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Extremal results for odd cycles in sparse pseudorandom graphs

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EXTREMAL RESULTS FOR ODD CYCLES IN SPARSE PSEUDORANDOM GRAPHS

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Abstract. We consider extremal problems for subgraphs of pseudorandom graphs. For graphs F and Γ the generalized Turán density $\pi_F(\Gamma)$ denotes the maximum edge density of a spanning subgraph of Γ, which contains no copy of F. Extending the Erdős-Stone theorem for odd cycles, we show that $\pi_F(\Gamma) = 1/2$ provided F is an odd cycle and Γ is a sufficiently pseudorandom graph.

In particular, for (n, d, λ)-graphs Γ, i.e., n-vertex, d-regular graphs with all non-trivial eigenvalues in the interval $[-\lambda, \lambda]$, our result holds for odd cycles of length ℓ, if

$$\lambda^{\ell-2} \log(n)^{\ell-3} \ll d^{\ell-1}/n\,.$$

For triangles this condition is best possible and this result was obtained by Sudakov, Szabó, and Vu, who addressed the case, when F is a complete graph. A construction of Alon and Kahale (based on an earlier construction of Alon for triangle-free (n, d, λ)-graphs) asserts that our assumption on Γ is best possible up to the polylog-factor for every odd $\ell \geq 5$.

1. Introduction and main result

For two graphs G and H, the *generalized Turán number*, denoted $\mathrm{ex}(G, H)$, is defined to be the largest number of edges an H-free subgraph of G may have. Here, a graph G is H-free if it contains no copy of H as a (not necessarily induced) subgraph. With this notation, the well known Erdős-Stone theorem reads

$$\mathrm{ex}(K_n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2} \tag{1}$$

where $\chi(H)$ denotes the chromatic number of H.

The systematic study of extensions of the Erdős–Stone theorem arising from replacing K_n in (1) with a sparse random or a pseudorandom graph was initiated by Kohayakawa and collaborators (see, e.g., [9, 10, 12, 13]). For random graphs such extensions were obtained recently in [8, 15].

Here, we continue the study for pseudorandom graphs. Roughly speaking, a pseudorandom graph is a graph whose edge distribution closely resembles that of a truly random graph of the same edge density. One way to formally capture this notion of pseudorandomness is through *eigenvalue separation*. A graph G on n vertices may be associated with a Boolean $n \times n$ adjacency matrix A. This matrix

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is symmetric and, hence, all its eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are real. If G is d-regular, then $\lambda_1 = d$ and $|\lambda_n| \leq d$ by the Perron-Frobenius theorem. The difference in order of magnitude between d and the *second eigenvalue* $\lambda(G) = \max\{\lambda_2, |\lambda_n|\}$ of G is often called the *spectral gap* of G. It is well known [1, 17] that the spectral gap provides a measure of control over the edge distribution of G. Roughly, the larger is the spectral gap the stronger is the resemblance between the edge distribution of G and that of the random graph $G(n,p)$, where $p = d/n$. This phenomenon led to the notion of (n,d,λ)-*graphs* by which we mean d-regular n-vertex graphs satisfying $|\lambda(G)| \leq \lambda$.

Turán type problems for sparse pseudorandom graphs were addressed in [7, 13, 16]. In this paper, we continue in studying extensions of the Erdős-Stone theorem for sparse host graphs and determine upper bounds for the generalized Turán number for odd cycles in sparse pseudorandom host graphs, i.e., $\mathrm{ex}(G, C_{2k+1})$ where G is a pseudorandom graphs and C_{2k+1} is the odd cycle of length $2k+1$. Our work is related to work of Sudakov, Szabó, and Vu [16] who determined $\mathrm{ex}(G, K_t)$ for a pseudorandom graph G and $t \geq 3$. Their result may be viewed as the pseudorandom counterpart of Turán's theorem [6].

For any graph G, the trivial lower bound $\mathrm{ex}(G, C_{2k+1}) \geq e(G)/2$, where $e(G) = |E(G)|$, follows from the fact that every graph G contains a bipartite subgraph with at least half the edges of G. For $G \cong K_n$, this bound is essentially tight, by the Erdős-Stone theorem. Our result asserts that this bound remains essentially tight for sufficiently pseudorandom graphs.

Theorem 1. *Let $k \geq 1$ be an integer. If Γ is an (n,d,λ)-graph satisfying*

$$\lambda^{2k-1} \ll \frac{d^{2k}}{n} (\log n)^{-4(k-1)^2}, \tag{2}$$

then

$$\mathrm{ex}(\Gamma, C_{2k+1}) = \left(\frac{1}{2} + o(1)\right) \frac{dn}{2}. \tag{3}$$

The asymptotic notation in (2) means that λ and d may be functions of n; and for two functions $f(n), g(n) > 0$ we write $f(n) \ll g(n)$ whenever $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$.

For $k = 1$, the same problem was studied in [16]. In this case, we obtain the same result which is known to be best possible due to the construction of Alon [2]. For $k \geq 2$, Alon's construction can be extended as to fit for general odd cycles; see [3] and [14, Example 10 and page 125], implying that that for any $k \geq 2$ the condition (2) is best possible up to the polylog-factor.

1.1. **Our main result.**
Theorem 1 is a consequence of Theorem 3 stated below for the so called *jumbled* graphs. We recall this notion of pseudorandomness which can be traced back to Thomason [18].

Given a graph Γ and two not necessarily disjoint sets $X, Y \subset V(\Gamma)$. By $\mathrm{vol}(X,Y)$ we denote the number of all possible edges with one end in X and the other in Y and by $e_\Gamma(X,Y)$ we denote the number of actual edges $xy \in E(\Gamma)$ satisfying $x \in X$ and $y \in Y$. As usual, let $e_\Gamma(X) = e_\Gamma(X,X)$.

Definition 2. *Let $p = p(n)$ be a sequence of densities, i.e., $0 \leq p \leq 1$, and let $\beta = \beta(n)$. An n-vertex graph Γ is called* (p, β)-jumbled *if*

$$|e_\Gamma(X, Y) - p\,\mathrm{vol}(X, Y)| \leq \beta\,\mathrm{vol}(X, Y)^{1/2},$$

for all $X, Y \subseteq V(\Gamma)$.

In particular, for disjoint sets X, Y

$$|e_\Gamma(X, Y) - p|X||Y|| \leq \beta(|X||Y|)^{1/2}, \tag{4}$$

and for $X = Y$ we have

$$\left|e_\Gamma(X) - p\binom{|X|}{2}\right| \leq \beta|X| \tag{5}$$

The following is our main result. The rest of the paper is dedicated to its proof.

Theorem 3. *For every integer $k \geq 1$ and every $\delta > 0$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exists an n_0 such that for any $n \geq n_0$ the following holds.*

If Γ is an n-vertex (p, β)-jumbled graph satisfying

$$\beta \leq \gamma \frac{p^{1+\frac{1}{(2k-1)}} n}{\log^{2(k-1)} n}, \tag{6}$$

then

$$\mathrm{ex}(\Gamma, C_{2k+1}) < \left(\frac{1}{2} + \delta\right) p \binom{n}{2}.$$

By the so called *expander mixing lemma* [4, 17] (see also [5, Prop. 9.2.1]), an (n, d, λ)-graph as in Theorem 1 is a (p, β)-jumbled graph with $p = d/n$ and $\beta = o(p^{1+1/(2k-1)} n \log^{-2(k-1)} n)$ showing that Theorem 3 indeed implies Theorem 1.

2. Proof of Theorem 3

Our proof of Theorem 3 relies on Lemmas 4 and 5 stated below. In this section, we state these lemmas while defering their proofs to Sections 3 and 4, respectively. We then show how these two lemmas imply Theorem 3.

To state Lemma 4, we employ the following notation. For a graph G and disjoint vertex sets $X, Y \subseteq V(G)$, we write $G[X, Y]$ to denote the bipartite subgraph of G whose vertex set is $X \cup Y$ and whose edge set, denoted $E_G(X, Y)$, consists of all edges of G with one end in X and the other in Y. Also, we write $E_G(X)$ to denote the edge set of $G[X]$.

For a graph R and a positive integer m, we write $R(m)$ to denote the graph obtained by replacing every vertex $i \in V(R)$ with a set of vertices V_i of size m and adding the complete bipartite graph between V_i and V_j whenever $ij \in E(R)$. A spanning subgraph of $R(m)$ is called an *$R(m)$-graph*. In addition, such a graph, say $G \subseteq R(m)$, is called (α, p, ε)-*degree-regular* if $\deg_{G[V_i, V_j]}(v) = (\alpha \pm \varepsilon)pm$ holds whenever $ij \in E(R)$ and $v \in V_i \cup V_j$. Throughout, the notation $R(m')$ for a positive real m' is shorthand for $R(\lceil m' \rceil)$; such conventions do not effect our asymptotic estimates.

The following lemma essentially asserts that under a certain assumption of jumbledness, a relatively dense subgraph of a sufficiently large (p, β)-jumbled graph contains a degree-regular C_ℓ(m)-graph with large m.

Lemma 4. *For any integer $\ell \geq 3$, all $\varrho > 0$, $\alpha_0 > 0$ and $0 < \varepsilon < \alpha_0$ there exist a $\nu > 0$ and a $\gamma > 0$ such that for every sequence of densities $p = p(n) \gg \log n/n$ there exists an n_0 such that for every $n \geq n_0$ the following holds.*

Let Γ be an n-vertex (p, β)-jumbled graph with $\beta = \beta(n) \leq \gamma p^{1+\varrho} n$ and let $G \subset \Gamma$ be a subgraph of Γ satisfying $e(G) \geq \alpha_0 p\binom{n}{2}$. Then, there exists an $\alpha \geq \alpha_0$ such that G contains an (α, p, ε)-degree-regular $C_\ell(\nu n)$-graph as a subgraph.

Equipped with Lemma 4, we focus on large degree-regular $C_\ell(m)$-graphs hosted in a sufficiently jumbled graph Γ. In this setting, we shall concentrate on odd cycles in Γ that have all but one of their edges in the hosted $C_\ell(m)$-graph. The remaining edge belongs to Γ. The first part of Lemma 5 stated below provides a lower bound for the number of such configurations (see (8)). We now make this precise.

We require some additional notation introduced next. Fix a vertex labeling of C_{2k+1}, say, $(u_k, \dots, u_1, w, v_1, \dots, v_k)$. For a jumbled graph Γ (as in Lemma 5), let $H \subseteq \Gamma$ be a $C_{2k+1}(m)$-graph with the following corresponding vertex partition $(U_k, \dots, U_1, W, V_1, \dots, V_k)$. By $\mathcal{C}(H, \Gamma)$ we denote the set of all cycles of length $(2k + 1)$ of the form $(u'_k, \dots, u'_1, w', v'_1, \dots, v'_k)$ such that $w' \in W, v'_i \in V_i$, $u'_i \in U_i$, $v'_k u'_k \in E(\Gamma)$, and all edges other than $v'_k u'_k$ in $E(H)$. In other words, a member of $\mathcal{C}(H, \Gamma)$ is a cycle of Γ of length $2k + 1$ with the additional requirement that the labeled edge $v'_k u'_k$ connects the ends of the path of length $2k$ in H.

For a real number $\mu > 0$, an edge of $\Gamma[V_k, U_k]$ is called *μ-saturated* if such is contained in at least $p(\mu p m)^{2k-1}$ members of $\mathcal{C}(H, \Gamma)$. A cycle in $\mathcal{C}(H, \Gamma)$ containing a μ-saturated edge is called a *μ-saturated cycle*. We write $\mathcal{S}(\mu, H, \Gamma)$ to denote the set of μ-saturated cycles in $\mathcal{C}(H, \Gamma)$.

To motivate the definition of μ-saturated edges, note that we expect that an edge of $\Gamma[U_k, V_k]$ extends to $(\alpha p)^{2k} m^{2k-1}$ members of $\mathcal{C}(H, \Gamma)$. For $\mu \approx \alpha$, a μ-saturated edge overshoots this expectation by a factor of $1/\alpha$.

The following lemma asserts for sufficiently jumbled graphs the number of α-saturated cycles is negligible compared to $|\mathcal{C}(H, \Gamma)|$ (see (8) and (9)).

Lemma 5. *For any integer $k \geq 1$ and all reals $0 < \nu, \alpha_0 \leq 1$, and $0 < \varepsilon \leq \alpha_0/3$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exists an n_0 such that for any $n \geq n_0$ the following holds.*

If Γ is (p, β)-jumbled n-vertex graphs with

$$\beta = \beta(n) \leq \gamma \frac{p^{1+\frac{1}{2k-1}} n}{\log^{2(k-1)} n}, \tag{7}$$

then for any $m \geq \nu n$ and any $\alpha \geq \alpha_0$ a (α, p, ε)-degree-regular $C_{2k+1}(m)$-graph $H \subseteq \Gamma$ satisfies

$$|\mathcal{C}(H, \Gamma)| \geq (\alpha - 2\varepsilon)^{2k} (pm)^{2k+1} \tag{8}$$

and

$$|\mathcal{S}(\alpha + 2\varepsilon, H, \Gamma)| \leq (3\varepsilon)^{2k} (pm)^{2k+1}. \tag{9}$$

With Lemmas 4 and 5 stated above (and proved in Sections 3 and 4, respectively) we proceed by showing how these imply Theorem 3; our main result.

Proof of Theorem 3. Let $k \geq 1$ and $\delta > 0$ be given. Without loss of generality, we may assume that $\delta \leq 1/2$. We set

$$\ell = 2k+1, \quad \varrho = \ell^{-1}, \quad \varepsilon = \frac{\delta}{4+32k+6^{2k+1}}, \quad \text{and} \quad \alpha_0 = 1/2+\delta, \tag{10}$$

and let ν and γ_1 be those obtained by applying Lemma 4 with $\ell, \varrho, \varepsilon$, and α_0 as these are set in (10). Next, let γ_2 be that obtained by applying Lemma 5 with k, ν, α_0 and ε, and set

$$\gamma = \min\{\gamma_1, \gamma_2, \delta\nu/4\}. \tag{11}$$

From this point on, Theorem 3 and Lemmas 4 and 5 are quantified in a similar manner. Hence, given $p = p(n)$, let n_0 be sufficiently large as to accommodate Lemmas 4 and 5.

Let Γ be a (p,β)-jumbled n-vertex graph with β satisfying (6). To prove Theorem 3, it is sufficient to show that every subgraph G of Γ satisfying $e(G) \geq \alpha_0 p\binom{n}{2}$ contains a C_{2k+1}. To that end, let G be such a subgraph of Γ and let $H \subseteq G$ be an (α, p, ε)-degree-regular $C_{2k+1}(m)$-graph given by Lemma 4, where $\alpha \geq \alpha_0$ and $m \geq \nu n$. Next, let $F = F(U_k, V_k) \subseteq E_\Gamma(U_k, V_k)$ denote those edges of $\Gamma[U_k, V_k]$ met by a member of $\mathcal{C}(H,\Gamma)$. Every edge in F completes a path of length $2k$ in H into a cycle of length $2k+1$. In what follows, we prove that $F \cap E(U_k, V_k) \neq \emptyset$ which then implies that $C_{2k+1} \subseteq H \subseteq G$ completing our proof of Theorem 3.

To this end, it is sufficient to show

$$|F| \geq \left(\alpha - \frac{\delta}{2}\right) pm^2. \tag{12}$$

Indeed, as H is (α, p, ε)-degree-regular and since $\varepsilon < \delta/2$, we obtain

$$e_H(U_k, V_k) = \sum_{v \in U_k} \deg_{H[U_k,V_k]}(v) \geq \left(\alpha - \frac{\delta}{2}\right) pm^2.$$

On the other hand, jumbledness of Γ combined with $m \geq \nu n$ and $\gamma \leq \delta\nu/4$ guarantees

$$e_\Gamma(U_k, V_k) \leq pm^2 + \beta m \leq pm^2 + \gamma p^{1+\varrho} nm \leq (1+\delta/2)\, pm^2.$$

Hence, $e_\Gamma(U_k, V_k) < |F| + e_H(U_k, V_k)$ implying that $|F|$ and $E_H(U_k, V_k)$ have non-empty intersection.

It remains to show (12). By definition, each member of F is contained in at most $p\,(\alpha+2\varepsilon)^{2k-1}\,(pm)^{2k-1}$ members of $\mathcal{C}' = \mathcal{C}(H,\Gamma) \setminus \mathcal{S}(\alpha+2\varepsilon, H, \Gamma)$ so that

$$|F| \geq \frac{|\mathcal{C}'|}{p\,(\alpha+2\varepsilon)^{2k-1}\,(pm)^{2k-1}}. \tag{13}$$

Next, owing to Lemma 5, we obtain the estimates

$$|\mathcal{C}(H,\Gamma)| \geq (\alpha - 2\varepsilon)^{2k}\,(pm)^{2k+1} \tag{14}$$

and

$$|\mathcal{S}(\alpha+2\varepsilon, H, \Gamma)| \leq (3\varepsilon)^{2k}(pm)^{2k+1}. \tag{15}$$

Combining (13), (14), and (15) we obtain that

$$|F| \geq \frac{(\alpha-2\varepsilon)^{2k} - (3\varepsilon)^{2k}}{(\alpha+2\varepsilon)^{2k-1}} pm^2.$$

First, we consider the term

$$T = \frac{(\alpha-2\varepsilon)^{2k}}{(\alpha+2\varepsilon)^{2k-1}} = (\alpha-2\varepsilon)\left(\frac{\alpha-2\varepsilon}{\alpha+2\varepsilon}\right)^{2k-1}.$$

As $\frac{a-b}{a+b} \geq 1 - \frac{2}{a}b$ for any $a,b>0$ and $\alpha \geq 1/2$, we attain

$$T \geq (\alpha-2\varepsilon)\left(1-\frac{2}{\alpha}2\varepsilon\right)^{2k-1} \geq (\alpha-2\varepsilon)(1-8\varepsilon)^{2k-1}.$$

By the Bernoulli inequality $(1-8\varepsilon)^{2k-1} \geq (1-16k\varepsilon)$. Then, since $\delta \leq 1/2$ it follows that

$$T \geq \alpha - 2\varepsilon - 16k\varepsilon. \tag{16}$$

Combining the fact that

$$\frac{(3\varepsilon)^{2k}}{(\alpha+2\varepsilon)^{2k-1}} \leq 2^{2k-1}3^{2k}\varepsilon \leq 6^{2k}\varepsilon$$

with (16), we arrive at

$$|F| \geq \left(\alpha - 2\varepsilon - 16k\varepsilon - 6^{2k}\varepsilon\right)pm^2.$$

As $2\varepsilon + 16k\varepsilon + 6^{2k}\varepsilon \leq \delta/2$, by the choice of ε, (12) follows. □

3. Proof of Lemma 4

In this section, we prove Lemma 4. This lemma follows from Lemma 6 below. Roughly speaking, Lemma 6 asserts that under certain assumptions, a jumbled graph contains a large subgraph with all its vertices having almost the same degree.

Lemma 6. *For all $\varrho \geq 0$, $\mu > 0$, $0 < \varepsilon < \alpha \leq 1$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exists an n_0 such that for any $n \geq n_0$ the following holds.*

Let Γ be a (p,β)-jumbled n-vertex graph with $\beta = \beta(n) \leq \gamma p^{1+\varrho} n$ and let $G \subset \Gamma$ be a subgraph of Γ satisfying $n' = |V(G)| \geq \mu n$ and $e(G) \geq \alpha p\binom{n'}{2}$. Then, there exists a subset $U \subset V(G)$ of size at least $\varepsilon n'/50$ such that one of the following holds:

(I) *$e(G[U]) \geq \left(\alpha + 2\varepsilon^2/25\right)p\binom{|U|}{2}$ or*
(II) *$\deg_{G[U]}(u) = (\alpha \pm \varepsilon)p|U|$ for all $u \in U$.*

To prove Lemma 6, we shall make use of the following property immediately deduced from jumbledness.

Proposition 7. *Let Γ be a (p,β)-jumbled graph. If $X, Y \subseteq V(\Gamma)$ are disjoint and satisfy $e_\Gamma(X,Y) \geq k|X| \neq p|Y||X|$, then*

$$|X| \leq \frac{\beta^2|Y|}{(k-p|Y|)^2}. \tag{17}$$

Proof of Lemma 6. Given ϱ, μ, ε, and α, set

$$t^* = \frac{1}{2\varrho} + 1, \quad \xi = \frac{8\varepsilon^2}{25^2}, \quad \text{and} \quad \gamma = \frac{\mu^2\varepsilon^2\xi}{2^{4t^*+1}} \tag{18}$$

Given $p = p(n)$, let n_0 be sufficiently large, and let Γ and G be as stated. We suppose that (I) is not satisfied and show that (II) holds.

We begin by passing to a large subgraph with a lower bound on its minimum degree. More precisely, we show that

$$\text{there exists a } W \subset V(G) \text{ with } |W| \geq \sqrt{\tfrac{\xi}{2}}n' \text{ and } \delta(G[W]) \geq (\alpha - \xi)p|W|, \quad (19)$$

where $\delta(J)$ denotes the minimum degree of a graph J.

To this end, let $\{V_{n'}, V_{n'-1}, \ldots, V_m\}$ be a maximum sequence of vertex sets where $V_{n'} = V(G)$ and V_{t-1} is obtained from V_t by removing a single vertex in V_t of degree less than $(\alpha-\xi)p|V_t| = (\alpha-\xi)pt$. Given such a sequence, we set $W = V_m$. It remains to show that $|W| = m \geq (\xi/2)^{1/2}\, n'$. Indeed,

$$\alpha p \binom{n'}{2} \leq e(G) \leq (\alpha - \xi)p \left(\sum_{t \in [n']} t \right) + e_\Gamma(W);$$

and since $e_\Gamma(W) \leq p\binom{|W|}{2} + \beta|W|$, we attain that

$$\alpha \binom{n'}{2} \leq (\alpha - \xi) \binom{n'+1}{2} + \binom{|W|}{2} + \gamma p^{\varrho} n |W|.$$

As $n' \geq \mu n$, then $\gamma p^{\varrho} n|W| \leq \frac{\gamma}{\mu}(n')^2$. Hence, for n' sufficiently large (i.e., n_0 sufficiently large) we have

$$\frac{\xi}{3}(n')^2 \leq \frac{|W|^2}{2} + \frac{\gamma}{\mu}(n')^2.$$

Isolating $|W|^2$ then yields

$$2\left(\frac{\xi}{3} - \frac{\gamma}{\mu}\right)(n')^2 \leq |W|^2.$$

Assertion (19) now follows as $\gamma \leq \xi\mu/12$.

Next, we repeatedly delete vertices with too high degree showing that this hardly effects the size of W and the minimum degree condition obtained above. To handle the maximum degree, we repeatedly discard vertices from W whose degree in $G[W]$ exceeds a certain limit. It turns out that although the number of such discarded vertices might be of order $\Omega(|W|)$ we may still ensure that a significant portion of $G[W]$ remains. We make this precise now.

Recall that $m = |W|$, let

$$X_1 = \big\{u \in W\colon\ \deg_{G[W]}(u) \geq (\alpha + \varepsilon/5)pm\big\},$$

and note that

$$|X_1| \leq \frac{\varepsilon}{25} m. \quad (20)$$

Indeed, $e_G(W) \leq \big(\alpha + 2\varepsilon^2/25\big)\, p\binom{m}{2}$ as assumption (I) does not hold. Then,

$$\begin{aligned}\big(\alpha + 2\varepsilon^2/25\big)\, pm^2 \geq 2|E_G(W)| &= \sum_{u \in W \setminus X_1} \deg_{G[W]}(u) + \sum_{x \in X_1} \deg_{G[W]}(x) \\ &\geq (\alpha - \xi)pm(m - |X_1|) + (\alpha + \varepsilon/5)pm|X_1| \\ &= (\alpha - \xi)pm^2 + (\varepsilon/5 + \xi)pm|X_1|;\end{aligned}$$

so that

$$|X_1| \leq \left(\frac{\varepsilon^2/25 + \xi}{\varepsilon/25 + \xi}\right) m \overset{(18)}{\leq} \frac{\varepsilon}{25} m.$$

The maximum degree of $G[W \setminus X_1]$ is at most $(\alpha+\varepsilon/5)pm$ and $|W \setminus X_1| \geq \frac{4\varepsilon}{5}m$. However, the minimum degree of a vertex in G_2 may now be lower than the bound requested in (II) and in the remainder of the proof we focus on handling this issue. Indeed, for a vertex $v \in W$, $|N_{G[W]}(v) \cap X_1|$ might be large to the extent that discarding X_1 has an effect on the degree of v in the resulting graph. As X_1 might be linear in $|W|$ such an event is possible. In the remainder of the proof we focus on handling this issue.

To this end, we define a sequence of sets $(X_2, X_3, \dots)$. The set of vertices $X_2 \subseteq W \setminus X_1$ consists of those vertices whose degree in $G[W]$ into X_1 is "too large". In a similar manner, for $t > 2$, we define X_t to be the set consisting of the vertices having "too high" degree into X_{t-1}. We will show that such a sequence has constant length and that $\sum_t |X_t|$ is negligible with respect to $(\alpha+\varepsilon/5)pm$ so that discarding all these sets does not affect the maximum degree of the remaining vertices.

For $t \geq 2$, put

$$X_t = \left\{ u \in W \setminus \bigcup_{j=1}^{t-1} X_j : \deg_{G[W]}(u, X_{t-1}) \geq \frac{\varepsilon}{2^{t+2}} pm \right\}.$$

In what follows, we prove by induction on i that

$$|X_t| \leq \gamma^{t-1} p^{(t-1)2\varrho} m, \text{ for all } 2 \leq t \leq t^*. \tag{21}$$

Indeed, $|X_2|$ satisfies,

$$|X_2| \overset{(17)}{\leq} \frac{\beta^2 |X_1|}{\left(\frac{\varepsilon}{8}pm - p|X_1|\right)^2} \leq \frac{\frac{\varepsilon}{25}\gamma^2 p^2 p^{2\varrho} n^2 m}{\left(\frac{\varepsilon}{8} - \frac{\varepsilon}{25}\right)^2 p^2 m^2} \leq \left(\frac{\gamma}{\mu \frac{\varepsilon}{25}\sqrt{\frac{\xi}{2}}} \right)^2 p^{2\varrho} m \overset{(18)}{\leq} \gamma p^{2\varrho} m,$$

where the third inequality is since $n^2 \leq \frac{m^2}{\mu^2 \xi/2}$. Consequently, (21) holds for $t = 2$.

For $t \geq 2$ we obtain that

$$|X_{t+1}| \leq \frac{\beta^2 |X_t|}{\left(\frac{\varepsilon}{2^{t+3}}pm - p|X_t|\right)^2}$$

holds, by (17). Substituting $|X_t|$ with the inductive hypothesis yields

$$|X_{t+1}| \leq \frac{\gamma^2 p^2 p^{2\varrho} n^2 \gamma^{t-1} p^{2(t-1)\varrho} m}{\left(\frac{\varepsilon}{2^{t+3}} - \gamma^{t-1} p^{2(t-1)\varrho}\right)^2 p^2 m^2} \leq \frac{\gamma^2 \gamma^{t-1} p^{2t\varrho} m}{\left(\frac{\varepsilon}{2^{t+3}} - \gamma^{t-1} p^{2(t-1)\varrho}\right)^2 \mu^2 \frac{\xi}{2}}.$$

As by the choice of γ, the inequality

$$\frac{\varepsilon}{2^{t+3}} - \gamma^{t-1} p^{2(t-1)\varrho} \geq \frac{\varepsilon}{2^{t+4}}$$

holds for each $2 \leq t \leq t^*$, we reach

$$|X_{t+1}| \leq \frac{\gamma^2 \gamma^{t-1}}{\left(\mu \frac{\varepsilon}{2^{t+4}} \sqrt{\frac{\xi}{2}}\right)^2} p^{2t\varrho} m \overset{(18)}{\leq} \gamma^t p^{2t\varrho} m.$$

This concludes our proof of (21).

Next, we show that the length of the sequence (X_t) is constant. In particular, we show that this sequence has length at most $t^* = \frac{1}{2\varrho} + 1$. To see this, observe that X_{t+1} is empty if $|X_t| < \frac{\varepsilon}{2^{t+2}} pm$. By (21), the latter is satisfied if

$$\gamma^{t-1} p^{2(t-1)\varrho} m < \frac{\varepsilon}{2^{t+2}} pm. \tag{22}$$

For t^*, (22) is satisfied provided

$$\gamma^{t^*-1} \leq \frac{\varepsilon}{2^{t^*+2}},$$

which indeed holds due to the choice of γ.

In the remainder of the proof we show that we may choose $U = W \setminus \bigcup_{t=1}^{t^*} X_t.$, i.e., $|U| \geq \frac{\varepsilon}{50} n'$ and $G[U]$ satisfies (II). Observe that

$$\begin{aligned} |U| = |W| - |X_1| - \sum_{t=2}^{t^*} |X_t| &\geq |W| - |X_1| - (t^* - 1)|X_2| \\ &\geq \sqrt{\frac{\xi}{2}} n' - \frac{\varepsilon}{25} m - \frac{1}{2\varrho} \gamma p^{2\varrho} m \geq \left(\sqrt{\frac{\xi}{2}} - \frac{\varepsilon}{25} - \frac{\gamma}{2\varrho} \right) n'. \end{aligned}$$

Recalling the choice of ξ and observing that $\frac{\gamma}{2\varrho} \overset{(18)}{\leq} \frac{\varepsilon}{50}$, we have that $|U| \geq \frac{\varepsilon}{50} n'$ as required. By almost the same argument we also observe that

$$|U| \geq m - |X_1| - (t^* - 1)|X_2| \geq \left(1 - \frac{3\varepsilon}{50}\right) m.$$

It remains to verify that $G[U]$ satisfies (II). We begin with the maximum degree of a vertex $v \in U$. Such a vertex satisfies

$$\deg_{G[U]}(v) < (\alpha + \varepsilon/5) pm \leq (\alpha + \varepsilon/5) p \frac{|U|}{\left(1 - \frac{3\varepsilon}{50}\right)} \leq (\alpha + \varepsilon) p |U|$$

as $\alpha + \varepsilon/5 \leq \left(1 - \frac{3\varepsilon}{50}\right)(\alpha + \varepsilon)$ since $\alpha + \varepsilon \leq 2$.

Finally, we consider the minimum degree of a vertex $v \in U$. We have that

$$\deg_{G[U]}(v) \geq (\alpha - \xi) pm - |X_{t^*}| - \frac{\varepsilon}{4} pm \left(\sum_{t=2}^{t^*} \frac{1}{2^t} \right).$$

The choice of γ and t^* and (21) yield $|X_{t^*}| \leq \frac{\varepsilon}{50} pm$. In addition, $\sum_{t=2}^{t^*} 2^{-t} \leq 1/2$. Consequently,

$$\deg_{G[U]}(v) \geq \left(\alpha - \xi - \frac{\varepsilon}{50} - \frac{\varepsilon}{4}\right) pm \overset{(18)}{\geq} (\alpha - \varepsilon) pm \geq (\alpha - \varepsilon) p |U|.$$

This concludes our proof of Lemma 6. □

Lemma 4 follows from Lemma 6 and a standard concentration result for the hypergeometric distribution.

Proof of Lemma 4. Given $\ell, \varrho, \varepsilon$, and α_0, set

$$\varepsilon_1 = \varepsilon/4, \quad \mu = (\varepsilon_1/50)^{100\varepsilon_1^{-2}}, \quad \text{and} \quad \nu = \mu/\ell,$$

let γ' be that obtained by applying Lemma 6 with ϱ, ε_1, and α_0, and put

$$\gamma = \min\{\gamma', \mu\}.$$

Given $p = p(n)$, let n_0 be sufficiently large as to accommodate Lemma 6. For $n \geq n_0$, let Γ be an n-vertex (p, β)-jumbled graph, where β is as specified in Lemma 4, and let $G \subseteq \Gamma$ be a subgraph of Γ satisfying $e(G) \geq \alpha_0 p\binom{n}{2}$.

To prove Lemma 4, we shall first pass to a subgraph of G that is essentially degree regular and of order linear in n. We then show that a random equipartition of such a subgraph is highly likely to be an (α, p, ε)-degree-regular $C_\ell(\nu n)$-graph for some $\alpha \geq \alpha_0$.

In what follows, we show that there exists an $\alpha \geq \alpha_0$ and a set $U \subseteq V(G)$ satisfying

(i) $|U| \geq \mu n$ and
(ii) $\deg_{G[U]}(u) = (\alpha \pm \varepsilon_1)p|U|$ for each $u \in U$.

Roughly speaking, to prove this we shall repeatedly apply Lemma 6 starting from $G_0 = G$ and nested subgraphs thereof until assertion (II) of that lemma holds. Due to jumbledness such an iteration gives rise to a sequence $(G_0, \ldots, G_t)$ of nested subgraphs of G_0 where t is a constant. We shall then set $U = V(G_t)$. We now make this precise.

Set $G_0 = G$. For $i > 0$, let $G_i \subseteq G_{i-1}$ be the subgraph of G_{i-1} obtained by assertion (I) of Lemma 6 so that

$$|V(G_i)| \geq \frac{\varepsilon_1}{50}|V(G_{i-1})|$$

and

$$e(G_i) \geq \left(\alpha_0 + i\frac{(\varepsilon_1)^2}{25}\right) p\binom{|V(G_i)|}{2}.$$

Owing to jumbledness and the assumption that $e(G_0) \geq \alpha_0 p\binom{|V(G_0)|}{2}$, it holds that $|V(G_0)| \geq \mu n$. Consequently, sequences of the form $(G_0, G_1, \ldots)$ exist. Let then $(G_0, \ldots, G_t)$ be a maximal such sequence. Jumbledness yields

$$\left(\alpha_0 + t\frac{\varepsilon_1^2}{25}\right) p\binom{\mu n}{2} \leq p\binom{\mu n}{2} + \beta\mu n,$$

so that $t \leq \frac{100\gamma}{\mu\varepsilon_1^2} \leq \frac{100}{\varepsilon_1^2}$, where $\gamma \leq \mu$ is used for the last inequality. The existence of α as required is clear.

In the remainder of the proof, we show that a random equipartition of $G[U]$ is highly likely to be an (α, p, ε)-degree-regular $C_\ell(\nu n)$-graph. Without loss of generality, we may assume that $|U| = \ell\nu n$ (and thus divisible by ℓ) and that $\deg_{G[U]}(u) = (\alpha \pm 2\varepsilon_1)p|U|$ for each $u \in U$. Indeed, a set $U' \subseteq U$ with $|U'| = \ell\nu n \geq \frac{\mu}{2}n$ can be obtained by removing at most $\ell - 1$ vertices from U. A vertex $u \in U'$ satisfies $\deg_{G[U']} = (\alpha \pm 2\varepsilon')p|U'|$ since $\ell \leq \varepsilon' p|U|$ for n (i.e., n_0) sufficiently large and $p = \Omega(n^{-1})$.

Let $U = U_1 \dot\cup \ldots \dot\cup U_\ell$ be a random equipartition of U consisting of ℓ sets each of size νn. Call a vertex $v \in U$ *bad* if there exists an index $j \in [\ell]$ such that $v \notin U_j$ and

$$\big|\deg(v, U_j) - \alpha p|U_j|\big| > \varepsilon p|U_j|.$$

Next, for two distinct indices $i, j \in [\ell]$ and a vertex $v \in U_i$, let $X = X_v^j = \deg(v, U_j)$ denote the degree of v into U_j in $G[U]$. The random variable X is hypergeometrically distributed with mean

$$\mathbb{E}X = \frac{(\alpha \pm 2\varepsilon_1)p|U||U_j|}{|U|} = (\alpha \pm 2\varepsilon_1)p|U_j|.$$

For hypergeometrically distributed random variable the following is a well-known concentration result (see e.g., [11, Theorem 2.10 and Equation (2.9)])

$$\mathbb{P}\left[|X - \mathbb{E}X| \geq \eta\mathbb{E}X\right] \leq 2\exp(-\eta\mathbb{E}X/3) \quad \text{for } \eta \leq 3/2.$$

Consequently, the probability that a fixed vertex v is bad is given by

$$\begin{aligned}\mathbb{P}\left[v \text{ is bad}\right] &\leq \ell\mathbb{P}\left[|X - \mathbb{E}X| > (\varepsilon - 2\varepsilon_1)p\nu n\right] \leq \ell\mathbb{P}\left[|X - \mathbb{E}X| > \frac{\varepsilon - 2\varepsilon_1}{\alpha + 2\varepsilon_1}\mathbb{E}X\right] \\ &\leq 2\ell\exp\left(-\frac{\varepsilon - 2\varepsilon_1}{\alpha + 2\varepsilon_1}\mathbb{E}X\right),\end{aligned}$$

Hence, for n sufficiently large since $p \gg \ln n/n$ we have that

$$\mathbb{P}\left[\text{there exists a bad vertex in } U\right] \leq 2\ell|U|\exp\left(-\frac{\varepsilon - 2\varepsilon_1}{\alpha + 2\varepsilon_1}\mathbb{E}X\right) < 1,$$

implying that there exists an equipartition of U yielding no bad vertices. Such a partition, with possible redundant edges removed, forms an (α, p, ε)-degree-regular $C_\ell(\nu n)$-graph, as required. □

4. Proof of Lemma 5

In this section we prove Lemma 5. Throughout this section, Γ denotes an n-vertex (p, β)-jumbled graph and H denotes an (α, p, ε)-degree-regular $C_{2k+1}(m)$-graph that is a subgraph of Γ. We assume that the graph H has a partition $(U_k, \ldots, U_1, W, V_1, \ldots, V_k)$ of its vertex set (see Section 2).

Lemma 5 has two parts the first of which concerns $\mathcal{C}(H, \Gamma)$. Recall that this set consists of all $(2k+1)$-cycles of the form $(u_k, \ldots, u_1, w, v_1, \ldots, v_k)$, where $u_i \in U_i$, $w \in W$, $v_i \in V_i$, the edge $u_k v_k$ is an edge of $\Gamma[U_k, V_k]$, and the remaining edges are that of H. The first part of the lemma (see (8)) asserts that

$$|\mathcal{C}(H, \Gamma)| \geq (\alpha - 2\varepsilon)^{2k}(pm)^{2k+1}.$$

Observe that the fact that H is almost degree regular implies that the number of paths of the form $(u_k, \ldots, u_1, w, v_1, \ldots, v_k)$ in H is $((\alpha \pm \varepsilon)pm)^{2k}$. As H is an arbitrary subgraph of Γ it may occur that this set of paths "clusters" on a small number of pairs of vertices $(u_k, v_k) \in U_k \times V_k$. The lower bound stated in (8) asserts that this is not the case. In fact, a p proportion of these paths extend to cycles in $\mathcal{C}(H, \Gamma)$ as one would expect in a purely random setting.

The second part of Lemma 5 concerns the set $\mathcal{S}(\alpha+2\varepsilon, H, \Gamma)$ of $(\alpha+2\varepsilon)$-saturated cycles. This consists of all cycles $(u_k, \ldots, u_1, w, v_1, \ldots, v_k)$ in $\mathcal{C}(H, \Gamma)$ for which the edge $u_k v_k \in E(\Gamma[U_k, V_k])$ is $(\alpha + 2\varepsilon)$-saturated, meaning that it is contained in at least $p((\alpha + 2\varepsilon)pm)^{2k-1}$ members of $\mathcal{C}(H, \Gamma)$. In (9), the second part of the lemma, an upper bound is put forth for $|\mathcal{S}(\alpha + 2\varepsilon, H, \Gamma)|$ asserting that the latter is negligible compared to $|\mathcal{C}(H, \Gamma)|$. The point here is that for $|\mathcal{C}(\Gamma, H)| \geq (\alpha - 2\varepsilon)^{2k}(pm)^{2k+1}$ ((8) will yield this) and $e_\Gamma[U_k, V_k]$ is approximately pm^2, we expect an edge of $\Gamma[U_k, V_k]$ to be contained in roughly at least $p(\alpha - 2\varepsilon)^{2k}(pm)^{2k-1}$

members of $\mathcal{C}(H,\Gamma)$. Hence, for a small ε, the number of $(\alpha+2\varepsilon)$-saturated cycles overshoots this expectation by a factor of roughly $1/\alpha$. In particular, we will show that $|\mathcal{S}(\alpha+2\varepsilon,H,\Gamma)|\le(3\varepsilon)^{2k}(pm)^{2k+1}$.

In the sequel we introduce a framework that will enable us to handle estimates for both $|\mathcal{C}(H,\Gamma)|$ and $|\mathcal{S}(\mu,H,\Gamma)|$ using essentially the same type of arguments. Prior to this framework, we include here a brief and rough sketch of our approach for $k=3$ in which H has the partition $(U_3,U_2,U_1,W,V_1,V_2,V_3)$.

For a vertex $w\in W$, we write $\mathcal{C}(H,\Gamma,w)$ to denote the cycles in $\mathcal{C}(H,\Gamma)$, so that $|\mathcal{C}(H,\Gamma,w)|$ is the *contribution* of w to $|\mathcal{C}(H,\Gamma)|$. Clearly, an estimate for such a contribution will translate into an estimate for $|\mathcal{C}(H,\Gamma)|$. To estimate $\mathcal{C}(H,\Gamma,w)$, we shall repeatedly apply the jumbledness condition (4) to pairs of subsets (L,T) where $L\subseteq U_3$ and $T\subseteq V_3$. The framework to be introduced will ensure us that in each such application of the jumbledness condition to a pair (L,T), each of the edges in $\Gamma[L,T]\subseteq\Gamma[U_k,V_k]$ thus obtained is contained in roughly the same number of cycles in $\mathcal{C}(H,\Gamma,w)$.

To achieve this level of control, consider, first, the neighborhood $X_w=N_H(w)\cap U_1$ of w in U_1. We partition the set U_2 according to the "backward" degrees of its vertices into X_w. More precisely, the ith partition class will consist of all vertices $u\in U_2$ satisfying $(1+\eta)^{i-1}\le|N_H(u)\cap X_w|<(1+\eta)^i$ for some small η to be chosen later on. Some of these classes may be empty. Nevertheless, these classes cover U_2, and there are at most $\lceil\log_{1+\eta}n\rceil+1$ such classes.

We proceed to U_3 in a similar manner. Each partition class of U_2 will define a partition of U_3. The latter is defined in a similar manner to the partition just defined for U_2 using X_w. More precisely, given the i-th partition class of U_2, say, $Z_\eta(i,X_w)$, we assign a vertex $u\in U_3$ to the j-th partition class of U_3 if it satisfies $(1+\eta)^{j-1}\le|N_H(u)\cap Z_\eta(i,X_w)|<(1+\eta)^j$. The resulting partition class is denoted $Z_\eta(i,j,X_w)$.

We partition the sets V_2 and V_3 in a similar manner where we use $Y_w=N_H(w)\cap V_1$, the neighborhood of w in V_1, instead of X_w.

The advantage of this kind of partitioning is that the number of paths between w and any vertex $u\in Z_\eta(i,j,X_w)\subset U_3$, confined to the set $Z_\eta(i,X_w)$ - i.e. paths of the form (w,u',u) where $u'\in Z_\eta(i,X_w)$ - is at least $(1+\eta)^{i+j-2}$ and at most $(1+\eta)^{i+j}$. As a result, this number is known up to a factor of $(1+\eta)^2$. Naturally, the same bounds hold for w and any vertex $v\in Z_\eta(i',j',Y_w)\subset V_3$, where $Z_\eta(i',j',Y_w)$ is the set obtained by the partitioning procedure with respect to Y_w, V_2, and V_3.

Now, if we take a path from w to $u\in Z_\eta(i,j,X_w)$, confined to $Z_\eta(i,X_w)$, and another path from w to $v\in Z_\eta(i',j',Y_w)$, confined to $Z_\eta(i',X_w)$, then these two paths yield a path from u to v. Hence, the number of such (u,v)-paths is at least $(1+\eta)^{i+i'+j+j'-4}$ and at most $(1+\eta)^{i+i'+j+j'}$.

Since u and v were arbitrary vertices from $Z_\eta(i,j,X_w)$ and $Z_\eta(i',j',Y_w)$, respectively, we conclude that if uv is an edge of Γ, then $(1+\eta)^{i+i'+j+j'\pm4}$ is the number of cycles in $\mathcal{C}(H,\Gamma,w)$ that contain the edge uv and are confined to $Z_\eta(i,X_w)$ and $Z_\eta(i',X_w)$.

On the other hand, jumbledness of Γ yields that the number of edges in Γ between $Z_\eta(i,j,X_w)$ and $Z_\eta(i',j',Y_w)$ is

$$p|Z_\eta(i,j,X_w)||Z_\eta(i',j',Y_w)|\pm\beta\sqrt{|Z_\eta(i,j,X_w)||Z_\eta(i',j',Y_w)|}.$$

Summing over all $1 \le i, j, i', j' \le \lceil \log_{1+\eta} n \rceil + 1$, we obtain a good estimate for $|\mathcal{C}(H, \Gamma, w)|$. Indeed, the main term of the contribution of w is

$$\sum_{i,j,i',j'} p|Z_\eta(i, j, X_w)||Z_\eta(i', j', Y_w)|(1 + \eta)^{i+j+i'+j' \pm 4}.$$

This, we will show to be

$$|X_w||Y_w|p((\alpha \pm \varepsilon)pm)^4, \tag{23}$$

by a simple argument. Note that we also obtain an upper bound here which will turn out important later. The main obstacle will be to show that the error term is negligible compared to the main term, i.e., that

$$\sum_{i,j,i',j'} \beta \sqrt{|Z_\eta(i, j, X_w)||Z_\eta(i', j', Y_w)|}(1 + \eta)^{i+j+i'+j'+4} = o(p(pm)^6), \tag{24}$$

provided Γ is sufficiently jumbled. This will be done in Claim 8.

So far we have discussed our approach for establishing the first part of Lemma 5, i.e., (8). For the second part of this lemma, i.e., (9), we are to estimate of $|\mathcal{S}(\alpha + 2\varepsilon, H, \Gamma)|$. This will be done by employing similar arguments to those above that will be applied to a rearrangement of the partition $(U_3, U_2, U_1, W, V_1, V_2, V_3)$. In particular we shall use the rearrangement

$$(\tilde{U}_3, \tilde{U}_2, \tilde{U}_1, \tilde{W}, \tilde{V}_1, \tilde{V}_2, \tilde{V}_3) = (W, U_1, U_2, U_3, V_3, V_2, V_1).$$

This is a valid partition of the $C_7(m)$-graph H.

The interest here is to estimate the number of $(\alpha + 2\varepsilon)$-saturated cycles. The $(\alpha + 2\varepsilon)$-saturated edges that these cycles contain now lie between the sets $\tilde{W} = U_3$ and $\tilde{V}_1 = V_3$. For a given vertex $\tilde{w} \in \tilde{W}$ we set, as before, $\tilde{X}_{\tilde{w}} = N_H(\tilde{w}) \cap \tilde{U}_1$. Unlike before, we shall define the set $\tilde{Y}_{\tilde{w}} \subset \tilde{V}_1$ to consist of those vertices of $\tilde{V}_1$ that are incident to $\tilde{w}$ through $(\alpha + 2\varepsilon)$-saturated edges.

The same arguments as above, yield bounds corresponding to (23) and (24) that will then lead to an upper bound on the number of $(\alpha + 2\varepsilon)$-saturated cycles containing $\tilde{w}$. In particular, we shall have that since every $(\alpha + 2\varepsilon)$-saturated edge is contained in at least $p((\alpha + 2\varepsilon)pm)^5$ cycles containing $\tilde{w}$, then

$$|\tilde{Y}_{\tilde{w}}|p((\alpha + 2\varepsilon)pm)^5 \le |\tilde{X}_w||\tilde{Y}_w|p((\alpha + \varepsilon)pm)^4 + o(p(pm)^6). \tag{25}$$

Now, as $|\tilde{X}_w| \le (\alpha + \varepsilon)pm$, we conclude that this last inequality can hold provided that $|\tilde{Y}_{\tilde{w}}| = o(pm)$, implying that the number of $(\alpha + 2\varepsilon)$-saturated cycles containing $\tilde{w}$ is bounded from above by the right hand side of (25), which is $o(p(pm)^6)$. Summing over all $\tilde{w} \in \tilde{W}$ yields the desired bound.

4.1. **Preparation for the proof of Lemma 5.** In what follows we make the above discussion as to our approach precise and make it fit for a general k. As already mentioned, here Γ is a (p, β)-jumbled graph and H is a subgraph of Γ that is (α, ε, p)-degree-regular $C_{2k+1}(m)$-graph. The latter we assume to have the partition $(U_k, \dots, U_1, W, V_1, \dots, V_k)$ of its vertex set.

Partitioning the neighborhoods. For a real $\eta > 0$, set $L_\eta = \lceil \log_{1+\eta} 2pm \rceil + 1$, and let

$$\mathcal{I}_\eta = \{0\} \times [L_\eta]^{k-1},$$

where the use of zero here will be made clear shortly. By $\boldsymbol{s}$ we mean a tuple of integers $(s_1, \dots, s_k) \in \mathcal{I}_\eta$ (so that s_1 is always zero), and write $\boldsymbol{s}_j$ to denote the prefix $(s_1, \dots, s_j)$, where $j \in [k]$, and write $\boldsymbol{s}$ instead of $\boldsymbol{s}_k$. We put $Z_\eta(\boldsymbol{s}_1, X) = Z_\eta(0, X) = X \subset U_1$, and for $j = 2, \dots, k$ we define

$$Z_\eta(\boldsymbol{s}_j, X) = \{x \in U_j \colon (1+\eta)^{s_j - 1} \le |N_H(x) \cap Z_\eta(\boldsymbol{s}_{j-1}, X)| < (1+\eta)^{s_j}\}, \quad (26)$$

so that $Z_\eta(\boldsymbol{s}_j, X) \subseteq U_j$ for each $j \in [k]$. For future reference, it will be convenient for us to stress that for $\eta \in (0,1]$, the value $(1+\eta)^{s_j}$ is essentially bounded by the maximum degree of H for any $j \in [k]$, in particular, it holds that

$$\text{if } \eta \in (0,1], \text{ then } (1+\eta)^{s_j} \le 8pm, \text{ for any } s_j \in [L_\eta]; \quad (27)$$

indeed,

$$L_\eta \le \log_{1+\eta}(2pm) + 2 \le \log_{1+\eta}(2pm) + \log_{1+\eta} 4 = \log_{1+\eta} 8pm.$$

Observe, in addition, that $(1+\eta)^{L_\eta - 1} \ge 2pm \ge (\alpha + \varepsilon)pm$, and that the maximum degree of a vertex in H is $(\alpha+\varepsilon)pm$. This means that (26) defines a partition of the neighborhood of $Z_\eta(\boldsymbol{s}_{j-1}, X)$ in U_j, i.e.,

$$\dot{\bigcup}_{1 \le i \le L_\eta} Z_\eta((s_1, \dots, s_{j-1}, i), X) = N_H(Z_\eta(\boldsymbol{s}_{j-1}, X)) \cap U_j, \quad (28)$$

where some of these sets may possibly be empty.

Counting paths. We exploit the above partitioning scheme in order to count (X, U_k)-paths in H. Throughout, by *paths* we always mean *shortest paths*. Also, if L and R are two subsets of vertices, we write (L, R)-path to denote a path with one end in L and the other in R. With these conventions, an (X, U_k)-path in H, where $X \subset U_1$, has a single vertex in each set U_i, $i \in [k]$. Finally, instead of $(X, \{y\})$-*path* we write (X, y)-*path*.

For $j \in \{2, \dots, k\}$ and a tuple $\boldsymbol{s} \in \mathcal{I}_\eta$, we write $\sum \boldsymbol{s}_j$ to denote the sum $\sum_{i=1}^{j} s_i = \sum_{i=2}^{j} s_i$, and we write $\sum \boldsymbol{s}$ instead of $\sum \boldsymbol{s}_k$. Further, the subgraph of H induced by the vertex sets $\{Z_\eta(\boldsymbol{s}_1, X), Z_\eta(\boldsymbol{s}_2, X) \dots, Z_\eta(\boldsymbol{s}_j, X)\}$ is denoted by $H(\boldsymbol{s}_j)$.

For a vertex $z \in Z_\eta(\boldsymbol{s}_j, X)$ the number $\pi_H(X, \boldsymbol{s}_j, z)$ of (X, z)-paths confined to $H(\boldsymbol{s}_j)$ clearly satisfies

$$\prod_{i=2}^{j} (1+\eta)^{s_i - 1} \le \pi_H(X, \boldsymbol{s}_j, z) \le \prod_{i=2}^{j} (1+\eta)^{s_i}.$$

Since $s_1 = 0$, we may write

$$(1+\eta)^{-(j-1)} (1+\eta)^{\sum \boldsymbol{s}_j} \le \pi_H(X, \boldsymbol{s}_j, z) \le (1+\eta)^{\sum \boldsymbol{s}_j}. \quad (29)$$

Recall the benefit of the above partitioning scheme, we observe that for any two vertices $z, z' \in Z_\eta(\boldsymbol{s}_j, X)$, the variation between $\pi_H(X, \boldsymbol{s}_j, z)$ and $\pi_H(X, \boldsymbol{s}_j, z')$ is bounded by a factor of $(1+\eta)^{j-1}$. Hence, the number

$$\pi_H(X, \boldsymbol{s}_j) = \sum_{z \in Z_\eta(\boldsymbol{s}_j, X)} \pi_H(X, \boldsymbol{s}_j, z)$$

of $(X, Z_\eta(\boldsymbol{s}_j, X))$-paths confined to $H(\boldsymbol{s}_j)$ satisfies

$$(1+\eta)^{-(j-1)} |Z_\eta(\boldsymbol{s}_j, X)| (1+\eta)^{\sum \boldsymbol{s}_j} \le \pi_H(X, \boldsymbol{s}_j) \le |Z_\eta(\boldsymbol{s}_j, X)| (1+\eta)^{\sum \boldsymbol{s}_j}.$$

By (28), every (X, U_j)-path is contained in $H(\boldsymbol{s}_j)$ for exactly one $\boldsymbol{s}_j$. Hence, summing over all $\boldsymbol{s}_j$, we obtain the following inequality for the number $\pi_H(X, U_j)$

of (X, U_j)-paths:

$$(1+\eta)^{-(j-1)} \sum_{\boldsymbol{s}_j} |Z_\eta(\boldsymbol{s}_j, X)|(1+\eta)^{\sum \boldsymbol{s}_j} \leq \pi_H(U_j, X) \leq \sum_{\boldsymbol{s}_j} |Z_\eta(\boldsymbol{s}_j, X)|(1+\eta)^{\sum \boldsymbol{s}_j}. \quad (30)$$

On the other hand, owing to the degree-regularity of H, we obviously have

$$|X| \left((\alpha - \varepsilon)pm\right)^{j-1} \leq \pi_H(U_j, X) \leq |X| \left((\alpha + \varepsilon)pm\right)^{j-1} \quad (31)$$

for all $j \in [k]$

We conclude this section by mentioning that for a set $Y \subseteq V_1$ and a tuple $\boldsymbol{t} \in \mathcal{I}_\eta$, we define the sets $\{Z_\eta(\boldsymbol{t}_j, Y)\}_{j=1}^k$ and the numbers $\pi_H(Y, \boldsymbol{t}_j, u)$, $\pi_H(Y, \boldsymbol{t}_j)$, $\pi_H(V_j, Y)$ in an analogous manner to the sets and numbers just defined. The properties (28) to (31) translate verbatim.

Counting cycles. Given $u \in U_k$, $v \in V_k$, $\pi_H(X, u)$, and $\pi_H(Y, v)$ (the number of (X, u)-paths and (Y, v)-paths, respectively), we have that

$$\mathcal{O}(X, Y) = \sum_{uv \in E_\Gamma(U_k, V_k)} \pi_H(X, u)\pi_H(Y, v) \quad (32)$$

is the number of composed paths each of which comprises a (X, u)-path and a (Y, v)-path (in H) connected by the edge $uv \in \Gamma[U_k, V_k]$. Let now $\mathcal{C}(H, \Gamma, w)$ denote the set of cycles in $\mathcal{C}(H, \Gamma)$ containing the vertex $w \in W$, and observe that

$$|\mathcal{C}(H, \Gamma, w)| = \mathcal{O}(N_H(w) \cap U_1, N_H(w) \cap V_1). \quad (33)$$

Next, let us consider $|\mathcal{S}(\mu, H, \Gamma)|$. To this end, we rearrange the partition $(U_k, \ldots, U_1, W, V_1, \ldots, V_k)$ to yield a partition $(\tilde{U}_k, \ldots, \tilde{U}_1, \tilde{W}, \tilde{V}_1, \ldots, \tilde{V}_k)$. This we obtain by simply renaming the partition classes as follows:

$$\tilde{W} = U_k,\ \tilde{U}_1 = U_{k-1}, \ldots, \tilde{U}_{k-1} = U_1,\ \tilde{U}_k = W,\ \tilde{V}_1 = V_k, \ldots, \tilde{V}_k = V_1. \quad (34)$$

Note that the new partition is still a valid partition of the $C_{2k+1}(m)$-graph H, and that the μ-saturated edges now lie between $\tilde{W}$ and $\tilde{V}_1$. For a vertex $\tilde{w} \in \tilde{W}$, let $D_\mu(\tilde{w})$ denote the set of vertices in $\tilde{V}_1$ adjacent to $\tilde{w}$ (in Γ) through a μ-saturated edge and let $\mathcal{S}(\mu, H, \Gamma, \tilde{w})$ denote the set of μ-saturated cycles containing $\tilde{w}$. Then,

$$|\mathcal{S}(\mu, H, \Gamma, \tilde{w})| \leq \tilde{\mathcal{O}}(N_H(\tilde{w}) \cap \tilde{U}_1, D_\mu(\tilde{w})), \quad (35)$$

where $\tilde{\mathcal{O}}(\tilde{X}, \tilde{Y})$ is defined in the same way as $\mathcal{O}(\tilde{X}, \tilde{Y})$ only with respect to the partition $(\tilde{U}_k, \ldots, \tilde{U}_1, \tilde{W}, \tilde{V}_1, \ldots, \tilde{V}_k)$, and where $\tilde{X} \subset \tilde{U}_1$ and $\tilde{Y} \subset \tilde{V}_1$. In (35), we obtain an upper bound only as cycles in $\tilde{\mathcal{O}}(N_H(\tilde{w}) \cap \tilde{U}_1, D_\mu(\tilde{w}))$ may involve edges in Γ between $\tilde{U}_k$ and $\tilde{V}_k$ which might not belong to H.

In view of (33) and (35), we focus on $\mathcal{O}(X, Y)$ in order to estimate $|\mathcal{C}(H, \Gamma, w)|$ and $|\mathcal{S}(\mu, H, \Gamma, \tilde{w})|$. For tuples $\boldsymbol{s}, \boldsymbol{t} \in \mathcal{I}_\eta$, we write $e_\Gamma(\boldsymbol{s}, \boldsymbol{t})$ for $e_\Gamma(Z_\eta(\boldsymbol{s}, X), Z_\eta(\boldsymbol{t}, Y))$, and observe that due to (30) we may write

$$\frac{\sum_{\boldsymbol{s}} \sum_{\boldsymbol{t}} e_\Gamma(\boldsymbol{s}, \boldsymbol{t})(1+\eta)^{\sum \boldsymbol{s} + \sum \boldsymbol{t}}}{(1+\eta)^{2(k-1)}} \leq \mathcal{O}(X, Y) \leq \sum_{\boldsymbol{s}} \sum_{\boldsymbol{t}} e_\Gamma(\boldsymbol{s}, \boldsymbol{t})(1+\eta)^{\sum \boldsymbol{s} + \sum \boldsymbol{t}}. \quad (36)$$

Here, we appeal to the jumbledness of Γ to estimate $e_\Gamma(\boldsymbol{s}, \boldsymbol{t})$ which asserts that

$$e_\Gamma(\boldsymbol{s}, \boldsymbol{t}) = p|Z_\eta(\boldsymbol{s}, X)||Z_\eta(\boldsymbol{t}, Y)| \pm \beta\sqrt{|Z_\eta(\boldsymbol{s}, X)||Z_\eta(\boldsymbol{t}, Y)|}. \quad (37)$$

Substituting this estimate for $e_\Gamma(\boldsymbol{s},\boldsymbol{t})$ in (36), we arrive at the following two bounds for $\mathcal{O}(X,Y)$.

$$(1+\eta)^{2(k-1)}\mathcal{O}(X,Y) \geq pP_\eta(X)P_\eta(Y) - \beta Q_\eta(X)Q_\eta(Y), \tag{38}$$

and

$$\mathcal{O}(X,Y) \leq pP_\eta(X)P_\eta(Y) + \beta Q_\eta(X)Q_\eta(Y), \tag{39}$$

where

$$P_\eta(X) = \sum_{\boldsymbol{s}} |Z_\eta(\boldsymbol{s},X)|(1+\eta)^{\sum \boldsymbol{s}} \text{ and } Q_\eta(X) = \sum_{\boldsymbol{s}} \sqrt{|Z_\eta(\boldsymbol{s},X)|}(1+\eta)^{\sum \boldsymbol{s}}, \tag{40}$$

and where $\tilde{P}_\eta(\tilde{X})$ and $Q_\eta(\tilde{X})$ are defined analogously.

Hence, for a small η, the size of $\mathcal{O}(X,Y)$ is essentially determined up to the additive error term $\beta Q_\eta(X)Q_\eta(Y)$. In the sequel, we show that this is dominated by the main term $pP_\eta(X)P_\eta(Y)$. The estimate of the error term is complicated and consequently delegated to Claim 8 below. The estimate of the main term, however, is almost trivial at this point. To see the latter, we rewrite (30) for $j=k$ as to obtain

$$P_\eta(X) \leq \pi_H(U_k,X)(1+\eta)^{k-1} \leq P_\eta(X)(1+\eta)^{k-1}.$$

With (31) this yields

$$|X|\left((\alpha-\varepsilon)pm\right)^{k-1} \leq P_\eta(X) \leq |X|\left((\alpha+\varepsilon)(1+\eta)pm\right)^{k-1}. \tag{41}$$

A similar assertion clearly holds for $P_\eta(Y)$.

We may now summarize all of the above discussion concisely as follows. For $w\in W$, the sets $X_w = N_H(w)\cap U_1$ and $Y_w = N_H(w)\cap V_1$ both have size $(\alpha\pm\varepsilon)pm$ due to the degree regularity of H. Owing to (33), (38), and (41) we have that

$$(1+\eta)^{2k}|\mathcal{C}(H,\Gamma,w)| \geq (\alpha-\varepsilon)^{2k}p(pm)^{2k} - \beta Q_\eta(X_w)Q_\eta(Y_w). \tag{42}$$

Next, for $\tilde{w}\in\tilde{W}$ and sets $X_{\tilde{w}} = N_H(\tilde{w})\cap\tilde{U}_1$ and $Y_{\tilde{w}} = D_\mu(\tilde{w})$ we have, owing to (35), (39), and (41), that

$$\begin{aligned}
|\mathcal{S}(\mu,H,\Gamma,\tilde{w})| &\overset{(35),(39)}{\leq} p\tilde{P}_\eta(X_{\tilde{w}})\tilde{P}_\eta(D_\mu(\tilde{w})) + \beta\tilde{Q}_\eta(X_{\tilde{w}})\tilde{Q}_\eta(D_\mu(\tilde{w})) \\
&\leq |X_{\tilde{w}}||D_\mu(\tilde{w})|p\left((\alpha+\varepsilon)pm\right)^{2(k-1)} + \beta\tilde{Q}_\eta(X_{\tilde{w}})\tilde{Q}_\eta(D_\mu(\tilde{w})) \\
&\leq |D_\mu(\tilde{w})|p\left((\alpha+\varepsilon)pm\right)^{2k-1} + \beta\tilde{Q}_\eta(X_{\tilde{w}})\tilde{Q}_\eta(D_\mu(\tilde{w})).
\end{aligned} \tag{43}$$

We conclude this section by stating the claim that will be used to control the error term $\beta Q_\eta(X)Q_\eta(Y)$ (and $\beta\tilde{Q}_\eta(\tilde{X})\tilde{Q}_\eta(\tilde{Y})$) discussed above.

Claim 8. *For any integer $k\geq 1$, and reals $0<\xi,\alpha,\eta,\nu\leq 1$, and $0<\varepsilon\leq\alpha/3$, there exists a $\gamma>0$ such that for every sequence of densities $p=p(n)>0$ there exists an n_0 such that for all $n>n_0$ the following holds.*

Let Γ be an n-vertex (p,β)-jumbled graph with

$$\beta < \gamma\frac{p^{1+\frac{1}{2k-1}}n}{\log^{2(k-1)} n}, \tag{44}$$

and let $H \subseteq \Gamma$ be an (α, p, ε)-degree-regular $C_{2k+1}(m)$-graph, $m \geq \nu n$, with the partition $(U_k, \dots, U_1, W, V_1, \dots, V_k)$. If $X \subset U_1$ and $Y \subset V_1$ both have size at most $(\alpha+\varepsilon)pm$, then

$$\beta Q_\eta(X) Q_\eta(Y) < \xi p (pm)^{2k}.$$

We postpone the proof of Claim 8 until Section 4.3. In the subsequent section we show how to derive Lemma 5 from Claim 8.

4.2. **Proof of Lemma 5.** Given k, ν, α_0, and ε, put

$$\xi = (\varepsilon/4)^{4k}, \quad \eta = \min\left\{\frac{\varepsilon}{4\alpha_0}, 1\right\}, \tag{45}$$

and let γ' be that obtained from Claim 8 applied with k, ξ, $\alpha = \alpha_0$, η, ν, and ε. For Lemma 5 we set

$$\gamma = \gamma',$$

and choose n_0 to be sufficiently large as to accommodate Claim 8. Finally, let Γ and H be as specified in Lemma 5.

Owing to (42)

$$|\mathcal{C}(G, \Gamma, w)| \geq \left(\frac{\alpha - \varepsilon}{1+\eta}\right)^{2k} p(pm)^{2k} - \frac{1}{(1+\eta)^{2k}} \beta Q_\eta(X_w) Q_\eta(Y_w),$$

for any $w \in W$, where $X_w = N_H(w) \cap U_1$ and $Y_w = N_H(w) \cap V_1$. Since H is (α, p, ε)-degree-regular both X_w and Y_w have size at most $(\alpha+\varepsilon)pm$. As a result, $\beta Q_\eta(X_w) Q_\eta(Y_w) \leq \xi p(pm)^{2k}$, by Claim 8 applied with $X = X_w$ and $Y = Y_w$. Then, owing to our choices for η and ξ in (45) we have that

$$\begin{aligned}
|\mathcal{C}(G, \Gamma, w)| &\geq \left(\frac{\alpha - \varepsilon}{1+\eta}\right)^{2k} p(pm)^{2k} - (1+\eta)^{-2k} \xi p (pm)^{2k} \\
&\geq \frac{(\alpha-\varepsilon)^{2k} - (\varepsilon/4)^{4k}}{(1+\eta)^{2k}} p(pm)^{2k} \\
&\geq \left(\frac{\alpha - (3/2)\varepsilon}{1+\eta}\right)^{2k} p(pm)^{2k} \\
&\geq (\alpha - 2\varepsilon)^{2k} p(pm)^{2k}
\end{aligned}$$

holds for any $w \in W$. Summing over all vertices in W yields (8), the first assertion of Lemma 5.

It remains to show (9), the second assertion of Lemma 5. Here, we use the partition of (34). It is sufficient to prove that for any $\tilde{w} \in \tilde{W}$ it holds that

$$|D_{\alpha+2\varepsilon}(\tilde{w})| \leq (3\varepsilon)^{2k} pm. \tag{46}$$

Indeed, assuming (46) yields

$$\begin{aligned}
|\mathcal{S}(\alpha+2\varepsilon, H, \Gamma, \tilde{w})| &\overset{(43)}{\leq} |D_{\alpha+2\varepsilon}(\tilde{w})| p \left((\alpha+\varepsilon)pm\right)^{2k-1} + \beta \tilde{Q}_\eta(X_{\tilde{w}}) \tilde{Q}_\eta(D_{\alpha+2\varepsilon}(\tilde{w})) \\
&\leq \varepsilon^{2k} (\alpha+\varepsilon)^{2k-1} p(pm)^{2k} + \xi p(pm)^{2k} \quad \text{(by Claim 8)} \\
&\overset{(45)}{\leq} \left(\varepsilon^{2k}(\alpha+\varepsilon)^{2k-1} + (\varepsilon/4)^{4k}\right) p(pm)^{2k} \\
&\leq (3\varepsilon)^{2k} p(pm)^{2k}.
\end{aligned}$$

With this (9) follows once we sum over all vertices in $\tilde{W}$.

It remains to prove (46). Suppose $|D_{\alpha+2\varepsilon}(\tilde{z})| > (3\varepsilon)^{2k}pm$ for some vertex $\tilde{z} \in \tilde{W}$, and choose $B \subseteq D_{\alpha+2\varepsilon}(\tilde{z}) \subseteq \tilde{V}_1$ of size $\lceil (3\varepsilon)^{2k}pm \rceil$. Observe that since $\varepsilon \le \alpha/3$ it holds that $(3\varepsilon)^{2k} \le \alpha+\varepsilon$ so that $|B| \le (\alpha+\varepsilon)pm$. Let us now count the number of members of $\mathcal{S}(\alpha+2\varepsilon, H, \Gamma, \tilde{z})$ with the $(\alpha+2\varepsilon)$-saturated edge of the form $\tilde{z}b$ where $b \in B$. We write $\mathcal{S}(B, \tilde{z})$ to denote this number. By the definition of an $(\alpha+2\varepsilon)$-saturated edge, we attain

$$|\mathcal{S}(B,\tilde{z})| \ge |B|p((\alpha+2\varepsilon)pm)^{2k-1} = (3\varepsilon)^{2k}(\alpha+2\varepsilon)^{2k-1}p(pm)^{2k}.$$

On the other hand, (43) with $X_{\tilde{z}} = N_H(\tilde{z}) \cap \tilde{U}_1$ and B instead of $D_\mu(\tilde{z})$, together with Claim 8 with $X = X_{\tilde{z}}$ and $Y = B$ yield

$$\begin{aligned} |\mathcal{S}(B,\tilde{z})| &\le |B|p((\alpha+\varepsilon)pm)^{2k-1} + \xi p(pm)^{2k} \text{ (by Claim 8 and (43))} \\ &\overset{(45)}{=} \left((3\varepsilon)^{2k}(\alpha+\varepsilon)^{2k-1} + (\varepsilon/4)^{4k}\right)p(pm)^{2k}. \end{aligned}$$

The contradiction here is that

$$(3\varepsilon)^{2k}(\alpha+\varepsilon)^{2k-1} + (\varepsilon/4)^{4k} < (3\varepsilon)^{2k}(\alpha+2\varepsilon)^{2k-1}.$$

To see this, observe that

$$(\alpha+\varepsilon)^{2k-1} + \frac{\varepsilon^{2k}}{3^{2k}4^{4k}} \le (\alpha+\varepsilon)^{2k-1} + (\varepsilon/12)^{2k-1} \le (\alpha+\varepsilon+\varepsilon/12)^{2k-1} < (\alpha+2\varepsilon)^{2k-1}.$$

This proves (46) and thus completes our proof of Lemma 5. □

4.3. **Proof of Claim 8.** Given k, ξ, α, ε, η, and ν, we set

$$\gamma = \frac{\xi\,(\log(1+\eta))^{2k}}{2^{8k}\nu}\,, \tag{47}$$

choose n_0 to be sufficiently large, and let Γ be a (p,β)-jumbled graph, where β satisfies (44).

Recall that we seek to show that $\beta Q_\eta(X)Q_\eta(Y) \le \xi p(pm)^{2k}$, that $Q_\eta(X) = \sum_{\boldsymbol{s}\in\mathcal{I}_\eta}\sqrt{|Z_\eta(\boldsymbol{s},X)|}(1+\eta)^{\sum \boldsymbol{s}}$, and that $Q_\eta(Y)$ is defined in a similar manner. To this end, we shall now consider the term

$$q_\eta(\boldsymbol{s},X) = \sqrt{|Z_\eta(\boldsymbol{s},X)|}(1+\eta)^{\sum \boldsymbol{s}}, \tag{48}$$

where $\boldsymbol{s} \in \mathcal{I}_\eta$. Below we shall prove that for any $\boldsymbol{s} \in \mathcal{I}_\eta$

$$q_\eta(\boldsymbol{s},X) \le 2^{4k}p^{k-\frac{1}{2(2k-1)}}m^{\frac{2k-1}{2}}. \tag{49}$$

This estimate will hold for the counterpart term $q_\eta(\boldsymbol{t},Y)$ with X replaced by Y and $\boldsymbol{t} \in \mathcal{I}_\eta$ as well due to symmetry.

Assuming (49), we prove that $\beta Q_\eta(X)Q_\eta(Y) \le \xi p(pm)^{2k}$ as follows. We observe the identity

$$\beta Q_\eta(X)Q_\eta(Y) = \beta\sum_{\boldsymbol{s},\boldsymbol{t}} q_\eta(\boldsymbol{s},X)q_\eta(\boldsymbol{t},Y), \tag{50}$$

and put

$$L = (2\log_{1+\eta} n)^{2(k-1)} \ge L_\eta^{2(k-1)},$$

which is an upper bound on the number of summands in (50) (for n sufficiently large). Then,

$$\begin{aligned}\beta Q_\eta(X)Q_\eta(Y) &\le \beta L\left(2^{4k}p^{k-\frac{1}{2(2k-1)}}m^{\frac{2k-1}{2}}\right)^2\\ &= \beta L 2^{8k}p^{2k-\frac{1}{2k-1}}m^{2k-1}\\ &\overset{(44)}{\le} \gamma 2^{8k}\left(\frac{1}{\log(1+\eta)}\right)^{2k-1}p^{2k+1}m^{2k-1}n\\ &\overset{(47)}{\le} \xi p(pm)^{2k}.\end{aligned}$$

It remains to prove (49). Fix now a tuple $\boldsymbol{s}\in\mathcal{I}_\eta$. We shall consider two cases. throughout these cases we shall use the estimates

$$\alpha+\varepsilon\le 2 \quad\text{and}\quad 1+\eta\le 2. \tag{51}$$

(1) Suppose, firstly, that $|Z_\eta(\boldsymbol{s}_j,X)|<p^{1/(2k-1)}m$ for all $2\le j\le k$. We shall show

$$q_\eta(\boldsymbol{s},X)=\sqrt{|Z_\eta(\boldsymbol{s},X)|}(1+\eta)^{\sum\boldsymbol{s}}\le(1+\eta)^k\sqrt{|X|}\prod_{j=2}^{k}M_j, \tag{52}$$

where

$$M_j=2\max\left\{\beta, p\sqrt{|Z_\eta(\boldsymbol{s}_j,X)||Z_\eta(\boldsymbol{s}_{j-1},X)|}\right\}<2p^{1+1/(2k-1)}m.$$

Then, (52) together with the assumption that $\sqrt{|X|}\le\sqrt{(\alpha+\varepsilon)pm}$ give

$$\begin{aligned}q_\eta(\boldsymbol{s},X) &\le (1+\eta)^k\sqrt{(\alpha+\varepsilon)pm}\left(2p^{1+\frac{1}{2k-1}}m\right)^{k-1}\\ &\overset{(51)}{\le} 2^{2k}\sqrt{pm}\left(p^{1+\frac{1}{2k-1}}m\right)^{k-1}\\ &= 2^{2k}p^{k-\frac{1}{2(2k-1)}}m^{\frac{2k-1}{2}},\end{aligned}$$

so that (49) holds in this case.

To verify (52), we first show for all $2\le j\le k$

$$\sqrt{|Z_\eta(\boldsymbol{s}_j,X)|}(1+\eta)^{s_j-1}\le M_j\sqrt{|Z_\eta(\boldsymbol{s}_{j-1},X)|}. \tag{53}$$

Note that (53) holds if $(1+\eta)^{s_j-1}\le 2p|Z_\eta(\boldsymbol{s}_{j-1},X)|$. On the other hand, if $(1+\eta)^{s_j}>2p|Z_\eta(\boldsymbol{s}_{j-1},X)|$ holds then

$$\sqrt{|Z_\eta(\boldsymbol{s}_j,X)|}\overset{(17)}{\le}\frac{\beta\sqrt{|Z_\eta(\boldsymbol{s}_{j-1},X)|}}{(1+\eta)^{s_j-1}-p|Z_\eta(\boldsymbol{s}_{j-1},X)|}\le\frac{\beta\sqrt{|Z_\eta(\boldsymbol{s}_{j-1},X)|}}{\frac12(1+\eta)^{s_j-1}}.$$

Repeating (53) for each $2\le j\le k$ yields (52). This leads to a $(1+\eta)^{k-1}$ multiplicative factor, here we take $(1+\eta)^k$. This concludes the proof of (52).

(2) Suppose, secondly, that $|Z_\eta(\boldsymbol{s}_j,X)|\ge p^{1/(2k-1)}m$ for some $2\le j\le k$. To prove (49) in this case, we express $q_\eta(\boldsymbol{s},X)$ as a product of two numbers, that is, we write

$$q_\eta(\boldsymbol{s},X)=\sqrt{|Z_\eta(\boldsymbol{s},X)|}(1+\eta)^{\sum\boldsymbol{s}}=R_1\times R_2, \tag{54}$$

where R_1 is given by

$$R_1 = \sqrt{|Z_\eta(\boldsymbol{s}_j, X)|}(1+\eta)^{\sum \boldsymbol{s}_j}, \tag{55}$$

and where R_2 is given by

$$R_2 = \prod_{r=j}^{k-1} \sqrt{\frac{|Z_\eta(\boldsymbol{s}_{r+1}, X)|}{|Z_\eta(\boldsymbol{s}_r, X)|}(1+\eta)^{s_{r+1}}}. \tag{56}$$

Before proceeding let us, first, observe that R_2^j is well-defined. Indeed, we are concerned with $q_\eta(\boldsymbol{s}, X)$ provided $Z_\eta(\boldsymbol{s}, X)$ is nonempty as otherwise $q_\eta(\boldsymbol{s}, X) = 0$ and does not contribute to the sum (50). Now, by the definition of the Z_η-sets, the set $Z_\eta(\boldsymbol{s}, X)$ being nonempty implies that every set $Z_\eta(\boldsymbol{s}_r, X)$ is nonempty for each $r \in [k]$. Consequently, it is valid to divide by $|Z_\eta(\boldsymbol{s}_r, X)|$ for each $r \in [k]$. Second, let us also note that $R_1 \times R_2$ is a telescope product; the cardinalities of all Z_η-sets cancel each other with only $|Z_\eta(\boldsymbol{s}, X)|$ remaining after all cancelations.

Now, to upper bound $q_\eta(\boldsymbol{s}, X)$ as required in this case, we prove that

$$R_1 \le 4^j p^{j - \frac{1}{2(2k-1)}} m^{\frac{2j-1}{2}}, \tag{57}$$

and that

$$R_2 \le (6pm)^{k-j}. \tag{58}$$

Owing to (54), these two estimates imply (49) in this case. In what follows, we prove the estimates (57) and (58).

To see (57), observe first that

$$|Z_\eta(\boldsymbol{s}_j, X)|(1+\eta)^{\sum \boldsymbol{s}_j} \le |X| \left((1+\eta)(\alpha+\varepsilon)pm\right)^{j-1} \le (4pm)^j, \tag{59}$$

where the first inequality is due to the degree-regularity of H (see, (30) and (31)), and the second inequality is due to the assumption that $|X| \le (\alpha+\varepsilon)pm$. This together with the assumption of this case that $|Z_\eta(\boldsymbol{s}_j, X)| \ge p^{1/(2k-1)}m$ yield that

$$(1+\eta)^{\sum \boldsymbol{s}_j} \le \frac{(4pm)^j}{|Z_\eta(\boldsymbol{s}_j, X)|} \le 4^j p^{j - \frac{1}{2k-1}} m^{j-1}. \tag{60}$$

Rewriting (55), we arrive at

$$R_1 = \sqrt{|Z_\eta(\boldsymbol{s}_j, X)|(1+\eta)^{\sum \boldsymbol{s}_j}}(1+\eta)^{\frac{1}{2}\sum \boldsymbol{s}_j}.$$

Owing to (59) and (60), we then have

$$R_1 \le (4pm)^{j/2} \left(4^j p^{j - \frac{1}{2k-1}} m^{j-1}\right)^{1/2},$$

and (57) follows.

It remains to prove (58). To see this, let us first rewrite (56) as to attain the form

$$R_2 = \prod_{r=j}^{k-1} \left(\frac{|Z_\eta(\boldsymbol{s}_{r+1}, X)|}{|Z_\eta(\boldsymbol{s}_r, X)|}(1+\eta)^{s_{r+1}}\right)^{1/2} (1+\eta)^{\frac{s_{r+1}}{2}}. \tag{61}$$

Recall, first, that for any $r \in [k]$, it holds that $(1+\eta)^{s_r} \leq 8pm$, by (27). Second, for $r \in [k-1]$, observe that

$$|Z_\eta(\boldsymbol{s}_{r+1}, X)|(1+\eta)^{s_{r+1}} \leq (1+\eta)e_H(Z_\eta(\boldsymbol{s}_r, X), Z_\eta(\boldsymbol{s}_{r+1}, X)),$$

so that the term $|Z_\eta(\boldsymbol{s}_{r+1}, X)|(1+\eta)^{s_{r+1}}/|Z_\eta(\boldsymbol{s}_r, X)|$ exceeds the average degree of a vertex in $Z_\eta(\boldsymbol{s}_r, X)$ in the graph $H[Z_\eta(\boldsymbol{s}_r, X), Z_\eta(\boldsymbol{s}_{r+1}, X)]$ by a factor of at most $1+\eta$. Owing to the degree-regularity of H, this average degree is bounded by $2pm$. Consequently,

$$\frac{|Z_\eta(\boldsymbol{s}_{r+1}, X)|}{|Z_\eta(\boldsymbol{s}_r, X)|}(1+\eta)^{s_{r+1}} \leq (1+\eta)2pm \leq 4pm.$$

It follows that a single factor in (61) is at most $\sqrt{4pm}\sqrt{8pm} \leq 6pm$ and (58) follows.

This concludes our proof of Claim 8. □

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