

# Topological Group Cohomology with Loop Contractible Coefficients

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## Abstract

We show that for topological groups and loop contractible coefficients the cohomology groups of continuous group cochains and of group cochains that are continuous on some identity neighbourhood are isomorphic. Moreover, we show a similar statement for compactly generated groups and Lie groups holds and apply our results to different concepts of group cohomology for finite-dimensional Lie groups.

## Introduction

For a topological group  $G$  and a continuous  $G$ -module  $A$  exist various concepts of “topological group cohomology”, i.e., cohomology groups taking the topological properties of  $G$  appropriately into account [Seg70, Del74, Moo76]. When approaching from concrete cocycle models, there exist the naive notion of the cohomology  $H_c^*(G, A)$  of the complex  $C_c^*(G, A)$  of continuous group cochains and the more ad-hoc notion of the cohomology  $H_{lc}^*(G, A)$  of the complex  $C_{lc}^*(G, A)$  of group cochains which are continuous only on some identity neighbourhood. The inclusion  $C_c^*(G, A) \hookrightarrow C_{lc}^*(G, A)$  induces a homomorphism  $H_c^*(G, A) \rightarrow H_{lc}^*(G, A)$  in cohomology, comparing these two notions.

If  $G$  is connected, then the continuous group cohomology  $H_c^2(G, A)$  describes abelian Lie group extensions which admit continuous global sections, i.e. they are trivial principal  $A$ -bundles. The “locally continuous” group cohomology  $H_{lc}^2(G, A)$  describes abelian Lie group extensions which only admit local continuous sections, i.e. they are locally trivial principal  $A$ -bundles. For paracompact  $G$  and (loop) contractible  $A$  the Lie group extensions described by  $H_{ls}^2(G, A)$  also admit a continuous global section, hence the morphism  $i^2 : H_c^2(G, A) \rightarrow H_{lc}^2(G, A)$  is an isomorphism in this case. We show that this is true in every degree and that the same also holds for a Lie group  $G$ , a smooth  $G$ -module  $A$  and the inclusion  $C_s^*(G, A) \hookrightarrow C_{ls}^*(G, A)$  of smooth into “locally smooth” cochains:

**Theorem.** *If  $G$  is a topological group and  $A$  a continuous  $G$ -module, then the natural inclusion*

$$C_c^*(G, A) \hookrightarrow C_{lc}^*(G, A)$$

*induces an isomorphism in cohomology provided  $A$  is loop contractible (e.g. a topological vector space). If  $G$  is a Lie group whose finite products are smoothly paracompact and  $A$  is a smooth  $G$ -module, then*

$$C_s^*(G, A) \hookrightarrow C_{ls}^*(G, A)$$

*induces an isomorphism in cohomology if  $A$  is smoothly loop contractible. ■*

We also show that similar result also holds for compactly generated topological groups and modules. This has as a consequence that the “locally continuous” cohomology coincides with the other mentioned topological group cohomologies, see [WW]. As an application of our main result we show that it has as a consequence that “locally smooth” and “locally continuous” cohomology for finite-dimensional Lie groups coincide under a mild technical assumption.

## I Continuous and “Locally Continuous” Cochains

Let  $G$  be a topological group and  $A$  be a topological  $G$ -module<sup>1</sup>. In this section we introduce the complexes of continuous and “locally continuous” (group) cochains and a double complex that will do the work for us when showing the main theorem. We start with the *continuous standard complex* complex  $A_c^*(G, A)$  and the complex  $C_c^*(G, A)$  of continuous homogeneous group cochains.

Recall that the *standard complex* is the complex  $A^*(G, A) := \text{Map}(G^{*+1}, A)$  with differential

$$d: A^n(G, A) \rightarrow A^{n+1}(G, A), \quad df(g_0, \dots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \dots, \widehat{g}_i, \dots, g_{n+1}),$$

which is always exact. The continuous cochains form a sub-complex  $C(G^{*+1}, A)$ , which is also exact and which we denote by  $A_c^*(G, A)$ . Somewhere in between there exists a complex of “locally continuous” cochains, which we are going to define now:

**Definition I.1.** For each identity neighbourhood  $U$  of a topological group  $G$  we define the open diagonal neighbourhood  $\Gamma_U^0 := G$  in  $G$  and  $\Gamma_U^p$  in  $G^{p+1}$  by setting

$$\Gamma_U^p := \{(g_0, \dots, g_p) \in G^{p+1} \mid \forall 0 \leq i, j \leq p : g_i^{-1} g_j \in U\},$$

for  $p \geq 1$ . For each topological  $G$ -module  $A$  the sub-complex

$$A_{lc}^*(G, A) := \{f: G^{*+1} \rightarrow A \mid f \text{ is continuous on some } \Gamma_U\}$$

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<sup>1</sup>In contrast to Section III we thus assume that the multiplication and module map  $G \times G \rightarrow G$  and  $G \times A \rightarrow A$  are continuous for the product topology.

of the standard complex is called the *complex of locally continuous cochains*. ■

The topological group  $G$  acts on the open neighbourhood  $\Gamma_U^p$  of the diagonal of  $G^{p+1}$  via the diagonal action (i.e. by  $g.(g_0, \dots, g_p) = (gg_0, \dots, gg_p)$ ), and thus  $G$  acts on the cochain groups  $A^p(G, A)$ ,  $A_{lc}^p(G, A)$  and  $A_c^p(G, A)$  via

$$(g.f)(g_0, \dots, g_p) = g.(f(g^{-1}.(g_0, \dots, g_p)))$$

leaving the corresponding sub-complexes invariant. The  $G$ -fixed points of this action are the  $G$ -equivariant functions. The action intertwines the differentials on the complexes. As a consequence the subgroups

$$C^*(G, A) := A^*(G, A)^G, \quad C_{lc}^*(G, A) := A_{lc}^*(G, A)^G \quad \text{and} \quad C_c^*(G, A) := A_c^*(G, A)^G$$

form sub complexes of  $A^*(G, A)$ . These are the complexes of homogeneous group cochains, locally continuous homogeneous group cochains and continuous homogeneous group cochains respectively. We denote the corresponding cohomology groups by  $H^*(G, A)$ ,  $H_{lc}^*(G, A)$  and  $H_c^*(G, A)$ .

Let  $\mathcal{U}_1$  be the neighbourhood filter of the identity in  $G$  and consider the abelian groups

$$A_{lc}^{p,q}(G, A) := \left\{ f : G^{p+1} \times G^{q+1} \rightarrow A \mid \exists U \in \mathcal{U}_1 : f|_{G^{p+1} \times \Gamma_U^q} \text{ is continuous} \right\}.$$

The abelian groups  $A_{lc}^{p,q}(G, A)$  form a first quadrant double complex with vertical and horizontal differentials given by

$$\begin{aligned} d_h^{p,q} : A_{lc}^{p,q} &\rightarrow A_{lc}^{p+1,q}, & d_h^{p,q}(f^{p,q})(x_0, \dots, x_{p+1}, \vec{y}) &= \sum_{i=0}^{p+1} (-1)^i f^{p,q}(x_0, \dots, \widehat{x}_i, \dots, x_{p+1}, \vec{y}) \\ d_v^{p,q} : A_{lc}^{p,q} &\rightarrow A_{lc}^{p,q+1}, & d_v^{p,q}(f^{p,q})(\vec{x}, y_0, \dots, y_{q+1}) &= (-1)^p \sum_{i=0}^{q+1} (-1)^i f^{p,q}(\vec{x}, y_0, \dots, \widehat{y}_i, \dots, y_{q+1}). \end{aligned}$$

The subgroups  $A_c^{p,q}(G, A) := C(G^{p+1} \times G^{q+1}, A)$  of the groups  $A_{lc}^{p,q}(G, A)$  form a sub double complex. Furthermore the groups  $A_{lc}^{p,q}(G, A)^G$  and  $A_c^{p,q}(G, A)^G$  of equivariant locally continuous and equivariant continuous cochains form a sub double complex of  $A_{lc}^{*,*}(G, A)$  and  $A_c^{*,*}(G, A)$  respectively.

The rows  $A_{lc}^{*,q}(G, A)^G$  of the double complex  $A_{lc}^{*,*}(G, A)^G$  of equivariant cochains can be augmented by the cochain groups  $C_{lc}^q(G, A)$  of locally continuous homogeneous group cochains and the columns  $A_{lc}^{p,*}(G, A)^G$  can be augmented by the cochain groups  $C_c^q(G, A)$  of continuous homogeneous group cochains (cf. the computations in [Fuc11b, Section 2]):

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow d & & \uparrow d_v & & \uparrow d_v & & \uparrow d_v \\
C_{lc}^2(G, A) & \longrightarrow & A_{lc}^{0,2}(G, A)^G & \xrightarrow{d_h} & A_{lc}^{1,2}(G, A)^G & \xrightarrow{d_h} & A_{lc}^{2,2}(G, A)^G \xrightarrow{d_h} \dots \\
\uparrow d & & \uparrow d_v & & \uparrow d_v & & \uparrow d_v \\
C_{lc}^1(G, A) & \longrightarrow & A_{lc}^{0,1}(G, A)^G & \xrightarrow{d_h} & A_{lc}^{1,1}(G, A)^G & \xrightarrow{d_h} & A_{lc}^{2,1}(G, A)^G \xrightarrow{d_h} \dots \\
\uparrow d & & \uparrow d_v & & \uparrow d_v & & \uparrow d_v \\
C_{lc}^0(G, A) & \longrightarrow & A_{lc}^{0,0}(G, A)^G & \xrightarrow{d_h} & A_{lc}^{1,0}(G, A)^G & \xrightarrow{d_h} & A_{lc}^{2,0}(G, A)^G \xrightarrow{d_h} \dots \\
& & \uparrow & & \uparrow & & \uparrow \\
& & C_c^0(G, A) & \xrightarrow{d} & C_c^1(G, A) & \xrightarrow{d} & C_c^2(G, A) \xrightarrow{d} \dots
\end{array}$$

We denote the total complex of this double complex by  $\text{Tot}A_{lc}^{*,*}(G, A)^G$ . The augmentations of the rows induces a morphisms  $j_h^* : C_{lc}^*(G, A) \hookrightarrow \text{Tot}A_{lc}^{*,*}(G, A)^G$  of cochain complexes. Likewise, the augmentations of the columns induces a morphism  $j_v^* : C_c^*(G, A) \hookrightarrow \text{Tot}A_{lc}^{*,*}(G, A)^G$ . On each row  $A_{lc}^{*,q}(G, A)^G$  one can define a contracting homotopy  $h^*$  by setting

$$\begin{aligned}
h^p &: A_{lc}^{p,q}(G, A)^G \rightarrow A_{lc}^{p-1,q}(G, A)^G \\
h^p(f)(x_0, \dots, x_{p-1}, \vec{y}) &= (-1)^p f(x_0, \dots, x_{p-1}, y_0, \vec{y})
\end{aligned} \tag{1}$$

(cf. the computations in [Fuc11b, Lemmata 2.1, 2.3, 2.8] and [Fuc10, Proposition 14.3.3]). As a consequence we note:

**Lemma I.2.** *The inclusion  $j_h^* : C_{lc}^*(G, A) \hookrightarrow \text{Tot}A_{lc}^{*,*}(G, A)^G$  induces an isomorphism in cohomology.*

Note that this contraction does not work in vertical direction since it would violate the continuity assumptions on the elements of  $A_{lc}^{p,q-1}(G, A)^G$ .

## II Continuous and Locally Continuous Group Cohomology

In the following we will show that for loop contractible coefficients the inclusion of the complex  $C_c^*(G, A)$  of continuous group cochains into the complex  $C_{lc}^*(G, A)$  induces an isomorphism  $H_c^*(G, A) \cong H_{lc}^*(G, A)$ .

Recall that a topological group  $A$  is called *loop contractible* if it is contractible by a homotopy  $H : [0, 1] \times A \rightarrow A$  such that each  $H(t, \cdot)$  is a group homomorphism (cf. [Fuc10, Section 5]). It is the key observation of this article that this requirement on  $A$  allows to adapt the procedure from [Fuc11b] to our setting.

The following will rely on the row exactness of the double complexes  $A_c^{*,*}(G, A)^G$  and  $A_{lc}^{*,*}(G, A)^G$  from the previous section and the following observation:

**Proposition II.1.** *If the augmented column complexes  $A_c^p(G, A) \hookrightarrow A_{lc}^{p,*}(G, A)$  are exact, then the augmented sub column complexes  $C_c^p(G, A) \hookrightarrow A_{lc}^{p,*}(G, A)^G$  of equivariant cochains are exact as well.*

**Proof.** For the sake of completeness we recall the proof given in [Fuc11b, Proposition 3.1]. Assume that the augmented column complexes  $A_c^p(G, A) \hookrightarrow A_{lc}^{p,*}(G, A)$  are exact. Each equivariant vertical cocycle  $f_{eq}^{p,q} \in A_{lc}^{p,q}(G, A)^G$  is the vertical coboundary  $d_v f^{p,q-1}$  of a (not necessary equivariant) cocycle  $f^{p,q-1}$  in  $A_{lc}^{p,q-1}(G, A)$ . Define an equivariant cochain  $f_{eq}^{p,q-1}$  of bidegree  $(p, q-1)$  via

$$f_{eq}^{p,q-1}(\vec{x}, \vec{y}) := x_0 \cdot f^{p,q-1}(x_0^{-1} \cdot \vec{x}, x_0^{-1} \cdot \vec{y})$$

We assert that the vertical coboundary  $d_v f_{eq}^{p,q-1}$  of this equivariant cochain is the equivariant vertical cocycle  $f_{eq}^{p,q}$ . Indeed, since the differential  $d_v$  is equivariant, the vertical coboundary of  $f_{eq}^{p,q-1}$  computes to

$$\begin{aligned} d_v f_{eq}^{p,q-1}(\vec{x}, \vec{y}) &= x_0 \cdot [d_v f^{p,q-1}(x_0^{-1} \cdot \vec{x}, x_0^{-1} \cdot \vec{y})] \\ &= x_0 \cdot [f_{eq}^{p,q}(x_0^{-1} \cdot \vec{x}, x_0^{-1} \cdot \vec{y})] = f_{eq}^{p,q}(\vec{x}, \vec{y}). \end{aligned}$$

Thus every equivariant vertical cocycle  $f_{eq}^{p,q}$  is the vertical coboundary of an equivariant cochain  $f_{eq}^{p,q-1}$  of bidegree  $(p, q-1)$ . ■

**Corollary II.2.** *If the augmented column complexes  $A_c^p(G, A) \hookrightarrow A_{lc}^{p,*}(G, A)$  are exact, then the inclusion  $j_v^* : C_c^*(G, A) \hookrightarrow \text{Tot}A_{lc}^{*,*}(G, A)^G$  induces an isomorphism in cohomology and the cohomologies  $H_c^p(G, A)$ ,  $H^p(\text{Tot}A_{lc}^{*,*}(G, A)^G)$  and  $H_c(G, A)^p$  are isomorphic.*

It remains to show that in this case the isomorphism  $H_c^p(G, A) \cong H_{lc}^p(G, A)$  is actually induced by the inclusion  $i^* : C_c^*(G, A) \hookrightarrow C_{lc}^*(G, A)$ . Here the proof of [Fuc10, Proposition 14.3.8] carries over almost in verbatim, see also [Fuc11b, Proposition 2.4]:

**Proposition II.3.** *The image  $j_v^n(f)$  of a homogeneous continuous group  $n$ -cocycle  $f$  on  $G$  in  $\text{Tot}A_{lc}^{*,*}(G, A)^G$  is cohomologous to the image  $j_h^n i^n(f)$  of the locally continuous homogeneous group  $n$ -cocycle  $i^n(f)$  in  $\text{Tot}A_{lc}^{*,*}(G, A)^G$ .*

**Proof.** Consider a continuous homogeneous group  $n$ -cocycle  $f : G^{n+1} \rightarrow A$  on  $G$  and define for all  $p+q = n-1$  equivariant cochains  $\psi^{p,q} : G^{p+1} \times G^{q+1} \cong G^{n+1} \rightarrow A$  in  $A_{lc}^{p,q}(G, A)$  via  $\psi^{p,q}(\vec{x}, \vec{y}) = (-1)^p f(\vec{x}, \vec{y})$ . The vertical coboundary of the cochain  $\psi^{p,q}$  is given by

$$\begin{aligned} [d_v \psi^{p,q}](\vec{x}, y_0, \dots, y_{q+1}) &= (-1)^p \sum (-1)^i f(\vec{x}, y_0, \dots, \hat{y}_i, \dots, y_q) \\ &= - \sum (-1)^{p+1+i} f(x_0, \dots, \hat{x}_i, \dots, x_p, \vec{y}) \\ &= [d_h \psi^{p-1, q+1}](x_0, \dots, x_p, \vec{y}). \end{aligned}$$

The anti-commutativity of the horizontal and the vertical differential ensures that the coboundary of the cochain  $\sum_{p+q=n-1} (-1)^p \psi^{p,q}$  in the total complex is the cochain  $j_v^n(f) - j_h^n i^n(f)$ . Thus the cocycles  $j^n(f)$  and  $j_h^n i^n(f)$  are cohomologous in  $\text{Tot}A_{lc}^{*,*}(G, A)^G$ . ■

**Corollary II.4.** *The homomorphism*

$$H(j_h^p)^{-1}H(j_v^p) : H_c^p(G, A) \rightarrow H_{lc}^p(G, A)$$

is induced by the inclusion  $C_c^*(G, A) \hookrightarrow C_{lc}^*(G, A)$ .

Recalling the exactness condition on the augmented columns of the double complex  $A_{lc}^{*,*}(G, A)$  we have shown:

**Theorem II.5.** *If  $G$  is a topological group,  $A$  a topological  $G$ -module and the augmented columns  $A_c^*(G, A) \hookrightarrow A_{lc}^{p,*}(G, A)$  of the double complex  $A_{lc}^{*,*}(G, A)$  are exact, then the inclusion  $C_c^*(G, A) \hookrightarrow C_{lc}^*(G, A)$  induces an isomorphism in cohomology.*

We now turn to showing the previous exactness condition in the case of loop contractible  $A$ .

**Proposition II.6.** *If  $A$  is loop contractible, then the augmented column complexes  $A_c^p(G, A) \hookrightarrow A_{lc}^{p,*}(G, A)$  are exact*

**Proof.** We consider for any open identity neighbourhood  $U$  in  $G$  the open neighbourhoods  $\mathfrak{U}_U[n] := \bigcup g.U^{n+1}$  of the diagonal, which form a simplicial subspace of  $G^{*+1}$ . This allows us to consider the cosimplicial group

$$A^{p,*}(G, \mathfrak{U}_U; A) := \{f : G^{p+1} \times \mathfrak{U}_U[*] \rightarrow A \mid \forall \vec{g} \in \mathfrak{U}_U[*] : f(-, \vec{g}) \in C(G^{p+1}, A)\}$$

and its cosimplicial sub group  $A_c^{p,*}(G, \mathfrak{U}_U; A) := C(G^{p+1} \times \mathfrak{U}_U[*], A)$  as well as the cochain complex associated to both. By switching arguments the cochain group  $A^{p,q}(G, \mathfrak{U}_U; A)$  can be identified with the group  $f : \mathfrak{U}_U[q] \rightarrow C(G^{p+1}, A)$  of  $\mathfrak{U}$ -local cochains. Taking the colimit over all identity neighbourhoods yields the complex of Alexander-Spanier cochains

$$\text{colim}A^{p,*}(G, \mathfrak{U}_U; A) \cong A_{AS}^*(G, C(G^{p+1}, A))$$

of  $G$  with coefficients  $C(G^{p+1}, A)$ . It has been shown in [Fuc11b, Lemma 3.12]<sup>2</sup> that the augmented complex  $A_c^p(G, A) \hookrightarrow A_{lc}^{p,*}(G, A)$  is exact if and only if the inclusion of colimit complexes

$$\text{colim}A_c^{p,*}(G, \mathfrak{U}_U; A) \hookrightarrow \text{colim}A^{p,*}(G, \mathfrak{U}_U; A),$$

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<sup>2</sup>In the cited manuscript the complex  $A_{lc}^{p,q}(G, A)$  is denoted by  $A_{cg}^{p,q}(G, A)$  and one has to consider  $G$  acting on itself by left translation.

where  $U$  runs over all open identity neighbourhoods in  $G$ , induces an isomorphism in cohomology.

The latter can be shown by adapting the construction in [Fuc11a], where it is shown that the inclusion  $A_{AS,c}^*(G, A) \hookrightarrow A_{AS}^*(G, A)$  of the continuous into the abstract Alexander-Spanier complex induces an isomorphism in cohomology. (For paracompact spaces and vector space coefficients this is a well-known fact [Fuc11a, Corollary 2.10], see [Fuc11a, Corollary 2.14] for not necessarily paracompact  $G$ .) In the case of loop contractible coefficients we replace the Alexander-Spanier presheafs  $A_c^q(-, A)$  and  $A^q(-, A)$  in the proof of [Fuc11a, Corollary 5.20] by the presheafs  $A_c^{p,q}(G, -; A)$  and  $A^{p,q}(G, -; A)$  given by

$$A_c^{p,q}(G, U; A) := C(G^{p+1} \times U^{q+1}, A)$$

and

$$A^{p,q}(G, U; A) := \{f : G^{p+1} \times U^{q+1} \rightarrow A \mid \forall \vec{g}' \in U^{q+1} : f(-, \vec{g}') \in C(G^{p+1}, A)\}.$$

The arguments leading to [Fuc11a, Corollary 5.20] carry over to show that for each fixed  $p$  the inclusion  $A_c^{p,*}(G, \mathfrak{U}; A) \hookrightarrow A^{p,*}(G, \mathfrak{U}; A)$  induces an isomorphism in cohomology. We thus obtain the desired isomorphism

$$H(\operatorname{colim} A_c^{p,*}(G, \mathfrak{U}_U; A)) \cong H(\operatorname{colim} A^{p,*}(G, \mathfrak{U}_U; A)). \quad (2)$$

■

**Remark II.7.** In the case that  $G$  is locally compact, we can obtain the isomorphism (2) directly from [Fuc11a, Corollary 2.14], since then  $A_c^{p,q}(G, U; A) \cong A_c^q(U, C(G^p, A))$  and thus the inclusion

$$A_c^{p,*}(G, \mathfrak{U}; A) \cong A_c^*(\mathfrak{U}, C(G^p, A)) \hookrightarrow A^{p,q}(\mathfrak{U}, C(G^p, A)) \cong A^{p,q}(G, \mathfrak{U}; A)$$

induces an isomorphism in cohomology. ■

**Corollary II.8.** *If  $G$  is a topological group and  $A$  is a loop contractible topological  $G$ -module, then the inclusion  $C_c^*(G, A) \hookrightarrow C_{lc}^*(G, A)$  induces an isomorphism in cohomology.*

### III The compactly generated case

There exists another version of locally continuous cochains if one works solely in the category **kTop** of compactly Hausdorff generated spaces. First of all this leads to a different notion of topological group, where one requires the group multiplication  $G \times G \rightarrow G$  to be continuous in the compactly generated topology, which is in general finer than the product topology. Moreover, this also affects the notion of  $G$ -module, where one also requires the action map  $G \times A \rightarrow A$  to be continuous with respect to the compactly Hausdorff generated topology. We then call  $G$  a  $k$ -group and  $A$  a  $k$ -module for  $G$ .

The ‘k-ification’ of topological spaces is a functor  $k : \mathbf{kTop} \rightarrow \mathbf{Top}$ . Working in  $\mathbf{kTop}$  one can still define continuous and ‘locally continuous’ cochains by using products in the category  $\mathbf{kTop}$  instead of  $\mathbf{Top}$  (cf. [Fuc11b, Section 4]):

$$\begin{aligned} C_{c,k}^q(G, A) &:= C(kG^{q+1}, A)^G \\ C_{lc,k}^q(G, A) &:= \left\{ f \in A^q(G, A)^G \mid \exists U \in \mathcal{U}_1 : f|_{k\Gamma_U^q} \text{ is continuous} \right\} \end{aligned}$$

These subcomplexes of the standard complex lead to generally different versions of cohomology groups  $H_{c,k}^n(G, A)$  and  $H_{lc,k}^n(G, A)$  respectively, and the inclusion  $C_{c,k}^*(G, A) \rightarrow C_{lc,k}^*(G, A)$  induces a homomorphism  $H_{c,k}(G, A) \rightarrow H_{lc,k}(G, A)$ . Likewise (cf. [Fuc11b, Section 4]) one can define another bicomplex

$$A_{lc,k}^{p,q}(G, A) := \left\{ f : G^{p+1} \times G^{q+1} \rightarrow G \mid \exists U \in \mathcal{U}_1 : f|_{kG^{p+1} \times_k k\Gamma_U^q} \text{ is continuous} \right\},$$

where  $kG^{p+1} \times_k k\Gamma_U^q$  is the product in  $\mathbf{kTop}$ . As observed in [Fuc11b, Section 4]) the rows and columns of the double complex  $A_{lc,k}^{p,q}(G, A)^G$  can be augmented by the complexes  $C_{lc,k}^*(G, A)$  and  $C_{lc,k}^*(G, A)$  respectively. The proof of the preceding two sections carry over to yield the following results.

**Lemma III.1.** (cf. Lemma I.2) *The inclusion  $j_{h,k}^* : C_{lc,k}^*(G, A) \hookrightarrow \text{Tot}A_{lc,k}^{*,*}(G, A)^G$  induces an isomorphism in cohomology.*

**Proposition III.2.** (cf. Proposition II.1) *If the augmented column complexes  $A_{c,k}^p(G, A) \hookrightarrow A_{lc,k}^{p,*}(G, A)$  are exact, then the augmented sub column complexes  $C_{c,k}^p(X; V)^G \hookrightarrow A_{lc,k}^{p,*}(X; V)^G$  are exact as well.*

**Proposition III.3.** (cf. Proposition II.3) *The image  $j_{k,v}^n(f)$  of a homogeneous continuous group  $n$ -cocycle  $f$  on  $G$  in  $\text{Tot}A_{lc,k}^{*,*}(G, A)^G$  is cohomologous to the image  $j_{h,k}^n i^n(f)$  of the locally continuous homogeneous group  $n$ -cocycle  $i^n(f)$  in  $\text{Tot}A_{lc,k}^{*,*}(G, A)^G$ .*

**Corollary III.4.** *The homomorphism*

$$H(j_{h,k}^p)^{-1} H(j_{v,k}^p) : H_{c,k}^p(G, A) \rightarrow H_{lc,k}^p(G, A)$$

*is induced by the inclusion  $C_{c,k}^*(G, A) \hookrightarrow C_{lc,k}^*(G, A)$ .*

**Corollary III.5.** *If the augmented column complexes  $A_{c,k}^p(G, A) \hookrightarrow A_{lc,k}^{p,*}(G, A)$  are exact, then the inclusion  $C_{c,k}^*(G, A) \hookrightarrow C_{lc,k}^{*,*}(G, A)$  induces an isomorphism in cohomology.*

Up to here the procedure is exactly the same as in the Section II. However, we have to restrict the setting slightly for the main result of this sections that we now restrict to.

**Proposition III.6.** *If  $G$  is a  $k$ -group whose finite products are  $k$ -spaces for the product topology and  $A$  is a loop contractible continuous  $G$ -module, then  $C_{c,k}^*(G, A) \hookrightarrow C_{lc,k}^*(G, A)$  induces an isomorphism  $H_{c,k}^p(G, A) \cong H_{lc,k}^p(G, A)$  in cohomology.*



**Proof.** By the previous corollary we have to show that the augmented column complexes  $A_{c,k}^p(G, A) \hookrightarrow A_{lc,k}^{p,*}(G, A)$  are exact. The assumption that each  $G^k$ , endowed with the product topology, is a  $k$ -space ensures that the open diagonal neighbourhoods  $\mathfrak{U}[k]$  are cofinal in the directed system of all open diagonal neighbourhoods. With this observation the proof of Proposition II.6 of the exactness of  $A_{c,k}^p(G, A) \hookrightarrow A_{lc,k}^{p,*}(G, A)$  carries over to the compactly generated case if one replaces the results from [Fuc11b] and [Fuc11a] accordingly.

In more detail, the cosimplicial group  $A^{p,*}(G, \mathfrak{U}; A)$  has to be replaced with

$$A_k^{p,*}(G, \mathfrak{U}; A) := \{f : G^{p+1} \times \mathfrak{U}[*] \rightarrow A \mid \forall \vec{g} \in \mathfrak{U}[*] : f(-, \vec{g}) \in C(kG^{p+1}, A)\},$$

[Fuc11b, Lemma 3.12] has to be replaced with [Fuc11b, Lemma 5.12] and [Fuc11a, Corollary 5.20] has to be replaced with [Fuc11a, Corollary 6.20]. ■

## IV Smooth and Locally Smooth Group Cohomology

From now on we assume that  $G$  and  $A$  are Lie groups and that  $A$  is a smooth  $G$ -module<sup>3</sup>. Replacing the notion of continuity in Section II by smoothness we obtain the complex  $C_s^*(G, A)$  of smooth homogeneous group cochains and the complex  $C_{ls}^*(G, A)$  of homogeneous group cochains which are smooth only on some identity neighbourhood in  $G$ . These augment the columns and rows of the double complex  $A_{ls}^{*,*}(G, A)^G$  given by the  $G$ -invariants of the double complex

$$A_{ls}^{p,q}(G, A) := \left\{ f : G^{p+1} \times G^{q+1} \rightarrow G \mid \exists U \in \mathcal{U}_1 : f|_{G^{p+1} \times \Gamma_U^q} \text{ is smooth} \right\},$$

We denote the total complex by  $\text{Tot}A_{ls}^{*,*}(G, A)^G$ . The augmentations of the rows induces a morphisms  $j_h^* : C_{ls}^*(G, A) \hookrightarrow \text{Tot}A_{ls}^{*,*}(G, A)^G$  of cochain complexes. Likewise, the augmentations of the columns induces a morphism  $j_v^* : C_s^*(G, A) \hookrightarrow \text{Tot}A_{ls}^{*,*}(G, A)^G$ . On each row  $A_{ls}^{*,q}(G, A)^G$  one can define the same contracting homotopy  $h^*$  as in (1) showing

**Lemma IV.1.** (cf. Lemma I.2) *The inclusion  $j_h^* : C_{ls}^*(G, A) \hookrightarrow \text{Tot}A_{ls}^{*,*}(G, A)^G$  induces an isomorphism in cohomology.*

Following the same argumentation as in Section II we further obtain:

**Proposition IV.2.** (cf. Proposition II.1) *If the augmented complexes  $A_s^p(G, A) \hookrightarrow A_{ls}^{p,*}(G, A)$  are exact, then the augmented sub column complexes  $C_s^p(G, A) \hookrightarrow A_{ls}^{p,*}(G, A)^G$  of equivariant cochains are exact as well.*

**Proposition IV.3.** (cf. Proposition II.3) *The image  $j_v^n(f)$  of a homogeneous smooth group  $n$ -cocycle  $f$  on  $G$  in  $\text{Tot}A_{ls}^{*,*}(G, A)^G$  is cohomologous to the image  $j_h^n i^n(f)$  of the locally smooth homogeneous group  $n$ -cocycle  $i^n(f)$  in  $\text{Tot}A_{ls}^{*,*}(G, A)^G$ .*

<sup>3</sup>Manifolds are understood in the general infinite dimensional calculus from [BGN04]

**Corollary IV.4.** *The homomorphism  $H(j_h^p)^{-1}H(j_v^p) : H_s^p(G, A) \rightarrow H_{ls}^p(G, A)$  is induced by the inclusion  $C_s^*(G, A) \hookrightarrow C_{ls}^*(G, A)$ .*

**Corollary IV.5.** *If the augmented complexes  $A_s^p(G, A) \hookrightarrow A_{ls}^{p,*}(G, A)$  are exact, then the inclusion  $C_s^*(G, A) \hookrightarrow C_{ls}^*(G, A)$  induces an isomorphism in cohomology.*

Like in the previous section, the procedure was exactly the same as in Section II, but now comes the point where we have to impose an additional condition on  $G$ .

**Proposition IV.6.** *If  $G$  is a Lie group whose finite products are smoothly paracompact and  $A$  is a smoothly loop contractible smooth  $G$ -module, then the inclusion  $C_s^*(G, A) \hookrightarrow C_{ls}^*(G, A)$  induces an isomorphism  $H_s^p(G, A) \cong H_{ls}^p(G, A)$  in cohomology.*

**Proof.** Analogously to Proposition II.6 one shows that the augmented complexes  $A_s^p(G, A) \hookrightarrow A_{ls}^{p,*}(G, A)$  are exact. The assumption on each  $G^k$  to be smoothly paracompact allows us to replace the results from [Fuc11b] and [Fuc11a] accordingly.

In more detail, we replace the cosimplicial group  $A^{p,*}(G, \mathfrak{U}_U; A)$  with

$$A_s^{p,*}(G, \mathfrak{U}_U; A) := \{f : G^{p+1} \times \mathfrak{U}_U[*] \rightarrow A \mid \forall \vec{g} \in \mathfrak{U}_U[*] : f(-, \vec{g}) \in C^\infty(G^{p+1}, A)\},$$

[Fuc11b, Lemma 3.12] has to be replaced with [Fuc11b, 7.12] and [Fuc11a, Corollary 5.20] has to be replaced with [Fuc11a, Theorem 7.16]. ■

## V Application to finite-dimensional Lie groups

In this section we show which impact the results of the previous sections have for finite-dimensional Lie groups. We first recall the following fact from [BW00, Lemma IX.5.2] or [HM62, Theorem 5.1].

**Theorem V.1.** *If  $G$  is finite-dimensional,  $\mathfrak{a}$  is a quasi complete locally convex space<sup>4</sup> and a smooth  $G$ -module, then the inclusion  $C_s^*(G, \mathfrak{a}) \hookrightarrow C_c^*(G, \mathfrak{a})$  induces an isomorphism in cohomology.*

**Lemma V.2.** *If  $\Gamma$  is a discrete  $G$ -module, then the inclusion  $C_s^*(G, \Gamma) \hookrightarrow C_c^*(G, \Gamma)$  is an isomorphism. In particular, it induces an isomorphism in cohomology.*

**Proof.** If  $A$  is discrete, then smooth maps are the same thing as continuous maps. ■

**Corollary V.3.** *If  $G$  is finite-dimensional,  $\mathfrak{a}$  is a quasi complete locally convex space and a smooth  $G$ -module and  $\Gamma \subset \mathfrak{a}$  is a discrete submodule, then the inclusion  $C_{ls}^*(G, A) \hookrightarrow C_{lc}^*(G, A)$  induces an isomorphism in cohomology, where  $A$  denotes the smooth  $G$ -module  $\mathfrak{a}/\Gamma$ .*

<sup>4</sup>A locally convex space is said to be quasi-complete if each bounded Cauchy net converges.

**Proof.** The exact sequence  $\Gamma \hookrightarrow \mathfrak{a} \rightarrow A$  of coefficients admits a smooth local section and thus induces long exact sequences in locally smooth and locally continuous cohomology (the argument of [Nee04, Appendix E] carries over literally to locally continuous group cohomology if the coefficient sequence admits a section which is continuous on some identity neighbourhood.). Together with the inclusions  $C_{ls}^*(G, A) \hookrightarrow C_{lc}^*(G, A)$  this gives rise to the commuting diagram

$$\begin{array}{ccccccccc} H_{ls}^n(G, \Gamma) & \rightarrow & H_{ls}^n(G, \mathfrak{a}) & \rightarrow & H_{ls}^n(G, A) & \rightarrow & H_{ls}^{n+1}(G, \Gamma) & \rightarrow & H_{ls}^{n+1}(G, \mathfrak{a}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{lc}^n(G, \Gamma) & \rightarrow & H_{lc}^n(G, \mathfrak{a}) & \rightarrow & H_{lc}^n(G, A) & \rightarrow & H_{lc}^{n+1}(G, \Gamma) & \rightarrow & H_{lc}^{n+1}(G, \mathfrak{a}) \end{array}$$

with exact rows. The first and last two vertical morphisms are isomorphisms by the preceding results of this section and Corollary II.8 and Proposition IV.6. Thus the middle one is also an isomorphism by the five lemma. ■

The previous result does not hold in for infinite-dimensional  $G$ , as the case  $G = C(S^1, K)$  for  $K$  a compact, simple and simply connected Lie group shows. The inclusion  $C^\infty(S^1, K) \hookrightarrow C(S^1, K)$  is a homotopy equivalence by [Nee02, Remark A.3.8] and thus the universal central extension of  $C^\infty(S^1, K)$  induces a topological non-trivial bundle

$$U(1) \rightarrow P \rightarrow C(S^1, K).$$

Now  $P$  can be equipped with the structure of a topological group, since this is invariant under homotopy equivalences [Gr72, Prop. VII.1.3.5]. This gives rise to a non-trivial element in  $H_{lc}^2(C(S^1, K), U(1))$ . However,  $C(S^1, K)$  is simply connected and thus does not have any non-trivial central Lie group extension by [Nee02, Theorem 7.12] and Corollary 13 and Theorem 16 from [Mai02]. Thus  $H_{ls}^2(C(S^1, K), U(1))$  vanishes. However,  $C(S^1, K)$  is not smoothly paracompact, which would be the natural framework under which one would expect that  $H_{ls}^*(G, A)$  is isomorphic to  $H_{lc}^*(G, A)$ . To the knowledge of the authors this is an open problem for smoothly paracompact  $G$

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