A Cocycle Model for for Topological and Lie Group Cohomology

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Abstract

We propose a unified framework in which the different constructions of cohomology groups for topological and Lie groups can all be treated on equal footings. In particular, we show that the cohomology of "locally continuous" cochains (respectively "locally smooth" in the case of Lie groups) fits into this framework, which provides an easily accessible cocycle model for topological and Lie group cohomology. We illustrate the use of this unified framework and the relation between the different models in various applications. This includes the construction of cohomology classes characterizing the string group and a direct connection to Lie algebra cohomology.

Introduction

It is a common pattern in mathematics that things that are easy to define are hard to compute and things that are hard to define come with lots of machinery to compute them[1]. On the other hand, mathematics can be very enjoyable if these different definitions can be shown to yield isomorphic objects. In the present article we want to promote such a perspective towards topological group cohomology, along with its specialization to Lie group cohomology.

It has become clear in the last decade that concretely accessible cocycle models for cohomology theories (understood in a broader sense) are as important as abstract constructions. Examples for this are differential cohomology theories (cocycle models come for instance from (bundle) gerbes, an important concept in topological and conformal field theory), elliptic cohomology (where cocycle models are yet conjectural but have nevertheless already been quite influential) and Chas-Sullivan's string topology operations (which are subject to certain very well behaved representing cocycles). This article describes an easily accessible cocycle model for the more complicated to define cohomology theories of topological and Lie groups [Seg70, Wig73, Del74, Bry00]. The cocycle model is a seemingly obscure mixture of (abstract) group cohomology, added in a continuity condition only around the identity. Its smooth analogue has been used in the context of Lie group cohomology and its relation to Lie algebra cohomology [TW87, WX91, Nee02, Nee04, Nee06, Nee07], which is where our original motivation stems from. The basic message will be that all the above concepts of topological and Lie group cohomology coincide for finite-dimensional Lie groups and coefficients modeled on quasi-complete locally convex spaces. Beyond finite-dimensional Lie groups the smooth and the continuous concepts begin to diverge, but still all continuous concepts agree.

There is a naïve notion of topological group cohomology for a topological group G and a continuous G-module A. It is the cohomology of the complex of continuous cochains with respect to the usual group differential. This is what we call "globally continuous" group cohomology and denote it by $H^n_{\mathrm{glob},c}(G, A)$. It cannot encode the topology of G appropriately, for instance $H^2_{\mathrm{glob},c}(G, A)$ can only describe abelian extensions which are topologically trivial bundles. However, in case G is contractible it will turn out that the more elaborate cohomology groups from above coincide with $H^n_{\mathrm{glob},c}(G, A)$. In this sense, the deviation from the above cohomology groups from being the globally continuous ones measures the non-triviality of the topology of

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[1] Quote taken from a lecture by Janko Latschev.

G. On the other hand, the comparison between $H^n_{\mathrm{glob},c}(G, A)$ and the other cohomology groups for topologically trivial *coefficients* A will give rise to a comparison theorem between the other cohomology groups. It is this circle of ideas that the present article is about.

The paper is organized as follows. In the first section we review the construction and provide the basic facts of what we call locally continuous group cohomology $H^n_{\mathrm{loc},c}(G, A)$ (respectively the locally smooth cohomology $H^n_{\mathrm{loc},s}(G, A)$ for G a Lie group and A a smooth G-module). Since it will become important in the sequel we highlight in particular that for loop contractible coefficients these cohomology groups coincide with the globally continuous (respectively smooth) cohomology groups $H^n_{\mathrm{glob},c}(G, A)$ (respectively $H^n_{\mathrm{glob},s}(G, A)$). In the second section we then introduce what we call simplicial continuous cohomology $H^n_{\mathrm{simp},c}(G, A)$ and construct a comparison morphism $H^n_{\mathrm{simp},c}(G, A) \to H^n_{\mathrm{loc},c}(G, A)$. The third section explains how simplicial cohomology may be computed in a way similar to computing sheaf cohomology via Čech cohomology (the fact that this gives indeed $H^n_{\mathrm{simp},c}(G, A)$ will have to wait until the next section).

The first main point of this paper comes in Section IV, where we give the following axiomatic characterization of what we call a cohomology theory for topological groups.

Theorem (Comparison Theorem). *Let G be a compactly generated topological group and let* **G-Mod** *be the category of locally contractible G-modules. Then there exists, up to isomorphism, exactly one sequence of functors $(H^n \colon \mathbf{G\text{-}Mod} \to \mathbf{Ab})_{n \in \mathbb{N}_0}$ admitting natural long exact sequences for short exact sequences in* **G-Mod** *such that*

1. *$H^0(A) = A^G$ is the invariants functor*

2. *$H^n(A) = H^n_{\mathrm{glob},c}(G, A)$ for contractible A.* ∎

There is one other way of defining cohomology groups $H^n_{\mathrm{SM}}(G, A)$ which it due to Segal and Mitchison [Seg70]. This construction will turn out to be the one which is best suited for establishing the Comparison Theorem. However, we then show that under some mild assumptions (guaranteed for instance by the metrizability of G) all cohomology theories that we had so far (except the globally continuous) obey these axiomatics. The rest of the section in then devoted to showing that almost all other concepts of cohomology theories for topological groups also fit into this scheme. This includes the ones considered by Flach in [Fla08], the measurable cohomology of Moore from [Moo76] and the mixture of measurable an locally continuous cohomology of Khedekar and Rajan from [KR10]. The only exception that we know not fitting into this scheme is the continuous bounded cohomology (see [Mon01, Mon06]), which differs from the above concepts by design.

The second main point comes with Section V, where we exploit the interplay between the different constructions. For instance, we construct cohomology classes that deserve to be named string classes, and we construct topological crossed modules associated to third cohomology classes. Moreover, we show how to extract the purely topological information contained in an element in $H^n_{\mathrm{loc},c}(G, A)$ by relating an explicit formula for this with a structure map for the spectral sequence associated to $H^n_{\mathrm{simp},c}(G, A)$. Furthermore, $H^n_{\mathrm{loc},s}(G, A)$ maps naturally to Lie algebra cohomology and we use the previous result to identify situations where this map becomes an isomorphism. Almost none of the consequences mentioned here could be drawn from one model on its own, so this demonstrates the strength of the unified framework.

In the last two sections, which are independent from the rest of the paper, we provide some details on the constructions that we use.

Acknowledgements

Conventions

Since we will be working in the two different regimes of compactly generated Hausdorff spaces and infinite-dimensional Lie groups we have to choose the setting with some care.

Unless further specified, G will throughout be a group in the category **CGHaus** of compactly generated Hausdorff spaces (cf. [Whi78, Mac98] or [Hov99]) and A will be a locally contractible G-module in this category[2]. This means that the multiplication (respectively action) map is continuous with respect to the *compactly generated topology* on the product. Note that the topology on the product may be finer than the product topology, so this may not be a topological group (respectively module) as defined below. To avoid confusion, we denote the compactly generated product by $X \times_{\boldsymbol{k}} Y$ ($X^{\times_{\boldsymbol{k}}^n}$ for the n-fold product) and the compactly generated topology on $C(X, Y)$ by $C_{\boldsymbol{k}}(X, Y)$ for X, Y in **CGHaus**.

If X and Y are arbitrary topological spaces, then we refer to the product topology by $X \times_{\boldsymbol{p}} Y$. With a topological group (respectively topological module) we shall mean a group (respectively module) in this category, i.e., the multiplication (respectively action) is continuous for the product topology.

Frequently we will assume, in addition, that G is a (possibly infinite dimensional) Lie group and that A is a smooth G-module[3]. With this we mean that G is a group in the category **Man** of manifolds, modeled modeled on locally convex vector spaces (see [Ham82, Mil84, Nee06] or [GN11] for the precise setting) and A is a G-modules in this category. This means in particular that the multiplication (respectively action) map is smooth for the product smooth structure. To avoid confusion we refer to the product in **Man** by $X \times_{\boldsymbol{m}} Y$ (and $X^{\times_{\boldsymbol{m}}^n}$).

Note that we set things up in such a way that the smooth setting is a specialization of the topological one, which is in turn a specialization of the compactly generated one. This is true since smooth maps are in particular continuous and since the product topology is coarser than the compactly generated one. Note also that all topological properties on G (except the existence of good covers) that we will assume are satisfied for metrizable G and all smoothness properties are satisfied for metrizable and smoothly paracompact G. The existence of good cover (as well as metrizability and smooth paracompactness) is in turn satisfied for large classes of infinite-dimensional Lie groups like mapping groups or diffeomorphism groups [KM97, SW10].

We shall sometimes have to impose topological conditions on the topological spaces underlying G and A. We will do so by leisurely adding the corresponding adjective. For instance, a contractible G-module A is a G-module such that A is contractible as a topological space.

I Locally continuous and locally smooth cohomology

One of our main objectives will be the relation of locally continuous and locally smooth cohomology for topological or Lie groups to other concepts of topological group cohomology. In this section, we recall the basic notions and properties of locally continuous and locally smooth cohomology. These concepts already appear in the work of Tuynman-Wiegerinck [TW87], of Weinstein-Xu [WX91] and have been popularized recently by Neeb [Nee02, Nee04, Nee06, Nee07]. There has also appeared a slight variation of this by measurable locally smooth cohomology in [KR10].

Definition I.1. For any pointed topological space (X, x) and abelian topological group A we set

$$C_{\mathrm{loc}}(X, A) := \{f \colon X \to A \mid f \text{ is continuous on some neighborhood of } x\}.$$

If, moreover, X is a smooth manifold and A a Lie group, then we set

$$C_{\mathrm{loc}}^\infty(X, A) := \{f \colon X \to A \mid f \text{ is smooth on some neighborhood of } x\}.$$

With this we set $C_{\mathrm{loc},c}^n(G, A) := C_{\mathrm{loc}}(G^{\times_{\boldsymbol{k}}^n}, A)$, where we choose the identity in G^n as base-point. We call these

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[2]From the beginning of Section IV we will also assume that A is locally contractible.

[3]This assumption seems to be quite restrictive for either side, but it is the natural playground on which homotopy theory and (infinite-dimensional) Lie theory interacts.

functions (by some abuse of language) *locally continuous group cochains*. The ordinary group differential

$$(\mathtt{d}_{\mathrm{gp}} f)(g_0, \ldots, g_n) = g_0.f(g_1, \ldots, g_n) +$$
$$+ \sum_{j=1}^{n} (-1)^j f(g_0, \ldots, g_{j-1} g_j, \ldots, g_n) + (-1)^{n+1} f(g_0, \ldots, g_{n-1}) \tag{1}$$

turns $(C_{\mathrm{loc},c}^n(G, A), \mathtt{d}_{\mathrm{gp}})$ into a cochain complex. Its cohomology will be denoted by $H_{\mathrm{loc},c}^n(G, A)$ and be called the *locally continuous group cohomology*.

If G is a Lie group and A a smooth G-module, then we also consider the sub complex $C_{\mathrm{loc},s}^n(G, A) := C_{\mathrm{loc}}^\infty(G^{\times_m^n}, A)$ and call its cohomology $H_{\mathrm{loc},s}^n(G, A)$ the *locally smooth group cohomology*. ∎

These two concepts should not be confused with the continuous *local cohomology* (respectively the smooth *local cohomology*) of G, which is given by the complex of germs of continuous (respectively smooth) A-valued functions at the identity (which is isomorphic to the Lie algebra cohomology for a finite-dimensional Lie group G, see Remark V.14). It is crucial that the cocycles in the locally continuous cohomology actually are extensions of locally defined cocycles and this extension is extra information they come along with. Note for instance, that not all locally defined homomorphisms of a topological groups extend to global homomorphisms and that not all locally defined 2-cocycles extend to globally defined cocycles [Smi51a, Smi51b, Est62a, Est62b].

Remark I.2. (cf. [Nee04, App. E]) Let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{2}$$

be a *short exact sequence of G-modules in* **CGHaus**, i.e., the underlying sequence of abstract abelian groups is exact and β (or equivalently α) has a continuous local section. The latter is equivalent to demanding that (2) is a locally trivial principal A-bundle. Then composition with α and β induces a sequence

$$C_{\mathrm{loc},c}^n(G, A) \xrightarrow{\alpha_*} C_{\mathrm{loc},c}^n(G, B) \xrightarrow{\beta_*} C_{\mathrm{loc},c}^n(G, C), \tag{3}$$

which we claim to be a short exact sequence of chain complexes. Injectivity of α_* and $\mathrm{im}(\alpha_*) \subseteq \ker(\beta_*)$ is clear. Since a local trivialization of the bundle induces a continuous left inverse to α on some neighborhood of $\ker(\beta)$, we also have $\ker(\beta_*) \subseteq \mathrm{im}(\alpha_*)$. To see that β_* is surjective, we choose a local continuous section $\sigma \colon U \to B$ which we extend to a global (but not necessarily continuous) section $\sigma \colon C \to B$. Thus if $f \in C_{\mathrm{loc},c}^n(G, C)$, then $\sigma \circ f \in C_{\mathrm{loc},c}^n(G, B)$ with $\beta_*(\sigma \circ f) = \beta \circ \sigma \circ f = f$ and β_* is surjective. Since (3) is exact, it induces a long exact sequence

$$\cdots \to H_{\mathrm{loc},c}^{n-1}(G, C) \to H_{\mathrm{loc},c}^n(G, A) \to H_{\mathrm{loc},c}^n(G, B) \to H_{\mathrm{loc},c}^n(G, C) \to H_{\mathrm{loc},c}^{n+1}(G, A) \to \cdots \tag{4}$$

in the locally continuous cohomology.

If, in addition, G is a Lie group and (2) is a *short exact sequence of smooth G-modules*, i.e., a smooth locally trivial principal A-bundle, then the same argument shows that α_* and β_* induce a long exact sequence

$$\cdots \to H_{\mathrm{loc},s}^{n-1}(G, C) \to H_{\mathrm{loc},s}^n(G, A) \to H_{\mathrm{loc},s}^n(G, B) \to H_{\mathrm{loc},s}^n(G, C) \to H_{\mathrm{loc},s}^{n+1}(G, A) \to \cdots$$

in the locally smooth cohomology. ∎

Remark I.3. The low-dimensional cohomology groups $H_{\mathrm{loc},c}^0(G, A)$, $H_{\mathrm{loc},c}^1(G, A)$ and $H_{\mathrm{loc},c}^2(G, A)$ have the usual interpretations. $H_{\mathrm{loc},c}^0(G, A) = A^G$ are the G-invariants of A, $H_{\mathrm{loc},c}^1(G, A)$ (respectively $H_{\mathrm{loc},s}^1(G, A)$) is the group of equivalence classes of continuous (respectively smooth) crossed homomorphisms modulo principal crossed homomorphisms. If G is connected, then $H_{\mathrm{loc},c}^2(G, A)$ (respectively $H_{\mathrm{loc},s}^2(G, A)$) is the group of equivalence classes of abelian extensions

$$A \to \widehat{G} \to G \tag{5}$$

which are continuous (respectively smooth) locally trivial principal A-bundles over G [Nee04, Sect. 2]. ∎

Remark I.4. The cohomology groups $H^n_{\mathrm{loc},c}(G, A)$ and $H^n_{\mathrm{loc},s}(G, A)$ are variations of the more naïve *globally continuous* cohomology groups $H^n_{\mathrm{glob},c}(G, A)$ and *globally smooth* cohomology groups $H^n_{\mathrm{glob},s}(G, A)$, which are the cohomology groups of the chain complexes

$$C^n_{\mathrm{glob},c}(G, A) := C(G^{\times \frac{n}{k}}, A) \quad \text{and} \quad C^n_{\mathrm{glob},s}(G, A) := C^\infty(G^{\times \frac{n}{m}}, A),$$

endowed with the differential (1). We obviously have

$$H^0_{\mathrm{loc},c}(G, A) = H^0_{\mathrm{glob},c}(G, A) \quad \text{and} \quad H^0_{\mathrm{loc},s}(G, A) = H^0_{\mathrm{glob},s}(G, A).$$

Since crossed homomorphisms are continuous (respectively smooth) if and only if they are so on some identity neighborhood (see for example [Nee04, Lemma III.1]), we also have

$$H^1_{\mathrm{loc},c}(G, A) = H^1_{\mathrm{glob},c}(G, A) \quad \text{and} \quad H^1_{\mathrm{loc},s}(G, A) = H^1_{\mathrm{glob},s}(G, A).$$

Moreover, the argument from Remark I.2 also shows that we have a long exact sequence

$$\cdots \to H^{n-1}_{\mathrm{glob},c}(G, C) \to H^n_{\mathrm{glob},c}(G, A) \to H^n_{\mathrm{glob},c}(G, B) \to H^n_{\mathrm{glob},c}(G, C) \to H^{n+1}_{\mathrm{glob},c}(G, A) \to \cdots$$

if the exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ has a global continuous section (and respectively for the globally smooth cohomology if $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ has a global smooth section).

Now assume that A is contractible (respectively smoothly contractible) and that G is connected and paracompact (respectively smoothly paracompact). In this case, the bundle (5) has a global continuous (respectively smooth) section and thus the extension (5) has a representative in $H^2_{\mathrm{glob},c}(G, A)$ (respectively $H^2_{\mathrm{glob},s}(G, A)$), cf. [Nee04, Prop. 6.2]. Moreover, the argument in [Nee04, Prop. 6.2] also shows that two extensions of the form (5) are in this case equivalent if and only if the representing globally continuous (respectively smooth) cocycles differ by a globally continuous (respectively smooth) coboundary, and thus the canonical homomorphisms

$$H^2_{\mathrm{glob},c}(G, A) \to H^2_{\mathrm{loc},c}(G, A) \quad \text{and} \quad H^2_{\mathrm{glob},s}(G, A) \to H^2_{\mathrm{loc},s}(G, A)$$

are isomorphisms in this case. ∎

It will be crucial in the following that the latter observation also holds for a large class of contractible coefficients in arbitrary dimension (and in the topological case also for not necessarily paracompact G). For this, recall that A is called *loop-contractible* if there exists a contracting homotopy $\rho \colon [0, 1] \times A \to A$ such that $\rho_t \colon A \to A$ is a group homomorphism for each $t \in [0, 1]$. If A is a Lie group, then it is called *smoothly loop-contractible* if ρ is, in addition, smooth. In particular, vector spaces are smoothly loop-contractible, but in the topological case there exist more elaborate and important examples (see Section IV).

Proposition I.5. *If A is loop-contractible, and the product topology on all G^n is compactly generated, then the inclusion $C^n_{\mathrm{glob},c}(G, A) \hookrightarrow C^n_{\mathrm{loc},c}(G, A)$ induces an isomorphism $H^n_{\mathrm{glob},c}(G, A) \cong H^n_{\mathrm{loc},c}(G, A)$.*

If G is a Lie group such that all $G^{\times \frac{n}{m}}$ are smoothly paracompact and A is a smooth G-module which is smoothly loop-contractible, then $C^n_{\mathrm{glob},s}(G, A) \hookrightarrow C^n_{\mathrm{loc},s}(G, A)$ induces an isomorphism $H^n_{\mathrm{glob},s}(G, A) \cong H^n_{\mathrm{loc},s}(G, A)$.

Proof. This is [FW11, Prop. III.6, Prop. IV.6]. ∎

In the case of discrete A we note that there is no difference between the locally continuous and locally smooth cohomology groups. This is immediate since continuous and smooth maps into discrete spaces are both the same thing as constant maps on connected components.

Lemma I.6. *If G is a Lie group and A is a discrete G-module, then the inclusion $C^n_{\mathrm{loc},s}(G, A) \hookrightarrow C^n_{\mathrm{loc},c}(G, A)$ induces an isomorphism in cohomology $H^n_{\mathrm{loc},s}(G, A) \to H^n_{\mathrm{loc},c}(G, A)$.*

In the finite-dimensional case, we also note that there is no difference between the locally continuous and locally smooth cohomology groups.

Proposition I.7. *Let G be a finite-dimensional Lie group, $\mathfrak{a}$ be a quasi-complete locally convex space[4] on which G acts smoothly, $\Gamma \subseteq \mathfrak{a}$ be a discrete submodule and set $A = \mathfrak{a}/\Gamma$. Then the inclusion $C_{\mathrm{loc},s}^n(G, A) \hookrightarrow C_{\mathrm{loc},c}^n(G, A)$ induces an isomorphism $H_{\mathrm{loc},s}^n(G, A) \cong H_{\mathrm{loc},c}^n(G, A)$.*

Proof. (cf. [FW11, Cor. V.3]) If $\Gamma = \{0\}$, then this is implied by Proposition I.5 and [HM62, Thm. 5.1]. The general case then follows from the previous lemma, the short exact sequence for the coefficient sequence $\Gamma \to \mathfrak{a} \to A$ and the Five Lemma. ∎

Remark I.8. Proposition I.7 does not remain valid in infinite dimensions. In fact, the group of continuous free loops $G = C(\mathbb{S}^1, K)$ into a simple compact 1-connected Lie group K provides a counterexample. The group G is a Banach Lie group in the compact-open topology modeled on its Lie algebra $\mathfrak{g} = C(\mathbb{S}^1, \mathfrak{k})$ where $\mathfrak{k}$ is the Lie algebra of K.

On the one hand, $H_{\mathrm{loc},s}^2(G, \mathrm{U}(1))$ vanishes. This is true, for instance, since its Lie algebra $\mathfrak{g}$ has trivial second continuous Lie algebra cohomology $H_{\mathrm{Lie},c}^2(\mathfrak{g}, \mathbb{R})$ [Mai02, Cor. 13, Thm. 16] and the sequence

$$\mathrm{Hom}(\pi_1(G), U(1)) \to H_{\mathrm{loc},s}^2(G, \mathrm{U}(1)) \to H_{\mathrm{Lie},c}^2(\mathfrak{g}, \mathbb{R})$$

is exact [Nee02, Thm. 7.12]. Since $\pi_1(G) \cong \pi_2(K) \oplus \pi_1(K) = 0$ this shows that $H_{\mathrm{loc},s}^2(G, \mathrm{U}(1))$ vanishes.

On the other hand, the 2-connected space $G\langle 2 \rangle$ (second step in the Whitehead tower) gives rise to a non-trivial element in $H_{\mathrm{loc},c}^2(G, \mathrm{U}(1))$, since it may be realized as a central extension

$$U(1) \to G\langle 2 \rangle \to G \tag{6}$$

of topological groups which is a locally trivial principal $U(1)$-bundle. In order to verify this, we note that a bundle $U(1) \to Q \to G$ admits on Q the structure of a topological group turning it into a central extension if and only if it is multiplicative, i.e., if $\mathrm{pr}_2^*(Q) \otimes m^*(Q) \otimes \mathrm{pr}_1^*(Q)$ is trivial as a bundle on $G \times G$ [Gro72, Prop. VII.1.3.5]. The inclusion $C^\infty(\mathbb{S}^1, K) \hookrightarrow G$ is a homotopy equivalence [Nee02, Remark A.3.8] and thus the universal central extension of $C^\infty(\mathbb{S}^1, K)$ gives rise to a multiplicative bundle $U(1) \to P \to G$, which carries the structure of central extension of topological groups. Since $\pi_2(G) \to \pi_1(U(1))$ is an isomorphism, this cannot not be trivial, and thus $H_{\mathrm{loc},c}^2(G, U(1))$ does not vanish. This phenomenon might be related to the missing smooth paracompactness of G or to the lack of "automatic smoothness" in infinite dimensions (see for example [Glö05]). ∎

Remark I.9. For a topological group G and a topological G-module A there also exists a variation of the locally continuous group cohomology, which are the cohomology groups of the cochain complex $(C_{\mathrm{loc},c}(G^{\times_p^n}, A), \mathrm{d}_{\mathrm{gp}})$ (note the difference in the topology that we put on G^n). We denote this by $H_{\mathrm{loc,top}}^n(G, A)$. The same argument as above yields long exact sequences from short exact sequences of topological G-modules that are locally trivial principal bundles. Moreover, they coincide with the corresponding globally continuous cohomology groups $H_{\mathrm{glob,top}}^n(G, A)$ of $(C(G^{\times_p^n}, A), \mathrm{d}_{\mathrm{gp}})$ if A is loop contractible [FW11, Cor. II.8]. We will use these cohomology groups very seldom. ∎

II   Simplicial group cohomology

The cohomology groups that we introduce in this section date back to [Wig73, Sect. 3] and have also been worked with for instance in [Del74, Fri82, Bry00, Con03]. Since the simplicial cohomology groups are defined in terms of sheaves on simplicial space, we first recall some facts about it. The material is largely taken from [Del74, Fri82] and [Con03].

Definition II.1. Let $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathbf{Top}$ be a simplicial space, i.e., a collection of topological spaces $(X_k)_{k \in \mathbb{N}_0}$, together with continuous face maps maps $d_k^i \colon X_k \to X_{k-1}$ for $i = 0, \ldots, k$ and continuous degeneracy maps

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[4]A locally convex space is said to be quasi-complete if each bounded Cauchy net converges.

$s_k^i \colon X_k \to X_{k+1}$ for $i = 0, \ldots, k$ satisfying the simplicial identities

$$
\begin{cases}
d_k^i \circ d_{k+1}^j = d_k^{j-1} \circ d_{k+1}^i & \text{if } i < j \\
d_{k+1}^i \circ s_k^j = s_{k-1}^{j-1} \circ d_k^i & \text{if } i < j \\
d_{k+1}^i \circ s_k^j = \mathrm{id}_{X_k} & \text{if } i = j \text{ or } i = j+1 \\
d_{k+1}^i \circ s_k^j = s_{k-1}^j \circ d_k^{i-1} & \text{if } i > j+1 \\
s_{k+1}^i \circ s_k^j = s_{k+1}^{j+1} \circ s_k^i & \text{if } i \le j.
\end{cases}
$$

(cf. [GJ99]). Then a sheaf $E^\bullet$ on $X_\bullet$ consists of sheaves E^k of abelian groups on each space X_k and a collection of morphisms $D_i^k \colon d_k^{i\,*} E^{k-1} \to E^k$ (for $k \ge 1$) and $S_i^k \colon s_k^{i\,*} E^{k+1} \to E^k$, obeying the simplicial identities

$$
\begin{cases}
D_j^k \circ d_k^{j\,*} D_i^{k+1} = D_i^k \circ d_k^{i\,*} D_{j-1}^{k+1} & \text{if } i < j \\
S_j^{k+1} \circ s_{k+1}^{j\,*} D_i^k = D_{k+1}^i \circ d_{k+1}^{i\,*} S_{j-1}^{k+2} & \text{if } i < j \\
S_j^{k+1} \circ s_{k+1}^{j\,*} D_i^k = \mathrm{id}_{E^k} & \text{if } i = j \text{ or } i = j+1 \\
S_j^{k+1} \circ s_{k+1}^{j\,*} D_i^k = D_{k+1}^{i-1} \circ d_{k+1}^{i-1\,*} S_k^j & \text{if } i > j+1 \\
S_j^{k+1} \circ s_k^{j\,*} S_i^k = S_i^{k+1} \circ s_{k+1}^{i\,*} S_{j+1}^k & \text{if } i \le j.
\end{cases}
$$

A morphism of sheaves $u \colon E^\bullet \to F^\bullet$ consists of morphisms $u^k \colon E^k \to F^k$ compatible with D_i^k and S_i^k (cf. [Del74, 5.1]). ∎

Note that $E^\bullet$ is *not* what one usually would call a simplicial sheaf since the latter usually refers to a sheaf (on some arbitrary site) with values in simplicial sets or, equivalently, to a simplicial object in the category of sheaves (again, on some arbitrary site). However, one can interpret sheaves on $X_\bullet$ as sheaves on a certain site [Del74, 5.1.8], [Con03, Def. 6.1].

Remark II.2. Sheaves on $X_\bullet$ and their morphisms constitute a category $\mathrm{Sh}(X_\bullet)$. Since morphisms in $\mathrm{Sh}(X_\bullet)$ consist of morphisms of sheaves on each X_k, $\mathrm{Sh}(X_\bullet)$ has naturally the structure of an abelian category (sums of morphisms, kernels and cokernels are simply taken space-wise). Moreover, $\mathrm{Sh}(X_\bullet)$ has enough injectives, since simplicial sheaves on sites do so [Mil80, Prop. II.1.1, 2nd proof], [Con03, p. 36]. ∎

Definition II.3. ([Del74, 5.1.13.1]) The *section functor* is the functor

$$
\Gamma \colon \mathrm{Sh}(X_\bullet) \to \mathbf{Ab}, \quad F^\bullet \mapsto \ker(D_0^1 - D_1^1),
$$

where D_i^1 denotes the homomorphism of the groups of global sections $\Gamma(E^0) \to \Gamma(E^1)$, induced from the morphisms of sheaves $D_i^1 \colon d_1^{i\,*} E^0 \to E^1$. ∎

Lemma II.4. *The functor Γ is left exact.*

Definition II.5. ([Del74, 5.2.2]) The cohomology groups $H^n(X_\bullet, E^\bullet)$ are the right derived functors of the section functor Γ. ∎

Since injective (or acyclic) resolutions in $\mathrm{Sh}(X_\bullet)$ are not easily dealt with (cf. [Con03, p. 36] or the explicit construction in [Fri82, Prop. 2.2]), the groups $H^n(X_\bullet, E^\bullet)$ are notoriously hard to access. However, the following proposition provides an important link to cohomology groups of the sheaves on each single space of $X_\bullet$.

Proposition II.6. *([Del74, 5.2.3.2], [Fri82, Prop. 2.4]) If $E^\bullet$ is a sheaf on $X_\bullet$, then there is a first quadrant spectral sequence with*

$$
E_1^{p,q} = H_{\mathrm{Sh}}^q(X_p, E^p) \Rightarrow H^{p+q}(X_\bullet, E^\bullet). \tag{7}
$$

Remark II.7. We will need the crucial step from the proof of this proposition, so we repeat it here. It is the fact that the spectral sequence arises from a double complex

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
F_2^\bullet \quad & \Gamma(F_2^0) \xrightarrow{d_2^0} \Gamma(F_2^1) \xrightarrow{d_2^1} \Gamma(F_2^2) & \cdots \\
\uparrow \quad & \uparrow \qquad \uparrow \qquad \uparrow \\
F_1^\bullet \quad & \Gamma(F_1^0) \xrightarrow{d_1^0} \Gamma(F_1^1) \xrightarrow{d_1^1} \Gamma(F_1^2) & \cdots \\
\uparrow \quad & \uparrow \qquad \uparrow \qquad \uparrow \\
E^\bullet \quad & \Gamma(E^0) \xrightarrow{d^0} \Gamma(E^1) \xrightarrow{d^1} \Gamma(E^2) & \cdots \\
\end{array}
$$
$$
X_0 \Longleftarrow X_1 \Longleftarrow X_2 \quad \cdots
$$

where each $F_q^\bullet$ is a sheaf on $X_\bullet$, $E^q \to F_\bullet^q$ is an injective resolution in $\mathrm{Sh}(X_q)$ [Con03, Lemma 6.4] and d_q^p is the alternating sum of morphisms induced from the D_i^p, respectively for each sheaf $F_q^\bullet$. Now taking the vertical differential first gives the above form of the E_1-term of the spectral sequence. ∎

Corollary II.8. *If $E^\bullet$ is a sheaf on $X_\bullet$ such that each E^k is acyclic on X_k, then $H^n(X_\bullet, E^\bullet)$ is the cohomology of the Moore complex of the cosimplicial group of sections of $E^\bullet$. More precisely, it is the cohomology of the complex $(\Gamma(X_k, E^k), d)$ with differential given by*

$$
d^k \gamma = \sum_{i=0}^{k} (-1)^i D_i^k {d_k^i}^*(\gamma) \quad for \quad \gamma \in \Gamma(X_k, E^k).
$$

Proof. The E_1-term of the spectral sequence from the previous proposition is concentrated in the first column due to the acyclicity of E^k and yields the described cochain complex. ∎

Remark II.9. The simplicial space that we will work with is the classifying space[5] $BG_\bullet$, associated to G. It is given by setting $BG_n := G^{\times_k^n}$ for $n \geq 1$ and $BG_0 = \mathrm{pt}$ and the standard simplicial maps given by multiplying adjacent elements (respectively dropping the outermost off) and inserting identities.

On $BG_\bullet$ we consider the sheaf $A_{\mathrm{glob},c}^\bullet$, given on $BG_n = G^n$ as the sheaf of continuous A-valued functions $\underline{A}_{G^n}^c$. We turn this into a sheaf on $BG_\bullet$ by introducing the following morphisms D_i^n and S_i^n. The structure maps on $BG_\bullet$ are in this case given by inclusions and projections. Indeed, the face maps factor through projections

$$
G^n \xrightarrow{\cong} G^{n-1} \times_k G \xrightarrow{\mathrm{pr}} G^{n-1} .
$$
$$
\underbrace{\phantom{G^n \xrightarrow{\cong} G^{n-1} \times_k G \xrightarrow{\mathrm{pr}}}}_{d_n^i}
$$

Thus ${d_n^i}^* \underline{A}_{G^n}^c(U) = C(d_n^i(U), A)$ and we may set

$$
(D_i^n f)(g_0, \ldots, g_n) = \begin{cases} f(d_n^i(g_0, \ldots, g_n)) & \text{if } i > 0 \\ g_0.f(g_1, \ldots, g_n) & \text{if } i = 0 \end{cases} .
$$

Similarly,

$$
{s_n^i}^* \underline{A}_{G^{n+1}}^c(U) = \varinjlim_V C(V, A),
$$

where V ranges through all open neighborhoods of $s_n^i(U)$, has a natural homomorphism S_i^n to $\underline{A}_{G^n}^c(U) = C(U, A)$, given by precomposition with s_n^i.

If, in addition, G is a Lie group, then we also consider the slightly different simplicial space $BG_\bullet^\infty$ with $BG_n^\infty = G^{\times_m^n}$ and the same maps. If A is a smooth G-module, we obtain in the same way the sheaf $A_{\mathrm{glob},s}^\bullet$ on $BG_\bullet^\infty$ by considering on each BG_n^∞ the sheaf $\underline{A}_{G^n}^s$ of smooth A-valued functions (in order to make sense out of the latter we have to consider $BG_\bullet^\infty$ instead of $BG_\bullet$). ∎

---

[5] The geometric realization of $BG_\bullet$ yields a model for the (topological) classifying space of G [Seg68], whence the name.

Definition II.10. The *continuous simplicial group cohomology* of G with coefficients in A is defined to be $H^n_{\mathrm{simp},c}(G, A) := H^n(BG_\bullet, A^\bullet_{\mathrm{glob},c})$. If G is a Lie group and A a smooth G-module, then the *smooth simplicial group cohomology* of G with coefficients in A is defined to be $H^n_{\mathrm{simp},s}(G, A) := H^n(BG^\infty_\bullet, A^\bullet_{\mathrm{glob},s})$. ∎

Lemma II.11. *If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is a short exact sequence of G-modules in* **CGHaus***, then composition with α and β induces a long exact sequence*

$$\cdots \to H^{n-1}_{\mathrm{simp},c}(G, C) \to H^n_{\mathrm{simp},c}(G, A) \to H^n_{\mathrm{simp},c}(G, B) \to H^n_{\mathrm{simp},c}(G, C) \to H^{n+1}_{\mathrm{simp},c}(G, A) \to \cdots.$$

If, moreover G is a Lie group and $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is a short exact sequence of smooth G-modules, then α and β induce a long exact sequence

$$\cdots \to H^{n-1}_{\mathrm{simp},s}(G, C) \to H^n_{\mathrm{simp},s}(G, A) \to H^n_{\mathrm{simp},s}(G, B) \to H^n_{\mathrm{simp},s}(G, C) \to H^{n+1}_{\mathrm{simp},s}(G, A) \to \cdots$$

Proof. Since kernels and cokernels of a sheaf $E^\bullet$ are simply the kernels and cokernels of E^k, this follows from the exactness of the sequences of sheaves of continuous functions $\underline{A}^c \to \underline{B}^c \to \underline{C}^c$ (and similarly for the smooth case). ∎

Proposition II.12. *If $G^{\times^n_k}$ is paracompact for each $n \geq 1$ and A is contractible, then*

$$H^n_{\mathrm{simp},c}(G, A) \cong H^n_{\mathrm{glob},c}(G, A).$$

If, moreover, G is a Lie group, A is a smoothly contractible[6] smooth G-module and if $G^{\times^n_m}$ is smoothly paracompact for each $n \geq 1$, then

$$H^n_{\mathrm{simp},s}(G, A) \cong H^n_{\mathrm{glob},s}(G, A).$$

Proof. In the case of contractible A the sheaves $\underline{A}$ are soft and thus acyclic on paracompact spaces [Bre97, Thm. II.9.11]. The first claim thus follows from Corollary II.8. In the smooth case, the requirements are necessary to have the softness of the sheaf of smooth A-valued functions on each G^k as well, since we then can extend sections from closed subsets (cf. (8)) by making use of smooth partitions of unity. ∎

Remark II.13. The requirement on $G^{\times^n_k}$ to be paracompact for each $n \geq 1$ is for instance fulfilled if G is metrizable, since then $G^{\times^n_k} = G^{\times^n_p}$ is so and metrizable spaces are paracompact. If G is, in addition, a smoothly paracompact Lie group, then [KM97, Cor. 16.17] shows that $G^{\times^n_m}$ is also smoothly paracompact.

However, metrizable topological groups are not the most general compactly generated topological groups that can be of interest. Any G that is a CW-complex has the property that $G^{\times^n_k}$ is a CW-complex and thus is in particular paracompact. ∎

We now introduce a second important sheaf on $BG_\bullet$.

Remark II.14. For an arbitrary pointed topological space (X, x) and an abelian topological group A, we denote by $A^{\mathrm{loc},c}_X$ the sheaf

$$U \mapsto \begin{cases} C_{\mathrm{loc}}(U, A) & \text{if } x \in U \\ \mathrm{Map}(U, A) & \text{if } x \notin U \end{cases}$$

and call it the *locally continuous sheaf* on X. If X is a manifold and A an abelian Lie group, then we similarly set

$$A^{\mathrm{loc},s}_X(U) = \begin{cases} C^\infty_{\mathrm{loc}}(U, A) & \text{if } x \in U \\ \mathrm{Map}(U, A) & \text{if } x \notin U. \end{cases}$$

Obviously, these sheaves have the sheaves of continuous functions $\underline{A}$ and of smooth functions $\underline{A}^s$ as sub sheaves.

As in Remark II.9, the sheaves $A^{\mathrm{loc},c}_{G^k}$ assemble into a sheaf $A^\bullet_{\mathrm{loc},c}$ on $BG_\bullet$. Likewise, if G is a Lie group and A is smooth, the sheaves $A^{\mathrm{loc},s}_{G^k}$ assemble into a sheaf $A^\bullet_{\mathrm{loc},s}$ on $BG^\infty_\bullet$. ∎

---

[6]By this we mean that there exists a contraction of A which is smooth as a map $[0, 1] \times_m A \to A$

We learned the importance of the following fact from [SP09].

Proposition II.15. *If X is regular, then $A_X^{\mathrm{loc},c}$ and $A_X^{\mathrm{loc},s}$ are soft sheaves. In particular, these both sheaves are acyclic if X is paracompact.*

Proof. In order to show that $A_X^{\mathrm{loc},c}$ is soft we have to show that sections extend from closed subsets. Let $C \subseteq X$ be closed and

$$[f] \in A_X^{\mathrm{loc},c}(C) = \varinjlim_U A_X^{\mathrm{loc},c}(U) \tag{8}$$

be a section over C, where the limit runs over all open neighborhoods of C (cf. [Bre97, Th. II.9.5]). Thus $[f]$ is represented by some $f \colon U \to A$ for U an open neighborhood of C. The argument now distinguishes the relative position of the base point x which enters the definition of $A_X^{\mathrm{loc},c}$ with respect to U.

If $x \in U$, then we may extend f arbitrarily to obtain a section on X which restricts to $[f]$. If $x \notin U$, then we choose $V \subseteq X$ open with $C \subseteq V$ and $x \notin \overline{V}$ and define $\tilde{f}$ to coincide with f on $U \cap V$ and to vanish elsewhere. This defines a section on X restricting to $[f]$. This argument works for $A_X^{\mathrm{loc},s}$ as well. Since soft sheaves on paracompact spaces are acyclic [Bre97, Thm. II.9.11], this finishes the proof. ∎

Together with Corollary II.8, this now implies

Corollary II.16. *If $G^{\times_k^n}$ is paracompact for all $n \geq 1$, then*

$$H^n(BG_\bullet, A_{\mathrm{loc},c}^\bullet) \cong H_{\mathrm{loc},c}^n(G, A). \tag{9}$$

If G is a Lie group and $G^{\times_p^n}$ is paracompact for all $n \geq 1$, then

$$H^n(BG_\bullet, A_{\mathrm{loc},s}^\bullet) \cong H_{\mathrm{loc},s}^n(G, A).$$

Note that the second of the previous assertions does not require each $G^{\times_m^n}$ to be *smoothly* paracompact, plain paracompactness of the underlying topological space suffices.

Remark II.17. From the isomorphisms (9) we also obtain natural morphisms

$$H_{\mathrm{simp},c}^n(G, A) \to H_{\mathrm{loc},c}^n(G, A) \quad \text{and} \quad H_{\mathrm{simp},s}^n(G, A) \to H_{\mathrm{loc},s}^n(G, A),$$

induced from the morphisms of sheaves $A_{\mathrm{glob},c}^\bullet \to A_{\mathrm{loc},c}^\bullet$ and $A_{\mathrm{glob},s}^\bullet \to A_{\mathrm{loc},s}^\bullet$ on $BG_\bullet$ and $BG_\bullet^\infty$. ∎

III   Čech cohomology

In this section, we will explain how to compute the cohomology groups introduced in the previous section in terms of Čech cocycles. This will also serve as a first touching point to the locally continuous (respectively smooth) cohomology from the first section in degree 2. The proof that all these cohomology theories are isomorphic in all degrees (under some technical conditions) will have to wait until Section IV.

Definition III.1. Let $X_\bullet$ be a *semi-simplicial space*, i.e., a collection of topological spaces $(X_k)_{k \in \mathbb{N}_0}$, together with continuous face maps maps $d_k^i \colon X_k \to X_{k-1}$ for $i = 0, \ldots, k$ such that $d_{k-1}^i \circ d_k^j = d_{k-1}^{j-1} \circ d_k^i$ if $i < j$. Then a *semi-simplicial cover* (or simply a *cover*) of $X_\bullet$ is a semi-simplicial space $\mathcal{U}_\bullet$, together with a morphism $f_\bullet \colon \mathcal{U}_\bullet \to X_\bullet$ of semi-simplicial spaces such that

$$\mathcal{U}_k = \coprod_{j \in J_k} U_k^j$$

for $(U_k^j)_{j \in J_k}$ an open cover of X_k and $f_k|_{U_k^j}$ is the inclusion $U_k^j \hookrightarrow X_k$. The cover is called *good* if each $(U_k^j)_{j \in J_k}$ is a good cover, i.e., all intersections $U_k^{j_0} \cap \ldots \cap U_k^{j_l}$ are contractible. ∎

Remark III.2. It is easy to construct semi-simplicial covers from covers of the X_k. In particular, we can construct good covers in the case that each X_k admits good cover, i.e., each cover has a refinement which is a good cover. Indeed, given an arbitrary cover $(U^i)_{i \in I}$ of X_0, denote I by J_0 and the cover by $(U_0^j)_{j \in J_0}$. We then obtain a cover of X_1 by pulling the cover $(U_0^j)_{j \in J_0}$ back along d_1^0, d_1^1, d_1^2 and take a common refinement $(U_1^j)_{j \in J_1}$ of the three covers. By definition, J_1 comes equipped with maps $\varepsilon_1^{1,2,3} \colon J_1 \to J_0$ such that $d_1^i(U_1^j) \subseteq U_0^{\varepsilon_1^i(j)}$. We may thus define the face maps of

$$\mathcal{U}_1 := \coprod_{j \in J_1} U_1^j$$

to coincide with d_1^i. In this way we then proceed to arbitrary k. In the case that each X_k admits good covers, we may refine the cover on each X_k before constructing the cover on X_{k+1} and thus obtain a good cover of $X_\bullet$.

The previous construction can be made more canonical in the case that $X_\bullet = BG_\bullet$ for a compact Lie group G. In this case, there exists a bi-invariant metric on G, and we let r be the injectivity radius of the exponential map at the identity (and thus at each $g \in G$). Then $(U^{g,r})_{g \in G}$ is a good open cover of G, where $U^{g,r}$ denotes the open ball of radius r around g. Now the triangle inequality shows that $U^{g_1,r/2} \cdot U^{g_2,r/2} = U^{g_1 g_2, r}$, which is obviously true for $g_1 = g_2 = e$ and thus for arbitrary g_1 and g_2 by the bi-invariance of the metric. Thus $(U^{g_1,r/2} \times U^{g_2,r/2})_{(g_1,g_2) \in G^2}$ gives a cover of G^2 compatible with the face maps $d_1^i \colon G^2 \to G$. Likewise,

$$(U^{g_1,r/2^k} \times \ldots \times U^{g_k,r/2^k})_{(g_1,\ldots,g_k) \in G^k}$$

gives a cover of G^k compatible with the face maps $d_k^i \colon G^k \to G^{k-1}$. Since each cover of G^k consists of geodesically convex open balls in the product metric, this consequently comprises a canonical good open cover of $BG_\bullet$. ∎

Definition III.3. Let $\mathcal{U}_\bullet$ be a cover of the semi-simplicial space $X_\bullet$ and $E^\bullet$ be a sheaf on $X_\bullet$.[7] Then the *Čech complex* associated to $\mathcal{U}_\bullet$ and $E^\bullet$ is the double complex

$$\check{C}^{p,q}(\mathcal{U}_\bullet, E^\bullet) := \prod_{i_0,\ldots,i_q \in I_p} E^p(U_{i_0,\ldots,i_q}),$$

where we set, as usual, $U_{i_0,\ldots,i_q} := U_{i_0} \cap \ldots \cap U_{i_q}$. The two differentials

$$d_h := \sum_{i=0}^{p} (-1)^{i+q} D_i^p \circ d_p^{i\,*} \colon \check{C}^{p,q}(\mathcal{U}_\bullet, E^\bullet) \to \check{C}^{p+1,q}(\mathcal{U}_\bullet, E^\bullet) \quad \text{and} \quad d_v := \check{\delta} \colon \check{C}^{p,q}(\mathcal{U}_\bullet, E^\bullet) \to \check{C}^{p,q+1}(\mathcal{U}_\bullet, E^\bullet)$$

turn $\check{C}^{p,q}(\mathcal{U}_\bullet, E^\bullet)$ into a double complex. We denote by $\check{H}^n(\mathcal{U}_\bullet, E^\bullet)$ the cohomology of the associated total complex and call it the *Čech Cohomology* of $E^\bullet$ with respect to $\mathcal{U}_\bullet$. ∎

Proposition III.4. *Suppose $G^{\times_k^n}$ is paracompact for each $n \geq 1$ and that $\mathcal{U}_\bullet$ is a good cover of $BG_\bullet$.[8] If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is a short exact sequence of G-modules in* **CGHaus**, *then composition with α and β induces a long exact sequence*

$$\cdots \to \check{H}^{n-1}(\mathcal{U}_\bullet, C_{\mathrm{glob},c}^\bullet) \to \check{H}^n(\mathcal{U}_\bullet, A_{\mathrm{glob},c}^\bullet) \to \check{H}^n(\mathcal{U}_\bullet, B_{\mathrm{glob},c}^\bullet) \to \check{H}^n(\mathcal{U}_\bullet, C_{\mathrm{glob},c}^\bullet) \to \check{H}^{n+1}(\mathcal{U}_\bullet, A_{\mathrm{glob},c}^\bullet) \to \cdots.$$

Moreover, for each sheaf $E^\bullet$ on $BG_\bullet$ there is a first quadrant spectral sequence with

$$E_1^{p,q} \cong \check{H}^q(G^{\times_k^p}, E^p) \Rightarrow \check{H}^{p+q}(\mathcal{U}_\bullet, E^\bullet).$$

In particular, if A is contractible, then

$$\check{H}^n(\mathcal{U}_\bullet, A_{\mathrm{glob},c}^\bullet) \cong H_{\mathrm{glob},c}^n(G, A).$$

---

[7] Sheaves on semi-simplicial spaces are defined likewise by omitting the degeneracy morphisms.

[8] We may also interpret $BG_\bullet$ as a semi-simplicial space by forgetting the degeneracy maps.

Proof. Each short exact sequence $A \to B \to C$ induces a short exact sequence of the associated double complexes and thus a long exact sequence between the cohomologies of the total complexes. The columns of the double complex $\check{C}^{p,q}(\mathcal{U}_\bullet, E^\bullet)$ are just the Čech complexes of the sheaf E^p on G^p for the open cover $\mathcal{U}_p$. Since the latter is good by assumption, the cohomology of the columns is isomorphic to the Čech cohomology of G^p with coefficients in the sheaf $\underline{A}$.

If A is contractible, then the sheaf $\underline{A}$ is soft on each $G^{\times_k^n}$ and thus acyclic. Hence the E_1-term of the spectral sequence is concentrated in the first column. Since $E_1^{0,q} = C(G^q, A)$ and the horizontal differential is just the standard group differential, this shows the claim. ∎

Remark III.5. For a connected topological group G and a topological G-module A we will now explain how to construct an isomorphism $H^2_{\mathrm{loc,top}}(G, A) \cong \check{H}^2(\mathcal{U}_\bullet, A^\bullet_{\mathrm{glob},c})$ in quite explicit terms (where $\mathcal{U}_\bullet$ now is a good cover of the semi simplicial space $(G^{\times_p^n})_{n \in \mathbb{N}_0}$). To a cocycle $f \in C_{\mathrm{loc},c}(G \times_p G, A)$ with $\mathrm{d}_{\mathrm{gp}} f = 0$ we associate the group $A \times_f G$ with underlying set $A \times G$ and multiplication $(a, g) \cdot (b, h) = (a + g.b + f(g, h), gh)$. Assuming that $U \subseteq G$ is such that $f|_{U \times U}$ is continuous and $V \subseteq U$ is an open identity neighborhood with $V = V^{-1}$ and $V^2 \subseteq U$, there exists a unique topology on $A \times_f G$ such that $A \times V \hookrightarrow A \times_f G$ is an open embedding. In particular, $\mathrm{pr}_2 \colon A \times_f G \to G$ is a continuous homomorphism and $x \mapsto (0, x)$ defines a continuous section thereof on V. Consequently, $A \times_f G \to G$ is a continuous principal A-bundle.

The topological type of this principal bundle is classified by a Čech cocycle $\tau(f)$, which can be obtained from the system of continuous sections

$$\sigma_g \colon gV \to A \times_f G, \quad x \mapsto (0, g) \cdot \sigma(g^{-1}x) = (f(g, g^{-1}x), x),$$

the associated trivializations $A \times gV \ni (a, x) \mapsto \sigma_g(x) \cdot (a, e) = (f(g, g^{-1}x) + x.a, x) \in \mathrm{pr}_2^{-1}(gV)$ and is thus given on the cover $(gV)_{g \in G}$ by

$$\tau(f)_{g_1, g_2} \colon g_1 V \cap g_2 V \to A, \quad x \mapsto f(g_2, g_2^{-1}x) - f(g_1, g_1^{-1}x) = g_1.f(g_1^{-1}g_2, g_2^{-1}x) - f(g_1, g_1^{-1}g_2).$$

The multiplication $\mu \colon (A \times_f G) \times (A \times_f G) \to A \times_f G$ may be expressed in terms of these local trivializations (although it might not be a bundle map in the case of non-trivial coefficients). For this, we pull back the cover $(gV)_{g \in G}$ via the multiplication to $G \times G$ and take a common refinement of this with the cover $(gV \times hV)_{(g,h) \in G \times G}$, over which the bundle $(A \times_f G) \times (A \times_f G) \to G \times G$ trivializes. A direct verification shows that $(V_{g,h})_{(g,h) \in g \times G}$ with

$$V_{g,h} := \{(x, y) \in G \times G : x \in gV, y \in hV, xy \in ghV\}$$

and the obvious maps does the job. Expressing μ in terms of these local trivializations, we obtain the representation

$$((a, x), (b, y)) \mapsto \big((xy)^{-1}.\big[f(g, g^{-1}x) + a.x + x.f(h, h^{-1}y) + xy.b + f(x, y) - f(gh, (gh)^{-1}xy)\big], xy\big)$$

for $(x, y) \in V_{(g,h)}$. Since this is a continuous map $A^2 \times V_{(g,h)} \to A \times V_{gh}$ and since G acts continuously on A it follows that

$$\mu(f)_{g,h} \colon V_{g,h} \to A, \quad (x, y) \mapsto f(g, g^{-1}x) + x.f(h, h^{-1}y) + f(x, y) - f(gh, (gh)^{-1}xy)$$

is indeed a continuous map. A straight forward computation with the definitions of d_v, d_h from Definition III.3 and the definitions of D_i^k from Remark II.9 shows that $d_h(\tau(f)) = d_v(\mu(f))$ in this case. Moreover, the cocycle identity for f shows that $d_h(\mu(f)) = 0$. Thus $(\mu(f), \tau(v))$ comprise a cocycle in the total complex of $\check{C}^{p,q}(\mathcal{U}_\bullet, E^\bullet)$ if we extend $(gV)_{g \in G}$ and $(V_{g,h})_{(g,h) \in G \times G}$ to a cover of $BG_\bullet$ as described in Remark III.2.

The reverse direction is more elementary. One associates to a cocycle (Φ, τ) in the total complex of $\check{C}^{p,q}(\mathcal{U}_\bullet, E^\bullet)$ a principal bundle $A \to P_\tau \to G$ clutched from the Čech cocycle τ. Then Φ defines a map $P_\tau \times P_\tau \to P_\tau$ (not necessarily a bundle map, if G acts non-trivially on A) whose continuity and associativity may be checked directly in local coordinates. Thus $P_\tau \to G$ is an abelian extension given by an element in $H^2_{\mathrm{loc},c}(G, A)$. By making the appropriate choices, one sees that these constructions are inverse to each other on the nose. ∎

IV   The Comparison Theorem via soft modules

We now describe a method for deciding whether certain cohomology groups are the same. The usual, and frequently used technique for this is to invoke Buchsbaum's criterion [Buc55], which also runs under the name universal δ-functor or "satellites" [CE56, Gro57, Wei94]. The point of this section is that a more natural requirement on the various cohomology groups, which can often be checked right away for different definitions, implies this criterion. The reader who is unfamiliar with these techniques might wish to consult the independent Section VI before continuing.

In order to make the comparison accessible, we have to introduce yet another definition of cohomology groups $H^n_{\mathrm{SM}}(G, A)$ for a G-module A in **CGHaus** due to Segal and Mitchison [Seg70]. We give some detail on this in Section VII; for the moment it is only important to recall that $A \mapsto H^n_{\mathrm{SM}}(G, A)$ is a δ-functor for exact sequences of locally contractible G-modules that are principal bundles [Seg70, Prop. 2.3] and that for contractible A, one has natural isomorphisms $H^n_{\mathrm{SM}}(G, A) \cong H^n_{\mathrm{glob},c}(G, A)$ [Seg70, Prop. 3.1].

Remark IV.1. In what follows, we will consider a special kind of classifying space functor, introduced by Segal in [Seg68]. The classifying space BG and the universal bundle EG are constructed by taking $BG = |BG_\bullet|$ (where $|\cdot|$ denotes geometric realization), and $EG = |EG_\bullet|$, where $EG_\bullet$ denotes the simplicial space obtained from the nerve of the pair groupoid of G. The resulting EG is contractible. The nice thing about this construction of BG is that it is functorial and that the natural map $E(G \times_{\boldsymbol{k}} G) \to EG \times_{\boldsymbol{k}} EG$ is a homeomorphism. In particular, EG and BG are again abelian groups in **CGHaus** provided that G is so. ∎

Definition IV.2. (cf. [Seg70]) On $C_{\boldsymbol{k}}(G, A)$, we consider the G-action $(g.f)(x) := g.(f(g^{-1} \cdot x))$[9], which obviously turns $C_{\boldsymbol{k}}(G, A)$ into a G-module in **CGHaus**. If A is contractible, then we call the module $C_{\boldsymbol{k}}(G, A)$ a *soft module*. Moreover, for arbitrary A we set $E_G(A) := C_{\boldsymbol{k}}(G, EA)$ and $B_G(A) := E_G(A)/i_A(A)$, where $i_A : A \hookrightarrow C_{\boldsymbol{k}}(G, EA)$ is the closed embedding $A \hookrightarrow EA$, composed with the closed embedding $EA \hookrightarrow C_{\boldsymbol{k}}(G, EA)$ of constant functions. ∎

Lemma IV.3. *The sequence $A \to E_G(A) \to B_G(A)$ has a local continuous section. If A is contractible, then it has a global continuous section.*

Proof. The first claim is contained in [Seg70, Prop. 2.1], the second follows from [Seg70, App. (B)]. ∎

Proposition IV.4. *Soft modules are acyclic for the globally continuous group cohomology, i.e., $H^n_{\mathrm{glob},c}(G, C_{\boldsymbol{k}}(G, A))$ vanishes for contractible A and $n \geq 1$.*

Proof. This is already implicitly contained in [Seg70, Prop. 2.2]. See also [SP09, Prop. 17] and Section VII. ∎

The following theorem now shows that all cohomology theories considered so far are in fact isomorphic, at least if the topology of G is sufficiently well-behaved.

Theorem IV.5 (Comparison Theorem). *Let* **G-Mod** *be the category of locally contractible G-modules in* **CGHaus**. *We call a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in* **G-Mod** *short exact if the underlying exact sequence of abelian groups is short exact and α (or equivalently β) has a local continuous section. If $(H^n : \mathbf{G\text{-}Mod} \to \mathbf{Ab})_{n \in \mathbb{N}_0}$ is a δ-functor such that*

1. *$H^0(A) = A^G$ is the invariants functor*

2. *$H^n(A) = H^n_{\mathrm{glob},c}(G, A)$ for contractible A,*

then $(H^n)_{n \in \mathbb{N}_0}$ is equivalent to $(H^n_{\mathrm{SM}}(G, \cdot))_{n \in \mathbb{N}_0}$ as δ-functor. Moreover, each morphism between δ-functors with properties 1. and 2. that is an isomorphism for $n = 0$ is automatically an isomorphism of δ-functors.

Proof. The functors $I(A) := E_G(A)$ and $U(A) := B_G(A)$ make Theorem VI.2 applicable. In order to check the requirements of the first part, we have to show that $H^{n \geq 1}(E_G(A))$ vanishes, which in turn follows from property 2. and Proposition IV.4.

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[9]This really is the action one wants to consider, as one sees in [Seg70, Prop. 3.1]. Some calculations in [Seg70, Ex. 2.4] seem to use the action $(g.f)(x) = f(g^{-1} \cdot x)$, we clarify this in Section VII.

To check the requirements of the second part of Theorem VI.2 we observe that if $f\colon A \to B$ is a closed embedding with a local continuous section, then $f(A)$ is also closed in $E_G(B)$ and thus we may set $Q_f := E_G(B)/f(A)$. The local sections of $f\colon A \to B$ and $B \to E_G(B)$ then also provide a section of the composition $A \to E_G(B)$, and $A \to E_G(B) \to Q_f$ is short exact. The morphism $B_G(A) \to Q_f$ can now be taken to be induced by $f_*\colon E_G(A) \to E_G(B)$, since it maps A to $f(A)$ by definition. Likewise, ι_B maps $f(A) \subseteq B$ into $f(A) \subseteq E_G(B)$, so induces a morphism $\gamma_f\colon B/f(A) \cong C \to Q_f = E_G(B)/f(A)$. The diagrams (17) thus commute by construction. ∎

The property of G-modules to be locally contractible is essential in order to provide a local section of the embedding $A \to E_G(A)$ [Seg70, Prop. A.1]. We will assume this from now on without any further reference.

Remark IV.6. Property 2. of the Comparison Theorem may be weakened to

$$H^n(A) = H^n_{\mathrm{glob},c}(G, A) \text{ for } loop \text{ contractible } A,$$

where loop contractible means that there exists a contracting homotopy $\rho\colon [0,1] \times A \to A$ such that each ρ_t is a group homomorphisms.

If this is the case, then one may still apply Theorem VI.2: Since EA is loop contractible ([Fuc11a, Ex. 5.5.] and [BM78, Rem. on p. 217]) so is $E_G = C_{\boldsymbol{k}}(G, EA)$ and thus $H^{n \geq 1}(E_G(A))$ still vanishes. In this case, it is then a consequence of Theorem VI.2 that $H^n(A) = H^n_{\mathrm{glob},c}(G, A)$ for *all* contractible modules A. ∎

Corollary IV.7. *If $G^{\times_{\boldsymbol{k}}^n}$ is paracompact for each $n \geq 1$, then $H^n_{\mathrm{SM}}(G, A) \cong H^n_{\mathrm{simp},c}(G, A)$.*

Corollary IV.8. *If $G^{\times_{\boldsymbol{p}}^n}$ is compactly generated for each $n \geq 1$, then we have $H^n_{\mathrm{loc},c}(G, A) \cong H^n_{\mathrm{SM}}(G, A)$[10]. If, moreover each $G^{\times_{\boldsymbol{p}}^n}$ is paracompact, then the morphisms*

$$H^n_{\mathrm{simp},c}(G, A) \to H^n_{\mathrm{loc},c}(G, A),$$

from Remark II.17 are isomorphisms.

Corollary IV.9. *Let G be a finite-dimensional Lie group, $\mathfrak{a}$ be a quasi-complete locally convex space on which G acts smoothly, $\Gamma \subseteq \mathfrak{a}$ be a discrete submodule and set $A = \mathfrak{a}/\Gamma$. Then the natural morphisms*

$$H^n_{\mathrm{simp},s}(G, A) \to H^n_{\mathrm{loc},s}(G, A) \to H^n_{\mathrm{loc},c}(G, A) \leftarrow H^n_{\mathrm{simp},c}(G, A) \tag{10}$$

are all isomorphisms.

Proof. The second is an isomorphism by Proposition I.7 and the third by the preceding corollary. Since $H^n_{\mathrm{simp},s}(G, \Gamma) \to H^n_{\mathrm{simp},c}(G, \Gamma)$ is an isomorphism by definition and $H^n_{\mathrm{simp},s}(G, \mathfrak{a}) \to H^n_{\mathrm{simp},c}(G, \mathfrak{a})$ is an isomorphism by Proposition II.12 and [HM62], the fist one in (10) is an isomorphism by the Five Lemma. ∎

Corollary IV.10. *If $G^{\times_{\boldsymbol{k}}^n}$ is paracompact for each $n \geq 1$, and $\mathcal{U}_\bullet$ is a good cover of $BG_\bullet$, then $H^n_{\mathrm{SM}}(G, A) \cong \check{H}^n(\mathcal{U}_\bullet, A^\bullet_{\mathrm{glob},c})$.*

Remark IV.11. Analogous to Corollary II.8 one sees that if each $G^{\times_{\boldsymbol{k}}^n}$ is paracompact, $\mathcal{U}_\bullet$ is a good cover of $BG_\bullet$ and $E^\bullet$ is a sheaf on $BG_\bullet$ with each E^n is acyclic, then $\check{H}^n(\mathcal{U}_\bullet, E^\bullet)$ is the cohomology of the first column of the E_1-term. This shows in particular that $\check{H}^n(\mathcal{U}_\bullet, A^\bullet_{\mathrm{loc},c}) \cong H^n_{\mathrm{loc},c}(G, A)$. Moreover, the morphism of sheaves $A^\bullet_{\mathrm{glob},c} \to A^\bullet_{\mathrm{loc},c}$ induces a morphism

$$\check{H}^n(\mathcal{U}_\bullet, A^\bullet_{\mathrm{glob},c}) \to \check{H}^n(\mathcal{U}_\bullet, A^\bullet_{\mathrm{loc},c}) \xrightarrow{\cong} H^n_{\mathrm{loc},c}(G, A). \tag{11}$$

This morphism can be constructed in (more or less) explicit terms by the standard staircase argument for double complexes with acyclic rows (note that by the acyclicity of $A^n_{\mathrm{loc},c}$ we may choose for each locally smooth Čech q-cocycle $\gamma_{i_0,\dots,i_q}\colon U_{i_0} \cap \dots \cap U_{i_q} \to A$ on G^p a locally smooth Čech cochain $\eta_{i_0,\dots,i_{q-1}}$ such that $\check{\delta}(\eta) = \gamma$). It is obvious that (11) defines a morphism of δ-functors. From the previous results and the uniqueness assertion of Theorem VI.2 it now follows that (11) is in fact an isomorphism provided $G^{\times_{\boldsymbol{p}}^n}$ is compactly generated and paracompact for each $n \geq 1$. ∎

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[10]This is also the main theorem in [SP09], whose proof remains unfortunately incomplete.

Remark IV.12. In [Fla08, Prop. 5.1] it is shown that for G a topological group and A a G-module, such that the sheaf of continuous functions has no cohomology, the cohomology group of [Fla08] coincide with $H^n_{\text{glob},c}(G, A)$. By [Fla08, Lem. 6] we also have long exact sequences, so the cohomology groups from [Fla08, Sect. 3] (which are anyway very similar to $H^n_{\text{simp},c}(G, A)$, see also [Lic09]) also agree with $H^n_{\text{SM}}(G, A)$.

There is a slight variation of the latter cohomology groups by Schreiber [Sch11] in the smooth setting and over the big topos of all cartesian smooth spaces. The advantage of this approach is that it is embedded in a general setting of differential cohomology. In the case that G is compact and A is discrete or $A = \mathfrak{a}/\Gamma$ for $\mathfrak{a}$ finite-dimensional, $\Gamma \subseteq \mathfrak{a}$ discrete and G acts trivially on A it has been shown in [Sch11, Prop. 3.3.12] that the cohomology groups $H^n_{\text{Smooth}\infty\text{Grpd}}(\mathbf{B}G, A)$ from [Sch11] are isomorphic to[11] $\check{H}^n(\mathcal{U}_\bullet, A^\bullet_{\text{glob},s})$ (where $\mathcal{U}_\bullet$ is a good cover of $BG^\infty_\bullet$). ∎

Remark IV.13. We now compare $H^n_{\text{loc},c}(G, A)$ with the cohomology groups from [Moo76]. For this we assume that G is a second countable locally compact group of finite covering dimension. A Polish G-module is a separable complete metrizable[12] abelian topological group A together with a jointly continuous action $G \times A \to A$. Morphisms of Polish G-modules are continuous group homomorphisms intertwining the G-action. If G is a locally compact group and A is a Polish G-module, then $H^n_{\text{Moore}}(G, A)$ denotes the cohomology of the cochain complex

$$C^n_\mu(G, A) := \{f \colon G^n \to A : f \text{ is Borel measurable}\}$$

with the group differential d_{gp} from (1). It has already been remarked in [Wig73] that these are isomorphic to $H^n_{\text{simp},c}(G, A)$, we give here a detailed proof of this and extend the result slightly.

On the category of Polish G-modules we consider as short exact sequences those sequences $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ for which the underlying sequence of abstract abelian groups is exact, α is an (automatically closed) embedding and β is open. From [Moo76, Prop. 11] it follows that from this we obtain natural long exact sequences, i.e., $H^n_{\text{Moore}}(G, \cdot)$ is a δ-functor. Moreover, it follows from [Wig73, Prop. 3] and from the remarks before [Wig73, Thm. 2] that each locally continuous cochain $f \colon G^{\times^n_p} \to C$ can be lifted to a locally continuous cochain $\tilde{f} \colon G^{\times^n_p} \to B$. This is due to the assumption on G to be finite-dimensional. From this it follows as in Remark I.2 that $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ also induces a long exact sequence for $H^n_{\text{loc},c}(G, \cdot)$ (this is the reason why we chose $H^n_{\text{loc},c}(G, A)$ for this comparison).

On the category of Polish G-modules we now consider the functors

$$A \mapsto \overline{E}_G(A) := C(G, U(I, A)),$$

where $U(I, A)$ is the group of Borel functions from the unit interval I to A. Moreover, $U(I, A)$ is a Polish G-module [Moo76, Sect. 2] and coincides with the completion of the metric abelian topological group A-valued step-functions on the right-open unit interval $[0, 1)$, endowed with the metric

$$d(f, g) := \int_0^1 d_A(f(t)g(t))\, dt,$$

see also [BM78, Kee73, HM58]. In particular, $U(I, A)$ inherits the structure of a G-module and so does $\overline{E}_G(A)$. Moreover, it is contractible and thus $\overline{E}_G(A)$ is soft. Since G is σ-compact we also have that $C(G, U(I, A))$ is completely metrizable.

Now A embeds as a closed submodule into $\overline{E}_G(A)$ and we set $\overline{B}_G(A) := \overline{E}_G(A)/A$. Thus

$$A \to \overline{E}_G(A) \to \overline{B}_G(A)$$

becomes short exact since orbit projection of continuous group actions are automatically open. In virtue of Theorem VI.2 and the fact that the locally continuous cohomology vanishes for soft modules this furnishes a morphism of δ-functors from $H^n_{\text{loc},c}(G, \cdot)$ to $H^n_{\text{Moore}}(G, \cdot)$ (the constructions of Q_f, β_f and γ_f from Theorem IV.5 carry over to the present setting). Moreover, the functors $A \mapsto I(A)$ and $A \mapsto U(A)$ that Moore constructs

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[11]This assertion is not stated explicitly but follows from [Sch11, Prop. 3.3.12] by the vanishing of $H^n_{\text{glob},s}(G, \mathfrak{a})$ [Est55, Thm. 1] and the long exact coefficient sequence.

[12]We will throughout assume that the metric is bounded. This is no lose of generality since we may replace each invariant metric $d(x, y)$ with the topologically equivalent bounded invariant metric $\frac{d(x,y)}{1+d(x,y)}$.

in [Moo76, Sect. 2] satisfy $H_{\text{Moore}}^n(I(A)) = 0$ [Moo76, Thm. 4]. Thus Remark VI.3 shows that $H_{\text{loc},c}^n(G, \cdot)$ and $H_{\text{Moore}}^n(G, \cdot)$ are isomorphic (even as δ-functors) on the category of Polish G-modules. This also extends [Aus10, Thm. C] to arbitrary locally contractible coefficients.

In addition, this shows that the mixture of measurable and locally continuous cohomology groups $H_{lcm}^n(G, A)$ from [KR10] does also coincide with $H_{\text{Moore}}^n(G, A)$. Indeed, the morphism $H_{lcm}^n(G, A) \to H_{\text{Moore}}^n(G, A)$ of δ-functors [KR10, Cor. 1] is surjective for each n and contractible A (since then $H_{\text{glob},c}^n(G, A) \to H_{\text{Moore}}^n(G, A)$ is surjective) and also injective (since $H_{\text{glob},c}^n(G, A) \to H_{lcm}^n(G, A) \to H_{\text{loc},c}^n(G, A)$ is so). Thus $H_{lcm}^n(G, A) \cong H_{\text{Moore}}^n(G, A) \cong H_{\text{glob},c}^n(G, A)$ for each n and contractible A and the Comparison Theorem shows that $H_{lcm}^n(G, \cdot)$ is isomorphic to $H_{\text{loc},c}^n(G, \cdot)$, also as δ-functor. ∎

Remark IV.14. Whereas all preceding cohomology theories fit into the framework of the Comparison Theorem, bounded continuous cohomology [Mon01, Mon06] does not. First of all, this concept considers locally compact G and Banach space coefficients A, whence all of the above cohomology theories agree to give $H_{\text{glob},c}^n(G, A)$. The bounded continuous cohomology $H_{bc}^n(G, A)$ is the cohomology of the sub complex of bounded continuous functions $(C_{bc}(G^n, A), \mathsf{d}_{\text{gp}})$. Thus there is a natural comparison map

$$H_{bc}^n(G, A) \to H_{\text{glob},c}^n(G, A)$$

which is obviously an isomorphism for compact G. However, bounded cohomology unfolds its strength not before considering non-compact groups, where the above map is in general not an isomorphism [Mon01, Ch. 9], even not for Lie groups [Mon01, Ex. 9.3.11]. In fact, bounded cohomology is *designed* to make the above map *not* into an isomorphism in order to measure the deviation of G from being compact. ∎

Despite the last example, the properties of the Comparison Theorem seem to be the essential ones for a large class of important concepts of cohomology groups for topological groups. We thus give it the following name.

Definition IV.15. A *cohomology theory for G* is a δ-functor $(F^n \colon \mathbf{G\text{-}Mod} \to \mathbf{Ab})_{n \in \mathbb{N}}$ satisfying conditions 1. and 2. of the Comparison Theorem. ∎

Remark IV.16. We end this section with listing properties that any cohomology theory for G has. We will always indicate the concrete model that we are using, the isomorphisms of the models are then due to the corollaries of this section. Parts of these facts have already been established for the various models in the respective references.

1. If A is discrete and each $G^{\times_k^n}$ is paracompact, then $H_{\text{SM}}^n(G, A) \cong H_{\pi_1(BG)}^n(BG, \underline{A})$ is the cohomology of the topological classifying space twisted by the $\pi_1(BG) \cong \pi_0(G)$-action on A (note that G_0 acts trivially since A is discrete). This follows from [Seg70, Prop. 3.3], cf. also [Del74, 6.1.4.2]. If, moreover, G is $(n-1)$-connected, then $H_{\text{SM}}^{n+1}(G, A) \cong \text{Hom}(\pi_n(G), A)$.

2. If G is contractible and each $G^{\times_p^n}$ is compactly generated, then $H_{\text{SM}}^n(G, A) \cong H_{\text{loc},c}^n(G, A) \cong H_{\text{glob},c}^n(G, A)$. This follows from [Fuc11b, Thm. 5.16].

3. If G is compact and $A = \mathfrak{a}/\Gamma$ for $\mathfrak{a}$ a quasi-complete locally convex space which is a continuous G-module and Γ a discrete submodule, then $H_{\text{SM}}^n(G, A) \cong H_{\pi_1(BG)}^{n+1}(BG, \Gamma)$. This follows from the vanishing of $H_{\text{SM}}^n(G, \mathfrak{a}) \cong H_{\text{glob},c}^n(G, \mathfrak{a})$ (cf. [Hu52, Thm. 2.8] or [BW00, Lem. IX.1.10]) and the long exact sequence induced from the short exact sequence $\Gamma \to \mathfrak{a} \to A$. In particular, if G is a compact Lie group and A is finite-dimensional, then

$$H_{\text{loc},c}^n(G, A) \cong H_{\text{loc},s}^n(G, A) \cong H_{\pi_1(BG)}^{n+1}(BG, \Gamma).$$ ∎

V   Examples and applications

The main motivation for this paper is that locally continuous and locally smooth cohomology are somewhat easy to handle, but lacked so far a conceptual framework. On the other hand, the simplicial cohomology groups or the ones introduced by Segal and Mitchison are hard to handle in degrees ≥ 3. We will give some results that one can derive from the interaction of these different concepts.

Example V.1. There is a bunch of cocycles which (or, more precisely, whose cohomology classes) deserve to be named "String Cocycle" (or, more precisely, "String Class"). For this example, we assume that G is a compact simple and 1-connected Lie group (which is thus automatically 2-connected). There exists for each $g \in G$ a path $\alpha_g \in C^\infty([0,1], G)$ with $\alpha_g(0) = e$, $\alpha_g(1) = g$ and for each $g, h \in G$ a filler[13] $\beta_{g,h} \in C^\infty(\Delta^2, G)$ for the triangle $(\mathsf{d}_{\mathrm{gp}}\alpha)(g,h,k) = g.\alpha_h - \alpha_{gh} + \alpha_g$ (Figure 1).
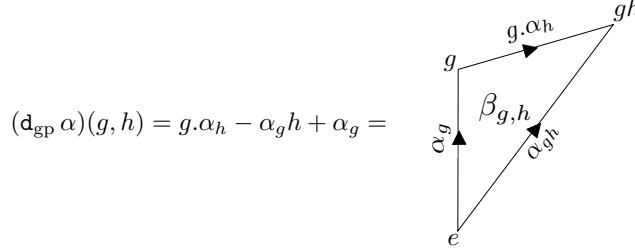
$$(\mathsf{d}_{\mathrm{gp}}\alpha)(g,h) = g.\alpha_h - \alpha_g h + \alpha_g =$$

Figure 1: $\beta_{g,h}$ fills $(\mathsf{d}_{\mathrm{gp}}\alpha)(g,h)$

Moreover, $(\mathsf{d}_{\mathrm{gp}}\beta)(g,h,k) = g.\beta_{h,k} - \beta_{gh,k} + \beta_{g,hk} - \beta_{g,h}$ bounds a tetrahedron which can be filled with $\gamma_{g,h,k} \in C^\infty(\Delta^3, G)$ (Figure 2).
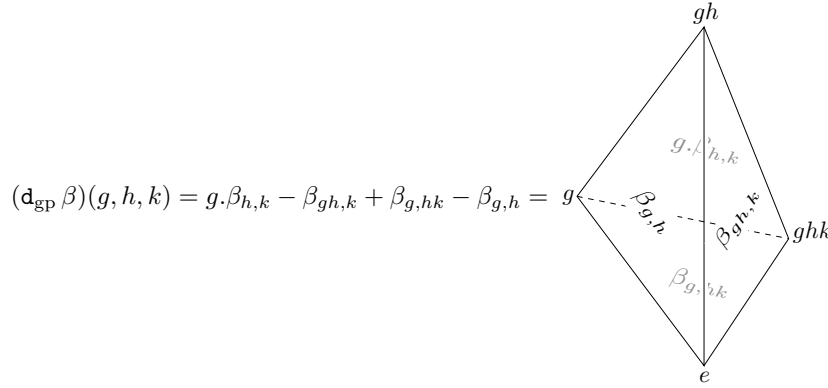
$$(\mathsf{d}_{\mathrm{gp}}\beta)(g,h,k) = g.\beta_{h,k} - \beta_{gh,k} + \beta_{g,hk} - \beta_{g,h} = g$$

Figure 2: $\gamma_{g,h,k}$ fills $(\mathsf{d}_{\mathrm{gp}}\beta)(g,h,k)$

In addition, α, β and γ, interpreted as maps $G^n \to C^\infty(\Delta^n, G)$ for $n = 1, 2, 3$, can be chosen to be smooth on some identity neighborhood. From these choices we can now construct the following cohomology classes (which in turn are independent of the above choices as a straight-forward check shows, cf. [Woc11, Rem. 1.12]).

1. Since $\partial\, \mathsf{d}_{\mathrm{gp}}(\gamma) = \mathsf{d}_{\mathrm{gp}}(\partial\gamma) = \mathsf{d}_{\mathrm{gp}}{}^2\beta = 0$, the map

$$(g, h, k, l) \mapsto (\mathsf{d}_{\mathrm{gp}}\gamma)(g, h, k, l)$$

   takes values in the singular 3-cycles on G and thus gives rise to map $\theta_3 \colon G^4 \to H_3(G) \cong \pi_3(G) \cong \mathbb{Z}$ (see also Example V.2). This map is locally smooth since γ was assumed to be so and it is a cocycle since $\mathsf{d}_{\mathrm{gp}}(\mathsf{d}_{\mathrm{gp}}(\gamma)) = 0$ (note that it is not a coboundary since γ does not take values in the singular cycles but only in the singular chains).

2. The cocycle $\sigma_3 \colon G^3 \to U(1)$ from [Woc11, Ex. 4.10] obtained by setting

$$\sigma_3(g, h, k) := \exp\left(\int_{\gamma_{g,h,k}} \omega\right),$$

---

[13]From the 2-connectedness of G is only follows that there exist continuous fillers, that these can chosen to be smooth follows from the density of $C^\infty(\Delta^n, G)$ in $C(\Delta^n, G)$ [Woc09, Cor. 14].

where ω is the left-invariant 3-from on G with $\omega(e) = \langle [\cdot, \cdot], \cdot \rangle$ normalized such that $[\omega] \in H^3_{\mathrm{dR}}(G)$ gives a generator of $H^3_{\mathrm{dR}}(G, \mathbb{Z}) \cong \mathbb{Z}$ and $\exp \colon \mathbb{R} \to U(1)$ is the exponential function of $U(1)$ with kernel $\mathbb{Z}$. Since ω is in particular an integral 3-form, this implies that σ_3 is a cocycle because $\mathtt{d}_{\mathrm{gp}}(\gamma)(g, h, k, l)$ is a piece-wise smooth singular cycle and thus

$$\mathtt{d}_{\mathrm{gp}}\, \sigma_3(g, h, k, l) = \exp\left(\int_{\mathtt{d}_{\mathrm{gp}}\, \gamma(g, h, k, l)} \omega \right) = 1.$$

Since γ is smooth on some identity neighborhood, σ_3 is so as well. Now

$$\widetilde{\sigma}_3(g, h, k) := \int_{\gamma(g, h, k)} \omega$$

provides a locally smooth lift of σ_3 to $\mathbb{R}$. Thus the homomorphism $\delta \colon H^3_{\mathrm{loc},s}(G, U(1)) \to H^4_{\mathrm{loc},s}(G, \mathbb{Z})$ maps $[\sigma_3]$ to $[\theta_3]$ since

$$\mathtt{d}_{\mathrm{gp}}\, \widetilde{\sigma}_3 = \int_{\mathtt{d}_{\mathrm{gp}}\, \gamma} \omega$$

and integration of piece-wise smooth representatives along ω provides the isomorphism $\pi_3(G) \cong \mathbb{Z}$. We will justify calling σ_3 a sting cocycle in Remark V.13.

3. The locally smooth cocycles arising as characteristic cocycles [Nee07, Lem. 3.6.] from the strict models [BCSS07, NSW11] of the string 2-group. In the case of the model from [BCSS07] this gives precisely σ_3.

Suppose $\mathcal{U}_\bullet$ is a good cover of $BG_\bullet$. The model from [SP11] is constructed by showing that $\check{H}^3(\mathcal{U}_\bullet, U(1)^\bullet_{\mathrm{glob},s})$ classifies central extensions of finite-dimensional group stacks

$$[*/U(1)] \to [\Gamma] \to G$$

and then taking the isomorphism

$$\check{H}^3(\mathcal{U}_\bullet, U(1)^\bullet_{\mathrm{glob},s}) \cong H^3_{\mathrm{SM}}(G, U(1)) \cong H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$$

(cf. Remark IV.16), yielding for each generator a model for the string group. We will see below that the classes from above are also generators in the respective cohomology groups and thus represent the various properties of the string group. For instance, we expect that the class $[\sigma_3]$ will be the characteristic class for representations of the string group. ∎

The previous construction can be generalized as follows.

Example V.2. Let G be a $(n-1)$-connected Lie group and denote by $C^\infty_*(\Delta^k, G)$ the group of based smooth k-simplices in G (the same construction works for locally contractible topological groups and the continuous k-simplices). Then we may choose for each $1 \le k \le n$ maps

$$\alpha_k \colon G^k \to C^\infty_*(\Delta^k, G),$$

such that each α_k is smooth on some identity neighborhood and that

$$\partial \alpha_k(g_1, \ldots, g_k) = \mathtt{d}_{\mathrm{gp}}(\alpha_{k-1})(g_1, \ldots, g_k).$$

In the latter formula, we interpret $C^\infty_*(\Delta^k, G)$ as a subset of the group $\langle C(\Delta^k, G) \rangle_{\mathbb{Z}}$ of singular k-chains in G, which becomes a G-module if we let G act by left multiplication. Since G is $(n-1)$-connected, we can inductively choose α_k, starting with $\alpha_0 \equiv e$.

Now consider the map

$$\theta_n := \mathtt{d}_{\mathrm{gp}}(\alpha_n) \colon G^{n+1} \to \langle C(\Delta^n, G) \rangle_{\mathbb{Z}}.$$

Since

$$\partial \theta_n = \partial\, \mathtt{d}_{\mathrm{gp}}(\alpha_n) = \mathtt{d}_{\mathrm{gp}}(\partial \alpha_n) = \mathtt{d}_{\mathrm{gp}}{}^2(\alpha_{n-1}) = 0, \tag{12}$$

θ_n takes values in the singular n-cycles on G and thus gives rise to a map $\theta_n \colon G^{n+1} \to H_n(G) \cong \pi_n(G)$. Moreover, θ_n is a group cocycle and it is locally smooth since α_n is so. Of course, this means here that θ_n even vanishes on some identity neighborhood (in the product topology). It is straight forward to show that different choices for α_k yield equivalent cocycles.

These are the characteristic cocycles for the n-fold extension

$$\pi_n(G) \to \widetilde{\Omega^n G} \to P_e \Omega^{n-1} G \to \cdots \to P_e \Omega G \to P_e G \to G, \tag{13}$$

(P_e denoted pointed paths and Ω pointed loops) of topological groups spliced together from the short exact sequences

$$\pi_n(G) \to \widetilde{\Omega^n G} \to \Omega^n G \quad \text{and} \quad \Omega^n G \to P_e \Omega^{n-1} G \to \Omega^{n-1} G \text{ for } n \geq 0.$$

Moreover, the exact sequence

$$\widetilde{\Omega^n G} \to \Omega^{n-1} G \to \cdots \to \Omega G \to P_e G$$

gives rise to a simplicial topological group $\Pi_n(G)$ and we have canonical morphisms

$$B^n \pi_n(G) \to \Pi_n(G) \to \underline{G}.$$

Here, $B^n \pi_n(G)$ is the nerve of the $(n-1)$-groupoid with only trivial morphisms up to $(n-2)$ and $\pi_n(G)$ as $(n-1)$-morphisms and $\underline{G}$ is the nerve of the groupoid with objects G and only identity morphisms. Taking the geometric realization $|\cdot|$ gives (at least for metrizable G) now an extension of groups in **CGHaus**

$$K(n, \pi_n(G)) \simeq |B^n \pi_n(G)| \to |\Pi_n(G)| \to |G| = G,$$

which can be shown to be an n-connected cover $G\langle n \rangle \to G$ with the same methods as in [BCSS07]. ∎

Remark V.3. Recall that a crossed module $\mu \colon M \to N$ is a group homomorphism together with an action by automorphisms of N on M such that μ is equivariant and such that for all $m, m' \in M$, the Peiffer identity

$$\mu(m).m' = mm'm^{-1}$$

holds. Taking into account topology, we suppose that M and N are groups in **CGHaus**, μ continuous and $(n, m) \mapsto n.m$ is continuous. We call a closed subgroup H of a group in **CGHaus** *split* if the multiplication map $G \times_{\boldsymbol{k}} H \to G$ defines a topological H-principal bundle (see [Nee07, Def. 2.1]). We will throughout use the constructions in the smooth setting from [Nee07], which carry over to the present topological setting. In this case, we have in particular that $G \to G/H$ has a continuous local section. In order to avoid extensions coming from non-complemented topological vector spaces, we suppose that all our crossed modules are *topologically split*, i.e., we suppose that $\ker(\mu)$ is a split topological subgroup of M, that $\mathrm{im}(\mu)$ is a split topological subgroup of N, and that μ induces a homeomorphism $M/\ker(\mu) \cong \mathrm{im}(\mu)$. In case M and N are (possibly infinite dimensional) Lie groups, this implies that the corresponding sequence of Lie algebras is topologically split as a sequence of topological vector spaces. ∎

Using the above methods, we can now show the following:

Theorem V.4. *If each $G^{\times_{\mathcal{P}}^n}$ is compactly generated, then the set of equivalence classes of crossed modules with cokernel G and kernel A is in bijection with the cohomology space $H^3_{\mathrm{loc},c}(G, A)$.*

Proof. It is standard to associate to a (topologically split) crossed module a locally continuous 3-cocycle (see [Nee07, Lem. 3.6]). To show that this defines an injection of the set of equivalence classes into $H^3_{\mathrm{loc},c}(G, A)$, we use the continuous version of [Nee07, Th. 3.8]. Namely, the existence of an extension $A \to \hat{N} \xrightarrow{q} N$ such that M is N-equivariantly isomorphic to $q^{-1}(\mathrm{im}(\mu))$ provides two morphisms of four term exact sequences linking $A \to M \to N \to G$ to the trivial crossed module $A \to A \xrightarrow{0} G \to G$.

Therefore we focus here on surjectivity, i.e. we reconstruct a crossed module from a given locally continuous 3-cocycle. For this, embed the G-module A in a soft G-module:

$$0 \to A \to E_G(A) \to B_G(A) \to 0$$

. Observe that $H^n_{\mathrm{SM}}(G, E_G(A)) \cong H^n_{\mathrm{glob},c}(G, E_G(A))$ vanishes for $n \geq 1$ (Proposition IV.4). The vanishing shows now that the connecting homomorphism of the associated long exact sequence induces an isomorphism

$$\delta : H^2_{\mathrm{loc},c}(G, B_G(A)) \cong H^3_{\mathrm{loc},c}(G, A),$$

where we have used the isomorphism of H^n_{SM} and $H^n_{\mathrm{loc},c}$. Thus for the given 3-cocycle γ, there exists a locally continuous 2-cocycle α with values in $B_G(A)$ such that $\delta[\alpha] = [\gamma]$. Using α, we can form an abelian extension

$$0 \to B_G(A) \to B_G(A) \times_\alpha G \to G \to 1.$$

Now splicing together this abelian extension with the short exact coefficient sequence

$$0 \to A \to E_G(A) \to B_G(A) \to 0$$

gives rise to a crossed module $\mu : E_G(A) \to B_G(A) \times_\alpha G$ which is topologically split in the above sense. Indeed, the coefficient sequence is topologically split by assumption, and the abelian extension has a continuous local section by construction.

Finally, the fact that the 3-class associated to this crossed module is $[\gamma]$ follows from $\delta[\alpha] = [\gamma]$. Some details for this kind of construction can also be found in [Wag06]. ∎

Remark V.5. In the case of locally compact second countable G and metrizable A the module EA is metrizable [BM78] and since G is in particular σ-compact $C(G, EA) = E_G(A)$ is also metrizable. Thus the above crossed module is a crossed module of metrizable topological groups. In particular, if we take a generator $[\alpha] \in H^3_{SM}(G, U(1)) \cong H^4(G, \mathbb{Z}) \cong \mathbb{Z}$ for G a simple compact 1-connected Lie group, then the crossed module

$$U(1) \to E_G(U(1)) \to B_G(U(1)) \times_\alpha G \to G$$

gives yet another (topological) model for the string 2-group. ∎

Remark V.6. (cf. [SP09, Def. 19]) The locally continuous cohomology can be topologized as follows. For an open identity neighborhood $U \subseteq G^{\times_k^n}$ we have the bijection

$$C^n_U(G, A) := \{f : G^n \to A : f|_U \text{ is continuous}\} \cong C(U, A) \times A^{G^n \setminus U}.$$

This carries a natural topology coming from $C_k(U, A) \times_k A^{G^n \setminus U}$, when first endowing $A^{G^n \setminus U}$ with the product topology and then taking the induced compactly generated topology. If $U \subseteq V$, then the inclusion $C^n_U(G, A) \hookrightarrow C^n_V(G, A)$ is continuous so that the direct limit

$$\varinjlim_{U \in \mathfrak{U}} C^n_U(G, A) \cong C^n_{\mathrm{loc},c}(G, A)$$

carries a natural topology. The differential d_{gp} is continuous and the cohomology groups $H^n_{\mathrm{loc},c}(G, A)$ inherits the corresponding quotient topology. ∎

Remark V.7. There is a classical way of constructing *products* for some of the cohomology theories which we have considered here. Let us recall these definitions. The easiest product is the usual *cup product* for the locally continuous (respectively the locally smooth) group cohomology $H^n_{\mathrm{loc},c}(G, A)$ (respectively $H^n_{\mathrm{loc},s}(G, A)$) [Mac63, Ch. VIII.9]. In the following, we will stick to $H^n_{\mathrm{loc},c}(G, A)$, noting that all constructions carry over word by word to $H^n_{\mathrm{loc},s}(G, A)$ for a Lie group G and a smooth G-module A.

Suppose that the two G-modules A and A' have a tensor product in **CGHaus**. The simplicial cup product (see [Mac63] equation (9.7) p. 246) in group cohomology yields a homomorphism

$$\cup : H^p_{\mathrm{loc},c}(G, A) \otimes H^q_{\mathrm{loc},c}(G, A') \to H^{p+q}_{\mathrm{loc},c}(G, A \otimes A'),$$

where the G-module $A \otimes A'$ is given the diagonal action.

In case the G-module A has its tensor product $A \otimes A$ in **CGHaus** and has a product, i.e. a homomorphism of G-modules $\alpha : A \otimes A \to A$, we obtain an *internal cup product*

$$\cup : H^p_{\mathrm{loc},c}(G, A) \otimes H^q_{\mathrm{loc},c}(G, A) \to H^{p+q}_{\mathrm{loc},c}(G, A)$$

by post composing with α. The product reads then explicitly for cochains $c \in C_{\mathrm{loc},c}^p(G,A)$ and $c' \in C_{\mathrm{loc},c}^q(G,A)$

$$c \cup c'(g_0,\ldots,g_{p+q}) = \alpha(c(g_0,\ldots,g_p), c'(g_p,\ldots,g_{p+q})).$$

On the other hand, Segal-Mitchison cohomology $H_{\mathrm{SM}}^n(G,A)$ is a (relative) derived functor, and therefore the setting of [Mac63, Sect. XII.10] is adapted. Observe that our choice of exact sequences does not satisfy all the demands of a *proper class* of exact sequences [Mac63, Sect. XII.4] (it does not satisfy the last two demands) and we neither have automatically enough proper injectives or projectives. Nevertheless, we have explicit acyclic resolutions for each module in **CGHaus** which are exact sequences in our sense. We have the universality property for the functor $H_{\mathrm{SM}}^n(G,A)$ [Mac63, Sect. XII.8] by Theorem VI.2. Therefore we obtain products for Segal-Mitchison cohomology by universality as in [Mac63, Th. XII.10.4] for two G-modules A and A' which have a tensor product in **CGHaus**.

By the uniqueness statement in [Mac63, Th. XII.10.4], the isomorphism $H_{\mathrm{SM}}^n(G,A) \cong H_{\mathrm{loc},c}^n(G,A)$ respects products. Note also that the differentiation homomorphism $D_n \colon H_{\mathrm{loc},s}^n(G,A) \to H_{\mathrm{Lie},c}^n$ that we will turn to in Remark V.14 is compatible with products. ∎

We now give an explicit description of the purely topological information contained in a locally continuous cohomology class. If G is a connected topological group and A is a topological G-module, then there is an exact sequence

$$0 \to H_{\mathrm{glob},\mathrm{top}}^2(G,A) \to H_{\mathrm{loc},\mathrm{top}}^2(G,A) \xrightarrow{\tau_2} \check{H}^1(G,\underline{A}) \tag{14}$$

[Woc10, Sect. 2], where τ_2 assigns to an abelian extension $A \to \hat{G} \to G$ the characteristic class of the underlying principal A-bundle. By definition, we have that $\mathrm{im}(\tau_2)$ are those classes in $\check{H}^1(G,\underline{A})$ whose associated principal A-bundles admit a compatible group structure. These are precisely the bundles $A \to P \to G$ for which the bundle $\mathrm{pr}_1^*(P) \otimes \mu^*(P) \otimes \mathrm{pr}_2^*(P)$ on $G \times G$ is trivial [Gro72, Prop. VII.1.3.5].

We will now establish a similar behavior of the map τ_n for arbitrary n.

Proposition V.8. *Let G be a connected topological group and A be a topological G-module. Suppose that the cocycle $f \in C_{\mathrm{loc},\mathrm{top}}^n(G,A)$ is continuous on the identity neighborhood $U \subseteq G^n$ and let $V \subseteq G$ be open such that $e \in V$ and $V^2 \times \ldots \times V^2 \subseteq U$. Then the map*

$$\tau(f)_{g_1,\ldots,g_n}\colon g_1 V \cap \ldots \cap g_n V \to A, \quad x \mapsto g_1.f(g_1^{-1}g_2,\ldots,g_{n-1}^{-1}g_n,g_n^{-1}x) - (-1)^n f(g_1,g_1^{-1}g_2,\ldots,g_{n-1}^{-1}g_n)$$

defines a continuous Čech $(n-1)$-cocycle on the open cover $(gV)_{g\in G}$. Moreover, this induces a well-defined map

$$\tau_n\colon H_{\mathrm{loc},\mathrm{top}}^n(G,A) \to \check{H}^{n-1}(G,\underline{A}), \quad [f] \mapsto [\tau(f)]$$

which is a morphism of δ-functors.

Proof. We first note that $\tau(f)_{g_1,\ldots,g_n}$ depends continuously on x. Indeed, the first term depends continuously on x since $g_1 V \cap \ldots \cap g_n V \neq \emptyset$ implies that $g_{k-1}^{-1}g_k \in V^2$ and f is continuous on $V^2 \times \ldots \times V^2$ by assumption. Since the second term does not depend on x, this shows continuity. Now the cocycle identity for f, evaluated on $(g_1, g_1^{-1}g_2,\ldots,g_{n-1}^{-1}g_n, g_n^{-1}x)$, shows that $\tau(f)_{g_1,\ldots,g_n}(x)$ may also be written as $(\check{\delta}(\kappa(f)))_{g_1,\ldots,g_n}(x)$, where

$$\kappa(f)_{g_2,\ldots,g_n}(x) := f(g_2, g_2^{-1}g_3, \ldots, g_n^{-1}x).$$

Note that $\kappa(f)_{g_2,\ldots,g_n}$ does not depend continuously on x and thus the above assertion does not imply that $\tau(f)$ is a coboundary. However, $\check{\delta}^2 = 0$ now implies that $\tau(f)$ is a cocycle.

It is clear that the class $[\tau(f)]$ in $\check{H}^{n-1}(G,\underline{A})$ does not depend on the choice of V since another such choice V' yields a cocycle given by the same formula on the refined cover $(g(V \cap V'))_{g\in G}$. Moreover, if f is a coboundary, i.e., $f = \mathsf{d}_{\mathrm{gp}} b$ for $b \in C_{\mathrm{loc},c}^{n-1}(G,A)$ (where we assume w.l.o.g. that b is also continuous on $V^2 \times \ldots \times V^2$), then we set

$$\rho(b)_{g_1,\ldots,g_{n-1}}(x) := g_1.b(g_1^{-1}g_2,\ldots,g_{n-1}^{-1}x) + (-1)^n b(g_1, g_1^{-1}g_2,\ldots,g_{n-2}^{-1}g_{n-1}).$$

As above, this defines a continuous function on $g_1 V \cap \ldots \cap g_{n-1} V \neq \emptyset$ and thus a Čech cochain. A direct calculation shows that $\check{\delta}(\rho(f)) = \tau(f)$ and thus that the class $[\tau(f)]$ only depends on the class of f.

We now turn to the second claim, for which we have to check that for each exact sequence $A \hookrightarrow B \xrightarrow{q} C$ of topological G-modules the diagram

$$
\begin{array}{ccc}
H_{\mathrm{loc},c}^n(G,C) & \xrightarrow{\delta_n} & H_{\mathrm{loc},c}^{n+1}(G,A) \\
\downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_{n+1}} \\
\check{H}^{n-1}(G,\underline{C}) & \xrightarrow{\delta_{n-1}} & \check{H}^n(G,\underline{A})
\end{array}
$$

commutes. For this, we recall that δ_n is constructed by choosing for $[f] \in H_{\mathrm{loc},c}^n(G,C)$ a lift $\widetilde{f} \colon G^n \to B$ and then setting $\delta_n([f]) = [\mathsf{d}_{\mathrm{gp}}\,\widetilde{f}]$. After possibly shrinking V, we can assume that f is continuous on $V^2 \times \ldots \times V^2$ (n factors) and that $\mathsf{d}_{\mathrm{gp}}\,\widetilde{f}$ is continuous on $V^2 \times \ldots \times V^2$ ($n+1$ factors).

Since q is a homomorphism, $\widetilde{f}$ also gives rise to lifts

$$\widetilde{\tau(f)}_{g_1,\ldots,g_n}(x) := g_1.\widetilde{f}(g_1^{-1}g_2,\ldots,g_{n-1}^{-1}g_n,g_n^{-1}x) - (-1)^n \widetilde{f}(g_1,g_1^{-1}g_2,\ldots,g_{n-1}^{-1}g_n)$$

of $\tau(f)_{g_1,\ldots,g_n}$, which obviously depends continuously on x on $g_1 V \cap \ldots \cap g_n V$. Thus we have that $\delta_{n-1}(\tau_n([f]))$ is represented by the Čech cocycle

$$\check{\delta}(\widetilde{\tau(f)})_{g_0,\ldots,g_n}.$$

On the other hand, $\tau_{n+1}(\delta_n([f]))$ is represented by $\tau(\mathsf{d}_{\mathrm{gp}}\,\widetilde{f})_{g_0,\ldots,g_n}$, whose value on x is given by

$$g_0.\mathsf{d}_{\mathrm{gp}}\,\widetilde{f}(g_0^{-1}g_1,\ldots,g_{n-1}^{-1}g_n,g_n^{-1}x) - (-1)^{n+1}\mathsf{d}_{\mathrm{gp}}\,\widetilde{f}(g_0,g_0^{-1}g_1,\ldots,g_{n-1}^{-1}g_n) =$$
$$g_0.\Big[g_0^{-1}g_1.\widetilde{f}(g_1^{-1}g_2,\ldots,g_n^{-1}x) \pm \ldots + (-1)^k \widetilde{f}(g_0^{-1}g_1,\ldots,g_{k-1}^{-1}g_{k+1},\ldots,g_n^{-1}x) \pm \ldots$$
$$+ \underline{(-1)^{n+1}\widetilde{f}(g_0^{-1}g_1,\ldots,g_{n-1}^{-1}g_n)}\Big] - (-1)^{n+1}\Big[\underline{g_0.\widetilde{f}(g_0^{-1}g_1,\ldots,g_{n-1}^{-1}g_n)} \pm \ldots$$
$$- (-1)^k \widetilde{f}(g_0,g_0^{-1}g_1,\ldots,g_{k-1}^{-1}g_{k+1},\ldots,g_{n-1}^{-1}g_n) \pm \ldots + (-1)^{n+1}\widetilde{f}(g_0,g_0^{-1}g_1,\ldots,g_{n-2}^{-1}g_{n-1})\Big]$$

The underlined terms cancel and the sum of the dashed terms gives $(-1)^k \widetilde{\tau(f)}_{g_0,\ldots,\widehat{g_k},\ldots,g_n}(x)$. This shows that

$$\check{\delta}(\widetilde{\tau(f)})_{g_1,\ldots,g_n}(x) = \tau(\mathsf{d}_{\mathrm{gp}}\,\widetilde{f})_{g_1,\ldots,g_n}(x). \qquad \blacksquare$$

We will now identify the map τ with one of the edge homomorphisms in the spectral sequence associated to $H_{\mathrm{simp},c}^n(G,A)$.

Proposition V.9. *For $n \geq 1$ the edge homomorphism of the spectral sequence* (7) *induces a homomorphism*

$$\mathrm{edge}_{n+1} \colon H_{\mathrm{simp},c}^{1+n}(G,A) \to H_{\mathrm{simp},c}^{1+n}(G,A)/\mathcal{F}^2 H_{\mathrm{simp},c}^{2+n}(G,A) \cong E_\infty^{1,n} \to E_1^{1,n} \cong H_{\mathrm{Sh}}^n(G,\underline{A}),$$

where $\mathcal{F}$ denotes the standard column filtration (cf. Remark II.7). If, moreover, $G^{\times_{\mathrm{p}}^n}$ is compactly generated, paracompact and admits good covers for all $n \geq 1$ and A is a topological G-module, then the diagram

$$
\begin{array}{ccc}
H_{\mathrm{simp},c}^{n+1}(G,A) & \xrightarrow{\mathrm{edge}_{n+1}} & H_{\mathrm{Sh}}^n(G,\underline{A}) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H_{\mathrm{loc},c}^{n+1}(G,A) & \xrightarrow{\tau_{n+1}} & \check{H}^n(G,\underline{A})
\end{array}
\qquad (15)
$$

commutes.

Proof. We first note that $H_{\mathrm{loc,top}}^n(G,A) = H_{\mathrm{loc},c}^n(G,A)$ under the above assumptions. Since $BG_0 = \mathrm{pt}$, we have $E_1^{0,q} = H_{\mathrm{Sh}}^q(\mathrm{pt}, \underline{A}) = 0$ for all $q \geq 1$ and thus the edge homomorphism $E_\infty^{1,p} \to E_1^{1,p}$. Since we have $\mathcal{F}^p H_{\mathrm{simp},c}^{p+q}(G,A) = H_{\mathrm{simp},c}^{p+q}(G,A)$ for $p = 0,1$, $q \geq 1$ this gives the desired form of edge_{q+1}. Since this construction commutes with the connecting homomorphisms, it is a morphism of δ-functors. Moreover, the isomorphism $H_{\mathrm{Sh}}^n(G, \underline{\,\cdot\,}) \cong \check{H}^n(G, \underline{\,\cdot\,})$ is even an isomorphism of δ-functors. By virtue of the uniqueness assertion for morphisms of δ-functors from Theorem VI.2, it thus remains to verify that that (15) commutes for $n = 1$.

The construction from Remark III.5 gives an isomorphism $H_{\mathrm{loc,top}}^2(G,A) \cong \check{H}^2(\mathcal{U}_\bullet, A_{\mathrm{loc},c}^\bullet)$, where $\mathcal{U}_\bullet$ is a good cover of $BG_\bullet$ chosen such that $\mathcal{U}_k$ refines the covers of G^k constructed there. Since this construction commutes with the connecting homomorphisms, the isomorphism $H_{\mathrm{loc,top}}^2(G,A) \cong \check{H}^2(\mathcal{U}_\bullet, A_{\mathrm{loc},c}^\bullet)$ is indeed the one from the unique isomorphism of the corresponding δ-functors. Now τ_2 coincides with the morphism $H_{\mathrm{loc,top}}^2(G,A) \cong \check{H}^2(\mathcal{U}_\bullet, A_{\mathrm{loc},c}^\bullet) \to \check{H}^1(G,A)$, given by projecting the cocycle (μ,τ) in the total complex of $\check{C}^{p,q}(\mathcal{U}_\bullet, E^\bullet)$ to the Čech cocycle τ. Since this is just the corresponding edge homomorphism, the diagram (15) commutes for $n = 1$. ∎

Remark V.10. In case the action of G on A is trivial, Proposition V.9 also holds for $n = 0$. Indeed, then the differential $A \cong E_1^{0,0} \to E_1^{1,0} \cong C^\infty(G,A)$, which is given by assigning the principal crossed homomorphism to an element of A, vanishes. This shows commutativity of (15) also for $n = 0$. ∎

Remark V.11. The other edge homomorphism is induced from the identification $C_{\mathrm{glob},c}^n(G,A) \cong H_{\mathrm{Sh}}^0(G^n, A) \cong E_1^{n,0}$, which shows $E_2^{n,0} \cong H_{\mathrm{glob},c}^n(G,A)$. It coincides with the morphism $H_{\mathrm{glob},c}^n(G,A) \to H_{\mathrm{loc},c}^n(G,A)$ induced by the inclusion $C_{\mathrm{glob},c}^n(G,A) \hookrightarrow C_{\mathrm{loc},c}^n(G,A)$ (cf. also [Seg70, Remarks in §3]). ∎

The following is a generalization of (14) in case A is discrete.

Corollary V.12. *If $n \geq 1$, G is $(n-1)$-connected and A is a discrete G-module, then $\tau_{n+1}\colon H_{\mathrm{loc},c}^{n+1}(G,A) \to \check{H}^n(G, \underline{A})$ is injective if $G^{\times_{\mathbb{R}}^n}$ is compactly generated, paracompact and admits good covers for all $n \geq 1$.*

Proof. If G is $(n-1)$-connected, and A is discrete, then $E_1^{p,q}$ of the spectral sequence (7) vanishes if $q \leq n-1$. Thus $E_\infty^{1,n-1} = \ker(d_1^{1,n-1}) \subseteq E_1^{1,n-1} \cong \check{H}^{n-1}(G, \underline{A})$ and edge_n coincides with the embedding

$$H_{\mathrm{loc},c}^n(G,A) \cong H_{\mathrm{simp},c}^n(G,A) \cong E_\infty^{1,n-1} \hookrightarrow E_1^{1,n-1} \cong \check{H}^{n-1}(G,A). \qquad \blacksquare$$

Remark V.13. An explicit analysis of the differentials of the spectral sequence (7) shows that for discrete A with trivial G-action and $(n-1)$-connected G the image of $\tau_{n+1}\colon H_{\mathrm{loc},c}^{n+1}(G,A) \to \check{H}^n(G, \underline{A})$ consists of those cohomology classes $c \in \check{H}^n(G, \underline{A})$ which are *primitive*, i.e., for which

$$\mathrm{pr}_1^* c \otimes \mu^* c \otimes \mathrm{pr}_2^* c = 0.$$

Since the primitive elements generate the rational cohomology of a compact Lie group G [GHV73, p. 167, Thm. IV], it follows that all non-torsion elements in the lowest cohomology degree are primitive in this case.

In particular, if G is a compact, simple and 1-connected (thus automatically 2-connected), the generator of $\check{H}^2(G, \underline{U(1)}) \cong \check{H}^3(G, \underline{\mathbb{Z}}) \cong \mathbb{Z}$ is primitive and thus $\tau_4\colon H_{\mathrm{loc},c}^4(G,\mathbb{Z}) \to \check{H}^3(G, \underline{\mathbb{Z}})$ is an isomorphism. Since the diagram

$$
\begin{array}{ccc}
H_{\mathrm{loc},c}^4(G,\mathbb{Z}) & \xrightarrow{\ \tau_4^{\mathbb{Z}}\ } & \check{H}^3(G, \underline{\mathbb{Z}}) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} \\
H_{\mathrm{loc},c}^3(G,U(1)) & \xrightarrow{\ \tau_3^{U(1)}\ } & \check{H}^2(G, \underline{U(1)})
\end{array}
$$

commutes by Proposition V.8, this shows that $\tau_3^{U(1)}$ is also an isomorphism. Since the string class $[\sigma_3]$ from Example V.1 maps under τ_3 to a generator [BM93], this shows that $[\sigma_3]$ gives indeed a generator of $H_{\mathrm{loc},c}^3(G,U(1))$, and $[\theta_3]$ gives a generator of $H_{\mathrm{loc},c}^4(G,\mathbb{Z})$. ∎

Remark V.14. One reason for the importance of *locally smooth* cohomology is that it allows for a direct connection to Lie algebra cohomology and thus may be computable in algebraic terms. This relation is induced by the differentiation homomorphism

$$H^n_{\mathrm{loc},s}(G, A) \xrightarrow{D_n} H^n_{\mathrm{Lie},c}(\mathfrak{g}, \mathfrak{a}),$$

where $H^n_{\mathrm{Lie},c}$ denotes the continuous Lie algebra cohomology, $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{a}$ the induced infinitesimal topological $\mathfrak{g}$-module (cf. [Nee06, Thm. V.2.6]).

Suppose G is finite-dimensional. Then the kernel of D_n consists of those cohomology classes $[f]$ that are represented by cocycles vanishing on some neighborhood of the identity. For $\Gamma = \{0\}$ this follows directly from [Świ71], where it is shown that the differentiation homomorphism from the cohomology of *locally defined* smooth group cochains to Lie algebra cohomology is an isomorphism. Thus if $[f] \in \ker(D_n)$, then there exists a locally defined smooth map b with $\mathrm{d}_{\mathrm{gp}} b - f = 0$ wherever defined. Since we can extend b arbitrarily to a locally smooth cochain this shows the claim. In the case of non-trivial Γ one may deduce the claim from the case of trivial Γ since $\mathfrak{a}$ and $A = \mathfrak{a}/\Gamma$ are isomorphic as local Lie groups so that A-valued local cochains can always be lifted to $\mathfrak{a}$-valued local cochains. If A^δ denotes A with the discrete topology, then the isomorphism $H^n_{\pi_1(BG)}(BG, \underline{A^\delta}) \cong H^n_{\mathrm{loc},s}(G, A^\delta)$ from Remark IV.16 induces an exact sequence

$$H^n_{\pi_1(BG)}(BG, \underline{A^\delta}) \to H^n_{\mathrm{loc},s}(G, A) \xrightarrow{D_n} H^n_{\mathrm{Lie},c}(\mathfrak{g}, \mathfrak{a})$$

(see also [Nee02, Nee04] for an exhaustive treatment of D_2 for general infinite-dimensional G). From the van Est spectral sequence [Est58] it follows that if G ist n-connected (more general G may be infinite-dimensional with split de Rham complex [Beg87]), then differentiation induces an isomorphism

$$H^n_{\mathrm{glob},s}(G, \mathfrak{a}) \to H^n_{\mathrm{Lie},c}(\mathfrak{g}, \mathfrak{a}).$$

For G an $(n-1)$-connected Lie group this is not true any more, for instance the Lie algebra 3-cocycle $\langle [\cdot, \cdot], \cdot \rangle$ from Example V.1 is non-trivial but $H^3_{\mathrm{glob},s}(G, \mathbb{R})$ vanishes by [Est55, Thm. 1] for compact and connected G.

However, there exists integrating cocycles when considering *locally smooth* cohomology: If G is an $(n-1)$-connected finite-dimensional Lie group and $A \cong \mathfrak{a}/\Gamma$ is a finite-dimensional smooth module for $\mathfrak{a}$ a finite-dimensional G-module and Γ a discrete submodule, then $D_n \colon H^n_{\mathrm{loc},s}(G, A) \to H^n_{\mathrm{Lie},c}(\mathfrak{g}, \mathfrak{a})$ is injective and its image consists of those cohomology classes $[\omega]$ whose associated period homomorphism $\mathrm{per}_{[\omega]}$ [Nee06, Def. V.2.12] has image in Γ. In fact, $H^n_{\mathrm{loc},s}(G, A^\delta)$ vanishes (by Corollary V.12), and thus D_n is injective. Surjectivity of D_n may be seen from the following standard integration argument. If ω is a Lie algebra n-cocycle, then the associated left-invariant n-form ω^l is closed [Nee02, Lem. 3.10]. If we make the choices of α_k for $1 \leq k \leq n$ as in Example V.2, then

$$\Omega(g_1, \ldots, g_n) := \int_{\alpha_n(g_1, \ldots, g_n)} \omega^l$$

defines

- a locally smooth group cochain on G, since α_n depends smoothly on $(g_1, \ldots, g_n)$ on an identity neighborhood and the integral depends smoothly $\alpha_n(g_1, \ldots, g_n)$.

- a group cocycle, since

$$\mathrm{d}_{\mathrm{gp}} \Omega(g_0, \ldots, g_n) = \int_{\mathrm{d}_{\mathrm{gp}} \alpha(g_0, \ldots, g_n)} \omega^l \in \mathrm{per}_\omega(\pi_n(G)) \subseteq \Gamma.$$

A straight forward calculation, similar to the ones in [Nee02] or [Nee04] now shows that $D_n([\Omega]) = [\omega]$. We expect that large parts of this remark can be generalized to arbitrary infinite-dimensional G with techniques similar to those of [Nee02, Nee02]. ∎

VI δ-Functors

In this section we recall the basic setting of (cohomological) δ-functors (sometimes also called "satellites"), as for instance exposed in [CE56, Chap. 3], [Buc55, Sect. III.5], [Gro57, Sect. 2] or [Moo76, Sect. 4]. It will be important that the arguments work in more general categories than abelian ones, the only thing one needs is a notion of short exact sequence.

Definition VI.1. A *category with short exact sequences* is a category $\mathbf{C}$, together with a distinguished class of composable morphisms $A \to B \to C$. The latter are called a short exact sequence. A morphisms between $A \to B \to C$ and $A' \to B' \to C'$ consists of morphisms $A \to A'$, $B \to B'$ and $C \to C'$ such that the diagram

$$
\begin{array}{ccc}
A & \longrightarrow B & \longrightarrow C \\
\downarrow & \downarrow & \downarrow \\
A' & \longrightarrow B' & \longrightarrow C'
\end{array}
$$

commutes.

A (cohomological) δ-functor on a category with short exact sequences is a sequence of functors

$$(H^n \colon \mathbf{C} \to \mathbf{Ab})_{n \in \mathbb{N}_0}$$

such that for each exact $A \to B \to C$ there exist morphisms $\delta_n \colon H^n(C) \to H^{n+1}(A)$ turning

$$H^0(A) \to H^0(B) \to H^0(C) \xrightarrow{\delta_0} \cdots \xrightarrow{\delta_{n-1}} H^n(A) \to H^n(B) \to H^n(C) \xrightarrow{\delta_n} \cdots$$

into an exact sequence[14] and that for each morphism of exact sequences the diagram

$$
\begin{array}{ccc}
H^n(C) & \xrightarrow{\delta_n} & H^{n+1}(A) \\
\downarrow & & \downarrow \\
H^n(C') & \xrightarrow{\delta_n} & H^{n+1}(A')
\end{array}
\tag{16}
$$

commutes. A morphisms of δ-functors from $(H^n)_{n \in \mathbb{N}_0}$ to $(G^n)_{n \in \mathbb{N}_0}$ is a sequence of natural transformations $(\varphi^n \colon H^n \Rightarrow G^n)_{n \in \mathbb{N}_0}$ such that for each short exact $A \to B \to C$ the diagram

$$
\begin{array}{ccc}
H^n(C) & \xrightarrow{\delta_n} & H^{n+1}(A) \\
\downarrow{\scriptstyle \varphi_C^n} & & \downarrow{\scriptstyle \varphi_A^{n+1}} \\
G^n(C) & \xrightarrow{\delta_n} & G^{n+1}(A)
\end{array}
\tag{17}
$$

commutes. An isomorphism of δ-functors is then a morphism for which all φ^n are natural isomorphisms of functors. ∎

Theorem VI.2. *Let $\mathbf{C}$ be a category with short exact sequences. Let $F \colon \mathbf{C} \to \mathbf{Ab}$, $I \colon \mathbf{C} \to \mathbf{C}$ and $U \colon \mathbf{C} \to \mathbf{C}$ be functors, $\iota_A \colon A \to I(A)$ and $\zeta_A \colon I(A) \to U(A)$ be natural such that $A \xrightarrow{\iota_A} I(A) \xrightarrow{\zeta_A} U(A)$ is short sequence and let $(H_n)_{n \in \mathbb{N}_0}$ and $(G_n)_{n \in \mathbb{N}_0}$ be two δ-functors.*

1. *If $\alpha \colon H^0 \Rightarrow G^0$ is a natural transformation and $H^n(I(A)) = 0$ for all A and all $1 \le n \le m$, then there exist natural transformations $\varphi^n \colon H^n \Rightarrow G^n$, uniquely determined by requiring that $\varphi^0 = \alpha$ and that*

$$
\begin{array}{ccc}
H^n(U(A)) & \xrightarrow{\delta_n} & H^{n+1}(A) \\
{\scriptstyle \varphi_{U(A)}^n}\downarrow & & \downarrow{\scriptstyle \varphi_A^{n+1}} \\
G^n(U(A)) & \xrightarrow{\overline{\delta}_n} & G^{n+1}(A)
\end{array}
$$

---

[14]Note that we do not require H^0 to be left exact.

commutes for $0 \leq n < m$. In particular, if $H^n(I(A)) = 0 = G^n(I(A))$ for all $n \geq 0$, then φ^n is an isomorphism of functors for all $n \in \mathbb{N}$ if and only if it is so for $n = 0$.

2. *Assume, moreover, that for any short exact sequence $A \xrightarrow{f} B \to C$ the morphism $A \to I(B)$ can be completed to a short exact sequence $A \to I(B) \to Q_f$ such that there exist morphisms $U(A) \xrightarrow{\beta_f} Q_f$ and $C \xrightarrow{\gamma_f} Q_f$ making*

$$
\begin{array}{ccccc}
A & \xrightarrow{\iota_A} & I(A) & \xrightarrow{\zeta_A} & U(A) \\
\| & & \downarrow{\scriptstyle I(f)} & & \downarrow{\scriptstyle \beta_f} \\
A & \longrightarrow & I(B) & \longrightarrow & Q_f
\end{array}
\quad and \quad
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \longrightarrow & C \\
\| & & \downarrow{\scriptstyle \iota_B} & & \downarrow{\scriptstyle \gamma_f} \\
A & \longrightarrow & I(B) & \longrightarrow & Q_f
\end{array}
\tag{18}
$$

commute. Then the diagram

$$
\begin{array}{ccc}
H^n(C) & \xrightarrow{\delta_n} & H^{n+1}(A) \\
\downarrow{\scriptstyle \varphi_C^n} & & \downarrow{\scriptstyle \varphi_A^k} \\
G^n(C) & \xrightarrow{\overline{\delta}_n} & G^{n+1}(A)
\end{array}
$$

also commutes for $0 \leq m < m$. In particular, if $H^n(I(A)) = 0$ for all A and all $n \geq 1$, then $(\varphi^n)_{n \in \mathbb{N}_0}$ is a morphism of δ-functors.

Proof. The proof of [Buc55, Thm. II.5.1] (cf. also [Moo76, Thm. 2]) carries over to this more general setting. The claims are shown by induction, so we assume that φ^n is constructed up to $n \geq 0$. Then we consider for arbitrary A the diagram (recall that $H^{n+1}(I(A)) = 0$)

$$
\begin{array}{ccccccc}
H^n(I(A)) & \longrightarrow & H^n(U(A)) & \xrightarrow{\delta_n} & H^{n+1}(A) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \varphi_{I(A)}^n} & & \downarrow{\scriptstyle \varphi_{U(A)}^n} & & \\
G^n(I(A)) & \longrightarrow & G^n(U(A)) & \xrightarrow{\overline{\delta}_n} & G^{n+1}(A) & &
\end{array}
,
$$

which shows that there is a unique $\varphi_A^{n+1} \colon H^{n+1}(A) \to G^{n+1}(A)$ making this diagram commute. To check naturality take $f \colon A \to B$. By the construction of φ_A^{n+1}, the induction hypothesis and the construction of φ_B^{n+1} the diagrams

$$
\begin{array}{ccc}
H^n(U(A)) & \xrightarrow{\delta_n^{U(A)}} & H^{n+1}(A) \\
\downarrow{\scriptstyle \varphi_{U(A)}^n} & & \downarrow{\scriptstyle \varphi_A^{n+1}} \\
G^n(U(A)) & \xrightarrow{\overline{\delta}_n^{U(A)}} & G^{n+1}(A)
\end{array}
\quad and \quad
\begin{array}{ccccc}
H^n(U(A)) & \xrightarrow{H^n(U(f))} & H^n(U(B)) & \xrightarrow{\delta_n^{U(B)}} & H^{n+1}(B) \\
\downarrow{\scriptstyle \varphi_{U(A)}^n} & & \downarrow{\scriptstyle \varphi_{U(B)}^n} & & \downarrow{\scriptstyle \varphi_B^{n+1}} \\
G^n(U(A)) & \xrightarrow{G^n(U(f))} & G^n(U(B)) & \xrightarrow{\overline{\delta}_n^{U(B)}} & G^{n+1}(B)
\end{array}
$$

commute. Since $(H_n)_{n \in \mathbb{N}_0}$ and $(G_n)_{n \in \mathbb{N}_0}$ are δ-functors we know that $H^{n+1}(f) \circ \delta_n^{U(A)} = \delta_n^{U(B)} \circ H^n(U(f))$ and that $G^{n+1}(f) \circ \overline{\delta}_n^{U(A)} = \overline{\delta}_n^{U(B)} \circ G^n(U(f))$. We thus conclude that

$$
\varphi_B^{n+1} \circ H^{n+1}(f) \circ \delta_n^{U(A)} = G^{n+1}(f) \circ \varphi_A^{n+1} \circ \delta_n^{U(A)}
$$

holds. Since $\delta_n^{U(A)}$ is an epimorphism this shows naturality of φ^{n+1} and finishes the proof of the first claim.

To show the second claim we note that the first diagram of (18) gives rise to a diagram

$$
\begin{array}{c}
\xymatrix{
H^n(U(A)) \ar[rr]^{\delta_n^{U(A)}} \ar[dd]_{\varphi_{U(A)}^n} \ar[dr]^{H^n(\beta_f)} & & \\
& H^n(Q_f) \ar[r]^{\delta_n^{Q_f}} \ar[d]^{\varphi_{Q_f}^n} & H^{n+1}(A) \ar[r] \ar[d]^{\varphi_A^{n+1}} & 0 \\
& G^n(Q_f) \ar[r]_{\overline{\delta}_n^{Q_f}} & G^{n+1}(A) \\
G^n(U(A)) \ar[ur]^{G^n(\beta_f)} \ar[rr]_{\overline{\delta}_n^{U(A)}} & &
}
\end{array}
\quad .
$$

The outer diagram commutes by construction of φ_A^{n+1} (see above), the already shown naturality of φ^n shows that the trapezoid on the left commutes and the two triangles are commutative because H and G are δ-functors. This implies that the whole diagram commutes. In particular, we have $\varphi_A^{n+1} \circ \delta_n^{Q_f} = \overline{\delta}_n^{Q_f} \circ \varphi_{Q_f}^n$. The latter now implies that

$$
\xymatrix{
H^n(C) \ar[r]^{H^n(\gamma_f)} \ar[d]_{\delta_n^C} & H^n(Q_f) \ar[r]^{\varphi_{Q_f}^n} \ar[d]^{\delta_n^{Q_f}} & G^n(Q_f) \ar[d]^{\overline{\delta}_n^{Q_f}} & G^n(C) \ar[l]_{G^n(\gamma_f)} \ar[d]^{\overline{\delta}_n^C} \\
H^{n+1}(A) \ar@{=}[r] & H^{n+1}(A) \ar[r]_{\varphi_A^{n+1}} & G^{n+1}(A) & G^n(A) \ar@{=}[l]
}
$$

commutes and since $G^n(\gamma_f) \circ \varphi_C^n = \varphi_{Q_f}^n \circ H^n(\gamma_f)$ we eventually conclude that

$$
\overline{\delta}_n^C \circ \varphi_C^n = \overline{\delta}_n^{Q_f} \circ G^n(\gamma_f) \circ \varphi_C^n = \overline{\delta}_n^{Q_f} \circ \varphi_{Q_f}^n \circ H^n(\gamma_f) = \varphi_A^{n+1} \circ \delta_n^{Q_f} \circ H^n(\gamma_f) = \varphi_A^{n+1} \circ \delta_n^C. \qquad \blacksquare
$$

Remark VI.3. The preceding theorem also shows the following slightly stronger statement. Assume that we have for each δ-functor $H = (H^n)_{n \in \mathbb{N}_0}$ and $G = (G^n)_{n \in \mathbb{N}_0}$ (defined on the *same* category with short exact sequences) *different* functors I, U and I', U' such that $H^n(I(A)) = 0 = G^n(I'(A))$ for all $n \geq 1$ and all A. Suppose that the assumptions of Theorem VI.2 (2.) are fulfilled for one of the functors I or I'.

If $\alpha \colon H^0 \to G^0$ is an isomorphism, then the natural transformations $\varphi^n \colon H^n \Rightarrow G^n$ (resulting from extending α) and $\psi^n \colon G^n \Rightarrow H^n$ (resulting from extending α^{-1}) are in fact isomorphisms of δ-functors. This follows immediately from the uniqueness assertion since the diagrams

$$
\xymatrix{
H^n(U(A)) \ar[r]^{\delta_n} \ar[d]_{\varphi_{U(A)}^n} & H^{n+1}(A) \ar[d]^{\varphi_A^k} \\
G^n(U(A)) \ar[r]_{\overline{\delta}_n} & G^{n+1}(A)
}
\qquad
\xymatrix{
G^n(U(A)) \ar[r]^{\overline{\delta}_n} \ar[d]_{\psi_{U(A)}^n} & G^{n+1}(A) \ar[d]^{\psi_A^k} \\
H^n(U(A)) \ar[r]_{\delta_n} & H^{n+1}(A)
}
$$

(and likewise for U') commute for arbitrary A due to the property of being a δ-functor. $\blacksquare$

Remark VI.4. Usually, one would impose some additional conditions on a category with short exact sequences, for instance that it is additive (with zero object), that for a short exact sequences $A \to B \to C$ the square

$$
\xymatrix{
A \ar[r] \ar[d] & 0 \ar[d] \\
B \ar[r] & C
}
$$

is a pull-back and a push-out, that short exact sequences are closed under isomorphisms and that certain pull-backs and push-outs exist [Bue09]. These assumptions will then help in constructing δ-functors. However, the above setting does not require this, all the assumptions are put into the requirements on the δ-functor. $\blacksquare$

Example VI.5. Suppose G is paracompact. On the category of G-modules in **CGHaus**, we consider the short exact sequences $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ such that β (or equivalently α) has a continuous local section and the functor $A \mapsto \check{H}^n(G, \underline{A})$ (or equivalently $A \mapsto H_{\mathrm{Sh}}^n(G, \underline{A})$). Then the functors $A \mapsto E_G(A)$ and $A \mapsto B_G(A)$ from Definition IV.2 satisfy $\check{H}^n(G, E_G(A)) = 0$ since $E_G(A)$ is contractible. ∎

Remark VI.6. The argument given in the proof of [Tu06, Prop. 6.1(b)] in order to draw the conclusion of the first part of Theorem VI.2 from weaker assumptions is false as one can see as follows. First note that the proof only uses $I(U(A)) \cong U(I(A))$, the more restrictive assumptions on the categories to be abelian and on the natural inclusion $A \hookrightarrow I(A)$ to satisfy $I(i_A) = i_{I(A)}$ may be replaced by this.

The requirements of [Tu06, Prop. 6.1(b)] are satisfied if we set $I(A) = E_G(A)$, $U(A) = B_G(A)$ and i_A as in Definition IV.2. In fact, the exactness of the functor E shows that

$$0 \to EA \to EC_{\boldsymbol{k}}(G, EA) \to EB_G(A) \to 0$$

is exact and since this sequence has a continuous section by [Seg70, Thm. B.2], we also have that

$$0 \to C_{\boldsymbol{k}}(G, EA) \to C_{\boldsymbol{k}}(G, EC_{\boldsymbol{k}}(G, EA)) \to C_{\boldsymbol{k}}(G, EB_G(A)) \to 0$$

is exact. Consequently, we have

$$E_G(B_G(A)) = C_{\boldsymbol{k}}(G, EB_G(A)) \cong C_{\boldsymbol{k}}(G, EC_{\boldsymbol{k}}(G, EA))/C_{\boldsymbol{k}}(G, EA) = B_G(E_G(A)).$$

However, the two sequences of functors $A \mapsto H_{\mathrm{SM}}^n(G, A) \cong H_{\mathrm{loc},c}^n(G, A)$ and $A \mapsto H_{\mathrm{glob},c}^n(G, A)$ vanish on $E_G(A)$ for $n = 1$, but are different:

- $H_{\mathrm{glob},c}^2(G, A)$ is not isomorphic to $H_{\mathrm{loc},c}^2(G, A)$, for instance for $G = C^\infty(\mathbb{S}^1, K)$ (K compact, simple and 1-connected) and $A = \mathrm{U}(1)$.

- For non-simply connected G, the universal cover gives rise to a an element in $H_{\mathrm{loc},c}^2(G, \pi_1(G))$, not contained in the image of $H_{\mathrm{glob},c}^2(G, \pi_1(G))$.

- The string classes from Example V.1 gives an element in $H_{\mathrm{loc},c}^3(K, \mathrm{U}(1))$, not contained in the image of $H_{\mathrm{glob},c}^3(K, \mathrm{U}(1))$. ∎

VII Supplements on Segal-Mitchison cohomology

We shortly recall the definition of the cohomology groups due to Segal and Mitchison from [Seg70]. Moreover, we also establish the acyclicity of the soft modules from above for the globally continuous group cohomology and show $H_{\mathrm{SM}}^n(G, A') \cong H_{\mathrm{glob},c}^n(G, A')$ for contractible A'. Consider the long exact sequence

$$A \to E_G A \to E_G(B_G A) \to E_G(B_G^2 A) \to E_G(B_G^3 A) \cdots . \tag{19}$$

This serves as a resolution of A for the invariants functor $A \mapsto A^G$ and the cohomology groups $H_{\mathrm{SM}}^n(G, A)$ are the cohomology groups of the complex

$$(E_G A)^G \to (E_G(B_G A))^G \to (E_G(B_G^2 A))^G \to (E_G(B_G^3 A))^G \cdots . \tag{20}$$

We now make the following observations:

1. [Seg70, Ex. 2.4] For an arbitrary short exact sequence $C_{\boldsymbol{k}}(G, A) \to B \to C$, the sequence

$$C_{\boldsymbol{k}}(G, A)^G \to B^G \to C^G$$

is exact, i.e., $B^G \to C^G$ is surjective. Indeed, for $y \in C^G$ choose an inverse image $x \in B$ and observe that $g.x - x$ may be interpreted as an element of $C_{\boldsymbol{k}}(G, A)$ for each $g \in G$. If we define

$$\psi(g, h) := (g.x - x)(h) \quad \text{and} \quad \xi(h) := h.\psi(h^{-1}, e)^{15},$$

then we have $g.\xi - \xi = g.x - x$ since

$$\begin{aligned}
(g.\xi - \xi)(h) =& g.(\xi(g^{-1}h)) - \xi(h) = h.(\psi(h^{-1}g, e)) - h.\psi(h^{-1}, e) \\
=& h.((h^{-1}.g.x - x)(e) - (h^{-1}.x - x)(e)) \\
=& h.((h^{-1}.(g.x - x))(e)) = (g.x - x)(h).
\end{aligned}$$

Thus $x - \xi$ is G-invariant and maps to y.

2. It is not necessary to work with the resolution (19), any resolution

$$A \to A_0 \to A_1 \to A_2 \to \cdots \tag{21}$$

(i.e., a long exact sequence of abelian groups such that the constituting short exact sequences have local continuous sections) with A_i of the form $C_{\boldsymbol{k}}(G, A_i')$ for some contractible A_i' would do the job. Indeed, then the double complex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
E_G(B_G^2 A) & \to & E_G(B_G^2(C_{\boldsymbol{k}}(G, A_0'))) & \to & E_G(B_G^2(C_{\boldsymbol{k}}(G^2, A_1'))) & \to & E_G(B_G^2(C_{\boldsymbol{k}}(G^3, A_2'))) & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
E_G(B_G A) & \to & E_G(B_G(C_{\boldsymbol{k}}(G, A_0'))) & \to & E_G(B_G(C_{\boldsymbol{k}}(G^2, A_1'))) & \to & E_G(B_G(C_{\boldsymbol{k}}(G^3, A_2'))) & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
E_G(A) & \longrightarrow & E_G(C_{\boldsymbol{k}}(G, A_0')) & \longrightarrow & E_G(C_{\boldsymbol{k}}(G^2, A_1')) & \longrightarrow & E_G(C_{\boldsymbol{k}}(G^3, A_2')) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
A & \longrightarrow & C_{\boldsymbol{k}}(G, A_0') & \longrightarrow & C_{\boldsymbol{k}}(G^2, A_1') & \longrightarrow & C_{\boldsymbol{k}}(G^3, A_2') & \longrightarrow & \cdots
\end{array}$$

has exact rows and columns (cf. [Seg70, Prop. 2.2]), which remain exact after applying the invariants functor to it by the observation from 1. Thus the cohomology of the first row is that of the first column, showing that the cohomology of 20 is the same as the cohomology of $A_0^G \to A_1^G \to A_2^G \to \cdots$.

In particular, for contractible A' we may replace (19) in the definition of $H_{\mathrm{SM}}^n(G, A')$ by

$$A' \to E_G' A \to E_G'(B_G' A') \to E_G'(B_G'^2 A) \to E_G'(B_G'^3 A') \cdots$$

with $E_G'(A') := C_{\boldsymbol{k}}(G, A')$ and $B_G'(A) := E_G'(A)/A$ (the occurrence of E in the definition $E_G(A) := C_{\boldsymbol{k}}(G, EA)$ only serves the purpose of making the target contractible).

3. Since A' is assumed to be contractible, the short exact sequence $A' \to E_G'(A') \to B_G'(A')$ has a global continuous section [Seg70, App. B], and thus the sequence

$$C_{\boldsymbol{k}}(G, A') \to C_{\boldsymbol{k}}(G, E_G'(A')) \to C_{\boldsymbol{k}}(G, B_G'(A'))$$

is exact. In particular, the isomorphism $C_{\boldsymbol{k}}(G, E_G'(A')) \cong E_G'(C_{\boldsymbol{k}}(G, A'))$ shows that

$$B_G'(C_{\boldsymbol{k}}(G, A')) := E_G'(C_{\boldsymbol{k}}(G, A'))/C_{\boldsymbol{k}}(G, A') \cong C_{\boldsymbol{k}}(G, E_G'(A'))/C_{\boldsymbol{k}}(G, A') \cong C_{\boldsymbol{k}}(G, B_G'(A'))$$

is again of the form $C_{\boldsymbol{k}}(G, A'')$ with A'' contractible.

These observations, together with an inductive argument, imply that the sequence

$$A^G \to (E_G' A)^G \to (E_G'(B_G A))^G \to (E_G(B_G'^2 A))^G \to (E_G'(B_G'^3 A))^G \cdots$$

---

[15] Note that the leading h is missing in [Seg70, Ex. 2.4].

is exact for $A = C_{\boldsymbol{k}}(G, A')$ and contractible A', and finally that $H_{\mathrm{SM}}^n(G, A)$ vanishes for $n \geq 1$. What also follows is that for contractible A', we have $H_{\mathrm{SM}}^n(G, A') \cong H_{\mathrm{glob},c}^n(G, A')$ (cf. [Seg70, Prop. 3.1]). Indeed, $C_{\boldsymbol{k}}(G^k, A') \cong C_{\boldsymbol{k}}(G, C_{\boldsymbol{k}}(G^{k-1}, A'))$ and thus

$$A' \to C_{\boldsymbol{k}}(G, A') \to C_{\boldsymbol{k}}(G^2, A') \to C_{\boldsymbol{k}}(G^3, A') \to \cdots$$

serves as a resolution of the form (21). Dropping A' and applying the invariants functor to it then gives the (homogeneous version of) the complex $C_{\mathrm{glob},c}^n(G, A')$.

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