Infinite matroid union I

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Abstract

We prove that the union of two finitary matroids is a matroid, and in fact finitary. On the other hand we show that the union of a finitary matroid with an arbitrary matroid need not be a matroid.

We extend the well-known base packing theorem for finite matroids to co-finitary matroids, implying the tree-packing results in infinite graphs of Tutte and of Diestel.

1 Introduction

For two finite matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$, the well-known *matroid union theorem* [7, 8] asserts that the set system

$$\mathcal{I}(M_1 \vee M_2) = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2\}$$

is the set of independent sets of their *union* matroid $M_1 \vee M_2$, whose bases are the unions of pairs of bases with least intersection.

In Example 3.4, we provide two infinite matroids M_1 and M_2 for which the set $\mathcal{I}(M_1 \vee M_2)$ does not define a matroid. Hence, an analogue of the finite matroid union theorem does not exist for arbitrary infinite matroids.

One of the matroids involved in Example 3.4 contains no infinite circuits, such a matroid is called *finitary*. Nevertheless, we prove the following.

Theorem 1.1. *If M_1 and M_2 are finitary matroids, then $M_1 \vee M_2$ is a matroid, and in fact finitary.*

The finite matroid union theorem asserts that the union of two disjoint bases of a finite matroid M is a base of $M \vee M$. For finitary matroids the same assertion is false. Indeed, there exists a finitary matroid N with two disjoint bases whose union is not a base of the matroid $N \vee N$, see Example 3.1.

An easy consequence of Theorem 1.1 is that the union of finitely many finitary matroids is a finitary matroid. We show that this is best possible in the sense that the union of countably many finite matroids need not be a matroid, see Example 3.2.

In [3], we use Theorem 1.1 to prove that the union of two 'nearly finitary' matroids is a 'nearly finitary' matroid. This then reduces the 'gap' between Theorem 1.1 and the counterexamples against matroid union.

The main difficulty in proving Theorem 1.1 arises from the need to verify that $\mathcal{I}(M_1 \vee M_2)$ satisfies the axioms (IM) and (I3), stated below. The former essentially asserts that each independent set is contained in a maximal one. To verify that this axiom is satisfied we use a topological argument (see Section 5) that shows that $\mathcal{I}(M_1 \vee M_2)$ is a *finitary set system*, meaning that an infinite set belongs to the system provided that each of its finite subsets does. It is then an easy consequence of Zorn's lemma that all finitary set systems satisfy (IM).

The axiom (I3) asserts that *whenever I is a non-maximal independent set and I' is base, then there exists an $x \in I' \setminus I$ such that $I + x$ is independent.* We prove the following.

Theorem 1.2. *If M_1 and M_2 are matroids, then $\mathcal{I}(M_1 \vee M_2)$ satisfies (I3).*

The main benefit of Theorem 1.2 is that it holds for arbitrary matroids (which need not be finitary); this reduces the problem of determining whether $\mathcal{I}(M_1 \vee M_2)$ defines a matroid to the problem of determining if the latter satisfies (IM).

To prove Theorem 1.2, we introduce the concept of *exchange chains*. These turn out to have further applications. For instance, in [3] we use these chains together with our union results to gain progress on the *infinite matroid intersection conjecture*, put forth by Nash-Williams [2].

Two well-known applications of the finite matroid union theorem are that of *base covering* (see Section 6) and *base packing* of finite matroids. The former extends to finitary matroids in a straightforward manner by Theorem 1.1 (see Corollary 6.1). We extend the latter to matroids whose dual is finitary, these are called *co-finitary* matroids. As a consequence, we obtain a single matroidal proof for the two well-known tree packing theorems in infinite graphs of Diestel [6, Theorem 8.5.7] and of Tutte [9] (see Section 6).

More specifically, the finite base packing theorem asserts that *a finite matroid M admits k disjoint bases if and only if $kr(X)+|E(M)\setminus X| \geq kr(M)$*

for every $X \subseteq E(M)$ [8], where r denotes the rank function of M. For infinite matroids, this rank condition is too crude. We reformulate it using the notion of *relative rank* [5] as follows.

Theorem 1.3. *A co-finitary matroid* M *with ground set* E *admits* k *disjoint bases if and only if* $|Y| \geq kr(E|E-Y)$ *for all finite sets* $Y \subseteq E$.

Theorem 1.3 does not extend naturally to arbitrary infinite matroids. Indeed, for every integer k there exists a finitary matroid with no three disjoint bases yet satisfying $|Y| \geq kr(E|E-Y)$ for every $Y \subseteq E$ [1, 6].

As far as we know, all known proofs in which the finite base packing theorem is derived from the finite matroid union theorem rely on the assumption that the matroid has finite rank. As this assumption is too restrictive in our setting, our proof of Theorem 1.3 does not appeal to Theorem 1.1.

This paper is organized as follows. Additional notation and terminology is set in Section 2. The examples mentioned above and a few others are detailed in Section 3. In Section 4, we prove Theorem 1.2. Our main result, Theorem 1.1, is proved in Section 5. In Section 6, we prove the Theorems 6.1 and 6. Finally, in Section 7, we pose some open problems.

2 Preliminaries

Throughout, notation and terminology for graphs are that of [6], for matroids that of [7, 5], and for topology that of [4]. M always denotes a matroid and $E(M)$, $\mathcal{I}(M)$, $\mathcal{B}(M)$, and $\mathcal{C}(M)$ denote its ground set, independent sets, bases, and circuits, respectively. It will be convenient to have a similar notation for set systems. That is, for a set system $\mathcal{I}$ over some ground set E, an element of $\mathcal{I}$ is called *independent*, a maximal element of $\mathcal{I}$ is called a *base* of $\mathcal{I}$, and a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ is called *circuit* of $\mathcal{I}$.

A set system $\mathcal{I}$ is the set of independent sets of a matroid if it satisfies the following *independence axioms* [5].

(I1) $\emptyset \in \mathcal{I}$.

(I2) $\lceil \mathcal{I} \rceil = \mathcal{I}$, that is, $\mathcal{I}$ is closed under taking subsets.

(I3) Whenever $I, I' \in \mathcal{I}$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$ has a maximal element.

In [5], a matroid is characterised in terms of its circuits. Of this characterisation we shall need the *circuit elimination axiom* phrased here for a matroid M.

(C3) Whenever $X \subseteq C \in \mathcal{C}(M)$ and $\{C_x \mid x \in X\} \subseteq \mathcal{C}(M)$ satisfies $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C \setminus \left(\bigcup_{x \in X} C_x\right)$ there exists a $C' \in \mathcal{C}(M)$ such that $z \in C' \subseteq \left(C \cup \bigcup_{x \in X} C_x\right) \setminus X$.

A circuit of M which contains a given set $X \subseteq E(M)$ is called an X-*circuit*. The *closure* (see [7]) of a set $X \subseteq E(M)$ is denoted by $\mathrm{cl}(X)$.

As mentioned in the Introduction, a set system is *finitary* if an infinite set belongs to the system provided each of its finite subsets does. If $\mathcal{I}(M)$ is finitary, then M is called a *finitary* matroid; as already mentioned, finitary matroids have no infinite circuits.

For a connected graph G, a maximal set of edges containing no finite cycles is called an *ordinary spanning tree*. These are the bases of the *finite cycle matroid*, denoted $M_F(G)$. Next, a maximal set of edges containing no finite cycles nor any double ray is called an *algebraic spanning tree*. These are the bases of the *algebraic cycle matroid*, denoted $M_A(G)$.

3 Examples

In this section, we demonstrate the significant difference between the union of finite matroids and that of infinite matroids by providing examples for the limitations of infinite matroid union.

Example 3.1. We present a finitary infinite matroid M admitting two disjoint bases whose union is properly contained in the union of some other two bases; so that in $M \vee M$ the union of the first pair of bases is not a base.

Consider the infinite one-sided ladder with every edge doubled, say H, and recall that the bases of $M_F(H)$ are the ordinary spanning trees of H. In Figure 1, (B_1, B_2) and (B_3, B_4) are both pairs of disjoint bases of $M_F(H)$. However, $B_3 \cup B_4$ properly covers $B_1 \cup B_2$ as it additionally contains the leftmost edge of H.

We remark that a direct sum of infinitely many copies of H gives rise to an infinite sequence of unions of disjoint bases, each properly containing the previous one, strengthening Example 3.1, that is, the failure of the 'finite intuition' mentioned above.

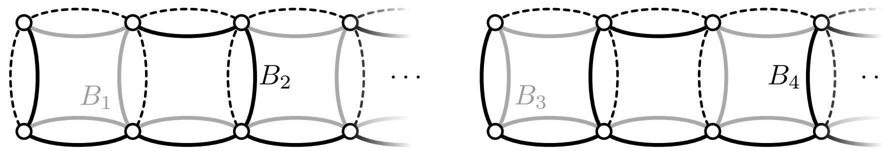


Figure 1: The disjoint bases B_1 and B_2 on the left are properly covered by the bases B_3 and B_4 on the right.

Example 3.2. A union of infinitely many matroids of finite rank need not be a matroid. For any integer $k \geq 1$, we have that

$$M := \bigvee_{n \in \mathbb{N}} U_{k,\mathbb{R}} \text{ is not a matroid.}$$

As usual, $U_{k,\mathbb{R}}$ denotes the k-uniform matroid with ground set $\mathbb{R}$. The set $\mathcal{I}(M)$ violates (IM) for $I = \emptyset$ and $X = \mathbb{R}$ as the independent sets of M are exactly the countable subsets of $\mathbb{R}$. Indeed, as a countable union of finite sets is countable, every independent set is countable. Conversely, an uncountable subset of $\mathbb{R}$ can never be written as a countable union of independent and thus finite sets.

In this example, we used the fact that $r(M)$ is countable and $E(M)$ is uncountable. For a subtler example, let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be disjoint and countable. Set $E_n := \{a_1, \ldots, a_n\} \cup \{b_n\}$. Then $M := \bigvee_{n \in \mathbb{N}} U_{1,E_n}$ is an infinite union of finite matroids and fails to satisfy (IM) for $I = A$ and $X = A \cup B = E(M)$.

Example 3.3. We construct two matroids M and N whose union is not a matroid. M is a direct sum of 1-uniform matroids and thus finitary and N is a direct sum of circuits and thus co-finitary (see Figure 2).

More precisely, the matroids M and N are as follows. Let $E = \mathbb{N} \times \mathbb{R}$. Let $M_n := U_{1,\{n\} \times \mathbb{R}}$ and set $M := \bigoplus_{n \in \mathbb{N}} M_n$. Let N_r be the matroid on $\mathbb{N} \times \{r\}$ whose only circuit is $\mathbb{N} \times \{r\}$ itself and set $N := \bigoplus_{r \in \mathbb{R}} N_r$. A base of M is countable as it consists of one element from each of the sets $\{n\} \times \mathbb{R}$. A base of N consists of all of E but one element from each of the sets $\mathbb{N} \times \{r\}$ and hence misses uncountably many elements of E.

We claim that $\mathcal{I}(M \vee N)$ violates (IM) for $I = \emptyset$ and $X = E$, that is, $\mathcal{I}(M \vee N)$ has no maximal element. For this, it is sufficient to show that a set $J \subseteq E$ is in $\mathcal{I}(M \vee N)$ if and only if it contains at most countably many circuits of N; since then, for any $J \in \mathcal{I}(M \vee N)$ and any circuit $C = \mathbb{N} \times \{r\}$ of N with $C \nsubseteq J$ (such exists) we have $J \cup C \in \mathcal{I}(M \vee N)$.

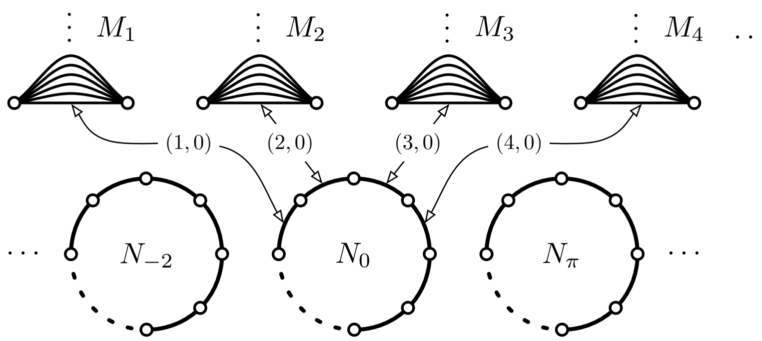


Figure 2: $M = \bigoplus_{n\in\mathbb{N}} M_n$ and $N = \bigoplus_{r\in\mathbb{R}} N_r$ from Example 3.3 as cycle matroids and how they relate.

Suppose that $J \subseteq E$ contains uncountably many circuits of N. Any $J_N \in \mathcal{I}(N)$ misses an element of each of these uncountably many disjoint circuits of N, so $D := J \setminus J_N$ is uncountable. On the other hand, any independent set of M is countable and hence cannot contain D. So $J \notin \mathcal{I}(M \vee N)$.

Suppose that $J \subseteq E$ contains countably many circuits of N. Clearly there is a set $J_N \in \mathcal{I}(N)$ containing all of J but one element from each of these circuits. As we are free to choose which element of each circuit is missed, we can make them lie in different M_n. In particular, we may assume that the set J_M of missed elements is independent in M and thus $J = J_M \cup J_N \in \mathcal{I}(M \vee N)$.

Example 3.4. Whereas Example 3.3 provides two matroids with uncountable ground sets whose union is not a matroid, in this example we construct two such matroids M and N with countable ground set.

Similarly to the previous example, M is a direct sum of finite 1-uniform matroids and thus finitary as well as co-finitary and N is a direct sum of circuits and thus co-finitary (see Figure 3). Note that although we prove later on that the union of two finitary matroids is indeed a matroid, this example shows that the union of two co-finitary matroids is not necessarily a matroid.

Starting the construction of M and N, let $E = (\mathbb{N} \times \mathbb{N}) \cup L$ where $L = \{l_1, l_2, \ldots\}$ is countable and disjoint to $\mathbb{N} \times \mathbb{N}$. For $r \in \mathbb{N}$, let N_r be the matroid on $\mathbb{N} \times \{r\}$ whose only circuit is $\mathbb{N} \times \{r\}$ itself and let N be the matroid obtained from $\bigoplus_{r\in\mathbb{N}} N_r$ by adding the elements of L as loops. Let M_n be the 1-uniform matroid on $(\{n\} \times \{1, 2, \ldots, n\}) \cup \{l_n\}$. We obtain M

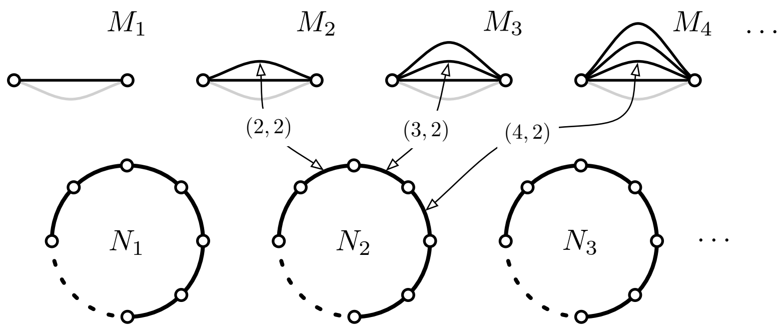


Figure 3: $M = \bigoplus_{n\in\mathbb{N}} M_n$ and $N = \bigoplus_{r\in\mathbb{N}} N_r$ from Example 3.4 as cycle matroids and how they relate. The set L in gray.

from $\bigoplus_{n\in\mathbb{N}} M_n$ by adding the remaining elements of E as loops.

We claim that $\mathcal{I}(M \vee N)$ violates (IM) for $I = \mathbb{N} \times \mathbb{N}$ and $X = E$. For this, it is sufficient to show $I \in \mathcal{I}(M \vee N)$ and that a set J with $I \subset J \subseteq E$ is in $\mathcal{I}(M \vee N)$ if and only if it misses infinitely many elements of L.

To see $I \in \mathcal{I}(M \vee N)$, note that the diagonal $J_M := \{(n, n) \mid n \in \mathbb{N}\}$ is independent in M and meets each circuit $\mathbb{N} \times \{r\}$ of N. In particular, $J_N := (\mathbb{N} \times \mathbb{N}) \setminus J_M$ is independent in N and therefore $I = J_M \cup J_N \in \mathcal{I}(M \vee N)$.

Suppose $J \in \mathcal{I}(M \vee N)$. There are $J_M \in \mathcal{I}(M)$ and $J_N \in \mathcal{I}(N)$ such that $J = J_M \cup J_N$. As J_N misses at least one element from each of the disjoint circuits of N in I, the set $D := I \setminus J_N$ is infinite. But $I \subseteq J$ and hence $D \subseteq J_M$. In particular, there is an infinite subset $L' \subseteq L$ such that $D + l$ contains a circuit of M for every $l \in L'$. This shows that J_M and L' are disjoint and thus J and L' are.

Now suppose that there is a sequence $i_1 < i_2 < \ldots$ such that J is disjoint from $L' = \{l_{i_r} \mid r \in \mathbb{N}\}$. We show that the superset $E \setminus L'$ of J is in $\mathcal{I}(M \vee N)$. Set $D := \{(i_r, r) \mid r \in \mathbb{N}\}$. Then D meets every circuit $\mathbb{N} \times \{r\}$ of N in I, so $J_N := \mathbb{N} \times \mathbb{N} \setminus D$ is independent in N. On the other hand, D contains exactly one element of each M_n with $n \in L'$. So $J_M := (L \setminus L') \cup D \in \mathcal{I}(M)$ and therefore $E \setminus L' = J_M \cup J_N \in \mathcal{I}(M \vee N)$.

4 Exchange chains – (I3) in unions

Throughout this section M_1 and M_2 are matroids and we show that $\mathcal{I}(M_1 \vee M_2)$ satisfies (I3).

Theorem 1.2. *Whenever M_1 and M_2 are matroids, then $\mathcal{I} = \mathcal{I}(M_1 \vee M_2)$ satisfies the following.*

(I3) Whenever $I, I' \in \mathcal{I}$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

For technical reasons, we rather show that the following mixture of (I3) and (B2) is satisfied.

Proposition 4.1. *The set $\mathcal{I} = \mathcal{I}(M_1 \vee M_2)$ satisfies*

(I3') For all $I, B \in \mathcal{I}$ where B is maximal and all $x \in I \setminus B$ there exists $y \in B \setminus I$ such that $(I + y) - x \in \mathcal{I}$.

We remark that unlike in (I3), the set I in (I3') may be maximal.

Proof of Theorem 1.2 assuming Proposition 4.1. Let $I \in \mathcal{I}$ be non-maximal and $B \in \mathcal{I}$ be maximal. As I is non-maximal there is an $x \in E \setminus I$ such that $I + x \in \mathcal{I}$. We may assume $x \notin B$ or the assertion follows by (I2). By (I3'), applied to $I + x$, B, and $x \in (I + x) \setminus B$ there is $y \in B \setminus (I + x)$ such that $I + y \in \mathcal{I}$. □

By a *representation of* a set $I \in \mathcal{I}(M_1 \vee M_2)$, we mean a pair (I_1, I_2) where $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$ such that $I = I_1 \cup I_2$. At the core of our proof of Proposition 4.1, is the following notion of an 'exchange chain'. For $I_1 \in \mathcal{I}(M_1)$, $I_2 \in \mathcal{I}(M_2)$, $x \in I_1 \cup I_2$, and an element $y \in E(M_1) \cup E(M_2)$ (possibly in $I_1 \cup I_2$), a tuple $Y = (y_0, \ldots, y_n)$ with $y_0 = y$ and $y_n = x$ is called an *even* (I_1, I_2, y, x)*-exchange chain* (or *even* (I_1, I_2, y, x)*-chain*) of *length* n if the following terms are satisfied.

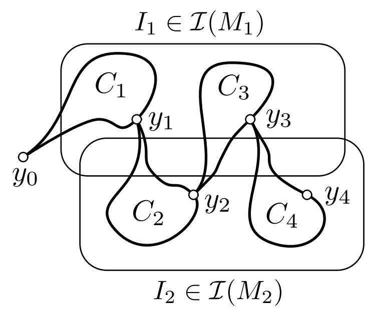
(X1) For even i, there exists a $\{y_i, y_{i+1}\}$-circuit $C_i \subseteq I_1 + y_i$ of M_1.

(X2) For odd i, there exists a $\{y_i, y_{i+1}\}$-circuit $C_i \subseteq I_2 + y_i$ of M_2.

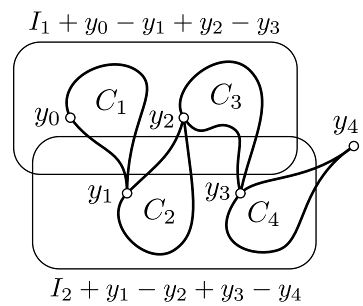
Observe that if $n \geq 1$, then (X1) and (X2) imply $y_0 \notin I_1$ and that, beginning with $y_1 \in I_1 \setminus I_2$, the y_i alternate between $I_1 \setminus I_2$ and $I_2 \setminus I_1$ up to y_n which may be in $I_1 \cap I_2$ as well. By an *odd exchange chain* (or *odd chain*) we mean an even chain with the words 'even' and 'odd' interchanged in the definition. Consequently, we say *exchange chain* (or *chain*) to refer to either of these. Furthermore, a subchain of a chain is also a chain; that is, given an (I_1, I_2, y_0, y_n)-chain $(y_0, \ldots, y_n)$, the tuple $(y_k, \ldots, y_l)$ is an (I_1, I_2, y_l, y_k)-chain for $0 \leq k \leq l \leq n$.

Lemma 4.2. *If there exists an (I_1, I_2, y, x)-chain, then $(I + y) - x \in \mathcal{I}(M_1 \vee M_2)$ where $I := I_1 \cup I_2$. Moreover, if $x \in I_1 \cap I_2$, then even $I + y \in \mathcal{I}(M_1 \vee M_2)$.*

In the proof of Lemma 4.2 we use chains to manipulate the sets I_1 and I_2 as depicted in Figure 4 such that they additionally cover $y_0 = y$ at the expense of not covering $y_n = x$ anymore (unless $x \in I_1 \cap I_2$). Although Lemma 4.2 is changing $I_1 \cup I_2$ by only one element, this might require an exchange chain of arbitrary length, for example in the configuration of Figure 4 a chain of length four is needed.



(a) the initial representation



(b) the obtained representation

Figure 4: An even exchange chain of length 4.

Proof. By induction on the length of the chain. The statement is trivial for chains of length 0. Assume $n \geq 1$ and that $Y = (y_0, \dots, y_n)$ is a shortest (I_1, I_2, y, x)-chain. Without loss of generality, let Y be an even chain. If $Y' := (y_1, \dots, y_n)$ is an (odd) (I_1', I_2, y_1, x)-chain where $I_1' := (I_1 + y_0) - y_1$, then $((I_1' \cup I_2) + y_1) - x \in \mathcal{I}(M_1 \vee M_2)$ by the induction hypothesis and the assertion follows, since $(I_1' \cup I_2) + y_1 = (I_1 \cup I_2) + y_0$. If also $x \in I_1 \cap I_2$, then either $x \in I_1' \cap I_2$ or $y_1 = x$ and hence $n = 1$. In the former case $I + y \in \mathcal{I}(M_1 \vee M_2)$ follows from the induction hypothesis and in the latter case $I + y = I_1' \cup I_2 \in \mathcal{I}(M_1 \vee M_2)$ as $x \in I_2$.

Since I_2 has not changed, (X2) still holds for Y', so to verify that Y' is an (I_1', I_2, y_1, x)-chain, it remains to show $I_1' \in \mathcal{I}(M_1)$ and to check (X1). To this end, let C_i be a $\{y_i, y_{i+1}\}$-circuit of M_1 in $I_1 + y_i$ for all even i. Such exist by (X1) for Y. Notice that any circuit of M_1 in $I_1 + y_0$ has to contain y_0 since $I_1 \in \mathcal{I}(M_1)$. On the other hand, two distinct circuits in $I_1 + y_0$ would give rise to a circuit contained in I_1 by the circuit elimination axiom applied to these two circuits, eliminating y_0. Hence C_0 is the unique circuit of M_1 in $I_1 + y_0$ and $y_1 \in C_0$ ensures $I_1' = (I_1 + y_0) - y_1 \in \mathcal{I}(M_1)$.

To see (X1), we show that there is a $\{y_i, y_{i+1}\}$-circuit C_i' of M_1 in $I_1' + y_i$ for every even $i \geq 2$. Indeed, if $C_i \subseteq I_1' + y_i$, then set $C_i' := C_i$; else, C_i

contains an element of $I_1 \setminus I_1' = \{y_1\}$. Furthermore, $y_{i+1} \in C_i \setminus C_0$; otherwise $(y_0, y_{i+1}, \ldots, y_n)$ is a shorter (I_1, I_2, y, x)-chain for, contradicting the choice of Y. Applying the circuit elimination axiom to C_0 and C_i, eliminating y_1 and fixing y_{i+1}, yields a circuit $C_i' \subseteq (C_0 \cup C_i) - y_1$ of M_1 containing y_{i+1}. Finally, as I_1' is independent and $C_i' \setminus I_1' \subseteq \{y_i\}$ it follows that $y_i \in C_i'$. $\square$

Central to our argument in the proof of Proposition 4.1 is the following set. For $I_1 \in \mathcal{I}(M_1)$, $I_2 \in \mathcal{I}(M_2)$, and $x \in I_1 \cup I_2$ let

$$A(I_1, I_2, x) := \{a \mid \text{there exists an } (I_1, I_2, a, x)\text{-chain}\}$$

Such has the property that

$$\text{for every } y \notin A, \text{ either } I_1 + y \in \mathcal{I}(M_1) \text{ or the unique circuit } C_y \text{ of } M_1 \text{ in } I_1 + y \text{ is disjoint from } A. \tag{1}$$

To see this, suppose $I_1 + y \notin \mathcal{I}(M_1)$. Then there is a unique circuit C_y of M_1 in $I_1 + y$. If $C_y \cap A = \emptyset$, then the assertion holds so we may assume that $C_y \cap A$ contains an element, a say. Hence there is an (I_1, I_2, a, x)-chain $(y_0 = a, y_1, \ldots, y_{n-1}, y_n = x)$. As $a \in I_1$ this chain must be odd or have length 0, that is, $a = x$. Clearly, $(y, a, y_1, \ldots, y_{n-1}, x)$ is an even (I_1, I_2, y, x)-chain, contradicting the assumption that $y \notin A$.

Proof of Proposition 4.1. Let $B \in \mathcal{I}(M_1 \vee M_2)$ maximal, $I \in \mathcal{I}(M_1 \vee M_2)$, and $x \in I \setminus B$. Recall that we seek a $y \in B \setminus I$ such that $(I+y)-x \in \mathcal{I}(M_1 \vee M_2)$. Let (I_1, I_2) and (B_1, B_2) be representations of I and B, respectively. We may assume $I_1 \in \mathcal{B}(M_1|I)$ and $I_2 \in \mathcal{B}(M_2|I)$. We may further assume that for all $y \in B \setminus I$ the sets $I_1 + y$ and $I_2 + y$ are dependent in M_1 and M_2, respectively, or it holds that $I + y \in \mathcal{I}(M_1 \vee M_2)$ so that the assertion follows. Hence, for every $y \in (B \cup I) \setminus I_1$ there is a circuit $C_y \subseteq I_1 + y$ of M_1; such contains y and is unique since otherwise the circuit elimination axiom applied to these two circuits eliminating y yields a circuit contained in I_1, a contradiction.

If $A := A(I_1, I_2, x)$ intersects $B \setminus I$, then the assertion follows from Lemma 4.2. Else, $A \cap (B \setminus I) = \emptyset$, in which case we derive a contradiction to the maximality of B. To this end, set (Figure 5)

$$B_1' := (B_1 \setminus b_1) \cup i_1 \qquad \text{where} \quad b_1 := B_1 \cap A \quad \text{and} \quad i_1 := I_1 \cap A$$

$$B_2' := (B_2 \setminus b_2) \cup i_2 \qquad \text{where} \quad b_2 := B_2 \cap A \quad \text{and} \quad i_2 := I_2 \cap A$$

Since A contains x but is disjoint from $B \setminus I$, it holds that $(b_1 \cup b_2) + x \subseteq i_1 \cup i_2$ and thus $B + x \subseteq B_1' \cup B_2'$. It remains to verify the independence of B_1' and B_2' in M_1 and M_2, respectively.

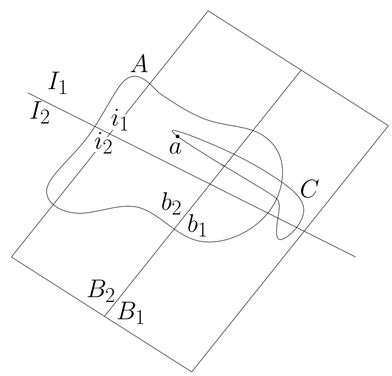


Figure 5: The independent sets I_1, at the top, and I_2, at the bottom, the bases B_1, on the right, and B_2, on the left, and their intersection with A.

Without loss of generality it is sufficient to show $B_1' \in \mathcal{I}(M_1)$. For the remainder of the proof 'independent' and 'circuit' refer to the matroid M_1. Suppose for a contradiction that the set B_1' is dependent, that is, it contains a circuit C. Since i_1 and $B_1 \setminus b_1$ are independent, neither of these contain C. Hence there is $a \in C \cap i_1 \subseteq A$. But $C \setminus I_1 \subseteq B_1 \setminus A$ and therefore no C_y with $y \in C \setminus I_1$ contains a by (1). Thus, applying the circuit elimination axiom on C eliminating all $y \in C \setminus I_1$ via C_y fixing a, yields a circuit in I_1, a contradiction. □

We conclude this section with a few remarks.

1. The maximality of B was only used to avoid the outcome $B + x \in \mathcal{I}(M_1 \vee M_2)$. So the proof actually shows a slightly stronger statement.

   Corollary 4.3. *For all $I, J \in \mathcal{I}(M_1 \vee M_2)$ and $x \in I \setminus J$, if $J + x \notin \mathcal{I}(M_1 \vee M_2)$, then there exists $y \in J \setminus I$ such that $(I + y) - x \in \mathcal{I}(M_1 \vee M_2)$.*

2. For any maximal representation (I_1, I_2) of I there is $y \in B \setminus I$ such that exchanging finitely many elements of I_1 and I_2 gives a representation of $(I + y) - x$.

3. To define the set $A(I_1, I_2, x)$, we consider chains whose last element is fixed. Alternatively, one may consider chains whose first element is fixed. More precisely, for $I_1 \in \mathcal{I}(M_1)$, $I_2 \in \mathcal{I}(M_2)$, and $y \notin I_1 \cup I_2$ let

$$Z(I_1, I_2, y) := \{z \mid \text{there exists an } (I_1, I_2, y, z)\text{-chain}\}$$

Then a similar argument as in the proof of Proposition 4.1, applied to $Z(I_1, I_2, y)$ instead of $A(I_1, I_2, y)$, yields the following.

Corollary 4.4. *For all $I, J \in \mathcal{I}(M_1 \vee M_2)$ and $y \in J \setminus I$, if $I + y \notin \mathcal{I}(M_1 \vee M_2)$, then there exists $x \in I \setminus J$ such that $(I + y) - x \in \mathcal{I}(M_1 \vee M_2)$.*

4. Even arbitrary unions of matroids satisfy (I3) (however, they may still violate (IM) as in Example 3.2). To see this, either adjust the proof of Proposition 4.1, in particular the notion of an exchange chain, to this more general setting, or reduce the general case to Proposition 4.1.

5 Finitary matroids

In this section, we prove our main result which reads as follows.

Theorem 1.1. *If M_1 and M_2 are finitary matroids, then $M_1 \vee M_2$ is a matroid; and in fact finitary.*

For countable matroids, Theorem 1.1 can be proved by means of König's infinity lemma. Here, to capture matroids on any infinite ground set, we resort to using a topological approach. See [4] for the required topological background needed here.

We recall the definition of the product topology on $\mathcal{P}(E)$. The usual base is formed by the system of all sets

$$C(A, B) := \{X \subseteq E \mid A \subseteq X, B \cap X = \emptyset\}$$

where $A, B \subseteq E$ are finite and disjoint. Note that these sets are closed as well. Throughout this section, $\mathcal{P}(E)$ is endowed with the product topology and *closed* is used in the topological sense only.

We will show that Theorem 1.1 can easily be deduced from Theorem 5.1 and Lemma 5.2, presented next.

Theorem 5.1. *Suppose $\mathcal{I} = \lceil \mathcal{I} \rceil \subseteq \mathcal{P}(E)$. Then the following are equivalent*

5.1.1. *$\mathcal{I}$ is finitary;*

5.1.2. *$\mathcal{I}$ is compact, in the subspace topology of $\mathcal{P}(E)$.*

Note that as $\mathcal{P}(E)$ is a compact Hausdorff space, 5.1.2 is equivalent to the assumption that $\mathcal{I}$ is closed in $\mathcal{P}(E)$, which we use quite often in the following proofs.

Proof. For the forward direction, we show that $\mathcal{I}$ is closed. Let $X \notin \mathcal{I}$. Since $\mathcal{I}$ is finitary, X has a finite subset $Y \notin \mathcal{I}$ and no superset of Y is in $\mathcal{I}$ as $\mathcal{I} = \lceil \mathcal{I} \rceil$. Therefore, $C(Y, \emptyset)$ is an open set containing X and avoiding $\mathcal{I}$ and hence $\mathcal{I}$ is closed.

For the converse direction, assume that $\mathcal{I}$ is compact and let X be a set such that all finite subsets of X are in $\mathcal{I}$. We show $X \in \mathcal{I}$ using the finite intersection property[1] of $\mathcal{P}(E)$. Consider the family $\mathcal{K}$ of pairs (A, B) where $A \subseteq X$ and $B \subseteq E \setminus X$ are both finite. The set $C(A, B) \cap \mathcal{I}$ is closed for every $(A, B) \in \mathcal{K}$, as $C(A, B)$ and $\mathcal{I}$ are closed. If $\mathcal{L}$ is a finite subfamily of $\mathcal{K}$, then

$$\bigcup_{(A,B)\in\mathcal{L}} A \in \bigcap_{(A,B)\in\mathcal{L}} (C(A,B) \cap \mathcal{I}) .$$

As $\mathcal{P}(E)$ is compact, the finite intersection property yields

$$\left(\bigcap_{(A,B)\in\mathcal{K}} C(A,B) \right) \cap \mathcal{I} = \bigcap_{(A,B)\in\mathcal{K}} (C(A,B) \cap \mathcal{I}) \neq \emptyset$$

But $\bigcap_{(A,B)\in\mathcal{K}} C(A,B) = \{X\}$, and hence $X \in \mathcal{I}$, as desired. □

Lemma 5.2. *If $\mathcal{I}$ and $\mathcal{J}$ are closed in $\mathcal{P}(E)$, then so is $\mathcal{I} \vee \mathcal{J}$.*

Proof. Equipping $\mathcal{P}(E) \times \mathcal{P}(E)$ with the product topology, yields that Cartesian products of closed sets in $\mathcal{P}(E)$ are closed in $\mathcal{P}(E) \times \mathcal{P}(E)$. In particular, $\mathcal{I} \times \mathcal{J}$ is closed in $\mathcal{P}(E) \times \mathcal{P}(E)$. In order to prove that $\mathcal{I} \vee \mathcal{J}$ is closed, we note that $\mathcal{I} \vee \mathcal{J}$ is exactly the image of $\mathcal{I} \times \mathcal{J}$ under the union map

$$f : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{P}(E), \quad f(A,B) = A \cup B.$$

It remains to check that f maps closed sets to closed sets; which is equivalent to showing that f maps compact sets to compact sets as $\mathcal{P}(E)$ is a compact Hausdorff space. As continuous images of compact spaces are compact, it suffices to prove that f is continuous, that is, to check that the pre-images of subbase sets $C(\{a\}, \emptyset)$ and $C(\emptyset, \{b\})$ are open:

$$f^{-1}(C(\{a\}, \emptyset)) = (C(\{a\}, \emptyset) \times \mathcal{P}(E)) \cup (\mathcal{P}(E) \times C(\{a\}, \emptyset))$$

[1] *The finite intersection property* ensures that an intersection over a family $\mathcal{C}$ of closed sets is non-empty if every intersection of finitely many members of $\mathcal{C}$ is.

$$f^{-1}(C(\emptyset, \{b\})) = C(\emptyset, \{b\}) \times C(\emptyset, \{b\})$$

□

Proof of Theorem 1.1. By Theorem 1.2 it remains to show that $\mathcal{I}(M_1) \vee \mathcal{I}(M_2)$ satisfies (IM). As all finitary set systems satisfy (IM), by Zorn's lemma, we show that $\mathcal{I}(M_1 \vee M_2)$ is finitary. By Theorem 5.1, $\mathcal{I}(M_1)$ and $\mathcal{I}(M_2)$ are both compact and thus closed in $\mathcal{P}(E)$, yielding, by Lemma 5.2, that $\mathcal{I}(M_1) \vee \mathcal{I}(M_2)$ is closed $\mathcal{P}(E)$, and thus compact. As $\mathcal{I}(M_1) \vee \mathcal{I}(M_2) = \lceil \mathcal{I}(M_1) \vee \mathcal{I}(M_2) \rceil$, Theorem 5.1 yields that $\mathcal{I}(M_1) \vee \mathcal{I}(M_2)$ is finitary, as desired. □

6 Base packing and covering

We begin by proving base packing for co-finitary matroids.

Theorem 1.3. *A co-finitary matroid $M = (E, \mathcal{I})$ has k disjoint bases if and only if $|Y| \geq k \cdot r(E|E - Y)$ for all finite sets $Y \subseteq E$.*

Proof. As the 'only if' direction is trivial, it remains to show the 'if' direction. For a matroid N and natural numbers k, c put

$$\mathcal{I}[N, k, c] := \{X \subseteq E(N) \mid \exists I_1, ..., I_k \in \mathcal{I}(N) \text{ with } g_c(I_1, ..., I_k) = X\},$$

where $g_c(I_1, ..., I_k) := \{e : |\{j : e \in I_j\}| \geq c\}$. As M has k disjoint spanning sets if and only if M^* has k independent sets such that every element of E is in at least $k - 1$ of those independent sets. Put another way, M has k disjoint bases if and only if

$$\mathcal{I}[M^*, k, k - 1] = \mathcal{P}(E) \quad (2)$$

As M^* is finitary, $\mathcal{I}[M^*, k, k - 1]$ is finitary by an argument similar to that in the proof of Lemma 5.2; here one may define

$$f : \mathcal{P}(E)^k \to \mathcal{P}(E); f(A_1, ..., A_k) = g_{k-1}(A_1, ..., A_k)$$

and repeat the above argument.

Thus, it suffices to show that every finite set Y is in $\mathcal{I}[M^*, k, k - 1]$. To this end, it is sufficient to find k independent sets of M^* such that every element of Y is in at least $k - 1$ of those; complements of which are M-spanning sets $S_1, ..., S_k$ such that these are disjoint if restricted to Y. To this end, we show that there are disjoint spanning sets $S'_1, ..., S'_k$ of $M.Y$ and set $S_i := S'_i \cup (E - Y)$. Since Theorem 1.3 is true for finite matroids [7], the sets $S'_1, ..., S'_k$ exist if and only if $|Z| \geq k \cdot r_{M.Y}(Y|Y - Z)$ for all $Z \subseteq Y$. As $|Z| \geq k \cdot r(E|E - Z)$, by assumption, and as $r(E|E - Z) = r_{M.Y}(Y|Y - Z)$ [5, Lemma 3.13], the assertion follows. $\square$

Diestel [6, Theorem 8.5.7] established that *a graph admits k disjoint topological spanning trees if and only if every partition of its vertex set into r classes has at least $k(r-1)$ edges between the classes*. A similar result to that of Diestel was established by Tutte [9] who proved a packing theorem for the algebraic spanning trees of a locally finite graph.

The topological spanning trees of a graph form the set of bases of a co-finitary matroid called the *topological cycle matroid* [5], and recall that the algebraic spanning trees of a locally finite graph form the set of bases the so called algebraic cycle matroid which is co-finitary. As mentioned in the Introduction, Theorem 1.3 provides then a short alternative matroidal proof of both results of Diestel and Tutte.

Finally, we use Theorem 1.1 to derive a base covering result for finitary matroids. The finite base covering theorem asserts that *a finite matroid M can be covered by k bases if and only if $r(X) \geq |X|/k$ for every $X \subseteq E(M)$* [8].

Corollary 6.1. *A finitary matroid M can be covered by k independent sets if and only if $r_M(X) \geq |X|/k$ for every finite $X \subseteq E(M)$.*

This claim is false if M is, say, an infinite circuit, implying that this result is best possible in the sense that M being finitary is necessary.

Proof. The 'only if' implication is trivial. Suppose then that each finite set $X \subseteq E(M)$ satisfies $r_M(X) \geq |X|/k$ and put $N = \bigvee_{i=1}^{k} M$; such is a finitary matroid by Theorem 1.1. If N is the free matroid, the assertion holds trivially. Suppose then that N is not the free matroid and consequently contains a circuit C; such is finite as N is finitary. Hence, $M|C$ cannot be covered by k independent sets of $M|C$ so that by the finite matroid covering theorem [7, Theorem 12.3.12] there exists a finite set $X \subseteq C$ such that $r_{M|C}(X) < |X|/k$ which clearly implies $r_M(X) < |X|/k$; a contradiction. $\square$

7 Union of a matroid with itself

In this paper and the second paper of this series, we solve the question of whether the union of two infinite matroids is a matroid almost completely.

However, we were not able to find a matroid whose union with itself is not a matroid. Specializing this question, we ask whether the union of a matroid having only infinite circuits with itself is the free matroid. In other words, we ask the following.

Question 7.1. *Suppose that each circuit of M is infinite. Is it true that the ground set of M can be covered with two bases of M?*

By Proposition 4.1, verifying that $\mathcal{I}(M \vee M)$ satisfies (IM) is sufficient in order to show that $M \vee M$ is a matroid. One approach to determine whether $\mathcal{I}(M \vee M)$ satisfies (IM) is to dualize (IM) and to resolve the resulting version in the duals. Such an approach brings forth the need to determine whether for a matroid M there exist two spanning sets with minimal intersection. For graphic matroids this reads as follows.

Question 7.2. *Is it true that every locally finite graph has two ordinary spanning trees whose intersection is minimal?*

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