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SUSSKIND FERMIONS ON A EUCLIDEAN LATTICE

by

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Abstract

A version of Euclidean lattice QCD obtained by introducing the fermions using Susskind's method is described and certain properties discussed. A U(1) axial current having the correct axial anomaly in the continuum limit is identified. We find $\frac{\Lambda_{MIN}}{\Lambda_{L,SUSS.}} = 28.78$ for SU(3) with 4 flavors.

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1. Introduction

Lattice gauge theories are thought to provide a convenient regularization of QCD in which non-perturbative phenomena can be systematically studied. Results of recent Monte-Carlo studies of Creutz [1], (which have been checked by many authors) are consistent with the properties of asymptotic freedom and confinement of static quarks.

The next important step is to analyse the effects of the introduction of dynamic quarks. Various proposals for Monte-Carlo calculations for actions with fermions have already been presented [2]. The results for models in 1-dimension are rather encouraging but improvements are probably necessary for a feasible programme on a sufficiently large lattice in 4-dimensions.

The question then arises as to which is an optimal lattice action to use. The most naive fermion lattice action (obtained by a naive discretization of the continuum Dirac action) leads in the continuum limit in 4-dimensions to a 16-fold degeneracy. Even for initial studies of Monte-Carlo calculations this is unsuitable because asymptotic freedom is lost for SU(2) color group and almost lost for SU(3). Wilson's modification [3] of the naive fermion action cures this problem. Wilson's action leads to the expected continuum limit [4,5] at least in renormalized perturbation theory. Moreover, Karsten and Smit [5] have shown that the correct U(1) anomaly is reproduced and there are also indications from Kawamoto's work [6] that chiral symmetry is realized in the Nambu-Goldstone mode in the continuum limit of Wilson's formulation.

Thus one may expect that Wilson's action will ultimately prove to be the best for a realistic study of quantum chromodynamics. However, a zero physical mass is not natural to the theory and the bare mass has to be carefully adjusted to extract physically relevant information. At least for initial Monte-Carlo studies one would prefer to have an action that has as few parameters to be varied as possible, especially when one is interested in studying the approach to the continuum limit.

Susskind [7] has proposed a method of putting fermions on a lattice in such a way that no mass counter-terms are needed for zero bare quark masses, which might therefore be more suitable for initial Monte-Carlo calculations [8]. All of the studies of Susskind's method (known to the present authors) have been made in the Hamiltonian formalism. It is the purpose of the present paper to present the formulation on a Euclidean lattice in a systematic way and to discuss some of the properties of the resulting theory.

The Susskind formulation on an Euclidean lattice is discussed in section 2. The action is the naive one on which constraints are imposed as discussed in the Hamiltonian formalism by Chodos and Healy [9]. Such an approach makes the formulation especially transparent and is more amenable to theoretical analysis than the original 1-component formulation of Susskind. By a naive extension of the approach of Chodos and Healy to the Euclidean lattice theory the degeneracy is reduced to $n_f = 2^{\nu/2}$ in ν dimensions, double that in the continuum time formalism. However, in a Euclidean formulation fermion fields χ and $\bar{\chi}$ are independent fields and by making them live on alternate sites the degeneracy is reduced to $2^{\nu/2-1}$.

In section 3 the relationship between the correlation functions of the naive theory and Susskind actions is analysed via the weak coupling expansion. The Susskind theory has a hidden cubic symmetry which allows all divergences in fermion self-energy to be absorbed by a rescaling of the fermion fields. It is argued that the expected continuum limit results at least in the case χ and $\bar{\chi}$ live on all sites. Complications in case χ and $\bar{\chi}$ live on alternate sites are pointed out but not completely analysed in this paper. In section 4 we obtain the transfer matrix for both cases. This exhibits a doubling of the fermion species with respect to the Hamiltonian formalism for the first system and no such doubling for the other.

In section 5 a U(1) current with the correct U(1) anomaly (appropriate to 2^{ν} flavors all with same axial charge) is identified even in the naive theory. This current involves ψ and $\bar{\psi}$ at non-nearest-neighbor lattice sites and has the same classical continuum limit as the naive U(1) current. However, since the fermion propagator has many poles, currents with different point splittings can give different phases to different fermion species. Indeed in the naive current, the axial charges add up to zero [5] whereas in the new current they add up to n_f.

In section 6 the ratio of the Susskind lattice Λ-parameter $\Lambda_{L,SUSS.}$ to the continuum Λ-parameter Λ_{min} (in the minimum subtraction scheme) is calculated. This ratio is relevant when making phenomenological estimates using Monte-Carlo [1] or strong coupling [10,11] methods.

Section 7 contains a short discussion of our conclusions.

2. Euclidean Lattice Formulation of Susskind Fermions

We work on a ν-dimensional Euclidean hypercubic lattice (ν = 2 or 4). The lattice spacing is a, and the lattice points are labelled with $x = n_\mu a$, $n_\mu = 0, \pm 1, \pm 2 \ldots$, $\mu = 1,2,\ldots \nu$. a_μ = vector along the μ-direction of length a. Fermions are represented by Grassmann spinor variables $\psi(x), \bar{\psi}(x)$. The Fourier transformed variables

$$\tilde{\psi}(k) = a^\nu \sum_x e^{-ikx} \psi(x),$$
$$\tilde{\bar{\psi}}(k) = a^\nu \sum_x e^{ikx} \bar{\psi}(x) \tag{2.1}$$

are periodic in each μ direction with period $2\pi/a$. Thus momenta of the independent fields are restricted to an interval of length $2\pi/a$ which can be chosen as

$$D = \left\{ k ; -\frac{\pi}{a} \le k < \frac{\pi}{a}, \forall \mu \right\} \tag{2.2}$$

Introducing the notation

$$\int_k = \int_{k \in D} \frac{d^\nu k}{(2\pi)^\nu} \tag{2.3}$$

we have

$$\psi(x) = \int_k e^{ikx} \tilde{\psi}(k),$$
$$\bar{\psi}(x) = \int_k e^{-ikx} \tilde{\bar{\psi}}(k) \tag{2.4}$$

The naive action for free fermions on the lattice, obtained from the continuum action by mere replacement of the derivative operator by the difference operator is

$$S_0(\psi) = -a^\nu \sum_x \Big[\sum_\mu \bar{\psi}(x) \gamma_\mu \frac{1}{2a} \left(\psi(x+a_\mu) - \psi(x-a_\mu) \right) + m \bar{\psi}(x) \psi(x) \Big] \tag{2.5}$$

where γ_μ are Euclidean Dirac matrices

$$\{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu},$$
$$\gamma_\mu^\dagger = \gamma_\mu \tag{2.6}$$

In terms of Fourier transformed variables the action is

$$S_0(\psi) = -\int_k \tilde{\bar{\psi}}(k)\, S(k)^{-1} \tilde{\psi}(k) \tag{2.7}$$

with propagator

$$S(k) = \Big[\sum_\mu i\gamma_\mu \frac{1}{a} \sin k_\mu a + m \Big]^{-1} \tag{2.8}$$

In the limit $a \to 0$ the propagator splits up into separate pieces one for each of the 2^ν sub-regions of D and hence the naive action describes 2^ν free fermions in this limit. The idea of Susskind [7] to reduce this degeneracy is to reduce the number of allowed spinor components at each lattice site to one. To discuss these points systematically it is useful to pause and make some further definitions.

First we divide the region D into 2^{ν} sectors in the following way. Let G be the set of ordered sets of different indices (including the empty set ϕ)

$$\mathcal{G} = \{ g \; ; \; g = (\mu_1, \mu_2, \cdots \mu_s) \; , \; 1 \leq \mu_1 < \mu_2 \cdots \leq \nu \} \tag{2.9}$$

Then for g ε G define π_g , the vector with components

$$(\pi_g)_\mu = \begin{cases} \frac{\pi}{a} & \text{for } \mu \in g \\ 0 & \text{for } \mu \notin g \end{cases} \tag{2.10}$$

and we decompose D according to

$$\mathcal{D} = \bigcup_{g \in \mathcal{G}} \mathcal{D}_g \tag{2.11}$$

with

$$\mathcal{D}_g = \{ k \; ; \; k = (k_\phi + \pi_g) \bmod \tfrac{2\pi}{a} \; , \; k_\phi \in \mathcal{D}_\phi \}$$
$$\mathcal{D}_\phi = \{ k \; ; \; -\tfrac{\pi}{2a} \leq k_\mu < \tfrac{\pi}{2a} \quad \forall \mu \} \tag{2.12}$$

Further it turns out convenient to give G the structure of an abelian group by introducing the product , $(g, g' \in \mathcal{G})$

$$g g' = g'' \in \mathcal{G}$$
$$\mu \in g'' \Rightarrow \mu \in g \cup g' \; , \; \mu \notin g \cap g' \tag{2.13}$$

and it follows

$$\pi_g + \pi_{g'} = \pi_{gg'} \mod \left(\tfrac{2\pi}{a} \right) \tag{2.14}$$

Now for g ε G we define new fields

$$\tilde{q}^g(k) = c M_g \tilde{\psi}(k + \pi_g)$$
$$\tilde{\bar{q}}^g(k) = \bar{c} \tilde{\bar{\psi}}(k + \pi_g) M_g^{\dagger} \tag{2.15}$$

where c is a normalization constant to be chosen later and M_g matrices defined by

$$M_g = M_{\mu_1} M_{\mu_2} \cdots M_{\mu_s} \; , \quad g \in \mathcal{G} \tag{2.16}$$

with

$$M_\mu = i \gamma_5 \gamma_\mu \tag{2.17}$$

where

$$\{ \gamma_5 , \gamma_\mu \} = 0 \; , \quad \gamma_5^{\dagger} = \gamma_5 \; , \quad \gamma_5^2 = 1 \tag{2.18}$$

The important properties of the matrices M_g are

i) $M_g^{+} M_g = 1$

ii) $M_g^{\dagger} \gamma_\mu M_g = e^{i a_\mu \pi_g} \gamma_\mu$ (2.19)

Obviously only the fields $\tilde{q}^g(k)$ with k ε D_ϕ are independent, fields with momenta in other regions are related

$$\tilde{q}^{g}(k+\pi_{g'}) = M_g M^{\dagger}_{gg'} \tilde{q}^{gg'}(k) \tag{2.20}$$

It is now easy to see the degeneracy, for using the definition (2.15) and the properties (2.19) the action can be written,

$$S_0(\psi) = -|c|^2 \sum_{g\in G} \int_{k,\varphi} \tilde{\bar{q}}^{g}(k)\, S(k)^{-1}\, \tilde{q}^{g}(k) \tag{2.21}$$

where S(k) has the behavior of the propagator for a single fermion in the region of integration. Here,

$$\int_{k,g} = \int_{k\in D_g} \frac{d^\nu k}{(2\pi)^\nu} \tag{2.22}$$

Each $q^g(k)$ represents an independent fermion species (which we shall call flavors) in the classical limit $a \to 0$.

Now the naive action $S_0(\psi)$ is invariant under the set $\mathcal{G}$ of 2^ν discrete transformations

$$\begin{aligned} \psi(x) &\to e^{ix\pi_g} M_g\, \psi(x) \\ \bar{\psi}(x) &\to e^{ix\pi_g}\, \bar{\psi}(x) M_g^{\dagger} \end{aligned} \tag{2.23a}$$

or in momentum space

$$\begin{aligned} \tilde{\psi}(k) &\to M_g\, \tilde{\psi}(k+\pi_g) \\ \tilde{\bar{\psi}}(k) &\to \tilde{\bar{\psi}}(k+\pi_g) M_g^{\dagger} \end{aligned} \tag{2.23b}$$

These (so-called doubling transformations) are simply (up to a sign) a permutation of the flavors,

$$\tilde{q}^{g'}(k) \to \varepsilon_{g',g}\, \tilde{q}^{g'g}(k) \tag{2.24}$$

where $\varepsilon_{g',g} = \pm 1$ is given by

$$M_{g'} M_g = \varepsilon_{g',g} M_{g'g} \tag{2.25}$$

The method of Susskind [7] to reduce the degeneracy amounts to a maximal diagonalization of the set $\mathcal{G}$. This has been discussed in the Hamiltonian formulation by Chodos and Healy [9]. One choses a maximal subgroup H⊂G such that

$$[M_h, M_{h'}] = 0 \quad \forall\, h, h' \in H \tag{2.26}$$

Any such H has $2^{\nu/2}$ elements. It is then consistent to impose the constraints (i.e. restrict the Grassmann 'measure') $\forall\, h \in H$

$$\begin{aligned} e^{ix\pi_h} \hat{M}_h\, \psi(x) &= \psi(x), \\ e^{ix\pi_h}\, \bar{\psi}(x) \hat{M}_h &= \bar{\psi}(x) \end{aligned} \tag{2.27}$$

where

$$\hat{M}_h = \frac{M_h}{\lambda_h} \tag{2.28}$$

with λ_h some set of eigenvalues of the M_h satisfying

$$\lambda_h \lambda_{h'} = \varepsilon_{h,h'} \lambda_{hh'} \qquad \forall \; h, h' \in H \tag{2.29}$$

We see then conditions (2.27) are equivalent to

$$P_x \psi(x) = \psi(x) \tag{2.30}$$

where P_x is a projector onto a 1-dimensional subspace

$$P_x = \frac{1}{2^{\nu/2}} \sum_{h \in H} e^{i x \pi_h} \hat{M}_h$$

$$P_x^2 = P_x \,, \qquad \mathrm{tr}\, P_x = 1 \tag{2.31}$$

In other words $\psi(x)$ is reduced to a form having only one independent Grassmann variable $\chi(x)$ per site

$$\psi_\alpha(x) = e_\alpha(x)\, \chi(x)$$

$$\bar{\psi}_\alpha(x) = e^{+}_\alpha(x)\, \bar{\chi}(x) \tag{2.32}$$

where e(x) is a c-number spinor satisfying (2.30) and $e(x + 2a_\mu) = e(x)$ and normalization $e^{+}(x)e(x) = 1$. The action is now reduced to the Susskind form

$$S_0'(\chi) = S_0(e\chi)$$

$$= -a^\nu \sum_x \Big[\sum_\mu \bar{\chi}(x) \frac{1}{2a} \left(\chi(x+a_\mu) - \chi(x-a_\mu)\right) \times c_\mu(x) + m\, \bar{\chi}(x)\chi(x) \Big] \tag{2.33}$$

with

$$c_\mu(x) = e^{+}(x)\, \gamma_\mu\, e(x+a_\mu) \tag{2.34}$$

In the appendix we give the $c_\mu(x)$ for a particular choice of H, $\{\lambda_h\}$ and realization of the Dirac matrices. This may be of use for application in Monte-Carlo programmes, but for weak coupling and other considerations it is generally more convenient to calculate in a representation independent manner.

The action $S_0'(\chi)$ now describes only $2^{\nu/2}$ free fermions in the continuum limit $a \to 0$ since not all the $\tilde{q}^g(k)$ are now independent. Indeed for $h \in H$

$$\tilde{q}^{gh}(k) = c_{g,h}\, \tilde{q}^g(k) \tag{2.35}$$

with the phase $c_{g,h}$ given by

$$c_{g,h} = M_{gh} \hat{M}_h M_g^{+} \tag{2.36}$$

The independent fields can be chosen as $\tilde{q}^f(k)$ with $f \in F$, where F is a subgroup of G such that

$$G = HF \tag{2.37}$$

and the constrained action becomes

$$S_c' = -2^{\nu/2} |c|^2 \sum_{f \in F} \int_{k,q} \bar{\tilde{q}}^f(k)\, S(k)^{-1}\, \tilde{q}^f(k) \tag{2.38}$$

The introduction of gauge fields is now straightforward. The naive lattice action is

$$S(\psi, U) = S(U) - a^\nu \sum_x \Big[\sum_\mu \bar\psi(x)\,\gamma_\mu \frac{1}{2a} \times \big(U_\mu(x)\,\psi(x+a_\mu) - U_\mu^{+}(x-a_\mu)\,\psi(x-a_\mu) \big) + m\,\bar\psi(x)\psi(x) \Big] \qquad (2.39)$$

with $S(U)$ some suitable lattice action for the pure gauge part involving matrices $U_\mu(x)$ belonging to the fundamental representation of the gauge group, associated with the directed link from x to $x + a_\mu$. The Susskind formulation [7] then amounts to constraining the Grassmann 'measure' precisely as for the free case described above.

The fermion degeneracy of the theory described so far is still a factor twice as large as in the Hamiltonian formulation. In fact we can reduce the degeneracy in the Euclidean formalism by half by restricting $\bar\psi(x)$ to live on sites with $\sum_\mu n_\mu$ even, (we will refer to these as 'even' sites) and $\psi(x)$ on sites with $\Sigma_\mu n_\mu$ odd, ('odd' sites), then

$$\tilde\psi(k) = -\tilde\psi(k+\pi_\xi)$$
$$\tilde{\bar\psi}(k) = +\tilde{\bar\psi}(k+\pi_\xi) \qquad (2.40)$$

where $(\pi_\xi)_\mu = \frac{\pi}{a}$ for every μ. The constraints (2.27) and the formalism described thereafter can be applied as before. The action is as in (2.39) with

$$\psi(x) \to P_-(x)\,\psi(x)\,,$$
$$\bar\psi(x) \to \bar\psi(x)\,P_+(x) \qquad (2.41)$$

where

$$P_\pm(x) = \frac{1}{2}\left(1 \pm (-1)^{\Sigma_\mu n_\mu}\right) P(x) \qquad (2.42)$$

Only $2^{\nu/2-1}$ of the fields $\tilde q^{f}(k)$ with $k \in D_\varphi$ are now independent. Note that the mass term $\bar\psi(x)\psi(x)$ vanishes identically with the extra constraint (2.40). Other mass terms which can be constructed are automatically momentum dependent before taking the classical continuum limit $a \to 0$.

3. Nature of the counterterms

In this section we will compare the structure of the amplitudes for the action with constraints with that for the naive action, i.e. one with a naive discretization in the fermion sector. For this it is convenient to first consider a weak coupling expansion in the coordinate space. Introducing the exponential parametrization

$$U_\mu(x) = \exp\, iag A_\mu(x)$$
$$A_\mu(x) = \frac{\lambda^i}{2} A_\mu^{\,i}(x) \qquad \text{(for gauge group SU(N))} \qquad (3.1)$$
$$[\lambda^i, \lambda^j] = 2i f^{ijk} \lambda^k\,, \quad \mathrm{tr}\, \lambda^i \lambda^j = 2\delta^{ij}$$

we may make an expansion in powers of g. A gauge fixing term is necessary and a ghost field is introduced as usual [4]. The difference from the naive Feynman rules is only in the fermion propagator. In any calculation, only the projected components $P_-(x)\psi(x)$ and $\bar\psi(y)P_+(y)$ enter and the free propagator is,

$$\langle P_-(x)\psi(x)\,\bar\psi(y)P_+(y)\rangle = P_-(x)\,S(x-y)\,P_+(y) \qquad (3.2)$$

where $S(x)$ is the naive propagator,

$$S(x) = \int_k e^{ikx} \Big[\sum_\mu i\gamma_\mu \frac{1}{a} \sin k_\mu a \Big]^{-1} \qquad (3.3)$$

Because of the presence of projection operators as in (3.2), translation invariance is not evident. As an illustrative example we will consider one piece of the fermion loop correction to the gauge boson propagator (Fig. 1a):

$$\bar{D}_{\mu\nu}(x,y) = -(iag)^2 \sum_{x\rho,w\sigma} D_{\mu\rho}(x-z)\, \mathrm{tr}\, P_+(z)\gamma_\rho \times P_-(z+a_\rho) S(z+a_\rho-w) P_+(w)\gamma_\sigma P_-(w+a_\sigma) \times S(w+a_\sigma-z) D_{\sigma\nu}(w-y) \tag{3.4}$$

We note the identities

$$P_\pm(x)\gamma_\mu = \gamma_\mu P_\mp(x+a_\mu) \qquad \text{(a)}$$

$$P_-(x) S(x-y) = S(x-y) P_+(y) \qquad \text{(b)} \qquad (3.5)$$

$$P_\pm^2(x) = P_\pm(x) \qquad \text{(c)}$$

To see (3.5b),

$$P_-(x) S(x-y) = 2^{-\nu/2-1}\left(1-e^{i\pi_\xi x}\right)\sum_h M_h e^{i\pi_h x} \times \int_k e^{ik(x-y)}\left(\sum_\mu i\gamma_\mu a^{-1}\sin k_\mu a\right)^{-1}$$

$$= 2^{-\nu/2-1}\sum_h \int_k e^{ik(x-y)}\left[\sum i\gamma_\mu a^{-1}\sin(k+\pi_h)_\mu a\right]^{-1} \times M_h e^{i\pi_h x}\left(1+e^{i\pi_\xi y}\right)$$

and after a shift $k \rightarrow k+\pi_h$,

$$= \int_k e^{ik(x-y)}\left(\sum i\gamma_\mu a^{-1}\sin k_\mu a\right)^{-1}\sum_h M_h e^{i\pi_h y} \times \left(1+e^{i\pi_\xi y}\right)$$

Using the identities (3.5), we may bring together all the projection operators in eqn. (3.4). We get

$$\bar{D}_{\mu\nu}(x,y) = -(iag)^2\sum_{z\rho,w\sigma} D_{\mu\rho}(x-z)\,\mathrm{tr}\,P_+(z) \times \gamma_\rho S(z+a_\rho-w)\gamma_\sigma S(w+a_\sigma-z) D_{\sigma\nu}(w-y) \tag{3.6}$$

After a relabelling we have

$$\bar{D}_{\mu\nu}(x,y) = -(iag)^2\sum_{z\rho,w\sigma} D_{\mu\rho}(x-y-z)\,\mathrm{tr}\,P_+(z+y) \times \gamma_\rho S(z+a_\rho-w)\gamma_\sigma S(w+a_\sigma-z) D_{\sigma\nu}(w)$$

Because of the explicit y dependence in P_+, translation invariance is not evident. For analysing this situation we will first consider the case in which ψ and $\bar{\psi}$ live on all sites (eqn. (2.30)). The present case shows some new features which will be mentioned later. When the constraint of eqn. (2.40) is absent, the only change in eqn. (3.6) is to replace $P_\pm(x)$ by P(x) of eqn. (2.30). With this change we will consider the self-energy part of eqn. (3.6) in momentum space:

$$\bar{\Pi}_{\rho\sigma}(\ell,\ell') = -(iag)^2\int_{k_1,k_2}\mathrm{tr}\, 2^{-\nu/2}\sum_h M_h\gamma_\rho \times S(k_1)\gamma_\sigma S(k_2)\int_{z,w}\exp i\{(-\ell+\pi_h+k_1-k_2)z + (-k_1+k_2+\ell')w + k_{1\rho}+k_{2\sigma}\}$$

$$= \int_{k_2}\mathrm{tr}\,2^{-\nu/2}\sum_h M_h\gamma_\rho S(k_2+\ell')\gamma_\sigma S(k_2) \times \exp i(k_{2\sigma}+(k_2+\ell')_\rho)\,\delta(\ell'-\ell+\pi_h) \tag{3.7}$$

Again momentum conservation $\ell' = \ell$ is not apparent. However, with $k_2 = k + \pi_g$ where $k \in D_\phi$ we get

$$\bar{\Pi}_{\rho\sigma}(\ell,\ell') = -(iag)^2 \int_{k,\phi} \mathrm{tr}\; 2^{-\nu/2} \sum_g M_g M_h M_g^\dagger$$
$$\times \gamma_\rho S(k+\ell')\gamma_\sigma S(k) \exp i(k_\sigma + (k+\ell')_\rho) \tag{3.8}$$
$$\times \delta(\ell' - \ell + \pi_h)$$

where we have used the identity (2.19). Since

$$\sum_g M_g M_h M_g^\dagger = 2^\nu \delta_{h\phi} \tag{3.9}$$

we finally get

$$\bar{\Pi}_{\rho\sigma}(\ell,\ell') = -(iag)^2\, 2^{\nu/2} \int_{k,\phi} \mathrm{tr}\, \gamma_\rho S(k+\ell')$$
$$\times \gamma_\sigma S(k) \exp i(k_\sigma + (k+\ell')_\rho)\, \delta(\ell' - \ell) \tag{3.10}$$

which explicitly shows momentum conservation. Note that the only difference with respect to the naive Feynman rules is to restrict the range of fermion loop momentum to D_ϕ after all momentum conservation δ-functions are integrated out, and to multiply the result by $2^{\nu/2}$, the number of "flavors".

We will now go back to the original case of ψ and $\bar{\psi}$ on every site. Eqn. (3.6) becomes

$$\bar{D}_{\mu\nu}(x,y) = -(iag)^2 \sum_{z\rho,\, w\sigma} D_{\mu\rho}(x-z)$$
$$\times \tfrac{1}{2}(1 + (-1)^{\xi\cdot z})\, \bar{\Pi}_{\rho\sigma}(z-w)\, D_{\sigma\nu}(w-y) \tag{3.11}$$

Note that the constraint that $\bar{\psi}$ lives only on even sites has entered into

this expression. As a consequence there is indeed no translation invariance at the level of one lattice spacing. In momentum space eqn. (3.11) becomes

$$\bar{D}_{\mu\nu}(\ell,\ell') = D_{\mu\rho}(\ell)\, \tfrac{1}{2}\left(\delta(\ell-\ell') + \delta(\ell-\ell'+\pi_\xi)\right)$$
$$\times \bar{\Pi}_{\rho\sigma}(\ell')\, D_{\sigma\nu}(\ell') \tag{3.12}$$

Thus because of vacuum polarization corrections, the gluon can change its momentum in steps of π_ξ. This is to be expected since $\tilde{\psi}(k)$ and $\tilde{\psi}(k+\pi_\xi)$ can propagate into one another (eqn. 2.40), so that $A_\mu(\ell)$ mixes with $A_\mu(\ell+\pi_\xi)$. We may form linear combinations which do not mix. These have the propagators

$$\tfrac{1}{2}a^2\left[\nu \pm \left\{\left(\sum_\rho \cos^2 a\ell_\rho\right)^2 + \pi^2(\ell)\right\}^{1/2}\right]^{-1} \delta_{\mu\nu} \tag{3.13}$$

$$+ \text{(gauge terms)}$$

For small ℓ the behavior is not appreciably altered from the usual case.

We will now consider the example of a second order correction to the fermion propagator (Fig. 2)

$$\bar{S}(x-y) = \tfrac{1}{2}(iag)^2 \sum_{z,w} P_-(x)\, S(x-z)\, P_+(z)$$
$$\times \gamma_\mu P_-(z+a_\mu)\, S(z+a_\mu-w)\, P_+(w)\, \gamma_\nu P_-(w+a_\nu)$$
$$\times S(w+a_\nu-y)\, P_+(y)\, D_{\mu\nu}(z-w)$$

(3.14)

We may again move the projection operators together. We get the naive amplitude sandwiched between $P_-(x)$ and $P_+(y)$. Thus the corrections to the propagating fermion components are exactly the same as in the naive theory. We will write the self-energy $\bar{\Sigma}(p)$ in momentum space with an integration over the range D_ϕ only.

$$\bar{\Sigma}(p) = \frac{1}{2}(iag)^2 \sum_g \int_{k,\phi} \gamma_\mu M_g S(k) M_g^\dagger \gamma_\nu D_{\mu\nu}(k+\pi_g+p) \quad (3.15)$$

We may regard $D_{\mu\nu}(k+p+\pi_g)$ for $g \neq \phi$ as the propagator for a new gauge boson which causes the flavor transition $f \to gf$. Since

$$D^{\dagger}_{\mu\nu}(k+\pi_g) = \delta_{\mu\nu}\Big(\sum_\rho 4a^{-2}\sin^2\frac{k_\rho a}{2} + 4a^{-2}\sum_\rho \delta_{(\pi_g)_\rho,\pi} \times \cos^2 k_\rho a\Big)^{-1} + \text{gauge fixing terms} \quad (3.16)$$

such gauge bosons have a $(\text{mass})^2 = 4a^{-2}\sum_\rho \delta_{(\pi_g)_\rho,\pi}$ which diverges in the continuum limit. Now all momenta of both fermions and gauge bosons lie in D_ϕ. For $g \in H$ there is no flavor transition, but nevertheless we may regard the exchanged gauge boson as new.

We may now generalize our results to arbitrary diagrams. We may assign a loop momentum to each fermion loop. Then in the case ψ and $\bar{\psi}$ live on all sites the effect of projection operators in fermion loops is to simply restrict the loop momentum to D_ϕ and to give a multiplicative factor n_f, the number of "flavors". In case $\bar{\psi}$ lives only on even sites, gluons when interacting with fermions end on even sites. As a result in any process involving fermion loops, gluon momentum is conserved only modulo π_ξ. However, in the continuum limit

such transitions are very weak and we expect that even with the divergences encountered in renormalized perturbation theory, usual continuum limit results. However, we will not pursue this problem here.

In case of open fermion lines, the projection operators can be moved to the ends and the naive Feynman rules are valid. The connection between the full fermion propagators of the one component formalism and the constraint formalism is

$$\langle \chi(x)\bar{\chi}(Y)\rangle = \langle e_\alpha(x)\psi_\alpha(x)\ \bar{\psi}_\beta(Y)e^*_\beta(Y)\rangle$$
$$= e_\alpha(x)\, S'_{\alpha\beta}(x-Y)\, e^*_\beta(Y) \quad (3.17)$$

which becomes in momentum space (for the massless case)

$$S'(p,p') = \int_q e_\alpha(p,q)(\gamma_\mu)_{\alpha\beta}\, e^*_\beta(q,p')\, q_\mu\, S'(q^2) \quad (3.18)$$

Here $e_\alpha(p,q)$ is a sum of δ-functions. Similar relations are valid for other correlation functions involving fermions. Note that the coefficients of q_μ in eqn. (3.18) are direction dependent. Hence we need a separate argument to show that all divergences (as $a \to 0$) in (3.18) can be absorbed by a single rescaling of the fermion fields. However, our arguments for the constrained formalism imply that the self-energy and vertex corrections are independent of the choice of the groups H and F. Thus the quantum corrections for all actions in the one-component formalism are same, independent of our choice of $e(x)$. In particular consider the naive fermion action with equal couplings along the different axes, viz

$$S = \frac{1}{2} a^3 \sum_{x\,\mu} \bar{\psi}(x)\, \gamma_\mu \left(\psi(x+a_\mu) - \psi(x-a_\mu) \right) \qquad (3.19)$$
$$+ \text{gauge interactions.}$$

This has the invariance under an interchange of axes, viz

$$x_\mu \to x_\nu\,, \quad x_\nu \to -x_\mu$$
$$\psi(x) \to \frac{1}{\sqrt{2}} \left(1 + \gamma_\mu \gamma_\nu\right) \psi(x) \qquad (3.20)$$
$$\bar{\psi}(x) \to \frac{1}{\sqrt{2}} \bar{\psi}(x) \left(1 - \gamma_\mu \gamma_\nu\right)$$

This suffices to imply that Z_2 defined via (ref. eqn. (3.20))

$$S'_{\alpha\beta}(p) = \sum_\mu Z_{2\mu} (\gamma_\mu)_{\alpha\beta}\, p_\mu + O(p^2) \qquad (3.21)$$

are same for all μ. If we impose constraints on the action (3.19), this feature therefore persists so that a single rescaling of the fermion field in the one-component formalism suffices to remove the divergences in the fermion self-energy. Thus all one-fermion-component actions of Susskind have the same renormalization constants and a hidden cubic symmetry.

We will now argue that to any order in renormalized perturbation theory there are no unwanted counterterms.

First consider the case m = 0 in (2.39). Then the action has a discrete symmetry,

$$\psi(x) \to i\gamma_5\, \psi(x+\xi)$$
$$\bar{\psi}(x) \to \bar{\psi}(x+\xi)\, i\gamma_5 \qquad (3.22)$$
$$U_\mu(x) \to U_\mu(x+\xi)\,, \quad \xi \in \Xi$$

where Ξ is the set of vectors,

$$\Xi = \{ \xi \,;\, \xi_\mu \in (a, -a) \ \forall \mu \} \qquad (3.23)$$

This transformation corresponds in momentum space to

$$\tilde{q}^f(k) \to e^{ik\xi}\, i\gamma_5\, \tilde{q}^f(k)\,, \quad \forall f \qquad (3.24)$$

and leaves the constraint equations (2.27) unchanged because

$$M_g^\dagger \gamma_5 M_g = e^{i\xi\pi_g} \gamma_5\,, \quad \forall g \in G\,, \ \xi \in \Xi \qquad (3.25)$$

The important consequence of this discrete symmetry is that mass counterterms will be absent in any order.

A counterterm of the form $\bar{\tilde{q}}^f(k)\, \Gamma\, \tilde{q}^{f'}(k)$ $(f \neq f')$(Fig. 3) is absent because momentum conservation forces the external legs to have the same momentum k whereas $q^f(k) = M_f M_{f'}^+ \; q^{f'}(k + \pi_{ff'})$ (see (2.20)). However, flavor changing vertices like in Fig. 4 will be non-vanishing for a non-zero lattice spacing. Even if the incoming momenta k_i are all in the region D_ϕ of the momentum space, the total momentum can lie outside it. However, in the continuum limit the incoming momenta are all infinitesimally close to the origin of D_ϕ and their sum will never lie outside it. There are also non-zero contributions [7] to the effective vertices in Fig. 5. However, in this case at least one gluon momentum is of order π/a and hence such vertices are not relevant for the continuum limit. There could be danger when such vertices appear as an internal part of a larger diagram (Fig. 6). In Fig. 6(a) at least one of the gluon propagators will be $a^2 \left(4 \sum_\mu \sin^2 k_\mu a/2\right)^{-1} = O(a^2)$

and hence the contribution vanishes as $a \to 0$. The case of Fig. 6(b) involving primitively divergent vertices is more tricky but the arguments of Ref. 4 imply that once the usual subtractions are made, all such diagrams vanish in the continuum limit to every order in renormalized perturbation series. Susskind [7] has also argued that because of asymptotic freedom, such induced vertices have no contribution as $a \to 0$ even in a non-perturvative context.

Finally we note that the theory is invariant under permutation of the flavors. This discrete invariance is transformed to a SU(4) invariance in the continuum limit.

4. The Transfer Matrix

In this section we establish the relation between the Lagrangian approach described in section 2 with the Hamiltonian formulation of Susskind [7]. To do this one has to construct the quantum mechanical space of states for the Euclidean lattice theory and extract a Hamiltonian by exhibiting the form of the transfer matrix. This has been thoroughly treated by Creutz [12] and Lüscher [13] for the case of Wilson's action [3]. Here we just point out the technical differences that arise because of the different way of introducing the fermions; in particular, in this section, we ignore gauge fields since their inclusion involves no new features with respect to the work in refs. (12,13). We first obtain the transfer matrix for the case in which the one component fermions $\chi(x)$ and $\bar{\chi}(x)$ both live on all sites and show how a doubling of the fermion species with respect to the continuous time formulation occurs. Next we will obtain the transfer matrix for the case where $\chi, \bar{\chi}$ live on alternate sites and show that there is no extra doubling and that the trans-

fer matrix in this case is positive.

We follow closely the work of Lüscher [13] and work on a finite lattice with specified boundary conditions. We first collect a few basic formulae summarized in the appendix of ref. 13. Consider a system of fermion operators $\hat{a}_1, \hat{a}_1^\dagger, \ldots \hat{a}_n, \hat{a}_n^\dagger$ satisfying canonical anti-commutation relations

$$\{\hat{a}_k, \hat{a}_\ell\} = \delta_{k\ell} \qquad \{\hat{a}_k, \hat{a}_\ell\} = \{\hat{a}_k^\dagger, \hat{a}_\ell^\dagger\} = 0 \tag{4.1}$$

The corresponding Fock space F is spanned by vectors

$$|k_1, \ldots k_j\rangle = \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_j} |0\rangle \qquad \hat{a}_k |0\rangle = 0 \quad \forall k \tag{4.2}$$

To every operator $\hat{\theta}$ acting on F one associates an element $\theta(a^+, a)$ of a Grassmann algebra generated by elements $a_1, a_1^+ \ldots a_n, a_n^+$ by the following rule

$$\theta(a^+, a) = \sum_{\substack{\{k_1 \cdots k_j\} \\ \{\ell_1 \cdots \ell_i\}}} \frac{1}{i!\, j!}\, a^+_{k_1} \cdots a^+_{k_j} \times \langle 0| \hat{a}_{k_j} \cdots \hat{a}_{k_1} \hat{\theta}\, \hat{a}^\dagger_{\ell_1} \cdots \hat{a}^\dagger_{\ell_i} |0\rangle\, a_{\ell_i} \cdots a_{\ell_1} \tag{4.3}$$

which implies

$$\hat{\theta} = :A(\hat{a}^\dagger, \hat{a}): \;\to\; \theta(a^+, a) = A(a^+, a)\, e^{\sum_i a_i^+ a_i} \tag{4.4}$$

where : : denotes the usual normal ordering. In particular it follows that

$$\hat{\theta} = e^{\sum_{i,j} \hat{a}_i^\dagger \Lambda_{ij} \hat{a}_j} \;\to\; \theta(a^+, a) = e^{\sum_{ij} a_i^+ (e^\Lambda)_{ij} a_j} \tag{4.5}$$

and if $\hat{B} = B(\hat{a}^+)$ and $C = C(\hat{a})$ are operators that depend on $\hat{a}^+$ resp. $\hat{a}$ only, then for any $\hat{\theta}$

$$\hat{B}\,\hat{\theta}\,\hat{C} \quad \rightarrow \quad B(a^+)\,\theta(a^+,a)\,C(a) \tag{4.6}$$

The tactic is now to rewrite the fermion functional integral in the form

$$Z = \int da_n^+ da_n \cdots da_1^+ da_1\, T(a_n^+, a_n)\, e^{a_n a_{n-1}^+} \times T(a_{n-1}^+, a_{n-1}) \cdots T(a_1^+, -a_1)\, e^{a_1 a_n^+} \tag{4.7}$$

where a_i may stand for all elements $a(x)$ at some fixed 'Euclidean time' x_i; then the operator T with Grassmann equivalent $T(a^+,a)$ is identified as the transfer matrix. Note that (4.7) differs from the analogous expression in ref. 13 by a relabelling of the indices. Lüscher identifies the exponential terms sandwiched between the T's with the mass terms in the Wilson action; whereas we will identify them with terms coupling the neighboring sites along the ν - axis. For massless theories we are forced to do this.

In the following we set $\nu = 4$ and lattice spacing a = 1. We first consider the Susskind action with the naive discretization along the 4 - direction (2.33) and χ, $\bar{\chi}$ live on all sites.

By a redefinition

$$\frac{1}{2}\,\bar{\chi}(x, x_4)\, C_4(x, x_4) = \chi^+_{x_4}(x) \tag{4.8}$$

where x labels the space coordinates only we get

$$\begin{aligned} S = \sum_{x_4} \sum_{x} \Big\{ & \chi(x)_{x_4+1}\, \chi^+(x)_{x_4} + \chi^+(x)_{x_4+1}\, \chi(x)_{x_4} \\ & + \sum_i \big[\chi^+(x+e_i)_{x_4}\, C_i'^{*}(x)\, \chi(x)_{x_4} + \chi^+(x)_{x_4} \\ & \times C_i'(x)\, \chi(x+e_i)_{x_4} \big] \Big\} \end{aligned} \tag{4.9}$$

Here $C_i'(x) = C_i(x)C_4(x)$ and we have presumed the reality of $C_4(x)$ and its independence with respect to the x_4 coordinate. This is true in the representation eqn. (A.16) of the Appendix. Otherwise our formulae are slightly altered. (To be precise, the definitions of the auxiliary fermion fields are different.) We will ignore the constant factors multiplying the functional integral due to the transformation (4.8).

Comparing with eqn. (4.7) and identifying x_4 with n in that equation we see that it is impossible to write the functional integral in that form unless we introduce a doubling of the Grassmann elements. The reason is clear: we have both $\chi^+_{x_4}\chi_{x_4+1}$ and $\chi_{x_4}\chi^+_{x_4+1}$ terms and the exponential cannot fit them both simultaneously. Introducing a set of auxiliary fields φ and φ^+ however, we get

$$\begin{aligned} T(\chi^+\varphi^+;\chi,\varphi) &= \exp \sum_x \big(\chi^+(x+e_i)\, c_i'^{*}(x)\,\chi(x) \\ &\quad + \chi^+(x)\, c_i'(x)\,\chi(x+e_i)\big) \prod_x \delta(\varphi(x) - \chi^+(x)) \\ &\quad \times \delta(\varphi^+(x) - \chi(x)) = \exp \sum_x \tfrac{1}{2}\big(\chi^+(x+e_i)\, c_i'^{*}(x) \\ &\quad \times \varphi^+(x) + \chi^+(x)\, c_i'(x)\, \varphi^+(x+e_i)\big) \prod_x \delta(\varphi(x) - \chi^+(x)) \\ &\quad \times \delta(\varphi^+(x) - \chi(x)) \exp \sum_x \tfrac{1}{2}\big(\varphi(x+e_i)\, c_i'^{*}(x)\,\chi(x) \\ &\quad + \varphi(x)\, c_i'(x)\,\chi(x+e_i)\big) \end{aligned} \tag{4.10}$$

An integration over φ and φ^+ using the δ-function immediately reproduces the original functional integral. Now T has the form of eqn. (4.6) and hence its form in the Fock space is immediately written down. Indeed using (4.4) and noting the Grassmann delta function is just given by

$$\delta(\varphi^+ - \chi)\,\delta(\varphi - \chi^+) = (\varphi^+ - \chi)(\varphi - \chi^+) \tag{4.11}$$

we obtain

$$\begin{aligned}\hat{T} &= \exp \sum_x \tfrac{1}{2}\left(\hat{\chi}^+(x+e_i)\, c_i'^*(x)\, \hat{\varphi}^\dagger(x) + \hat{\chi}^\dagger(x)\, c_i'(x)\, \hat{\varphi}^\dagger(x+e_i)\right)\\ &\times \prod_x :(\hat{\varphi}^\dagger(x) - \hat{\chi}(x))(\hat{\varphi}(x) - \hat{\chi}^\dagger(x)):\\ &\times \exp \sum_x \tfrac{1}{2}\left(\hat{\varphi}(x+e_i)\, c_i'^*(x)\, \hat{\chi}(x) + \hat{\varphi}(x)\, c_i'(x)\, \hat{\chi}(x+e_i)\right)\end{aligned} \tag{4.12}$$

This is explicitly Hermitian but not positive definite. In fact each normal product operator has the eigenvectors

$$\hat{\varphi}^\dagger(x)|0\rangle\ ,\quad \hat{\chi}^\dagger(x)|0\rangle\ ,\quad \frac{1}{\sqrt{2}}(1 \pm \hat{\varphi}^\dagger \hat{\chi}^\dagger)|0\rangle \tag{4.13}$$

with eigenvalues $(1, -1, \mp 1)$ respectively which are not all positive. However, $\hat{T}^2$ is positive and this suffices to define the Hamiltonian for the system: $H = -(2a)^{-1} \ln \hat{T}^2$. In fact since χ propagates to $\bar{\chi}$ and back to χ after two lattice spacings, $\hat{T}^2$ is the natural transfer matrix for the system.

We now consider the case with χ and $\bar{\chi}$ on alternate sites. Making the transformation as in eqn. (4.8) and labelling both χ and χ^+ by $\varphi(x)$ we get the action

$$\begin{aligned}S = \sum_{x_4, x} \varphi(x)_{x_4+1}\, \varphi(x)_{x_4} + \sum_{x_4, x_i} \big(&\varphi(x)_{2x_4}\, A_i(x)\\ &\times \varphi(x+e_i)_{2x_4} + \varphi(x)_{2x_4+1}\, B_i(x)\, \varphi(x+e_i)_{2x_4+1} \big)\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}A_i(x) &= c_i(x)\, P_e(x) + c_i(x+e_i)\, P_o(x)\\ B_i(x) &= c_i(x)\, P_o(x) + c_i(x+e_i)\, P_e(x)\end{aligned} \tag{4.15}$$

For obtaining the transfer matrix we will identify the exponential term in (4.7) with $\varphi(x)_{x_4+1}\, \varphi(x)_{x_4}$ with x_4 = even. The rest will be of the form $\prod_{x_4} T(\varphi_{2x_4}, \varphi_{2x_4-1})$. We get

$$\begin{aligned}T(\chi, \psi) &= \exp \sum_x \chi(x)\, A_i(x)\, \chi(x+e_i)\\ &\times \exp \sum_x \chi(x)\, \psi(x)\ \ \exp \sum_x \psi(x)\, B_i(x)\, \psi(x+e_i)\end{aligned} \tag{4.16}$$

This is again of the form eqn. (4.6). Since $\exp \chi(x)\psi(x)$ becomes 1 in Fock space, we get

$$\begin{aligned}\hat{T}(\varphi^\dagger, \varphi) = \exp \sum_x (\hat{\varphi}^\dagger(x)\, A_i(x)\, \hat{\varphi}^\dagger(x+e_i))\ \exp \sum_x (\hat{\varphi}(x)\\ \times B_i(x)\, \hat{\varphi}(x+e_i))\end{aligned} \tag{4.17}$$

We may now interchange $\hat{\varphi}$ and $\hat{\varphi}^\dagger$ for x = odd. (This is a canonical transformation for fermions.) We then get a form which is explicitly fermion number conserving. Note that $\hat{T}$ is Hermitian and moreover positive definite. In the naive limit of a zero lattice spacing in the 4th direction we may identify the Hamiltonian with the sum of the exponents in eqn. (4.17). We then get the naive Hamiltonian corresponding to the action (4.14) which is exactly that of Susskind [7]. There is no doubling of the fermion species due to discretization of time.

5. The U(1) Chiral Anomaly

In this section we will identify an axial current for the naive and Susskind modified theories that reproduces the correct U(1) axial anomaly. The current involves a particular point splitting and the method we use to show the anomaly is analogous to the external field method used by Jackiw and Johnson [14]. We first consider the naive theory. Karsten and Smit [5] have shown that the axial current

$$J^{(c)}_{\mu,5}(x) = \frac{i}{2}\left[\bar{\psi}(x+a_\mu)\, U^\dagger_\mu(x)\, \gamma_\mu \gamma_5\, \psi(x) + h.c.\right] \tag{5.1}$$

associated with the infinitesimal chiral transformation

$$\begin{aligned} \delta\psi(x) &= \varepsilon\, i\gamma_5\, \psi(x) \\ \delta\bar{\psi}(x) &= \varepsilon\, \bar{\psi}(x)\, i\gamma_5 \end{aligned} \tag{5.2}$$

is exactly conserved in the continuum limit. The absence of an anomaly can be traced back to the property that under the transformations (5.2) half of the $\tilde{q}^g$ transform with one sign and the other half with the opposite sign so that their contributions to the would-be anomaly cancel.

An infinitesimal transformation diagonal in the flavors under which all the $\tilde{q}^g$ behave alike is given by $(\xi \in \Xi)$

$$\begin{aligned} \delta\psi(x) &= \varepsilon\, i\gamma_5\, \psi(x+\xi) \\ \delta\bar{\psi}(x) &= \varepsilon\, \bar{\psi}(x+\xi)\, i\gamma_5 \end{aligned} \tag{5.3}$$

To extend (5.3) to a local chiral transformation that preserves gauge invariance we take

$$\begin{aligned} \delta\psi(x) &= \varepsilon(x)\, i\gamma_5\, \psi(x,\Xi), \\ \delta\bar{\psi}(x) &= \varepsilon(x)\, \bar{\psi}(x,\Xi)\, i\gamma_5, \\ \delta U_\mu(x) &= 0 \end{aligned} \tag{5.4}$$

where

$$\psi(x,\Xi) = \frac{1}{2^\nu} \sum_{\xi\in\Xi} U(x,\xi)\, \psi(x+\xi) \tag{5.5}$$

and $U(x,\xi) = U^\dagger(x+\xi,-\xi)$ is the product of U's along a path of length νa from x to x + ξ, (one may also average over different paths but this is not necessary for our purpose). Note that although the transformations (5.4) do not exponentiate simply to form a group, there will be an associated current. It is convenient to write the action in the form

$$S = S(U) - a^\nu \sum_{x',x} \bar{\psi}(x')\, S^{-1}(x',x)\, \psi(x) \tag{5.6}$$

with

$$S^{-1}(x',x) = \frac{1}{2a}\gamma_\mu \left(U_\mu(x')\, \delta_{x',x-a_\mu} - U^\dagger_\mu(x)\, \delta_{x',x+a_\mu} \right) + m\,\delta_{x',x} \tag{5.7}$$

writing $\int_\psi$ = usual Grassmann integral over $\psi, \bar{\psi}$, and noting that the Grassmann 'measure' is invariant to first order ε under the transformations (5.4), we obtain from these transformations the identity, for any observable Θ

$$0 = \int_\psi e^S \Big[-\sum_{x'} \big(\bar\Psi(x')\, S^{-1}(x',x)\, i\gamma_5\, \psi(x,\Xi)$$
$$+ \bar\Psi(x,\Xi)\, i\gamma_5\, S^{-1}(x,x')\, \psi(x')\big)\, \theta \tag{5.8}$$
$$+ a^{-\nu}\Big(\theta \frac{\overleftarrow{\partial}}{\partial\psi(x)}\, i\gamma_5\, \psi(x,\Xi) + \bar\Psi(x,\Xi)\, i\gamma_5 \frac{\partial\theta}{\partial\bar\psi(x)} \Big)\Big]$$

Defining the axial current (up to renormalization)

$$J_{\mu,5}(x) = \frac{i}{2}\Big(\bar\Psi(x+a_\mu,\Xi)\, U^\dagger_\mu(x)\, \gamma_\mu \gamma_5\, \psi(x) \tag{5.9}$$
$$+ \bar\Psi(x)\, U_\mu(x)\, \gamma_\mu \gamma_5\, \psi(x+a_\mu,\Xi)\Big)$$

we can rewrite (5.8) as

$$0 = \int_\psi e^S \Big[\Big\{ \sum_\mu \frac{1}{a}\big(J_{\mu,5}(x) - J_{\mu,5}(x-a_\mu)\big) - 2m\mathfrak{z}(x) \tag{5.10}$$
$$- a(x)\Big\}\theta + a^{-\nu}\Big(\theta \frac{\overleftarrow{\partial}}{\partial\psi(x)}\, i\gamma_5\, \psi(x,\Xi) + \bar\Psi(x,\Xi)\, i\gamma_5 \frac{\partial\theta}{\partial\bar\psi(x)} \Big)\Big]$$

with pseudoscalar density

$$\mathfrak{z}(x) = \frac{i}{2}\big(\bar\Psi(x)\, \gamma_5\, \psi(x,\Xi) + \bar\Psi(x,\Xi)\, \gamma_5\, \psi(x)\big) \tag{5.11}$$

and (after a little rearrangement) the additional term

$$\mathcal{A}(x) = (\mathcal{A}_1(x) + \mathcal{A}_2(x)) + \text{h.c} \tag{5.12}$$

with

$$a_1(x) = \frac{1}{2a}\frac{1}{2^\nu} \sum_{\xi\in\Xi} \sum_\mu \bar\Psi(x)\, i\gamma_\mu\gamma_5 \big\{ U_\mu(x)\, U(x+a_\mu,\xi) \tag{5.13}$$
$$- U(x,\xi)\, U_\mu(x+\xi)\big\}\, \psi(x+a_\mu+\xi)$$

and

$$a_2(x) = \frac{1}{2a}\frac{1}{2^\nu} \sum_{\xi\in\Xi} \sum_\mu \big(\bar\Psi(x)\, U(x,\xi)\, U_\mu(x+\xi) \tag{5.14}$$
$$\times\, i\gamma_\mu\gamma_5\, \psi(x+\xi+a_\mu) - [\, x \to x-\xi\,]\big)$$

We wish to calculate $\int_\psi e^S a_i(x)$ i = 1,2 in the continuum limit. Both contributions involve the propagator (in an external field),

$$\mathcal{S}(x,\mu,\xi)_{ab} = Z_\psi^{-1} \int_\psi e^S\, \bar\Psi_b(x)\, i\gamma_\mu\gamma_5\, \psi(x+a_\mu+\xi)$$
$$= -\mathrm{tr}\, i\gamma_\mu\gamma_5\, S_{ab}(x+a_\mu+\xi, x) \tag{5.15}$$
$$= -\mathrm{tr}\, i\gamma_\mu\gamma_5 \big[\delta_{ab} S_c - S_o V_{ab} S_c + \cdots \big](x+a_\mu+\xi, x)$$

where $Z_\psi = \int_\psi e^S$,

$$S_o(x,x')^{-1} = \sum_\rho \frac{1}{2a} \gamma_\rho \big(\delta_{x',x-a_\rho} - \delta_{x',x+a_\rho}\big) + m\,\delta_{x',x} \tag{5.16}$$

and

$$V_{ab}(x',x) = \sum_\nu \frac{1}{2a} \gamma_\nu \big([U_\nu(x')-1]\, \delta_{x',x-a_\nu} - [U^\dagger_\nu(x)-1] \tag{5.17}$$
$$\times\, \delta_{x',x+a_\nu}\big)_{ab}$$

First we consider the contribution from $\mathcal{A}_1$. The term in the curly brackets in (5.13) is $O(a^2)$, more precisely $(\hat\xi = \xi/a)$

$$U_\mu(x)\, U(x+a_\mu,\xi) - U(x,\xi)\, U_\mu(x+\xi) \tag{5.18}$$
$$= i a^2 g\, \hat\xi_\rho\, F_{\mu\rho}(x) + O(a^3)$$

where (in the continuum limit)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu] \tag{5.19}$$

Thus only the term of order a^{-1} in (5.15) contributes. In $\nu = 2$ dimensions, only the first term in the perturbation expansion yields such a contribution

$$\mathcal{S}(x,\mu,\xi)_{ab} \underset{\nu=2}{=} \frac{2i}{a}\,\varepsilon_{\mu\nu}\,\delta_{ab}\int_{-\pi}^{+\pi}\frac{d^2k}{(2\pi)^2}\,\frac{\sin k_\nu}{\sum_\sigma \sin^2 k_\sigma}$$

$$\times \exp i(k_\mu + \hat{\xi} k) + O(1)$$

$$= -\frac{2}{a}\,\varepsilon_{\mu\nu}\,\delta_{ab}\,n_F\,\hat{\xi}_\nu\int_{-\pi/2}^{+\pi/2}\frac{d^2k}{(2\pi)^2}\,\frac{\cos k_\mu\,(1+\hat{\xi}_\mu)\sin^2 k_\nu}{\sum \sin^2 k_\sigma} + O(1) \tag{5.20}$$

where here and in the following $n_f = 2^\nu$ until stated otherwise. Using the identity

$$\frac{1}{2^\nu}\sum_{\xi\in\Xi}\hat{\xi}_\rho\,\hat{\xi}_\epsilon = \delta_{\rho\epsilon} \tag{5.21}$$

we see for $\nu = 2$

$$\frac{1}{4}\sum_{\xi\in\Xi}\hat{\xi}_\rho\,\mathcal{S}(x,\mu,\xi)_{ab} = -\frac{2}{a}\,\varepsilon_{\mu\rho}\,\delta_{ab}\,n_f\,C_2 + O(1) \tag{5.22}$$

where C_ν are integrals

$$C_\nu = \int_{-\pi/2}^{\pi/2}\frac{d^\nu k}{(2\pi)^\nu}\,\frac{\sin^2 k_1\,\cos^2 k_2 \cdots \cos^2 k_\nu}{\left(\sum_\sigma \sin^2 k_\sigma\right)^{\nu/2}} \tag{5.23}$$

Thus for $\nu = 2$ and gauge group U(1) we have

$$Z_\psi^{-1}\int_\psi e^S\,a_1(x) \underset{\nu=2}{=} -i n_f\,g\,\varepsilon_{\mu\rho}\,F_{\mu\rho}\,C_2 + O(a) \tag{5.24}$$

For the case $\nu = 4$ the first term in the perturbation expansion (5.15) is identically zero due to the Dirac tracing. The second and third terms in (5.15) yield terms of order a^{-1}. For example the second term yields (for general ν)

$$\mathrm{tr}\; i\gamma_\mu\gamma_5\,(S_0 V_{ab} S_0)(x+a_\mu+\xi, x)$$

$$= n_f\,g\sum_\nu\int_p e^{ipx}\,A_{\nu ab}(p)\,K_{\mu\nu}(p,s) \tag{5.25}$$

with

$$K_{\mu\nu}(p,\xi) = \mathrm{tr}\;\gamma_5\gamma_\mu\int_{k,p} e^{ik(a_\mu+\xi)}\,S(k)\,\gamma_\nu$$

$$\times S(k-p)\cos[(k-p/2)a_\nu] \tag{5.26}$$

Thus for $\nu = 4$ dimensions

$$K_{\mu\nu}(p,\xi) = -4ia^{-1}\sum_{\epsilon,\lambda}\epsilon_{\mu\nu\epsilon\lambda}\,p_\lambda$$

$$\times\int_{-\pi/2}^{\pi/2}\frac{d^4k}{(2\pi)^\nu}\,\frac{\cos k_\nu\,\sin(k_\mu + k\hat{\xi})\sin k_\epsilon\,\cos k_\lambda}{\left(\sum_\sigma \sin^2 k_\sigma\right)^2} \tag{5.27}$$

using (5.21) we have then

$$\frac{1}{16}\sum_{\xi\in\Xi}\hat{\xi}_\rho\;\mathrm{tr}\; i\gamma_\mu\gamma_5\,(S_0 V_{ab} S_0)\,(x+a_\mu+\xi, x)$$

$$= -4a^{-1}n_f\,g\,c_4\sum_{\lambda,\nu}\epsilon_{\mu\rho\lambda\nu}\,\partial_\lambda A_\nu(x) + O(1) \tag{5.28}$$

The third term in the perturbative expansion combines with the second to yield

$$\frac{1}{16}\sum_{\xi\in\Xi}\hat{\xi}_\rho\,\Delta(x,\mu,\xi) \underset{\nu=4}{=} -in_f g^2 c_4\,\varepsilon_{\mu\rho\lambda\nu}F_{\lambda\nu}(x) + O(1) \tag{5.29}$$

From (5.18) and (5.29) now follows for SU(N) gauge group

$$Z_\psi^{-1}\int_\psi e^S a_1(x) \underset{\nu=4}{=} -in_f g^2 c_4\,\epsilon_{\mu\rho\lambda\nu}\,\mathrm{tr}\,F_{\mu\rho}F_{\lambda\nu}(x) + O(a) \tag{5.30}$$

Now we turn to a_2;

$$Z_\psi^{-1}\int_\psi e^S a_2(x) = \frac{1}{2^{\nu+1}}\sum_{\xi\in\Xi}\sum_\rho \hat{\xi}_\rho\,\partial_\rho\sum_\mu \mathrm{tr}\,\{1 + iga[A_\mu+\hat{\xi}A](x)+\cdots)\,\Delta(x,\mu,\xi)\} \tag{5.31}$$

In the case $\nu=2$ the second term in the perturbative expansion (5.25) yields a term of order a^0, indeed

$$\frac{1}{4}\sum_{\xi\in\Xi}\xi_\rho\sum_\mu K_{\mu\nu}(\rho,\xi) = -2i\,\epsilon_{\rho\nu}C_2 + O(a) \tag{5.32}$$

Noting

$$\sum_{\xi\in\Xi}\hat{\xi}_\rho\hat{\xi}_\sigma\sum_\mu \Delta(x,\mu,\xi) = O(1) \tag{5.33}$$

we see combining (5.31-33) and (5.22), (5.25) for $\nu=2$,

$$Z_\psi^{-1}\int_\psi e^S a_2(x) = O(a) \tag{5.34}$$

For $\nu=4$ dimensions we note the third and fourth terms in the perturbation expansion (5.21) contribute O(1) to give

$$\frac{1}{16}\sum_{\xi\in\Xi}\hat{\xi}_\rho\sum_\mu \mathrm{tr}\,\Delta(x,\mu,\xi) = 2in_f g^2 c_4 \times\sum_{\mu\lambda\nu}\epsilon_{\mu\rho\lambda\nu}\,\mathrm{tr}\,F_{\lambda\nu}A_\mu(x) + O(a) \tag{5.35}$$

Noting again (5.33) we see that (5.34) also holds for $\nu=4$ and hence $a_1(x)$ provides the only contribution in the continuum limit. It gives the correct anomaly since the integrals c_ν defined in (5.23) can be evaluated analytically

$$\begin{aligned} c_\nu &= -\lim_{\epsilon\to 0}\int_{-\pi/2}^{+\pi/2}\frac{d^\nu k}{(2\pi)^\nu}\,\theta(k^2-\epsilon^2)\cos^2 k_2\cdots\cos^2 k_\nu \times\partial_1\left(\frac{\cos k_1\sin k_1}{\left(\sum_\sigma \sin^2 k_\sigma\right)^{\nu/2}}\right) \\ &= 2\lim_{\epsilon\to 0}\int\frac{d^{\nu-1}k}{(2\pi)^\nu}\,\frac{\theta(\epsilon^2-k^2)\cos\sqrt{\epsilon^2-k^2}\,\sin\sqrt{\epsilon^2-k^2}\,\prod_i\cos^2 k_i}{\left(\sum_i \sin^2 k_i + \left(\sin\sqrt{\epsilon^2-k^2}\right)^2\right)^{\nu/2}} \\ &= 2\int\frac{d^{\nu-1}k}{(2\pi)^\nu}\,\theta(1-k^2)\sqrt{1-k^2} = \frac{1}{(4\pi)^{\nu/2}\,\Gamma\left(\frac{\nu+2}{2}\right)} \end{aligned} \tag{5.36}$$

Thus the chiral anomaly in the continuum limit is

$$Z_\psi^{-1}\int_\psi e^S a(x) = \begin{cases} -\frac{in_f}{2\pi}\,g\,\epsilon_{\rho\nu}F_{\rho\nu}(x), & \nu=2,\ U(1)\ \text{theory} \\ -\frac{i}{16\pi^2}\,n_f g^2\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma}(x), & \nu=4,\ SU(N)\ \text{theory} \end{cases} \tag{5.37}$$

In Susskind's modified theory with $2^{\nu/2}$ flavors the same analysis goes through since the transformation (5.4) is consistent with the constraints (2.30). The only modification in the analysis is that the projector $P(x)$ has to be introduced in the appropriate places. Thus we consider instead of (5.15)

$$\begin{aligned} \mathcal{S}'(x,\mu,\xi) &= Z_x^{-1} \int_x e^S\, \bar\psi_b(x)\, i\gamma_\mu\gamma_5\, \psi_a(x+a_\mu+\xi) \\ &= -\mathrm{tr}\, P(x)\, i\gamma_\mu\gamma_5\, S_{ab}(x+a_\mu+\xi, x) \end{aligned} \qquad (5.38)$$

The analysis from (5.15) onwards goes through as before (using repeatedly the identity (3.9)), with the only modification that n_f has to be set equal $2^{\nu/2}$ instead of 2^ν.

The analysis of the axial currents in the theory with $2^{\nu/2-1}$ flavors is more complicated and we have not yet considered this case in sufficient detail.

6. Calculation of $\Lambda_{MIN}/\Lambda_{L,SUSS}$

In this section we restrict consideration to $\nu = 4$ dimensions. The continuum limit of the lattice theory is expected to be obtained by taking the lattice spacing $a \to 0$ and the bare coupling $g \to 0$ such that the renormalization group invariant mass parameter

$$\Lambda_{L,Suss.} = a^{-1}\, e^{-\frac{1}{2\beta_0 g^2}} \left(\beta_0 g^2\right)^{-\frac{\beta_1}{2\beta_0^2}} \left(1+O(g^2)\right) \qquad (6.1)$$

tends to a finite non-zero limit. Here β_0, β_1 are the universal first two

coefficients of the Callan-Symanzik β-function, in particular

$$\beta_0 = \frac{1}{(4\pi)^2}\left(\frac{11}{3}N - \frac{4}{3}T(R)\, n_f\right) \qquad (6.2)$$

where $T(R) = \frac{1}{2}$ for fermions in the fundamental representation of SU(N). The calculation of Λ_L for $n_f = 0$ depends on the lattice action used for the pure gauge part. We take the Wilson action

$$S(U) = \frac{1}{g^2}\sum_{x,\mu\neq\nu} \mathrm{tr}\left(U_\mu(x)\, U_\nu(x+a)\, U_\nu^\dagger(x+a_\nu)\, U_\nu^\dagger(x) - 1\right) \qquad (6.3)$$

The calculation of Λ_{MIN}/Λ_L for this action has been first performed by Hasenfratz and Hasenfratz [15] and repeated by Dashen and Gross [16] using the background field method. Using the latter method the extension to the case $n_f = 0$ requires only the determination of the 1-loop contributions of the fermions to the gluon self energy, shown in fig. 1.

For the naive action one obtains for fig. 1a,

$$\begin{aligned} \Pi^{(a)}_{\mu\nu}(p) = -T(R)\, n_f\, g^2 \int_{k,\varphi} \mathrm{tr}\, V_\mu(k; k+p)\, S(k+p) \\ \times V_\nu(k+p, k)\, S(k) \end{aligned} \qquad (6.4)$$

and for fig. 1b

$$\Pi^{(b)}_{\mu\nu}(p) = T(R)\, n_f\, g^2 \int_{k,\varphi} \mathrm{tr}\, V_{\mu\nu}(k,k)\, S(k) \qquad (6.5)$$

with $n_f = 16$. For the Susskind modified theory these equations still hold with $n_f = 4$ as argued in section 3. It is easily shown that

$$\Pi^{(a)}_{\mu\nu}(0) + \Pi^{(b)}_{\mu\nu}(0) = 0 \qquad (6.6)$$

and in the continuum limit

$$\Pi^{(a)}_{\mu\nu}(p) + \Pi^{(b)}_{\mu\nu}(p) = (\delta_{\mu\nu} p^2 - p_\mu p_\nu)\,\pi^{(f)} + \left(1 - p_\rho \partial_\rho\big|_0 - \tfrac{1}{2} p_\sigma p_\rho \partial_\rho \partial_\sigma\big|_0\right) \Pi^{(b)\,cont}_{\mu\nu}(p) \qquad (6.7)$$

with

$$\pi^{(f)} = -T(R)\, n_f\, g^2\, 2 \int_{-\pi/2}^{\pi/2} \frac{d^4k}{(2\pi)^4} \frac{\cos^2 k_1 \cos^2 k_2 - \frac{1}{3}\cos 2k_1 \cos 2k_2}{[m^2 a^2 + \sum_\rho \sin^2 k_\rho]^2} \qquad (6.8)$$

$\pi^{(f)}$ is just the quantity which is relevant for the calculation of the contribution of the fermions to the lattice partition function in the weak background field, weak coupling approximation. Using the pure gauge part previously calculated [15,16] we obtain

$$\frac{\Lambda_{L,\,Suss}}{\Lambda_{Min}} = \exp\left[J + \frac{1}{\beta_0}\left\{\frac{1}{16N} - NP + T(R)\, n_f\, P_4\right\}\right] \qquad (6.9)$$

where

$$J = \tfrac{1}{2}(\ln 4\pi - \gamma) = 0.9769042 \qquad (6.10)$$

$$P = \frac{1}{48}\left(\frac{10}{3} P_1 + 88 P_2 + \frac{3}{2} - \frac{1}{2\pi^2}\right) \qquad (6.11)$$

$$P_4 = \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{[\cos^2 k_1 \cos^2 k_2 - \frac{1}{3}\cos 2k_1 \cos 2k_2]}{(\sum_\rho \sin^2 k_\rho)^2} \times \prod_{\mu=1}^{4} \theta\left(\frac{\pi}{2} - |k_\mu|\right) - \frac{2}{3}\left(\frac{1}{(k^2)^2} - \frac{1}{(k^2+1)^2}\right)\right\}$$

$$= \frac{2}{3}\left(P_2 - \frac{7}{96} P_1 - \frac{1}{8\pi^2} \ln 2\right) \qquad (6.12)$$

and P_1, P_2 are the integrals (evaluated in ref. (16))

$$P_1 = \int_0^\infty d\beta\; e^{-8\beta}\, I_0^4(2\beta) = 0.1549334\ldots \qquad (6.13)$$

$$P_2 = \int_0^\infty d\beta \left\{\beta\, e^{-8\beta}\, I_0^4(2\beta) - \frac{1 - e^{-\beta}}{16\pi^2 \beta}\right\} \qquad (6.14)$$

$$= 0.0240132\ldots$$

yielding

$$P = .0849780 \qquad (6.15)$$

$$P_4 = .0026248 \qquad (6.16)$$

and hence e.g. for $n_f = 4$,

$$\frac{\Lambda_{Min}}{\Lambda_{L,\,Suss}} = 34.44 \quad \text{for } N = 2 \qquad\qquad (6.17)$$
$$= 28.78 \quad \text{for } N = 3$$

which may be compared with the case without fermions

$$\left.\frac{\Lambda_{Min.}}{\Lambda_{L}}\right|_{n_f=0} = 7.46 \quad \text{for } N = 2 \qquad\qquad (6.18)$$
$$= 10.85 \quad \text{for } N = 3$$

The ratio $\left.\frac{\Lambda_{MOM}}{\Lambda_{L,SUSS}}\right|_{\alpha=1}$ varies less dramatically with n_f.

7. Conclusions

In this paper we have studied the Susskind method [7] of introducing fermions on a Euclidean lattice. We have argued that there are no unwanted counter-terms in renormalized perturbation theory, the correct U(1) anomaly is reproduced and a Hamiltonian with real eigenvalues can be identified. Thus we may expect that the theory is on a good theoretical footing. There is a discrete invariance which implies that no mass counter-term is required if one starts with a zero bare mass. This is a distinct advantage over Wilson's action for initial Monte-Carlo studies especially as the cut-off dependence of the bare fermion mass is not known. However, this is at the cost of requiring fermion degeneracy, four species in case χ and $\bar{\chi}$ live on all sites. Fortunately these species behave as flavors so far as strong interactions are concerned, though it is perhaps impossible to give separate electro-weak quantum numbers

to these flavors; and four massless flavors is not too far from the real world.

We have also considered the Euclidean action with only two fermion species in 4-dimensions. This is phenomenologically even more satisfactory. However, the periodicity of the action is now eight times the lattice spacing so that at least an 8^4 lattice is required for the Monte-Carlo studies. Also the renormalized perturbation theory presents new complications. Hence it appears that the action with 4 flavors is the most favorable.

Nielsen [17] has presented a fermion action on an Euclidean lattice which has the γ_5 -invariance and just two species (in 4-dimensions). The inverse propagator is $S^{-1}(p) = \sum_1^3 i\gamma_i a^{-1} \sin p_i a + i\gamma_4 a^{-1}\left(\sum^4 \cos p_\mu a - 3\right)$, which has two zeroes at $(0, 0, 0, \pm \pi/2a)$. However, the cubic symmetry is explicitly broken and this is not advantageous from the point of view of counter-terms. No-go theorems [18] suggest that a further reduction of the fermion species maintaining chiral symmetry and locality is not possible without a drastic change in our assumptions.

The Susskind action has only a discrete γ_5-invariance so that for a non-zero lattice spacing we do not get strictly massless pions even if the discrete symmetry gets spontaneously broken. The expectation is that the discrete symmetry is promoted to the continuous chiral symmetry in the continuum limit. However, in the strong coupling calculations of Banks et al. [10] of the spectrum of the Hamiltonian lattice gauge theory, the pion mass turned out to be too large even when extrapolated to the continuum limit. It is possible that this defect is due to the strong coupling extrapolation methods employed. However, there is the danger that it reflects a defect of the Susskind action itself, in that the approach to continuous chiral symmetry is too slow, which may be a disadvantage for Monte-Carlo studies.

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Appendix

2 dimensions

We consider the following realization of the Dirac matrices

$$\gamma_2 = \sigma_3 \quad , \quad \gamma_1 = \sigma_1 \tag{A.1}$$

with σ_j the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.2}$$

Then

$$\gamma_5 = -i\gamma_1\gamma_2 = \sigma_2 \tag{A.3}$$

We chose the maximal subgroup H (see (2.26)) as

$$H = \{ \phi, (1) \} \tag{A.4}$$

and the corresponding eigenvalues λ_h (see (2.29))

$$\lambda_\phi = \lambda_{(1)} = 1 \tag{A.5}$$

The c-number spinor e(x) satisfying (2.30) is given by

$$e(x) = \begin{pmatrix} P^{+}_{(1)}(x) \\ P^{-}_{(1)}(x) \end{pmatrix} s(x) \qquad (A.6)$$

where for $g \in G$, $P^{\pm}_{g}(x)$ are projectors

$$P^{\pm}_{g}(x) = \frac{1}{2}\left(1 \pm e^{i\sum_{\mu} \Pi_g a_\mu}\right) \qquad (A.7)$$

and $s(x)$ is a phase factor $\left(|s(x)| = 1\right)$. This yields coefficients $c_\mu(x)$ in (2.33)

$$c_1(x) = \overline{s(x)}\ s(x+a_1)\ ,$$
$$c_2(x) = \overline{s(x)}\ s(x+a_2)\ (-1)^{n_1} \qquad (A.8)$$

A convenient choice for $s(x)$ is obviously $s(x) = 1$

4 dimensions

We consider the following realization of Dirac matrices

$$\gamma_4 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1,2,3 \qquad (A.9)$$

Then

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = i\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \qquad (A.10)$$

We chose the maximal subgroup H as

$$H = \left\{\phi,\ (3),\ (1,2),\ (1,2,3)\right\} \qquad (A.11)$$

and the corresponding eigenvalues λ_h as

$$\lambda_\phi = \lambda_{(3)} = 1\ , \qquad \lambda_{(1,2)} = \lambda_{(1,2,3)} = i \qquad (A.12)$$

The c-number spinor $e(x)$ satisfying (2.30) is given by

$$e(x) = \begin{pmatrix} P^{+}_{(3)}(x)\ P^{+}_{(12)}(x) \\ P^{-}_{(3)}(x)\ P^{-}_{(12)}(x) \\ P^{-}_{(3)}(x)\ P^{+}_{(12)}(x) \\ P^{+}_{(3)}(x)\ P^{-}_{(12)}(x) \end{pmatrix} \qquad (A.13)$$

Yielding the coefficients

$$\begin{aligned} c_1(x) &= \overline{s(x)}\ s(x+a_1) \\ c_2(x) &= -i\,(-1)^{n_1+n_2}\ \overline{s(x)}\ s(x+a_2) \\ c_3(x) &= (-1)^{n_1+n_2}\ \overline{s(x)}\ s(x+a_3) \\ c_4(x) &= (-1)^{n_1+n_2+n_3}\ \overline{s(x)}\ s(x+a_4) \end{aligned} \qquad (A.14)$$

A convenient choice for $s(x)$ is for example

$$s(x) = (-1)^{n_1+n_2+n_3}\left(P^{+}_{(2)} - i P^{-}_{(2)}\right)\left(2P^{+}_{(1)}P^{+}_{(3)} - 1\right) \qquad (A.15)$$

giving

$$c_1(x) = (-1)^{n_3}\ ,\quad c_2(x) = (-1)^{n_1},\quad c_3(x) = (-1)^{n_2},\quad c_4(x) = (-1)^{n_1+n_2+n_3} \qquad (A.16)$$

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Figure Captions

Fig. 1 Diagrams for 1-loop fermionic contribution to gluon propagator.

Fig. 2 A diagram contributing to the second order correction to the fermion propagator.

Fig. 3 A flavor changing process that is strictly forbidden.

Fig. 4 A flavor changing process that is allowed for a finite lattice spacing but is not relevant for the continuum theory.

Fig. 5 A flavor changing process accompanied by the emission of one or more gluons carrying a momentum of order π/a.

Fig. 6 a) An anomalous flavor changing diagram that vanishes in the continuum limit in a straightforward manner.

b) An anomalous flavor changing diagram that vanishes once the usual subtractions for the primitively divergent subdiagrams are made.

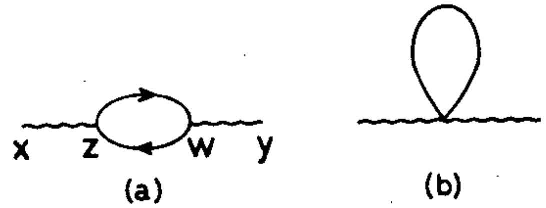

Fig.1

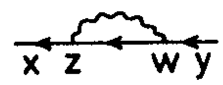

Fig.2

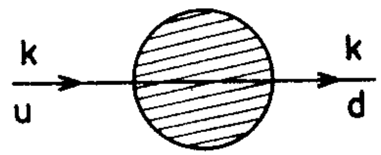

Fig.3

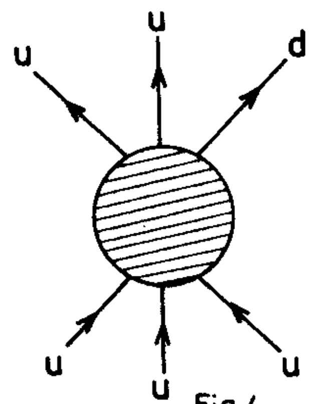

Fig.4

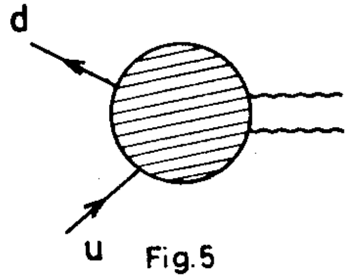

Fig.5

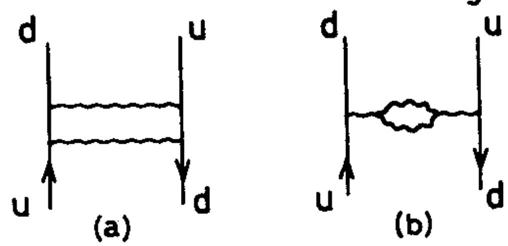

Fig.6