Infinite matroid union II

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Abstract

We introduce the *nearly finitary* matroids which form a superclass of the finitary matroids, and prove that the union of two nearly finitary matroids is a matroid and, in fact, nearly finitary. To prove the latter, we appeal to the finitary matroid union theorem established in the first paper of this series. We also characterize the nearly finitary graphic matroids.

Using the nearly finitary matroid union result, we establish that the *infinite matroid intersection conjecture* of Nash-Williams is true whenever the first matroid is nearly finitary and the second is the dual of a nearly finitary matroid.

From this we derive an alternative matroidal proof of the infinite Menger theorem for locally finite graphs. In addition, we show that the infinite matroid intersection conjecture for finitary implies the general infinite Menger theorem which was conjectured by Erdős, and proved only recently by Aharoni and Berger.

1 Introduction

For two finite matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$, the well-known *matroid union theorem* [10, 12] asserts that the set system

$$\mathcal{I}(M_1 \vee M_2) = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2\}$$

is the set of independent sets of their *union matroid* $M_1 \vee M_2$.

There exist two infinite matroids M_1 and M_2 for which the set $\mathcal{I}(M_1 \vee M_2)$ does not define a matroid [4]. One of the matroids involved contains no infinite circuits, such a matroid is called *finitary*. Hence, an analogue of the finite matroid union theorem does not exist for arbitrary infinite matroids. Nevertheless, the following is true.

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Theorem 1.1 ([4]). *If M_1 and M_2 are finitary matroids, then $M_1 \vee M_2$ is a matroid, and in fact finitary.*

The main results of this paper are an extension of this theorem to nearly finitary matroids, defined below, and its applications. For a given matroid M, the subsets of its ground set containing no finite circuit are the independent sets of a finitary matroid, which we call the *finitarization of M* (see Section 4). We call a matroid M *nearly finitary* if every base of its finitarization contains a base of M missing only finitely many elements from the base of the finitarization. Using Theorem 1.1, we prove the following.

Theorem 1.2. *If M_1 and M_2 are nearly finitary matroids, then $M_1 \vee M_2$ is a matroid and in fact nearly finitary.*

In view of the above mentioned counterexample against matroid union, this theorem is rather tight: the matroid involved in the counterexample that is not finitary is a countable sum of infinite circuits and of loops and thus it is the simplest example of a non-nearly finitary matroid. More generally, we show under an additional assumption that for every non-nearly finitary matroid there is a finitary matroid such that the union of these two is not a matroid (see Proposition 4.3).

A simple consequence of Theorem 1.2 is that $M_1 \vee \cdots \vee M_k$ is a nearly finitary matroid whenever $M_1, \ldots, M_k$ are nearly finitary. On the other hand, a countable union of finitary matroids need not be a matroid [4].

The class of nearly finitary matroids contains all finitary matroids but not only. One way to construct nearly finitary matroids that are not finitary is the following. Given any infinite-rank finitary matroid M, the set system $\mathcal{C}(M) \cup \mathcal{B}(M)$, consisting of the circuits of M together with the bases of M, respectively, is the set of circuits of a nearly finitary matroid that is not finitary (see Proposition 5.1). Nearly finitary matroids that are not finitary also arise from graphs. In this paper, we characterize the nearly finitary graphic matroids (see Propositions 5.2 and 5.3).

An appealing aspect of Theorem 1.2 is that it reveals new types of infinite matroids which do not seem to arise from graphs directly (see Example 3.2).

In finite matroid theory, the matroid union theorem has numerous applications; one striking application is an exceptionally short proof of the finite *matroid intersection* theorem [10]. The following conjecture, which was put forth by Nash-Williams and first appeared in [3], can be seen as the infinite analogue of matroid intersection. [1]

[1] An alternative notion of infinite matroid intersection was given by Christian [7].

Conjecture 1.3. *Any two matroids M_1 and M_2 on a common ground set E have a common independent set I admitting a partition $I = J_1 \cup J_2$ such that $cl_{M_1}(J_1) \cup cl_{M_2}(J_2) = E$.*

Originally, the above conjecture was put forth for finitary matroids. In [3], the connections between Conjecture 1.3 and the infinite analogues of König's and Hall's theorems are established. Aharoni and Ziv [3] showed that the conjecture is true whenever one matroid is finitary and the other is a countable direct sum of finite-rank matroids. We prove the following connection between infinite matroid union and infinite matroid intersection. Although this connection is well known for finite matroids, it requires more effort to prove it for for infinite matroids.

Theorem 1.4. *If M_1 and M_2 are matroids on a common ground set E and $M_1 \vee M_2^*$ is a matroid, then Conjecture 1.3 holds for M_1 and M_2.*

A consequence of Theorem 1.2 and Theorem 1.4 reads as follows.

Corollary 1.5. *Conjecture 1.3 holds for M_1 and M_2 whenever M_1 is nearly finitary and M_2 is the dual of a nearly finitary matroid.*

Corollary 1.5 does not imply the result of [3] nor is it implied by their results.

The infinite Menger theorem, conjectured by Erdős, reads as follows.

Theorem 1.6 (Aharoni and Berger [2])**.** *Let G be a connected graph. Then for any $S, T \subseteq V(G)$ there is a set $\mathcal{L}$ of vertex disjoint S–T paths and an S–T separator $X \subseteq \bigcup_{P \in \mathcal{L}} V(P)$ satisfying $|X \cap V(P)| = 1$ for each $P \in \mathcal{L}$.*

While Podewski and Steffens proved the infinite Menger theorem for countable rayless graphs in 1977 [11], it took ten more years until Aharoni solved the countable case [1]. This was the first proof of the infinite Menger theorem for locally finite graphs. Using Corollary 1.5, we provide an alternative matroidal proof of the infinite Menger theorem for locally finite graphs, see Section 6.

It is natural to ask whether one can also provide a matroidal proof of the general infinite Menger theorem. Towards this venue, we offer the following.

Theorem 1.7. *The infinite matroid intersection conjecture for finitary matroids implies the general infinite Menger theorem.*

Note that the counterexamples against matroid union show that the general infinite Menger theorem cannot be deduced from Theorem 1.4 and Theorem 1.7 directly, see Section 6.

This paper is organized as follows. Notation and terminology are set in Section 2. Certain instructive examples (some mentioned in the Introduction) are presented in Section 3. The proof of Theorem 1.2 can be found in Section 4. In Section 5, we construct nearly finitary matroids that are not finitary from finitary matroids and characterize the nearly finitary graphic matroids. In Section 6, we consider infinite matroid intersection and prove Theorem 1.4, Corollary 1.5, and Theorem 1.7 and finally provide an alternative matroidal proof of the infinite Menger theorem for locally finite graphs.

2 Preliminaries

Throughout, notation and terminology for graphs are that of [8] and are that of [6, 10] for matroids. M always denotes a matroid where $E(M)$, $\mathcal{C}(M)$, $\mathcal{I}(M)$, and $\mathcal{B}(M)$ denote its ground set, circuits, independent sets, and bases, respectively. For $X \subseteq E(M)$, we write $cl_M(X)$ to denote the *closure* of X [10]. G always denotes a graph where $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively.

Following [6], a set system $\mathcal{I}$ taken from the power set $\mathcal{P}(E)$ of a set E is the set of independent sets of a matroid provided it satisfies the following *independence axioms*.

(I1) $\emptyset \in \mathcal{I}$.

(I2) $\lceil \mathcal{I} \rceil = \mathcal{I}$, i.e., $\mathcal{I}$ is closed under taking subsets.

(I3) Whenever $I, I' \in \mathcal{I}$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{ I' \in \mathcal{I} \mid I \subseteq I' \subseteq X \}$ has a maximal element.

The following is one of the main results of [4].

Theorem 2.1. *([4, Theorem 1.2])*
If M_1 and M_2 are matroids, then $\mathcal{I}(M_1 \vee M_2)$ satisfies (I3).

Consequently, whenever we are confronted with the task of verifying that $M_1 \vee M_2$ is a matroid for some two matroids M_1 and M_2, it is sufficient to show that $\mathcal{I}(M_1 \vee M_2)$ satisfies (IM). But the proof of the above theorem also yields another result [4, Corollary 4.4]:

Lemma 2.2. *If M_1 and M_2 are matroids, then $\mathcal{I} = \mathcal{I}(M_1 \vee M_2)$ satisfies*

(*) *For all $I, J \in \mathcal{I}$ and all $y \in I \setminus J$ with $J + y \notin \mathcal{I}$ there exists $x \in J \setminus I$ such that $(J + y) - x \in \mathcal{I}$.*

Lemma 2.3. *Let M be a matroid and $I, B \in \mathcal{I}(M)$ with B maximal and $B \setminus I$ finite. Then $|I \setminus B| \leq |B \setminus I|$.*

Proof. The proof is by induction on $|B \setminus I|$. For $|B \setminus I| = 0$ we have $B \subseteq I$ and hence $B = I$ by maximality of B. Now suppose there is $y \in B \setminus I$. If $I + y \in \mathcal{I}$ then by induction

$$|I \setminus B| = |(I + y) \setminus B| \leq |B \setminus (I + y)| = |B \setminus I| - 1$$

and hence $|I \setminus B| < |B \setminus I|$. Otherwise there is a unique circuit C of M in $I + y$. Clearly C cannot be be contained in B an therefore has an element $x \in I \setminus B$. Then $(I + y) - x$ is independent, so by induction

$$|I \setminus B| - 1 = |((I + y) - x) \setminus B| \leq |B \setminus ((I + y) - x)| = |B \setminus I| - 1$$

and hence $|I \setminus B| \leq |B \setminus I|$. $\square$

Of the *circuit axioms* we require only the *circuit elimination axiom* phrased here for a matroid M.

(CE) Whenever $X \subseteq C \in \mathcal{C}(M)$ and $\{C_x \mid x \in X\} \subseteq \mathcal{C}(M)$[2] satisfies $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C \setminus \left(\bigcup_{x \in X} C_x\right)$ there exists a $C' \in \mathcal{C}(M)$ such that $z \in C' \subseteq \left(C \cup \bigcup_{x \in X} C_x\right) \setminus X$.

The finite graphic matroids have three natural extensions in the infinite scene [6]; each with its ground set $E(G)$. The most studied of which is $M_F(G)$, the *finite cycle matroid*, whose circuits are the finite cycles of G. This is a finitary matroid. The second extension is the *algebraic cycle matroid*, denoted $M_A(G)$, whose circuits are the finite cycles and double rays of G [6, 5][3].

The third extension of the finite graphic matroids are the *topological cycle matroids*, denoted $M_T(G)$[4], whose circuits are the finite and topological cycles[5] of G.

For a connected graph G, a maximal set of edges containing no finite cycles is called an *ordinary spanning tree*. A maximal set of edges containing no finite cycles nor any double ray is called an *algebraic spanning tree*. These are the bases of $M_F(G)$ and $M_A(G)$, respectively.

[2] $\mathcal{C}(M)$ is the set of circuits of M.
[3] $M_A(G)$ is not necessarily a matroid for any G; see [9].
[4] $M_T(G)$ is a matroid for any G; see [5].
[5] See Section 4 or [8, Chapter 8].

3 Examples

In this section, we collect various examples, in order to demonstrate the significant difference between the union of finite matroids and that of infinite matroids.

Example 3.1. Let G be the $(\mathbb{Z}\times\mathbb{Z})$-grid. We show that $\mathcal{I}(M_A(G)\vee M_A(G))$ admits a properly nested sequence $A_1 \cup B_1 \subsetneq A_2 \cup B_2 \subsetneq \ldots$ of unions of disjoint bases of $M_A(G)$. Moreover, not even $\bigcup_{k\in\mathbb{N}}(A_k \cup B_k)$ is a base of $M_A(G) \vee M_A(G)$ which is nevertheless a matroid.

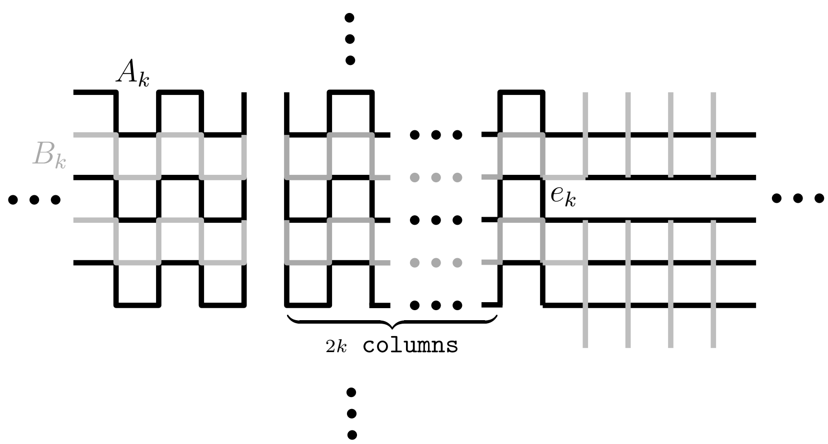

Figure 1: The sets A_k (black) and B_k (gray) are disjoint bases of $M_A(G)$.

The set $A_k \cup B_k$, as in Figure 1, covers all edges of the grid but the horizontal edges of one column and the vertical edges from one row that are on the right side of e_k. As the edge e_k moves right when k increases, $A_k \cup B_k$ is properly contained in $A_{k+1} \cup B_{k+1}$.

In fact, $M_A(G) \vee M_A(G)$ is the free matroid since it can be covered with two independent sets, see Figure 2. Another more subtle difference between finite and infinite matroid union is pointed out in the last paragraph of Section 6.

Example 3.2. In the following we look at a matroid obtained as a union of two graphic nearly finitary matroids. Let H be the infinite one-sided ladder with every edge doubled as depicted in Figure 3. More formally, H has the vertex set $A \cup B$ where $A = \{a_0, a_1, \ldots\}$ and $B = \{b_0, b_1, \ldots\}$ are disjoint and for any non-negative integer i there are

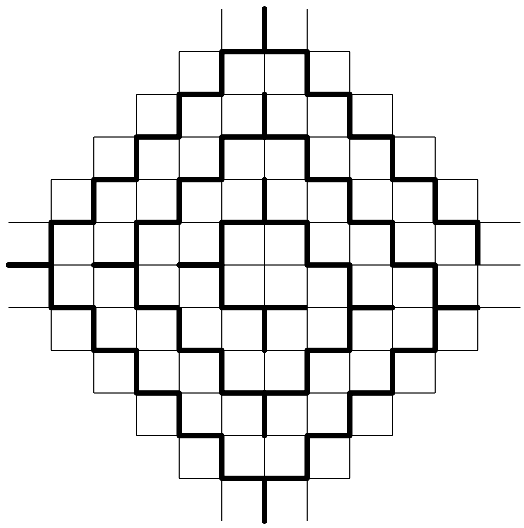

Figure 2: The set drawn and its complement are independent in $M_A(G)$.

- two edges between a_i and b_i which we call *rungs*,
- two edges between a_i and a_{i+1} which we call *upper* edges, and
- two edges between b_i and b_{i+1} which we call *lower* edges.

By H_n we denote the graph induced by H on the vertices $\{a_0, \dots, a_n\} \cup \{b_0, \dots, b_n\}$.

Note that $M_A(H)$ is nearly finitary and hence, by Theorem 1.2, $M := M_A(H) \vee M_A(H)$ is a matroid. The set of all non-rungs of H plus one of the two rungs between a_0 and b_0 is easily seen to be a circuit of M and we denote it by C. Removing from $E(H)$ one edge from each pair of parallel upper edges and one rung from each pair of parallel rungs apart from the first pair between a_0 and b_0 gives an independent set I of M. However, the contractions of both C and I to H_n are circuits of $M.E(H_n)$.

So in some sense, M does not resemble graphic matroids like M_F, M_A, and M_T as such a phenomenon does not occur there. For instance, in a countable graph G denote by G_n the graph obtained from G by contracting all but the first n edges. Then an edge set forms circuit of $M_T(G)$ if and only if, for every n, its restriction to G_n forms a circuit of $M_T(G_n) = M_T(G).E(G_n)$. This fact is often used to construct the infinite circuits of M_T [5]; such a tool is no longer available for M.

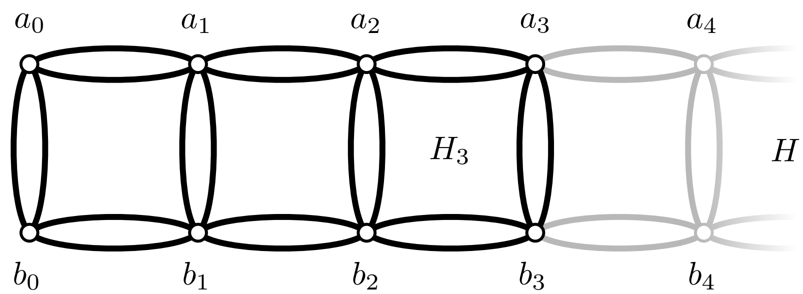

Figure 3: The subgraph H_3 of H.

4 Union of nearly finitary matroids

In this section, we prove Theorem 1.2.

4.1 Finitarizations of matroids

For a matroid M, let $\mathcal{I}^{\text{fin}}(M)$ denote the set of subsets of $E(M)$ containing no finite circuit of M, or equivalently, the set of subsets of $E(M)$ which have all their finite subsets in $\mathcal{I}(M)$. We call $M^{\text{fin}} = (E(M), \mathcal{I}^{\text{fin}}(M))$ the *finitarization* of M. With this notation, a matroid M is nearly finitary if it satisfies

(NEAR) for every $J \in \mathcal{I}(M^{\text{fin}})$ there is $I \in \mathcal{I}(M)$ such that $|J \setminus I| < \infty$.

In addition, if there is an integer k such that for every $J \in \mathcal{I}(M^{\text{fin}})$ there is an $I \in \mathcal{I}(M)$ with $|J \setminus I| \leq k$, then M is called *k-nearly finitary*.

For a set system $\mathcal{I}$ (not necessarily the independent sets of a matroid) we call a maximal element of $\mathcal{I}$ a *base* and a minimal element subject to not being in $\mathcal{I}$ a *circuit*. This allows us to extend the notions of *finitarization* and *nearly finitary* to $\mathcal{I}$; despite the fact that $\mathcal{I}$ is not necessarily a matroid.

Note that we cannot expect that, in a k-nearly finitary matroid M, for every base B^{fin} of M^{fin} we have $|B^{\text{fin}} \setminus B| = k$ for all $B \in \mathcal{B}(M)$. Consider for example $M_A(L)$ where L is the one-sided infinite ladder. Clearly, L has a spanning double ray D and a spanning ray R. The edge sets of both form bases of $M_A(L)^{\text{fin}}$ since they are spanning. In M, on the other hand, $E(D)$ is a circuit and $E(R)$ a base.

Let $\mathcal{I} = \lceil \mathcal{I} \rceil$. The finitarization $\mathcal{I}^{\text{fin}}$ of $\mathcal{I}$ has the following properties.

1. $\mathcal{I} \subseteq \mathcal{I}^{\text{fin}}$ with equality if and only if $\mathcal{I}$ is finitary.

2. $\mathcal{I}^{\text{fin}}$ is finitary and its circuits are exactly the finite circuits of $\mathcal{I}$.

3. $(\mathcal{I}|X)^{\text{fin}} = \mathcal{I}^{\text{fin}}|X$, in particular $\mathcal{I}|X$ is nearly finitary if $\mathcal{I}$ is.

The first two statements are obvious. For the third, we conclude as follows. Suppose $\mathcal{I}$ is nearly finitary and $J \in \mathcal{I}|X \subseteq \mathcal{I}$. By definition there is $I \in \mathcal{I}$ such that $J \setminus I$ is finite. As $J \subseteq X$ we also have that $J \setminus (I \cap X)$ is finite and clearly $I \cap X \in \mathcal{I}|X$.

Proposition 4.1. *M^{fin} is a finitary matroid, whenever M is a matroid.*

Proof. By construction $\mathcal{I}^{\text{fin}} = \mathcal{I}(M^{\text{fin}})$ satisfies (I1) and (I2) and is finitary, implying that it also satisfies (IM). So it remains to show that $\mathcal{I}^{\text{fin}}$ satisfies $(I3)$. By definition, a set $X \subseteq E(M)$ is not in $\mathcal{I}^{\text{fin}}$ if and only if it contains a finite circuit of M. Let $B, I \in \mathcal{I}^{\text{fin}}$ such that B is maximal and I is not. Hence there is $y \in E(M) \setminus I$ such that $I + y \in \mathcal{I}^{\text{fin}}$. If $I + x \in \mathcal{I}^{\text{fin}}$ for any $x \in B \setminus I$ then we are done, so suppose not. Then $y \notin B$ and for any $x \in B \setminus I$ there is a finite circuit C_x of M in $I + x$ containing x. By maximality of B, there is a finite circuit C of M in $B + y$ containing y. By circuit elimination in M applied to C and the C_x with $x \in X := C \cap (B \setminus I)$ there is a circuit

$$D \subseteq \left(C \cup \bigcup_{x \in X} C_x \right) \setminus X \subseteq I + y$$

of M containing $y \in C \setminus \bigcup_{x \in X} C_x$. But D is finite, since C and all the C_x are, contradicting $I + y \in \mathcal{I}^{\text{fin}}$. $\square$

Proposition 4.2. *For arbitrary matroids M_1 and M_2 it holds that*

$$\mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}}) = \mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}})^{\text{fin}} = \mathcal{I}(M_1 \vee M_2)^{\text{fin}}$$

Proof. By Proposition 4.1, M_1^{fin} and M_2^{fin} are finitary matroids and therefore also $M_1^{\text{fin}} \vee M_2^{\text{fin}}$ is a finitary matroid by Theorem 1.1. This establishes the first equality.

The second equality follows from the definition of the finitarization provided we show that the finite sets of $\mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}})$ and $\mathcal{I}(M_1 \vee M_2)$ are the same. Since $\mathcal{I}(M_1) \subseteq \mathcal{I}(M_1^{\text{fin}})$ and $\mathcal{I}(M_2) \subseteq \mathcal{I}(M_2^{\text{fin}})$ it holds that $\mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}}) \supseteq \mathcal{I}(M_1 \vee M_2)$. On the other hand, a finite set $I \in \mathcal{I}(M_1^{\text{fin}} \vee M_2^{\text{fin}})$ can be written as $I = I_1 \cup I_2$ with $I_1 \in \mathcal{I}(M_1^{\text{fin}})$ and $I_2 \in \mathcal{I}(M_2^{\text{fin}})$ finite. As I_1 and I_2 are finite, $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$, implying that $I \in \mathcal{I}(M_1 \vee M_2)$. $\square$

4.2 Proof of the nearly finitary union theorem

As mentioned in the Introduction, we prove the following.

Theorem 1.2. *If M_1 and M_2 are nearly finitary matroids, then $M_1 \vee M_2$ is a matroid; and in fact nearly finitary.*

Proof. By Theorem 2.1, in order to prove that $M_1 \vee M_2$ is a matroid, it is sufficient to prove that $\mathcal{I}(M_1 \vee M_2)$ satisfies (IM). This will be done in two steps. We first show that $\mathcal{I}(M_1 \vee M_2)$ is nearly finitary and then deduce that it satisfies (IM).

To see that $\mathcal{I}(M_1 \vee M_2)$ is nearly finitary, let $J \in \mathcal{I}(M_1 \vee M_2)^{\text{fin}}$. By Proposition 4.2 we may assume that $J = J_1 \cup J_2$ with $J_1 \in \mathcal{I}(M_1^{\text{fin}})$ and $J_2 \in \mathcal{I}(M_2^{\text{fin}})$. By assumption there are $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$ such that $J_1 \setminus I_1$ and $J_2 \setminus I_2$ are finite. Then $I = I_1 \cup I_2 \in \mathcal{I}(M_1 \vee M_2)$ and the assertion follows as $J \setminus (I_1 \cup I_2) \subseteq (J_1 \setminus I_1) \cup (J_2 \setminus I_2)$ is finite.

To prove that $\mathcal{I} = \mathcal{I}(M_1 \vee M_2)$ satisfies (IM), let $I \subseteq X \subseteq E(M_1 \vee M_2)$ with $I \in \mathcal{I}$. As $\mathcal{I}^{\text{fin}}$ satisfies (IM) there is a set $B^{\text{fin}} \in \mathcal{I}^{\text{fin}}$ which is maximal subject to $I \subseteq B^{\text{fin}} \subseteq X$ and being in $\mathcal{I}^{\text{fin}}$. By assumption there is $J \in \mathcal{I}$ such that $B^{\text{fin}} \setminus J$ is finite and we may assume that $J \subseteq X$. Then, $I \setminus J \subseteq B^{\text{fin}} \setminus J$ is finite so that we may choose a J minimizing $|I \setminus J|$. If there is a $y \in I \setminus J$, then, by Lemma 2.2, $J + y \in \mathcal{I}$ or there is an $x \in J \setminus I$ such that $(J + y) - x \in \mathcal{I}$. Both outcomes give a set containing more elements of I and hence contradicting the choice of J.

It remains to show that J can be extended to a maximal set B of $\mathcal{I}$ in X. For any superset $J' \in \mathcal{I}$ of J, we have $J' \in \mathcal{I}^{\text{fin}}$ and $B^{\text{fin}} \setminus J'$ is finite as it is a subset of $B^{\text{fin}} \setminus J$. As $\mathcal{I}^{\text{fin}}$ is a matroid, Lemma 2.3 implies

$$|J' \setminus B^{\text{fin}}| \leq |B^{\text{fin}} \setminus J'| \leq |B^{\text{fin}} \setminus J|.$$

Hence, $|J' \setminus J| \leq 2|B^{\text{fin}} \setminus J| < \infty$. Thus, we can greedily add elements to J to obtain the wanted set B after finitely many steps. $\square$

A matroid N is non-nearly finitary, if such has a set $I \in \mathcal{I}(N^{\text{fin}})$ with the property that no finite subset of I meets all (necessarily infinite) circuits of N in I. The following proposition asserts for certain non-nearly finitary matroids a union theorem is false. More precisely, we show that if a non-nearly finitary matroid N has a set I such that $N|I$ has only countably many infinite circuits, then $M \vee N$ is not a matroid for some matroid M; moreover, we show that there is a finitary such M. In this sense our main result Theorem 1.2 is best possible.

Proposition 4.3. *Let N be a non-nearly finitary matroid such that some $I \in \mathcal{I}(N^{\text{fin}})$ contains only countably many circuits and no finite subset of I meets all these circuits. Then there is a finitary matroid M such that $M \vee N$ is not a matroid.*

Proof. For N and I as above pick an enumeration $C_1, C_2, \ldots$ of the circuits of N in I. We may assume that $I = \bigcup_{n \in \mathbb{N}} C_n$. There exist countably many disjoint subsets $Y_1, Y_2, \ldots$ of I satisfying

1. $|Y_n| \leq n$ for all $n \in \mathbb{N}$; and

2. $Y_n \cap C_i \neq \emptyset$ for all $n \in \mathbb{N}$ and all $1 \leq i \leq n$.

Such sets can be constructed[6] as follows. Suppose $Y_1, \ldots, Y_n$ have already been defined. Let Y_{n+1} be a set of size at most $n+1$ disjoint to each of $Y_1, \ldots, Y_n$ and meeting the circuits $C_1, \ldots, C_{n+1}$; such exists as $\bigcup_{i=1}^{n} Y_i$ is finite and all circuits in I are infinite.

Let $L = \{l_1, l_2, \ldots\}$ be a countable set disjoint from $E(N)$. For each $n \in \mathbb{N}$ let M_n be the 1-uniform matroid on $Y_n \cup \{l_n\}$, i.e. $M_n := U_{1, Y_n \cup \{l_n\}}$. Then, $M := \bigoplus_{n \in \mathbb{N}} M_n$ is a direct sum of finite matroids and hence finitary.

We contend that $I \in \mathcal{I}(M \vee N)$ and that $\mathcal{I}(M \vee N)$ violates (IM) for I and $X := I \cup L$. By construction, Y_n contains some element d_n of C_n, for every $n \in \mathbb{N}$. So that $J_M = \{d_1, d_2, \ldots\}$ meets every circuit of N in I and is independent in M. This means that $J_N := I \setminus J_M \in \mathcal{I}(N)$ and thus $I = J_M \cup J_N \in \mathcal{I}(M \vee N)$.

It is now sufficient to show that some $I \subseteq J \subseteq X$ is in $\mathcal{I}(M \vee N)$ if and only if it misses infinitely many elements $L' \subseteq L$. Suppose that $J \in \mathcal{I}(M \vee N)$. There are sets $J_M \in \mathcal{I}(M)$ and $J_N \in \mathcal{I}(N)$ such that $J = J_M \cup J_N$. As $D := I \setminus J_N$ meets every circuit of N in I by independence of J_N, the set D is infinite. But $I \subseteq J$ and hence $D \subseteq J_M$. Let A be the set of all integers n such that $Y_n \cap D \neq \emptyset$. As Y_n is finite for every $n \in \mathbb{N}$, the set A must be infinite and so is $L' := \{l_n \mid n \in A\}$. Since J_M is independent in M and any element of L' forms a circuit of M with some element of J_M, we have $J_M \cap L' = \emptyset$ and thus $J \cap L' = \emptyset$ as no independent set of N meets L.

Suppose that there is a sequence $i_1 < i_2 < \ldots$ such that J is disjoint from $L' = \{l_{i_n} \mid n \in \mathbb{N}\}$. We show that the superset $X \setminus L'$ of J is in $\mathcal{I}(M \vee N)$. By construction, for every $n \in \mathbb{N}$, the set Y_{i_n} contains an elements d_n of C_n. Set $D := \{d_n \mid n \in \mathbb{N}\}$. Then D meets every circuit of N in I, so $J_N := I \setminus D$ is independent in N. On the other hand, D contains exactly

[6] Our construction here mimics Example 3.4 in [4].

one element of each M_n with $n \in L'$. So $J_M := (L \setminus L') \cup D \in \mathcal{I}(M)$ and therefore $X \setminus L' = J_M \cup J_N \in \mathcal{I}(M \vee N)$. $\square$

5 Nearly finitary matroids

It is natural to ask what the class of nearly finitary matroids consists of. In this section, we address this question.

This next construction is probably the most natural manner to construct nearly finitary matroids; as such can be obtained from finitary matroids as follows.

For a matroid M and an integer $k \geq 0$, set $M[k] := (E(M), \mathcal{I}[k])$, where

$$\mathcal{I}[k] := \{I \in \mathcal{I}(M) \mid \exists J \in \mathcal{I}(M) \text{ such that } I \subseteq J \text{ and } |J \setminus I| = k\}.$$

Proposition 5.1. *If* $\operatorname{rank}(M) \geq k$*, then* $M[k]$ *is a matroid.*

Clearly, $M[k]$ is k-nearly finitary, if M is finitary. And, if $E(M) \notin \mathcal{C}(M)$, then $\mathcal{C}(M[1]) = \mathcal{C}(M) \cup \mathcal{B}(M)$. We postpone the proof of Proposition 5.1 to Section 5.1.

In Propositions 5.2 and 5.3 (see below), we characterize the nearly finitary graphic matroids; We recall from Section 3 that for a graph G, the matroid $M_A(G)$ has $E(G)$ as its ground set and as circuits the edge sets of finite cycles and double rays of G. Similarly, $M_T(G)$ is the matroid on $E(G)$ whose circuits are edge sets of the finite and topological cycles of G.

The nearly finitary algebraic cycle matroids are characterized as follows.

Proposition 5.2. *$M_A(G)$ is a nearly finitary matroid if and only if G has only a finite number of vertex disjoint rays.*

A graph with each of its vertices having finite degree is called *locally finite*. In the same spirit, the nearly finitary topological cycle matroids are characterized as follows.

Proposition 5.3. *Suppose that G is 2-connected and locally finite. Then, $M_T(G)$ is a nearly finitary matroid if and only if G has only a finite number of vertex disjoint rays.*

The proofs of Propositions 5.2 and 5.3 can be found in the Sections 5.2 and 5.3, respectively. Both proofs require the following theorem from [8, Theorem 8.2.5].

Theorem 5.4 (Halin 1965)**.** *If an infinite graph G contains k disjoint rays for every $k \in \mathbb{N}$, then G contains infinitely many disjoint rays.*

5.1 $M[k]$ is a matroid

Here, we prove Proposition 5.1.

Proof of Proposition 5.1. (I1) holds as $\operatorname{rank}(M) \geq k$ and (I2) holds as it does in M.

For (I3) let $I', I \in \mathcal{I}(M[k])$ such that I' is maximal and I is not. There is a set $F' \subseteq E(M) \setminus I'$ of size k such that, in M, $I' \cup F'$ is not only independent but, by maximality of I', also a base. Similarly, there is a set $F \subseteq E(M) \setminus I$ of size k such that $I \cup F \in \mathcal{I}(M)$.

We claim that $I \cup F$ is non-maximal in $\mathcal{I}(M)$ for any such F. Suppose not and $I \cup F$ is maximal for some F as above. By assumption, I is contained in some larger set of $\mathcal{I}(M[k])$. Hence there is a set $F^+ \subseteq E(M) \setminus I$ of size $k+1$ such that $I \cup F^+$ is independent in M. Clearly $(I \cup F) \setminus (I \cup F^+) = F \setminus F^+$ is finite, so Lemma 2.3 implies that

$$\left|F^+ \setminus F\right| = \left|(I \cup F^+) \setminus (I \cup F)\right| \leq \left|(I \cup F) \setminus (I \cup F^+)\right| = \left|F \setminus F^+\right|.$$

In particular, $k+1 = |F^+| \leq |F| = k$, a contradiction.

Hence we can pick F such that $F \cap F'$ is maximal and, as $I \cup F$ is non-maximal in $\mathcal{I}(M)$, apply (I3) in M to obtain a $x \in (I' \cup F') \setminus (I \cup F)$ such that $(I \cup F) + x \in \mathcal{I}(M)$. This means $I + x \in \mathcal{I}(M[k])$. And $x \in I' \setminus I$ follows, as $x \notin F'$ by our choice of F.

To show (IM), let $I \subseteq X \subseteq E(M)$ with $I \in \mathcal{I}(M[k])$ be given. By (IM) for M, there is a $B \in \mathcal{I}(M)$ which is maximal subject to $I \subseteq B \subseteq X$. We may assume that $F := B \setminus I$ has at most k elements; for otherwise there is a superset $I' \subseteq B$ of I such that $|B \setminus I'| = k$ and it suffices to find a maximal set containing $I' \in \mathcal{I}(M[k])$ instead of I.

We claim that for any $F^+ \subseteq X \setminus I$ of size $k+1$ the set $I \cup F^+$ is not in $\mathcal{I}(M[k])$. For a contradiction, suppose it is. Then in $M|X$, the set $B = I \cup F$ is a base and $I \cup F^+$ is independent and as $(I \cup F) \setminus (I \cup F^+) \subseteq F \setminus F^+$ is finite, Lemma 2.3 implies

$$\left|F^+ \setminus F\right| = \left|(I \cup F^+) \setminus (I \cup F)\right| \leq \left|(I \cup F) \setminus (I \cup F^+)\right| = \left|F \setminus F^+\right|.$$

This means $k+1 = |F^+| \leq |F| = k$, a contradiction. So by successively adding single elements of $X \setminus I$ to I as long as the obtained set is still in $\mathcal{I}(M[k])$ we arrive at the wanted maximal element after at most k steps. $\square$

5.2 The nearly finitary algebraic cycle matroids

In this section, we prove Proposition 5.2. It will be instructive to note that a base of $M_A(G)$ is a maximal subset of $E(G)$ containing no finite cycle and no double ray of G; while a base of $M_A(G)^{\mathrm{fin}}$ is a maximal subset of $E(G)$ containing no finite cycle of G, i.e., an ordinary spanning tree of G.

Proof of Proposition 5.2. Suppose that G has k disjoint rays for every integer k; so that G has a set $\mathcal{R}$ of infinitely many disjoint rays by Theorem 5.4. We show that $M_A(G)$ is not nearly finitary.

The edge set of $\bigcup \mathcal{R} = \bigcup_{R \in \mathcal{R}} R$ is independent in $M_A(G)^{\mathrm{fin}}$ as it induces no finite cycle of G. Therefore there is a base of $M_A(G)^{\mathrm{fin}}$ containing it; such induces an ordinary spanning tree, say T, of G. We show that

$$T - F \text{ contains a double ray for any finite edge set } F \subseteq E(T). \qquad (1)$$

This implies that $E(T) \setminus I$ is infinite for every independent set I of $M_A(G)$ and hence $M_A(G)$ is not nearly finitary. To see (1), note that $T - F$ has $|F|+1$ components for any finite edge set $F \subseteq E(T)$ as T is a tree and successively removing edges always splits one component into two. So one of these components contains infinitely many disjoint rays from $\mathcal{R}$ and consequently a double ray.

Suppose next, that G has at most k disjoint rays for some integer k and let T be an ordinary spanning tree of G, that is, $E(T)$ is maximal in $M_A(G)^{\mathrm{fin}}$. To prove that $M_A(G)$ is nearly finitary, we need to find a finite set $F \subseteq E(T)$ such that $E(T) \setminus F$ is independent in $M_A(G)$, i.e. it induces no double ray of G. Let $\mathcal{R}$ be a maximal set of disjoint rays in T; such exists by assumption and $|\mathcal{R}| \leq k$. As T is a tree and the rays of $\mathcal{R}$ are vertex disjoint, it is easy to see that T contains a set F of $|\mathcal{R}| - 1$ edges such that $T - F$ has $|\mathcal{R}|$ components which each contain one ray of $\mathcal{R}$. By maximality of $\mathcal{R}$ no component of $T - F$ contains two disjoint rays, or equivalently, a double ray. $\square$

5.3 The nearly finitary topological cycle matroids

In this section, we prove Proposition 5.3. To this end, we shall require only the following notions for $M_T(G)$ which are presented in full detail in [5]. An *end of* G is an equivalence class of rays, where two rays are *equivalent* if they cannot be separated by a finite edge set. In particular, two rays meeting infinitely often are equivalent. Let the *degree* of an end ω be the size of a maximal set of vertex disjoint rays belonging to ω, which is well-defined [8]. We say that a double ray *belongs to* an end if the two rays which arise from the removal of one edge from the double ray belong to that end; this does not depend on the choice of the edge.

The finite circuits of $M_T(G)$ are the edge sets of finite cycles of G. An instance of an infinite circuit of $M_T(G)$ is the edge set of a double ray which is comprised of two rays from the same end. In fact, every infinite circuit of $M_T(G)$ induces at least one double ray; provided that G is locally finite [8].

A graph G has only finitely many disjoint rays if and only if G has only finitely many ends, each with finite degree. Also, note that

$$\text{every end of a 2-connected locally finite graph has degree at least 2.} \tag{2}$$

Indeed, applying Menger's theorem inductively, it is easy to construct in any k-connected graph for any end ω a set of k disjoint rays of ω.

Proof of Proposition 5.3. If G has only a finite number of vertex disjoint rays then $M_A(G)$ is nearly finitary by Proposition 5.2. Observing $M_A(G)^{\text{fin}} = M_T(G)^{\text{fin}}$ and $\mathcal{I}(M_A(G)) \subseteq \mathcal{I}(M_T(G))$, we conclude that $M_T(G)$ is nearly finitary as well.

Now, suppose that G contains k vertex disjoint rays for every $k \in \mathbb{N}$. If G has an end ω of infinite degree, then there is an infinite set $\mathcal{R}$ of vertex disjoint rays belonging to ω. As any double ray containing two rays of $\mathcal{R}$ forms a circuit of $M_T(G)$, the argument from the proof of Proposition 5.2 shows that $M_T(G)$ is not nearly finitary.

Assume then that G has no end of infinite degree. There are infinitely many disjoint rays, by Theorem 5.4. Hence, there is a countable set of ends $\Omega = \{\omega_1, \omega_2, \ldots\}$.

We inductively construct a set $\mathcal{R}$ of infinitely many vertex disjoint double rays, one belonging to each end of Ω. Suppose that for any integer $n \geq 0$ we have constructed a set $\mathcal{R}_n$ of n disjoint double rays, one belonging to each of the ends $\omega_1, \ldots, \omega_n$. Different ends can be separated by finitely many vertices so there is a finite set S of vertices such that $\bigcup \mathcal{R}_n$ has no vertex in the component C of $G - S$ which contains ω_{n+1}. Since ω_{n+1} has degree 2 by (2), there are two disjoint rays from ω_{n+1} in C an thus also a double ray D belonging to ω_{n+1}. Set $\mathcal{R}_{n+1} := \mathcal{R}_n \cup \{D\}$ and $\mathcal{R} := \bigcup_{n \in \mathbb{N}} \mathcal{R}_n$.

As $\bigcup \mathcal{R}$ contains no finite cycle of G, it can be extended to an ordinary spanning tree of G. Removing finitely many edges from this tree clearly leaves an element of $\mathcal{R}$ intact. Hence, the edge set of the resulting graph still contains a circuit of $M_T(G)$. Thus, $M_T(G)$ is not nearly finitary in this case as well. $\square$

We end this section with the following open problem.

Question 5.5. *Is it true that any nearly finitary matroid is k-nearly finitary for some $k \in \mathbb{N}$?*

Note that the argument in the proof of Proposition 5.2 (which we reuse for Proposition 5.3) shows that $M_A(G)$ and $M_T(G)$ are $(k-1)$-nearly finitary if G has at most k vertex disjoint rays. So for $M_A(G)$ and $M_T(G)$ the answer is yes when G is locally finite and 2-connected but both proofs use the non-trivial theorem of Halin for which no matroidal equivalent is known.

6 Infinite matroid intersection

In this section, we prove Theorem 1.4, Theorem 1.7 and provide an alternative matroidal proof of the infinite Menger Theorem for locally finite graphs.

6.1 From infinite matroid union to infinite matroid intersection

Here, we prove Theorem 1.4 which reads as follows.

Theorem 1.4. *If M_1 and M_2 are matroids on a common ground set E and $M_1 \vee M_2^*$ is a matroid, then there exists an $I \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)$ admitting a partition $I = J_1 \cup J_2$ such that $cl_{M_1}(J_1) \cup cl_{M_2}(J_2) = E$.*

Proof of Theorem 1.4. We start from the well known proof from finite matroid theory that matroid union implies a solution to the matroid intersection problem. Indeed, let $B_1 \cup B_2^* \in \mathcal{B}(M_1 \vee M_2^*)$ where $B_1 \in \mathcal{B}(M_1)$ and $B_2^* \in \mathcal{B}(M_2^*)$. Rut $B_2 = E \setminus B_2^* \in \mathcal{B}(M_2)$. Then, $I = B_1 \cap B_2 \in \mathcal{I}(M_1) \cap \mathcal{I}(M_2)$.

It remains to show that I admits the required partition. This problem can be rephrased (elegantly) as a directed graph coloring problem. We write $C_i(x)$ to denote the fundamental circuit of x into B_i in M_i whenever $x \notin B_i$ for $i = 1, 2$. Let $C_2^*(x)$ denote the fundamental circuit of x into B_2^* in M_2^* whenever $x \notin B_2^*$. Also, put $X = B_1 \cap B_2^*$, $Y = B_2 \setminus I$, and $Z = B_2^* \setminus X$, see Figure 4.

We shall also require the well known fact [10], which is easy to generalize to infinite matroids [6, Lemma 3.11], that for any matroid M

$$|C \cap C^*| \neq 1 \text{ whenever } C \in \mathcal{C}(M) \text{ and } C^* \in \mathcal{C}(M^*). \tag{3}$$

To prove that I has the required partition, we need the following mitigation:

$$cl_{M_1}(I) \cup cl_{M_2}(I) = E = I \cup X \cup Y \cup Z. \tag{4}$$

To see (4), note first that

$$X \subseteq cl_{M_2}(I). \tag{5}$$

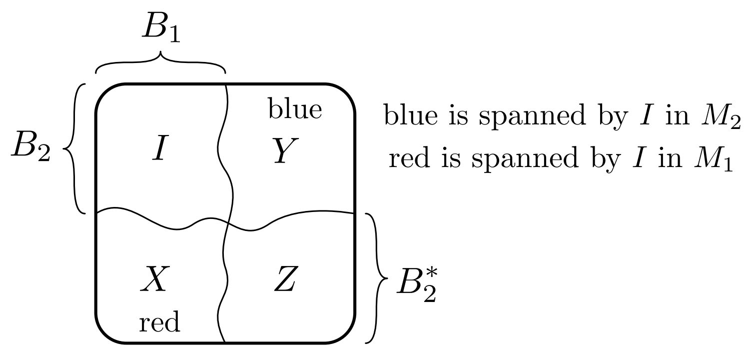

Figure 4: The sets X, Y, and Z and their colorings.

Clearly, no member of X is spanned by I in M_1. Assume then that $x \in X$ is not spanned by I in M_2 so that there exists a $y \in C_2(x) \cap Y$. Then, $x \in C_2^*(y)$, by (3). Consequently, $B_1 \cup B_2^* \subsetneq B_1 \cup (B_2^* + y - x) \in \mathcal{I}(M_1 \vee M_2^*)$; contradiction to the maximality of $B_1 \cup B_2^*$, implying (5).

A similar argument shows that

$$Y \subseteq cl_{M_1}(I). \tag{6}$$

To see that

$$Z \subseteq cl_{M_1}(I) \cup cl_{M_2}(I), \tag{7}$$

assume, towards contradiction, that some $z \in Z$ is not spanned by I neither in M_1 nor in M_2 so that there exist an $x \in C_1(z) \cap X$ and a $y \in C_2(z) \cap Y$. Then $B_1 - x + z$ and $B_2 - y + z$ are bases and thus $B_1 \cup B_2^* \subsetneq (B_1 - x + z) \cup (B_2^* - z + y)$ contradicts the maximality of $B_1 \cup B_2^*$.

Having proved (4), we translate the problem of finding a suitable partition $I = J_1 \cup J_2$ into a directed graph coloring problem. By (4), each $x \in E \setminus I$ satisfies $C_1(x) - x \subseteq I$ or $C_2(x) - x \subseteq I$. Define $G = (V, E)$ to be the directed graph whose vertex set is $V = E \setminus I$ and whose edge set is given by

$$E = \{(x, y) : C_1(x) \cap C_2(y) \cap I \neq \emptyset\}. \tag{8}$$

Recall that a *source* is a vertex with no incoming edges and a *sink* is a vertex with no outgoing edges. As $C_1(x)$ does not exist for any $x \in X$ and $C_2(y)$ does not exist for any $y \in Y$, it follows that

$$\text{the members of } X \text{ are sinks and those of } Y \text{ are sources in } G. \tag{9}$$

A 2-coloring of G, by say blue and red, is called *divisive* if such satisfies the following.

(D.1) I spans all the blue elements in M_1;

(D.2) I spans all the red elements in M_2; and

(D.3) $J_1 \cap J_2 = \emptyset$ where $J_1 := (\bigcup_{x \text{ blue}} C_1(x)) \cap I$ and $J_2 := (\bigcup_{x \text{ red}} C_2(x)) \cap I$.

Clearly, if G has a divisive coloring, then I admits the required partition.

We show that G admits a divisive coloring. Color with blue all the sources. These are the vertices that can only be spanned by I in M_1. Color with red all the sinks, that is, all the vertices that can only be spanned by I in M_2. This defines a partial coloring of G in which all members of X are red and those of Y are blue. Such a partial coloring can clearly be extended into a divisive coloring of G if

$$G \text{ has no } (y, x)\text{-path with } y \text{ blue and } x \text{ red.} \tag{10}$$

Indeed, given (10) and (9), color all vertices reachable by a path from a blue vertex with the color blue, color all vertices from which a red vertex is reachable by a path with red, and color all remaining vertices with, say, blue. The resulting coloring is divisive.

It remains to prove (10). We show that the existence of a path as in (10) contradicts the following property. Suppose M and N are matroids and $B \cup B'$ is maximal in $\mathcal{I}(M \vee N)$. Let $y \notin B \cup B'$ and let $x \in B \cap B'$. Then,

$$\text{there exists no } (B, B', y, x)\text{-chain} \tag{11}$$

by [4, Lemma 4.2]. In fact, the contradiction in the proofs of (5),(6), and (7) arose from simple instances of such forbidden chains. In what follows, we see that these also capture the more general setting.

Assume, towards contradiction, that P is a (y, x)-path with y blue and x red; the intermediate vertices of such a path are not colored since they are not a sink nor a source. In what follows we use P to construct a $(B_1, B_2^*, y_0, y_{2|P|})$-chain $(y_0, y_1, \ldots, y_{2|P|})$ such that $y_0 \in Y$, $y_{2|P|} \in X$, all odd indexed members of the chain are in $V(P) \cap Z$, and all even indexed elements of the chain other than y_0 and $y_{2|P|}$ are in I. Existence of such a chain would contradict (11).

Definition of y_0. As y is pre-colored blue then either $y \in Y$ or $C_2(y) \cap Y \neq \emptyset$. In the former case set $y_0 = y$ and in the latter choose $y_0 \in C_2(y) \cap Y$.

Definition of $y_{2|P|}$. In a similar manner, x is pre-colored red since either $x \in X$ or $C_1(x) \cap X \neq \emptyset$. In the former case, set $y_{2|P|} = x$ and in the latter case choose $y_{2|P|} \in C_1(x) \cap X$.

The remainder of the chain. Enumerate $V(P) \cap Z = \{y_1, y_3, \ldots, y_{2|P|-1}\}$ where the enumeration is with respect to the order of the vertices defined by P. Next, for an edge $(y_{2i-1}, y_{2i+1}) \in E(P)$, let $y_{2i} \in C_1(y_{2i-1}) \cap C_2(y_{2i+1}) \cap I$; such exists by the assumption that $(y_{2i-1}, y_{2i+1}) \in E$. As $y_{2i+1} \in C_2^*(y_{2i})$ for all relevant i, by (3), the sequence $(y_0, y_1, y_2, \ldots, y_{2|P|})$ is a $(B_1, B_2^*, y_0, y_{2|P|})$-chain in $\mathcal{I}(M_1 \vee M_2^*)$ as defined in [4]. $\square$

We note that in the above proof, we do not use the assumption that $M_1 \vee M_2^*$ is a matroid; in fact, we only need that $\mathcal{I}(M_1 \vee M_2^*)$ has a maximal element.

6.2 From infinite matroid intersection to the infinite Menger theorem

Here, we prove Theorem 1.7 and then proceed and provide an alternative matroidal proof for the infinite Menger theorem in the case that the graph is locally finite. Prior to all this, we recall the formulation of the general infinite Menger Theorem. Given a graph G and $S, T \subseteq V(G)$, a set $X \subseteq V(G)$ is called an *S–T separator* if $G - X$ contains no S–T path. Recall the infinite Menger theorem.

Theorem 1.6 (Aharoni and Berger [2])**.** *Let G be a connected graph. Then for any $S, T \subseteq V(G)$ there is a set $\mathcal{L}$ of vertex disjoint S–T paths and an S–T separator $X \subseteq \bigcup_{P \in \mathcal{L}} V(P)$ satisfying $|X \cap V(P)| = 1$ for each $P \in \mathcal{L}$.*

As already mentioned in the Introduction, here we prove the following.

Theorem 1.7. *The infinite matroid intersection conjecture for finitary matroids implies the general infinite Menger theorem.*

Note that infinite matroid union cannot be used to obtain the general infinite Menger Theorem directly via Theorem 1.4 and Theorem 1.7. Indeed, in [4, Example 3.4] a finitary matroid M and a co-finitary matroid N are constructed such that their union is not a matroid. So one cannot apply Theorem 1.4 to the finitary matroids M and N^* in order to obtain Conjecture 1.3 for them. However, it is easy to see that Conjecture 1.3 is true for M and N^*.

Proof of Theorem 1.7. Let G be a connected graph and let $S, T \subseteq V(G)$ be as in Theorem 1.6. We may assume that $G[S]$ and $G[T]$ are both connected. Indeed, an S–T separator with $G[S]$ and $G[T]$ connected gives rise to an S–T separator when these are not necessarily connected. Abbreviate $E(S) :=$

$E(G[S])$ and $E(T) := E(G[T])$, let M be the finite cycle matroid $M_F(G)$, and put $M_S := M/E(S) - E(T)$ and $M_T := M/E(T) - E(S)$; all three matroids are clearly finitary.

Assuming that the infinite matroid intersection conjecture holds for M_S and M_T, there exists a set $I \in \mathcal{I}(M_S) \cap \mathcal{I}(M_T)$ which admits a partition $I = J_S \cup J_T$ satisfying

$$\mathrm{cl}_{M_S}(J_S) \cup \mathrm{cl}_{M_T}(J_T) = E,$$

where $E = E(M_S) = E(M_T)$. We regard I as a subset of $E(G)$.

For the components of $G[I]$ we observe two useful properties. As I is disjoint from $E(S)$ and $E(T)$, the edges of a cycle in a component of $G[I]$ form a circuit in both, M_S and M_T, contradicting the independence of I in either. Consequently,

$$\text{the components of } G[I] \text{ are trees.} \tag{12}$$

Next, an S-path[7] or a T-path in a component of $G[I]$ gives rise to a circuit of M_S or M_T in I, respectively. Hence,

$$|V(C) \cap S| \leq 1 \text{ and } |V(C) \cap S| \leq 1 \text{ for each component } C \text{ of } G[I]. \tag{13}$$

Let $\mathcal{C}$ denote the components of $G[I]$ meeting both of S and T. Then by (12) and (13) each member of $\mathcal{C}$ contains a unique S–T path and we denote the set of all these paths by $\mathcal{L}$. Clearly, the paths in $\mathcal{L}$ are vertex-disjoint.

In what follows, we find a set X comprised of one vertex from each $P \in \mathcal{L}$ to serve as the required S–T separator. To that end, we show that one may alter the partition $I = J_S \cup J_T$ to yield a partition

$$I = K_S \cup K_T \text{ satisfying } cl_{M_S}(K_S) \cup cl_{M_T}(K_T) = E \text{ and (Y.1-4)}, \tag{14}$$

where (Y.1-4) are as follows.

(Y.1) Each component C of $G[I]$ contains a vertex of $S \cup T$.

(Y.2) Each component C of $G[I]$ meeting S but not T satisfies $E(C) \subseteq K_S$.

(Y.3) Each component C of $G[I]$ meeting T but not S satisfies $E(C) \subseteq K_T$.

(Y.4) Each component C of $G[I]$ meeting both, S and T, contains at most one vertex which at the same time

[7]A non-trivial path meeting $G[S]$ exactly in its end vertices.

(a) lies in S or is incident with an edge of K_S, and

(b) lies in T or is incident with an edge of K_T.

Postponing the proof of (14), we first show how to deduce the existence of the required S–T separator from (14). Define a pair of sets of vertices (V_S, V_T) of $V(G)$ by letting V_S consist of those vertices contained in S or incident with an edge of K_S and defining V_T in a similar manner. Then $V_S \cap V_T$ may serve as the required S–T separator. To see this, we verify below that (V_S, V_T) satisfies all of the terms (Z.1-4) stated next.

(Z.1) $V_S \cup V_T = V(G)$;

(Z.2) for every edge e of G either $e \subseteq V_S$ or $e \subseteq V_T$;

(Z.3) every vertex in $V_S \cap V_T$ lies on a path from $\mathcal{L}$; and

(Z.4) every member of $\mathcal{L}$ meets $V_S \cap V_T$ at most once.

To see (Z.1), suppose v is a vertex not in $S \cup T$. As G is connected, such a vertex is incident with some edge $e \notin E(T) \cup E(S)$. The edge e is spanned by K_T or K_S; say K_T. Thus, $K_T + e$ contains a circle containing e or $G[K_T + e]$ has a T-path containing e. In either case v is incident with an edge of K_T and thus in V_T, as desired.

To see (Z.2), let $e \in cl_{M_T}(K_T) \backslash K_T$; so that $K_T + e$ has a circle containing e or $G[K_T + e]$ has T-path containing e; in either case both end vertices of e are in V_T, as desired. The treatment of the case $e \in cl_{M_S}(K_S)$ is similar.

To see (Z.3), let $v \in V_S \cap V_T$; such is in S or is incident with an edge of K_S, and in T or is incident with an edge in K_T. Let C be the component of $G[I]$ containing v. By (Y.1-4), $C \in \mathcal{C}$, i.e. it meets both, S and T and therefore contains an S–T path $P \in \mathcal{L}$. Recall that every edge of C is either in K_S or K_T and consider the last vertex w of a maximal initial segment of P in $C - K_T$. Then w satisfies (Y.4a), as well as (Y.4b), implying $v = w$; so that v lies on a path from $\mathcal{L}$.

To see (Z.4), we restate (Y.4) in terms of V_S and V_T: each component of $\mathcal{C}$ contains at most one vertex of $V_S \cap V_T$. This clearly also holds for the path from $\mathcal{L}$ which is contained in C.

It remains to prove (14). To this end, we show that any component C of $G[I]$ contains a vertex of $S \cup T$. Suppose not. Let e be the first edge on a $V(C)$–S path Q which exists by the connectedness of G. Then $e \notin I$ but without loss of generality we may assume that $e \in \mathrm{cl}_{M_S}(J_S)$. So in $G[I] + e$ there must be a cycle or an S-path. The latter implies that C contains a vertex of S and the former means that Q was not internally disjoint to $V(C)$, yielding contradictions in both cases.

We define the sets K_S and K_T as follows. Let C be a component of $G[I]$.

1. If C meets S but not T, then include its edges into K_S.

2. If C meets T but not S, then include its edges into K_T.

3. Otherwise (C meets both of S and T) there is a path P from $\mathcal{L}$ in C. Denote by v_C the last vertex of a maximal initial segment of P in $C - J_T$. As C is a tree, each component C' of $C - v_C$ is a tree and there is a unique edge e between v_C and C'. For every such component C', include the edges of $C' + e$ in K_S if $e \in J_S$ and in K_T otherwise, i.e. if $e \in J_T$.

Note that, by choice of v_C, either v_C is the last vertex of P or the next edge of P belongs to J_T. This ensures that K_S and K_T satisfy (Y.4). Moreover, they form a partition of I which satisfies (Y.1-3) by construction. It remains to show that $\mathrm{cl}_{M_S}(K_S) \cup \mathrm{cl}_{M_T}(K_T) = E$.

As $K_S \cup K_T = I$, it suffices to show that any $e \in E \setminus I$ is spanned by K_S in M_S or by K_T in M_T. Suppose $e \in \mathrm{cl}_{M_S}(J_S)$, i.e. $J_S + e$ contains a circuit of M_S. Hence, $G[J_S]$ either contains an e-path R or two disjoint e–S paths R_1 and R_2. We show that $E(R) \subseteq K_S$ or $E(R) \subseteq K_T$ in the former case and $E(R_1) \cup E(R_2) \subseteq K_S$ in the latter.

The path R is contained in some component C of $G[I]$. Suppose $C \in \mathcal{C}$ and v_C is an inner vertex of R. By assumption, the edges preceding and succeeding v_C on R are both in J_S and hence the edges of both components of $C - v_C$ which are met by R plus their edges to v_C got included into K_S, showing $E(R) \subseteq K_S$. Otherwise $C \notin \mathcal{C}$ or $C \in \mathcal{C}$ but v_C is no inner vertex of R. In both cases the whole set $E(R)$ got included into K_S or K_T.

We apply the same argument to R_1 and R_2 except for one difference. If $C \notin \mathcal{C}$ or $C \in \mathcal{C}$ but v_C is no inner vertex of R_i, then $E(R_i)$ got included into K_S as R_i meets S.

Although the definitions of K_S and K_T are not symmetrical, a similar argument shows $e \in \mathrm{cl}_{M_S}(K_S) \cup \mathrm{cl}_{M_T}(K_T)$ if e is spanned by J_T in M_T. □

Combining Theorem 1.4 and a slight modification of the above proof, we now provide an alternative matroidal proof of the following.

Theorem 6.1. *The infinite Menger theorem is true for locally finite graphs.*

Before proving Theorem 6.1, we observe the following.

$$\text{If } G \text{ is locally finite, then } M_A(G) \text{ is co-finitary.} \tag{15}$$

Proof. Indeed, assume towards contradiction that $M_A(G)$ has an infinite co-circuit S. In graph language, S is an edge separator that does not separate two rays. By minimality of S, the graph $G - S$ has exactly two components and these meet every edge in S. Since G is locally finite and S is infinite, these two components are infinite and contain a ray, contradicting that S separates. □

Proof of Theorem 6.1. To obtain the assertion, we modify the proof of Theorem 1.7 as follows. Since G is locally finite and connected, it has only countably many vertices. In particular, $G[S]$ and $G[T]$ can still be assumed to be connected without forfeiting that G is locally finite. Furthermore, G does not contain a subdivision of the (non-locally-finite) Bean graph and hence $M_A(G)$ is a matroid by [6, Theorem 2.5]. By (15), $M_A(G)$ is co-finitary and this is preserved under taking minors. Hence $M_T := M_A(G)/E(T) - E(S)$ is co-finitary as well. In particular, $M_S \vee M_T^*$ is a matroid by Theorem 1.1 and thus Theorem 1.4 implies that M_S and M_T satisfy the infinite matroid intersection conjecture.

It remains to check that the proof of Theorem 1.7 works with the modified M_T as well. To reduce the number of cases to consider, we assume that T is infinite and hence contains a ray. Otherwise attach a ray at some vertex of T and add all its vertices to T. Hence no component of $G[I]$ meeting T may contain a ray. The proof of Theorem 1.7 can be applied if we replace (Y.1) by the following: Each component C of $G[I]$ not meeting $S \cup T$ is in K_T. □

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