P)$ on $\pi_1(K)$. In particular, the cohomology class of this crossed module leads to an element of $H^3_{\mathrm{grp}}(\pi_1(X), \pi_2(X))$ which can only be nonzero if $\partial_2^P \neq 0$.

Lemma 7.5. *Let $q_j \colon P_j \to X$ be K_j-bundles for $j = 1, 2$. If $\widehat{K}_j$, $j = 1, 2$, are central Z-extensions of K_j and $\mu_Z \colon Z \times Z \to Z$ is the multiplication map, then*

$$Q := (\mu_Z)_*(\widehat{K}_1 \times \widehat{K}_2)$$

is a central extension of $K_1 \times K_2$ by Z. For the obstruction class of the $K_1 \times K_2$-bundle $P_1 \times_X P_2$ over X we then have

$$\operatorname{obs}_{P_1 \times_X P_2}([Q]) = \operatorname{obs}_{P_1}([\widehat{K}_1]) + \operatorname{obs}_{P_2}([\widehat{K}_2]) \in \check{H}^2(X, \underline{Z}).$$

Proof. This is an immediate consequence of the definition of the obstruction class. □

The following proposition is a key tool which relates certain obstruction classes for general connected group K to obstruction classes for central extensions of $\pi_1(X)$.

Proposition 7.6. *If K is connected, $P \to X$ a K-bundle, $D := \mathrm{im}(\alpha) \subseteq \pi_1(P)$ and $K^+ := \widetilde{K}/\mathrm{im}(\partial_2^P)$, then*

$$\mathrm{obs}_P([K^+]) = -\,\mathrm{obs}_{\widetilde{X}}([\pi_1(P)]) \in \Lambda^2(X, D).$$

If $\partial_2^P = 0$, then $D = \pi_1(K)$ and we have in particular

$$\mathrm{obs}_P([\widetilde{K}]) = -\,\mathrm{obs}_{\widetilde{X}}([\pi_1(P)]) \in \Lambda^2(X, \pi_1(K)).$$

Proof. We apply Lemma 7.5 with $Z = D$, $K_1 = K$, $\widehat{K}_1 = K^+$, $P_1 = P$, $K_2 = \pi_1(X)$, $\widehat{K}_2 = \pi_1(P)$ and $P_2 = \widetilde{X}$ (Remark 7.4(b)). Then $Q \cong K^\sharp$ as a central extension of $K \times \pi_1(X)$ by D (Lemma 7.3).

As $\widetilde{P}/K^+ \sim \widetilde{X}$ as $\pi_1(X)$-bundles and $\widetilde{P}/\pi_1(P) \sim P$ as K-bundles, it follows that

$$\widetilde{P}/D \sim P \times_X \widetilde{X} = q_X^* P$$

as $(K \times \pi_1(X))$-bundle. In particular, the K-bundle $q_X^* P$ lifts to a K^+-bundle, so that

$$0 = \mathrm{obs}_{q_X^* P}([K^+]) = \mathrm{obs}_P([K^+]) + \mathrm{obs}_{\widetilde{X}}([\pi_1(P)]),$$

where the second equality follows from Lemma 7.5. □

We now have all the tools to give a complete description of the aspherical obstruction classes for the case where $Z = D$ is discrete. If X is 1-connected, then this has already been done in Corollary 5.3 and the subsequent Remark 5.4.

Proposition 7.7. *Suppose that K is connected, D is a discrete abelian group, $\widehat{K}$ is a central extension of K by D and that P is a K-bundle over X. Then $\widehat{K} \cong \gamma_* \widetilde{K}$ for $\gamma = \partial_1^{\widehat{K}}$ and $\delta_1([P])$ is aspherical if and only if $\mathrm{im}(\partial_2^P) \subseteq \ker \gamma$. In this case the homomorphism $\gamma \colon \pi_1(K) \to D$ factors through a homomorphism $\overline{\gamma} \colon \mathrm{coker}(\partial_2^P) \to D$ and*

$$\mathrm{obs}_P([\widehat{K}]) = -\overline{\gamma}_* \,\mathrm{obs}_{\widetilde{X}}([\pi_1(P)]) \in \Lambda^2(X, D),$$

where $\pi_1(P)$ is considered as a central extension of $\pi_1(X)$ by $\mathrm{im}(\partial_2^P)$.

Proof. From Remark 3.1(a) we know that $\widehat{K} \cong \gamma_* \widetilde{K}$ for some $\gamma = \partial_1^{\widehat{K}} \in \mathrm{Hom}(\pi_1(K), D)$. Since

$$\alpha_2(\delta_1([P])) = -\partial_1^{\widehat{K}} \circ \partial_2^P = -\gamma \circ \partial_2^P$$

by Theorem 1.1(b), the obstruction class $\delta_1([P])$ is aspherical if and only if $\mathrm{im}(\partial_2^P) \subseteq \ker \gamma$. In this case we obtain $\overline{\gamma}$ by factorization of γ, and for $K^+ := \widetilde{K}/\mathrm{im}(\partial_2^P)$ we obtain $\widehat{K} \cong \overline{\gamma}_* K^+$. Therefore we can use Proposition 7.6 to obtain

$$\mathrm{obs}_P([\widehat{K}]) = \mathrm{obs}_P([\overline{\gamma}_* K^+]) = \overline{\gamma}_* \,\mathrm{obs}_P([K^+]) = -\overline{\gamma}_* \,\mathrm{obs}_{\widetilde{X}}([\pi_1(P)]).$$ □

Remark 7.8. Keeping the K-bundle $P \to X$ fixed, the preceding proposition can be used as follows to decide if two central D-extensions $\widehat{K}_1$ and $\widehat{K}_2$ have the same obstruction class. A necessary condition is that

$$\partial_1^{\widehat{K}_1} \circ \partial_2^P = -\alpha_2(\mathrm{obs}_P([\widehat{K}_1])) = -\alpha_2(\mathrm{obs}_P([\widehat{K}_2])) = \partial_1^{\widehat{K}_2} \circ \partial_2^P.$$

If this condition is satisfied, then the central extension $[\widehat{K}_3] := [\widehat{K}_1] - [\widehat{K}_2]$ has an aspherical extension class which can be computed with Proposition 7.7.

The following theorem provides an obstruction theoretic version of Hopf's Theorem (cf. Example 2.8(c)).

Theorem 7.9. *For every abelian group D,*

$$\mathrm{obs}_{\widetilde{X}} \colon H^2_{\mathrm{grp}}(\pi_1(X), D) \to \Lambda^2(X, D)$$

is an isomorphism of abelian groups.

Proof. From Lemma 7.2 we know that $\mathrm{im}(\mathrm{obs}_{\widetilde{X}})) \subseteq \Lambda^2(X, D)$.

To see that $\mathrm{obs}_{\widetilde{X}}$ is injective, suppose that $\mathrm{obs}_{\widetilde{X}}([\widehat{\pi}_1(X)]) = 0$, i.e., that the $\pi_1(X)$-bundle $\widetilde{X}$ lifts to a $\widehat{\pi}_1(X)$-bundle $\widehat{q} \colon \widehat{X} \to X$ with $\widehat{X}/D \cong \widetilde{X}$. Since $\widehat{q}$ is a covering of X, it is associated to $\widetilde{X}$ by a homomorphism $\gamma \colon \pi_1(X) \to \widehat{\pi}_1(X)$, and this homomorphism splits the extension $\widehat{\pi}_1(X)$ of $\pi_1(X)$. Hence $\mathrm{obs}_{\widetilde{X}}$ is injective.

Now we show that $\mathrm{obs}_{\widetilde{X}}$ is surjective. Let Z be a $K(D,1)$-group. In view of Proposition 5.5, it suffices to show that every Chern class $\mathrm{Ch}(P)$, $[P] \in \mathrm{Bun}(X, Z)_0$, is contained in the range of $\mathrm{obs}_{\widetilde{X}}$. In view of $\mathrm{Ch}(P) = \mathrm{obs}_P([\widetilde{Z}])$ (Example 2.8(a)), this follows from Proposition 7.6. ☐

Corollary 7.10. *If K is connected, then we have a sequence of maps*

$$\mathrm{Bun}(X, \widetilde{K})_0 \to \mathrm{Bun}(X, K)_0 \xrightarrow{\zeta} H^2_{\mathrm{grp}}(\pi_1(X), \pi_1(K)), \quad \zeta([P]) = [\pi_1(P)], \tag{23}$$

which is exact in the sense that $\zeta([P]) = 0$ is equivalent to existence of a lift of P to a $\widetilde{K}$-bundle.

Proof. From Proposition 7.6 we know that $\mathrm{obs}_P([\widetilde{K}]) = -\mathrm{obs}_{\widetilde{X}}([\pi_1(P)])$, so that the assertion follows from Theorem 7.9. ☐

For abelian groups, the preceding observation can be refined as follows.

Proposition 7.11. *For a $K(\Gamma,1)$-group Z, the following assertions hold:*

(i) *Any $[P] \in \mathrm{Bun}(X, Z)_0$ satisfies $\mathrm{Ch}(P) = -\mathrm{obs}_{\widetilde{X}}([\pi_1(P)])$.*

(ii) *$\zeta \colon \mathrm{Bun}(X, Z)_0 \to H^2_{\mathrm{grp}}(\pi_1(X), \Gamma), \zeta([P]) = [\pi_1(P)]$ is an isomorphism of abelian groups.*

Proof. (i) follows from $\mathrm{Ch}(P) = \mathrm{obs}_P([\widetilde{Z}])$ (Example 2.8(a)) and Proposition 7.6.

(ii) From (i) we derive that

$$\mathrm{obs}_{\widetilde{X}} \circ \zeta = -\mathrm{Ch} \colon \mathrm{Bun}(X, Z)_0 \to \Lambda^2(X, D).$$

Since Ch and $\mathrm{obs}_{\widetilde{X}}$ are isomorphisms by Proposition 5.5 and Theorem 7.9, ζ also is an isomorphism. ☐

7.2 Flat central extensions

After the detailed discussion of central extensions by discrete groups, we now turn to the larger class of flat central exensions. Here the case where Z is a $K(\Gamma, 1)$-group leads to aspherical obstruction classes in $\Lambda^3(X, \Gamma)$. We start with the case of bundles P for which ∂_2^P vanishes and discuss general bundles over 1-connected spaces in the following subsection.

If $\widehat{K} = \gamma_* \widetilde{K}$ is a flat central extension of K by Z and $[P] \in \mathrm{Bun}(X, K)_0$, then we can use Proposition 7.7 and the naturality of the obstruction class (Lemma 2.2(c)) to obtain the relation

$$\mathrm{obs}_P([\widehat{K}]) = \gamma_* \, \mathrm{obs}_P([\widetilde{K}]) = -\gamma_* \, \mathrm{obs}_{\widetilde{X}}([\pi_1(P)]).$$

To evaluate this formula, we first recall that $\mathrm{obs}_{\widetilde{X}}([\pi_1(P)]) \in \check{H}^2(X, \pi_1(K))$ and that γ_* denotes the natural map

$$\gamma_* \colon \check{H}^2(X, \pi_1(K)) \to \check{H}^2(X, \underline{Z}) \cong \check{H}^3(X, \Gamma).$$

Writing $\iota^Z = \mathrm{id}_Z \colon Z_d \to Z$ for the continuous bijection, where Z_d denotes the discrete group Z, then γ can be factorized as $\gamma = \iota^Z \circ \gamma_d$, which accordingly leads to $\gamma_* = \iota^Z_* \circ (\gamma_d)_*$ with

$$(\gamma_d)_* \colon H^2(X, \pi_1(K)) \to H^2(X, Z) \quad \text{and} \quad \iota^Z_* \colon H^2(X, Z) \to H^3(X, \Gamma).$$

If Z is a $K(\Gamma, 1)$-group, then the functors $\mathrm{Ext}(\cdot, Z)$ and $\mathrm{Ext}(\cdot, \widetilde{Z})$ vanish, so that the Universal Coefficient Theorem implies that

$$H^2(X, Z) \cong \mathrm{Hom}(H_2(X), Z) \quad \text{and} \quad H^2(X, \widetilde{Z}) \cong \mathrm{Hom}(H_2(X), \widetilde{Z}).$$

Accordingly, the connecting map $\delta_2 \colon H^2(X, Z) \to H^3(X, \Gamma)$ leads to the injection

$$\mathrm{Ext}(H_2(X), \Gamma) \cong \mathrm{im}(\delta_2) \hookrightarrow H^3(X, \Gamma)$$

from the Universal Coefficient Theorem. This implies that

$$\mathrm{im}(\iota^Z_*) = \mathrm{Ext}(H_2(X), \Gamma) \subseteq H^3(X, \Gamma).$$

We thus arrive that the following theorem complementing Theorem 1.1 by providing similar information on aspherical obstruction classes for $\partial_2^P = 0$.

Theorem 7.12. *If K is connected, $[P] \in \mathrm{Bun}(X, K)_0$, Z is a $K(\Gamma, 1)$-group and $\widehat{K} = \gamma_* \widetilde{K}$ a flat extension of K by Z, then the corresponding obstruction class is given by*

$$\mathrm{obs}_P([\widehat{K}]) = -\iota^Z_* \, \mathrm{obs}_{\widetilde{X}}((\gamma_d)_*[\pi_1(P)]) \in \mathrm{Ext}(H_2(X), \Gamma) \subseteq H^3(X, \Gamma).$$

Remark 7.13. If the abelian group $\pi_1(K)$ is finitely generated, which is the case for a finite dimensional Lie group K, then $\pi_1(K) \cong \mathbb{Z}^d \oplus F$ for some $d \in \mathbb{N}_0$ and a finite group F. For $Z = \mathbb{T}$ and $\Gamma = \mathbb{Z}$ we thus obtain a homomorphism

$$\overline{\xi} \colon \mathrm{Ext}(\pi_1(K), \mathbb{Z}) = \mathrm{Ext}(F, \mathbb{Z}) \cong \mathrm{Hom}(F, \mathbb{T}) \cong F \to H^3(X, \Gamma).$$

Note that, for every finite CW-complex X, the groups $H_k(X)$ are finitely generated, so that

$$H^3(X, \mathbb{Z}) \cong \mathrm{Ext}(H_2(X), \mathbb{Z}) \oplus \mathrm{Hom}(H_3(X), \mathbb{Z}),$$

where $\mathrm{Hom}(H_3(X), \mathbb{Z})$ is free and

$$\begin{aligned}\mathrm{Tor}(H^3(X, \mathbb{Z})) &\cong \mathrm{Ext}(H_2(X), \mathbb{Z}) \cong \mathrm{Ext}(\mathrm{Tor}(H_2(X)), \mathbb{Z}) \\ &\cong \mathrm{Hom}(\mathrm{Tor}(H_2(X)), \mathbb{T}) \cong \mathrm{Tor}(H_2(X))\end{aligned}$$

is finite.

Remark 7.14. (Flat bundles) (a) Let $\gamma\colon \pi_1(X) \to K$ be a group homomorphism and

$$P^\gamma := \gamma_* \widetilde{X} = (\widetilde{X} \times K)/\pi_1(X)$$

denote the corresponding flat bundle, which is the quotient of $\widetilde{X} \times K$ by the right action of $\pi_1(X)$ by $(x, k).d = (xd, \gamma(d)^{-1}k)$. Then $\pi_1(P^\gamma)$ is a central extension of $\pi_1(X)$ by $\pi_1(K)$, and since $\widetilde{P^\gamma} \cong \widetilde{X} \times \widetilde{K}$, it follows that

$$\pi_1(P^\gamma) \cong \{(d, \widetilde{k}) \in \pi_1(X) \times \widetilde{K} \colon \gamma(d) = q_K(\widetilde{k})\} = \gamma^* \widetilde{K},$$

as a central extension of $\pi_1(X)$ by $\pi_1(K)$. In view of Corollary 7.10, the bundle P lifts to a $\widetilde{K}$-bundle over X if and only if $\zeta([P^\gamma]) = [\pi_1(P^\gamma)]$ vanishes, which in turn is equivalent to the existence of a lift $\widetilde{\gamma}\colon \pi_1(X) \to \widetilde{K}$. In particular, any flat K-bundle that lifts to some $\widetilde{K}$-bundle lifts to a flat $\widetilde{K}$-bundle.

(b) Let Z be a $K(\Gamma, 1)$-group. If $P^\gamma \to X$ is a flat Z-bundle defined by a homomorphism $\gamma\colon \pi_1(X) \to Z$, then the Hurewicz Theorem implies that γ factors through a homomorphism

$$\overline{\gamma}\colon H_1(X) \cong \pi_1(X)/(\pi_1(X), \pi_1(X)) \to Z.$$

Therefore

$$\zeta(P^\gamma) = [\pi_1(P^\gamma)] = \gamma^* \widetilde{Z}$$

lies in the subgroup $\mathrm{Ext}(H_1(X), \Gamma) \subseteq H^2(X, \Gamma)$, parametrizing those central extensions pulled back from abelian extensions of $H_1(X)$ by the quotient map $h_1\colon \pi_1(X) \to H_1(X)$. Since $\widetilde{Z}$ is divisible, $\mathrm{Ext}(H_1(X), \widetilde{Z})$ vanishes, and therefore the exact Hom-Ext sequence for abelian groups shows that the connecting map

$$\delta_1\colon\ \mathrm{Hom}(H_1(X), Z) \to \mathrm{Ext}(H_1(X), \Gamma)$$

is surjective, i.e., every class in $\mathrm{Ext}(H_1(X), \Gamma)$ can be represented by a homomorphism $\gamma\colon \pi_1(X) \to Z$ as $[\overline{\gamma}^* \widetilde{Z}]$. This implies that

$$\mathrm{Ch}(\mathrm{Bun}(X, Z)_{\mathrm{flat}}) = \mathrm{Ext}(H_1(X), \Gamma) \subseteq \mathrm{Ch}(\mathrm{Bun}(X, Z)_0) = \Lambda^2(X, \Gamma).$$

7.3 Aspherical classes for 1-connected spaces

If X is 1-connected, then $\Lambda^2(X,\Gamma) \cong H^2_{\rm grp}(\pi_1(X),\Gamma)$ vanishes. Therefore we only have non-zero aspherical obstruction classes if Z is not discrete. Let us first recall the structure of $\Lambda^3(X,\Gamma)$.

Remark 7.15. (a) If $\pi_2(X)$ vanishes, then it follows from [EML45, Thm. II] that

$$H_3(X)/\Sigma_3(X) \cong H_3(\pi_1(X))$$

is the third homology group of $\pi_1(X)$. If, in addition, X is 2-connected, then $h_3\colon \pi_3(X) \to H_3(X)$ is an isomorphism and $H_2(X)$ vanishes. Therefore $\Lambda^3(X,\Gamma) = \{0\}$.

(b) If X is 1-connected, then [EML45, Thm. II^m, p. 509] shows that

$$H_3(X)/\Sigma_3(X) \cong H_3(K(\pi_2(X),2)) = \{0\},$$

where the last equality is due to the fact that the homology algebra of any topological group of type $K(\pi_2(X),2)$ (which always exists by [Mi67, Thm. 4.1]) is generated by $\pi_2(X)$ in degree 2, so that all odd degree classes vanish. We conclude that

$$\Lambda^3(X,\Gamma) \cong \mathrm{Ext}(H_2(X),\Gamma). \tag{24}$$

Since the Hurewicz homomorphism $h_2\colon \pi_2(X) \to H_2(X)$ is an isomorphism, we also have

$$\Lambda^3(X,\Gamma) \cong \mathrm{Ext}(\pi_2(X),\Gamma). \tag{25}$$

The preceding discussion shows that aspherical obstruction classes for bundles over 1-connected spaces are most naturally represented by central extensions of $\pi_2(X)$. Therefore the problem is to find a natural way to describe this extension in terms of $\widehat{K}$ and P. From Theorem 1.1(a) we know that $\delta_1([P])$ is aspherical if and only if $\partial_2^{\widehat{K}} \circ \partial_3^P = 0$, which is always the case if K is a finite dimensional Lie group.

Proposition 7.16. *If X is 1-connected, K is connected and $\widehat{K} = \gamma_*\widetilde{K}$ a flat central extension of K by the $K(\Gamma,1)$-group Z, then*

$$\mathrm{obs}_P([\widehat{K}]) = -(\gamma_d \circ \partial_2^P)^*\widetilde{Z} \in \mathrm{Ext}(\pi_2(X),\Gamma) \cong \Lambda^3(X,\Gamma),$$

where $\gamma_d\colon \pi_1(K) \to Z_d$ is γ, considered as a homomorphism into the discrete group Z_d underlying Z.

Proof. By assumption, $\widehat{K} = \gamma_*\widetilde{K}$ for $\gamma = \partial_1^{\widehat{K}}\colon \pi_1(K) \to D$ (Remark 3.1). Since $\widetilde{K}$ is a central extension by a discrete group, Theorem 1.1(b) tells us that

$$\alpha_2(\mathrm{obs}_P([\widetilde{K}])) = -\partial_1^{\widetilde{K}} \circ \partial_2^P = -\partial_2^P$$

because $\partial_1^{\widetilde{K}} = \mathrm{id}_{\pi_1(K)}$. Since X is 1-connected,

$$H^2(X,\pi_1(K)) \cong \mathrm{Hom}(\pi_2(X),\pi_1(K)),$$

so that we have a complete description of $\mathrm{obs}_P([\widetilde{K}])$. Using the factorization $\gamma = \iota^Z \circ \gamma_d$ (cf. Subection 7.2), we now obtain

$$\mathrm{obs}_P([\widehat{K}]) = \mathrm{obs}_P([(\iota^Z)_*(\gamma_d)_*\widetilde{K}]) = (\iota^Z)_*(\gamma_d)_* \,\mathrm{obs}_P([\widetilde{K}]),$$

where

$$\begin{aligned}(\iota^Z)_* \colon \mathrm{Hom}(\pi_2(X), Z) &\cong \mathrm{Hom}(H_2(X), Z) \cong H^2(X, Z) \\ &\to \mathrm{Ext}(H_2(X), \Gamma) \subseteq H^3(X, \Gamma), \quad \alpha \mapsto [\alpha^* \widetilde{Z}]\end{aligned}$$

is the natural homomorphism which can be identified with the connecting homomorphism $\delta_2 \colon \mathrm{Hom}(H_2(X), Z) \to \mathrm{Ext}(H_2(X), \Gamma)$ in the exact Hom-Ext sequence obtained for $H_2(X)$ from the short exact sequence $\Gamma \hookrightarrow \widetilde{Z} \twoheadrightarrow Z$.

Identifying $H^2(X, Z_d)$ with $\mathrm{Hom}(H_2(X), Z_d)$ and $\Lambda^3(X, Z_d)$ with $\mathrm{Ext}(H_2(X), Z_d)$, we thus obtain

$$\mathrm{obs}_P([\widehat{K}]) = -(\iota^Z)_*(\gamma_d)_*\partial_2^P = -(\iota^Z)_*(\gamma_d \circ \partial_2^P) = -(\gamma_d \circ \partial_2^P)^* \widetilde{Z},$$

and this completes the proof. ☐

Remark 7.17. It is also instructive to take a closer look at the obstruction classes of the universal bundle $q \colon EK \to BK$ of a connected locally contractible topological group K. We assume that BK is locally contractible, which is the case for each metrizable Lie group ([NSW11, Lemma 4.4]). As BK is 1-connected, $H_2(BK) \cong \pi_2(BK) \cong \pi_1(K)$, so that the Universal Coefficient Theorem leads to

$$H^2(BK, \Gamma) \cong \mathrm{Hom}(\pi_1(K), \Gamma).$$

Moreover, $\pi_3(BK) \cong \pi_2(K)$ and $\Lambda^3(BK, \Gamma) \cong \mathrm{Ext}(H_2(BK), \Gamma) \cong \mathrm{Ext}(\pi_1(K), \Gamma)$ (Remark 7.15(b)). Therefore the characteristic data determining cohomology classes in degrees 2 and 3 consists of elements of

$$\mathrm{Hom}(\pi_1(K), \Gamma), \quad \mathrm{Hom}(\pi_2(K), \Gamma) \quad \text{and} \quad \mathrm{Ext}(\pi_1(K), \Gamma).$$

For a central extension of K by a discrete group D we obtain with Theorem 1.1(b) the obstruction class

$$\mathrm{obs}_{EK}([\widehat{K}]) = -\partial_1^{\widehat{K}} \in H^2(BK, D) \cong \mathrm{Hom}(\pi_1(K), D).$$

For a flat central extension $\widehat{K} = \gamma_* \widetilde{K}$ by the $K(\Gamma, 1)$-group Z, defined by the homomorphism $\gamma \colon \pi_1(K) \to Z$, we have

$$\mathrm{obs}_{EK}([\gamma_* \widetilde{K}]) = -\delta(\gamma) \in \mathrm{Ext}(\pi_1(K), \Gamma).$$

For a general central extension of K by Z, Theorem 1.1 implies that

$$\alpha_3(\mathrm{obs}_{EK}([\widehat{K}])) = \partial_2^{\widehat{K}} \colon \pi_2(K) \to \Gamma$$

because $\partial_3^P = \mathrm{id}_{\pi_2(K)}$. Note that these objects describe precisely the data discussed in Remark 3.4.

The class $\mathrm{obs}_{EK}([\widehat{K}])$ is aspherical if and only if $[\widehat{K}] \in \mathrm{Bun}(K, Z)_0$. If this is the case, then we know from Remark 3.5 that that $\mathrm{obs}_{EK}([\widehat{K}]) = \mathrm{obs}_{EK}([\gamma_* \widetilde{K}])$ for some homomorphism $\gamma \colon \pi_1(K) \to Z$, and in this case the obstruction class is $-\delta(\gamma) \in \mathrm{Ext}(\pi_1(K), \Gamma)$.

Since a general K-bundle $P \to X$ is equivalent to f^*EK for some continuous map $f \colon X \to BK$, we can obtain the corresponding obstruction classes as pullbacks of the classes of BK.

From Remark 7.15(b), Theorem 1.1 and the Universal Coefficient Theorem, we obtain in particular the following variant of [MS11].

Proposition 7.18. *If X is 1-connected and $\partial_2^{\widehat{K}} = 0$, then $\mathrm{obs}_P([\widehat{K}])$ is contained $\mathrm{Ext}(H_2(X), \Gamma) \cong \Lambda^3(X, \Gamma)$. If $H_2(X)$ and $H_3(X)$ are finitely generated and Γ is free, then it is a torsion element.*

If $H_2(X)$ and $H_3(X)$ are finitely generated, then

$$\mathrm{Tor}(H^3(X, \mathbb{Z})) \cong \mathrm{Ext}(H_2(X), \mathbb{Z}) \cong \mathrm{Ext}(\mathrm{Tor}(H_2(X)), \mathbb{Z})$$

follows from the Universal Coefficient Theorem, and this group is finite. Its elements can be considered as lifting obstructions of homomorphisms $\gamma \colon H_2(X) \to \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$.

Remark 7.19. For a 1-connected space X, a natural way to represent classes in $\mathrm{Ext}(H_2(X), \Gamma)$ as obstruction classes is to consider a $K(H_2(X), 1)$-group H and write $q \colon \widehat{X} \to X$ for the H-bundle with Chern class

$$\mathrm{Ch}(\widehat{X}) = \mathrm{id}_{H_2(X)} \in \mathrm{End}(H_2(X)) \cong H^2(X, H_2(X)).$$

(a) As $\mathrm{Ch}(\widehat{X}) = \mathrm{obs}^{\widetilde{H}}([\widehat{X}])$, we then obtain for a $K(\Gamma, 1)$-group Z, a homomorphism $\gamma \colon H_2(X) \to Z$, and the factorization $\gamma = \iota^Z \circ \gamma_d$, the relation

$$\begin{aligned} \mathrm{obs}_{\widehat{X}}(\gamma_* \widetilde{H}) &= \mathrm{obs}_{\widehat{X}}(\iota^Z_*(\gamma_d)_* \widetilde{H}) = \iota^Z_*(\gamma_d)_* \,\mathrm{obs}_{\widehat{X}}(\widetilde{H}) \\ &= \iota^Z_*(\gamma_d)_* \mathrm{id}_{H_2(X)} = \iota^Z_* \gamma_d = \gamma_d^* \widetilde{Z} \in \mathrm{Ext}(H_2(X), \Gamma). \end{aligned}$$

If $H_2(X)$ is free and finitely generated, then H can be realized as a torus but then $\mathrm{Ext}(H_2(X), \Gamma)$ vanishes. Therefore the preceding construction produces no non-trivial obstruction classes if H is an abelian Lie group.

(b) (Geometric interpretation of ∂_3^P) As $\partial_2^{\widehat{X}} \colon \pi_2(X) \cong H_2(X) \to \pi_1(H)$ is an isomorphism and $\pi_2(H)$ vanishes, the long exact homotopy sequence of $\widehat{X}$ implies that $\widehat{X}$ is 2-connected. Therefore $H^3(\widehat{X}, \Gamma) \cong \mathrm{Hom}(\pi_3(X), \Gamma)$.

Suppose that $\widehat{K}$ is a central Z-extension of K for which $\partial_2^{\widehat{K}}$ is an isomorphism. For a K-bundle $P \to X$ Theorem 1.1(a) now shows that

$$\alpha_3(\delta_1([q^*P])) = \alpha_3(\delta_1([P])) \circ \pi_3(q) = \partial_2^{\widehat{K}} \circ \partial_3^P \circ \pi_3(q),$$

and since $\pi_3(q)\colon \pi_3(\widehat{X}) \to \pi_3(X)$ is an isomorphism, we see that $\delta_1([q^*P])$ vanishes if and only if $\partial_3^P = 0$. This means that the vanishing of ∂_3^P is equivalent to the existence of a lift of the K-bundle q^*P to a $\widehat{K}$-bundle.

As we shall see below, one obtains non-trivial aspherical obstruction classes from non-abelian Lie groups K with finite fundamental group. As

$$H^2(X, \pi_1(K)) \cong \mathrm{Hom}(\pi_2(X), \pi_1(K))$$

for a 1-connected space X, the main point is to find K-bundles P for which ∂_2^P is non-trivial.

Remark 7.20. (a) If K is a connected Lie group for which $H^1(\mathfrak{k}, \mathbb{R})$ and $H^2(\mathfrak{k}, \mathbb{R})$ vanish, we have

$$\mathrm{Hom}(\pi_1(K), \mathbb{T}) \cong \mathrm{Ext}_s(K, \mathbb{T})$$

(cf. [Ne02, Thm. 7.12]). For $K = \mathrm{PU}_n(\mathbb{C}) \cong \mathrm{SU}_n(\mathbb{C})/C_n\mathbf{1}$ this implies that

$$\mathrm{Ext}_s(\mathrm{PU}_n(\mathbb{C}), \mathbb{T}) \cong \mathrm{Hom}(C_n, \mathbb{T}) \cong C_n$$

is cyclic of order n and the central $\mathbb{T}$-extension $\widehat{K} = \mathrm{U}_n(\mathbb{C})$ corresponds to a generator. For a K-bundle $P \to X$, the corresponding obstruction class $\mathrm{obs}_P([\widehat{K}])$ is therefore contained in $\mathrm{Tor}(H^3(X, \mathbb{Z})) \cong \mathrm{Ext}(H_2(X), \mathbb{Z})$. According to [Gr64], all torsion classes are of this form. Indeed, the *Brauer group* consists of equivalence classes of Azumaya algebras. Grothendieck identifies equivalence classes of Azumaya algebras and PGL_n-bundles which do not lift to GL_n. He cites then in [Gr64] a theorem of Serre which shows that for a finite CW complex X, the Brauer group is isomorphic to $\mathrm{Tor}(H^3(X, \mathbb{Z}))$.

(b) Let $K = \mathrm{PU}(\mathcal{H})$ be the projective unitary group of an infinite dimensional separable Hilbert space $\mathcal{H}$ and $\widehat{K} := \mathrm{U}(\mathcal{H})$, which is a non-trivial central $\mathbb{T}$-extension for which

$$\partial_2^{\widehat{K}}\colon \pi_2(\mathrm{PU}(\mathcal{H})) \to \pi_1(\mathbb{T}) \cong \mathbb{Z}$$

is an isomorphism.

For any compact locally contractible space X, it is known that any element of $H^3(X, \mathbb{Z})$ is the Dixmier–Douady invariant $\mathrm{obs}^{\mathrm{U}(\mathcal{H})}([P])$ of a principal $\mathrm{PU}(\mathcal{H})$-bundle $P \to X$ (see [Dix63] or [Sc09, Thm. 5.1]).[7] The obstruction class $\mathrm{obs}^{\mathrm{U}(\mathcal{H})}([P])$ is non-zero for each non-trivial $\mathrm{PU}(\mathcal{H})$-bundle over X.

Example 7.21. For $X = \mathbb{T}^3$ the group $\pi_3(X) \cong \pi_3(\widetilde{X}) = \pi_3(\mathbb{R}^3)$ is trivial and $\Lambda^3(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) \cong \mathbb{Z}$ because X is an orientable 3-manifold. Hence $\partial_3^P = 0 = \partial_2^P$ for every K-bundle $P \to X$. However, for $K = \mathrm{PU}(\mathcal{H})$, each element of $\Lambda^3(X, \mathbb{Z})$ can be obtained as the Dixmier–Douady class of a K-bundle $P \to X$.

[7] Since the classifying space $B\,\mathrm{PU}(\mathcal{H})$ is an Eilenberg–MacLane space of type $K(\mathbb{Z}, 3)$, Huber's Theorem ([Hu61]) also provides a bijection $\mathrm{Bun}(X, \mathrm{PU}(\mathcal{H})) \cong [X, B\,\mathrm{PU}(\mathcal{H})] \cong \check{H}^3(X, \mathbb{Z})$ for any k-space X.

Example 7.22. We construct an example of a non-trivial aspherical obstruction class $\delta_1([P])$ for a central group extension with $\partial_2^{\widehat{K}} \neq 0$ and a space with non-trivial π_3.

In view of Remark 7.20, it suffices to construct a compact topological space X such that $H^3(X, \mathbb{Z}) \cong \mathbb{Z}$ and $\pi_3(X)$ is a torsion group because the latter assumption implies that $\partial_3^P : \pi_3(X) \to \pi_2(K) \cong \mathbb{Z}$ vanishes for $K = \mathrm{PU}(\mathcal{H})$ and $\widehat{K} = \mathrm{U}(\mathcal{H})$.

We start with $A := \mathbb{S}^1 \times \mathbb{S}^2$, considered as the 3-skeleton of a CW-complex. We attach to A a 4-cell D^4 via the attaching map $f : \mathbb{S}^3 \to \mathbb{S}^2$, obtained by the composition of a map $\mathbb{S}^3 \to \mathbb{S}^3$ of degree $p > 1$ and the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ (which is the generator of $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$). We put

$$X := \mathbb{S}^1 \times (\mathbb{S}^2 \cup_f D^4).$$

The space X is then the 4-cellular extension of A in the sense of [Wh78, p. 48], and is compact by [Wh78, (1.1), p. 48]. The third homotopy group of X can be computed using [Wh78, Thm (1.1) p. 212], which implies that the relative homotopy group $\pi_4(X, A)$ is a free $\mathbb{Z}$-module generated by the image of f, i.e. $\pi_4(X, A) \cong p\mathbb{Z}$. The long exact sequence of the pair now yields

$$\pi_4(X, A) = p\mathbb{Z} \to \pi_3(A) = \mathbb{Z} \to \pi_3(X) \to \pi_3(X, A) = \{0\}$$

([Wh78, Thm. (2.4), p. 162]). It follows that $\pi_3(X) \cong \mathbb{Z}/p\mathbb{Z}$ as claimed.

The third cohomology of X is computed in an indirect way. It is clear that $H^*(A, \mathbb{Z})$ is isomorphic to an exterior algebra in one generator θ of degree 1 and one generator ω of degree 2:

$$H^*(A, \mathbb{Z}) = H^*(\mathbb{S}^1 \times \mathbb{S}^2, \mathbb{Z}) \cong H^*(\mathbb{S}^1, \mathbb{Z}) \otimes H^*(\mathbb{S}^2, \mathbb{Z}) \cong \Lambda^*[\theta, \omega].$$

Therefore $H^3(A, \mathbb{Z}) = \mathbb{Z}\theta \wedge \omega \cong \mathbb{Z}$. We only have to show that the attachment of the 4-cell does not change H^3.

The cohomology of a CW complex may be computed by cellular cohomology, see [Br97a, p. 297]. In order to compute $H^3(X, \mathbb{Z})$, we must add a 4-cell in the complex $C^*(A, \mathbb{Z})$ which computes the cellular cohomology of A. But the coboundary operator for cellular cohomology is zero on the degree 4-generator, because the 4-cell is attached to no 3-cell ([Br97a, p. 297]). Therefore the attachment of the 4-cell does not change H^3, which results in $H^3(X, \mathbb{Z}) \cong \mathbb{Z}$.

8 Concluding remarks and problems

Proposition 7.7 above gives a complete description of the aspherical obstruction classes for central extensions of K by discrete groups. For central extensions by $K(\Gamma, 1)$-groups our results are less complete. Theorem 7.12 provides a solution for flat central extensions and bundles P for which $\partial_2^P = 0$. The main open points concern the identification of aspherical obstructions on 1-connected spaces X with the corresponding central extensions of $\pi_2(X)$. Below we describe some open problems concerning the concrete identification of obstruction classes.

Problem 8.1. It would be instructive to have more explicit information on examples where $P = G$ is a Lie group, $K \subseteq G$ is a closed subgroup and $X = G/K$ is a homogeneous space. Then we have for $Z = (\mathfrak{z}/\Gamma) \times D$ a natural map

$$\delta_1 \colon \operatorname{Ext}_c(K, Z) \to H^3(G/K, \Gamma) \oplus H^2(G/K, D).$$

(a) For the special case where G is finite dimensional, the group $\pi_2(K)$ vanishes, so that the identity component of Z leads to no non-trivial obstruction classes (Proposition 3.3). This leaves us with the case where $Z = D$ is discrete. If K is connected, we thus obtain a homomorphism

$$\delta_1 \colon \operatorname{Ext}_c(K, Z) \cong \operatorname{Hom}(\pi_1(K), D) \to H^2(G/K, D).$$

As $\pi_2(G)$ and $\pi_2(K)$ are trivial, we have an exact sequence

$$\{0\} \to \pi_2(G/K) \to \pi_1(K) \to \pi_1(G) \to \pi_1(G/K) \to \pi_0(K) \to \cdots.$$

If G is 1-connected and K is connected, it follows that G/K is 1-connected with $H_2(G/K) \cong \pi_2(G/K) \cong \pi_1(K)$. This implies that δ_1 is an isomorphism.

(b) The case where $K \trianglelefteq G$ is normal is closely related to crossed modules (in the case that K is closed and $G \to G/K$ admits a local continuous section) because for any lift of the G-action on K to a central Z-extension $\widehat{K}$, the homomorphism $p \colon \widehat{K} \to G$ defines a crossed module of groups whose characteristic class is an element of the third locally continuous group cohomology $H^3_c(G/K, Z)$ (or locally smooth in the case of Lie groups). For an arbitrary topological group Q there is a homomorphism

$$\tau \colon H^3_c(Q, Z) \to \check{H}^2(Q, \underline{Z}),$$

given as follows. If $f \colon Q \times Q \times Q \to Z$ represents a class in $H^3_c(Q, Z)$, then it is a group cocycle and continuous on $U \times U \times U$ for some open $U \subseteq Q$ with $e \in U$. If $V \subseteq U$ is open, contains e and satisfies $V^2 \subseteq U$, then we have a Čech cocycle $\tau(f)_{g,h,k} \colon g \cdot V \cap h \cdot V \cap k \cdot V \to Z$,

$$\tau(f)_{g,h,k}(x) = f(h, h^{-1}k, k^{-1}x) - f(g, g^{-1}k, k^{-1}x) + f(g, g^{-1}h, h^{-1}x),$$

for the open cover $(g \cdot V)_{g \in Q}$ of Q (cf. [Wo10, Lemma 2.2]). Note that the cocycle identity for f tells us that

$$\tau(f)_{g,h,k}(x) = f(g, g^{-1}h, h^{-1}k) + f(g^{-1}h, h^{-1}k, k^{-1}x)$$

depends continuously on x, since $g^{-1}h$, $h^{-1}k$ and $k^{-1}x$ are in U if $g \cdot V \cap h \cdot V \cap k \cdot V \neq \emptyset$. Moreover, the class of $\tau(f)$ clearly only depends on the class of f.

If now $[c] \in H^3_c(G/K, Z)$ is the characteristic class of the crossed module $\widehat{K} \to G$ ([Ne07, Lemma 3.6]) and G is considered as a K-principal bundle over G/K, then we expect that

$$\tau([c]) = \pm\delta_1([G]),$$

where $\widehat{K} \to K$ is the central extension defining the crossed module.

Problem 8.2. (a) Consider K-bundles $q\colon P \to X$ for which ∂_2^P and ∂_3^P vanish, so that $\pi_2(P)$ is an abelian extension of $\pi_2(X)$ by $\pi_1(K)$. Composing with $\partial_2^{\widehat{K}}$, we obtain a class

$$(\partial_2^{\widehat{K}})_*[\pi_2(P)] \in \mathrm{Ext}(\pi_2(X), \Gamma).$$

If X is 1-connected, then $\Lambda^3(X, \Gamma) \cong \mathrm{Ext}(\pi_2(X), \Gamma)$ (Remark 7.15(b)). Is it true that $(\partial_2^{\widehat{K}})_*[\pi_2(P)]$ coincides with $\mathrm{obs}_P([\widehat{K}])$?

(b) If Z is a $K(\Gamma, 1)$-group and $\partial_2^{\widehat{K}}\colon \pi_2(K) \to \pi_1(Z)$ vanishes, then $\pi_1(\widehat{K})$ is an abelian extension of $\pi_1(K)$ by Γ and for any K-bundle $P \to X$ we obtain a class

$$(\partial_2^P)^*[\pi_1(\widehat{K})] \in \mathrm{Ext}(\pi_2(X), \Gamma).$$

Suppose that X is 1-connected, which implies $\Lambda^3(X, \Gamma) \cong \mathrm{Ext}(\pi_2(X), \Gamma)$. Is it true that $(\partial_2^P)^*[\pi_1(\widehat{K})] = \mathrm{obs}_P([\widehat{K}])$?

Problem 8.3. If X is connected and $q_X\colon \widetilde{X} \to X$ is a universal covering, then $\pi_2(\widetilde{X}) \cong \pi_2(X)$, and we obtain a homomorphism

$$q_X^*\colon \Lambda^3(X, \Gamma) \to \Lambda^3(\widetilde{X}, \Gamma) \cong \mathrm{Ext}(\pi_2(X), \Gamma).$$

It is an interesting problem to give an explicit description of the kernel of q_X^* in $\Lambda^3(X, \Gamma)$.

Problem 8.4. (Aspherical spaces) Suppose that X is aspherical, i.e., that $\widetilde{X}$ is contractible. Then X is a $K(\pi_1(X), 1)$-space, so that

$$H^k(X, \Gamma) \cong H^k_{\mathrm{grp}}(\pi_1(X), \Gamma) \quad \text{for} \quad k \in \mathbb{N}_0.$$

Since all homotopy groups $\pi_k(X)$ vanish for $k \geq 2$, the corresponding connecting maps ∂_k^P are trivial for any K-bundle P over X. Therefore all obstruction classes are aspherical by Theorem 1.1.

Write $q_X\colon \widetilde{X} \to X$ for the universal covering of X. Then, for every K-bundle $q\colon P \to X$, the pullback $q_X^* P \to \widetilde{X}$ is trivial because $\widetilde{X}$ is contractible. As $q_X^* P \sim \widetilde{X} \times K$ as K-bundles, the bundle P can be written as a quotient

$$P \cong (\widetilde{X} \times K)/\pi_1(X),$$

where $\pi_1(X)$ acts on the trivial K-bundle $\widetilde{X} \times K$ by bundle automorphism.

Any lift of the $\pi_1(X)$-right action on $\widetilde{X}$ to an action by bundle automorphisms on $\widetilde{X} \times K$ is of the form

$$(x, k).d = (xd, f_d^{-1}(xd)k),$$

where $f_d \in C(\widetilde{X}, K)$ and the map

$$f\colon \pi_1(X) \to C(\widetilde{X}, K), \quad d \mapsto f_d$$

is a 1-cocycle, i.e.,

$$f_{d_1 d_2} = f_{d_1} \cdot (d_1.f_{d_2}). \tag{26}$$

We write $P^f := (\widetilde{X} \times K)/\pi_1(X)$ for the quotient of $\widetilde{X} \times K$ defined by the action given by the cocycle $f\colon \pi_1(X) \to C(\widetilde{X}, K)$ and obtain a surjective map

$$H^1_{\rm grp}(\pi_1(X), C(\widetilde{X}, K)) \to \mathrm{Bun}(X, K), \quad [f] \mapsto [P^f].$$

Let $\widehat{K}$ be a topological Z-extension of K. Then we have a natural map $C(\widetilde{X}, \widehat{K}) \to C(\widetilde{X}, K)$ which is surjective because, for every continuous function $f\colon \widetilde{X} \to K$, the Z-bundle $f^*\widehat{K} \to \widetilde{X}$ is trivial. We thus have a central extension

$$\mathbf{1} \to C(\widetilde{X}, Z) \to C(\widetilde{X}, \widehat{K}) \to C(\widetilde{X}, K) \to \mathbf{1}$$

which leads to long exact sequence in group cohomology. In particular, if

$$\delta_1\colon H^1_{\rm grp}(\pi_1(X), C(\widetilde{X}, K)) \to H^2_{\rm grp}(\pi_1(X), C(\widetilde{X}, Z)).$$

is the connecting map, then $\delta_1([f])$ vanishes if and only if the K-bundle P^f lifts to a $\widehat{K}$-bundle. Next we consider the short exact sequence

$$\mathbf{1} \to C(\widetilde{X}, \Gamma) \cong \Gamma \to C(\widetilde{X}, \widetilde{Z}) \to C(\widetilde{X}, Z) \to \mathbf{1}$$

which in turn leads to a connecting map

$$\delta_2\colon H^2_{\rm grp}(\pi_1(X), C(\widetilde{X}, Z)) \to H^3_{\rm grp}(\pi_1(X), \Gamma) \cong H^3(X, \Gamma).$$

Is it true that

$$\delta_1([P^f]) = \pm\delta_2(\delta_1([f]))?$$

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