DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 80/130
December 1980

ON THE IMPOSSIBILITY OF EMBEDDING HORIZONTAL AND VERTICAL SUBGROUPS IN HIGHER SIMPLE GROUPS

by

F. Ghaboussi

II. Institut für Theoretische Physik der Universität Hamburg

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY 80/130
December 1980

ON THE IMPOSSIBILITY OF EMBEDDING HORIZONTAL AND VERTICAL SUBGROUPS IN HIGHER SIMPLE GROUPS

F. GHABOUSSI

II. Institut für Theoretische Physik der Universität Hamburg
Hamburg 50, Luruper Chaussee 149, Germany

ABSTRACT

It is shown that the embedding of the vertical SU(2) symmetry and a horizontal SU(2) symmetry group in higher simple symmetry groups SU(4) and SU(6) is not possible. This is proved in a direct way through the matrix algebra of the generators.

INTRODUCTION

After the experimental success of the Salam-Weinberg theory and its modification by the G-I-M mechanism during the early seventies, there still remain some basic unsolved problems like the masspattern of fermions and the related Cabbibo angles, which could be understood after the solution of the family [1] problem. Therefore there has been a great interest in the last two years in solving these problems in a direct way, so that with a minimum of new hypotheses one could find a natural [2] solution.

Most authors have tried to solve these connected problems through the introduction of some interfamily (horizontal) gauge symmetry [3] (discrete or continous). One of the most suitable of such horizontal symmetries, with which one can hope to explain most of the open problems with families is a horizontal SU(2) (called $SU_H(2)$). Then the natural way to extend the Salam-Weinberg $SU(2)_L \times U(1)$ theory in this direction is through $SU(2)_V \times U(1) \times SU_H(n)$ [4]. On the other hand we have the known grand unification [5] by the gauge symmetry $SU(5) \supset SU_C(3) \times SU(2)_L \times U(1)$. But in general we are interested in those theories which have a minimal number of coupling constants (ideally just one). In other words the mentioned extension $SU(2)_V \times U(1) \times SU_H(n)$ can be natural if we can find a higher simple group $\underline{G}$ which can break spontaneously into such a subgroup. After this one could include color SU(3). The groups which could be sensible for $\underline{G}$ in the first step are $SU(N \leqslant 8)$. An example with N = 4, $SU(4) \times U(1)$ symmetry unifying electroweak interactions, which accomodates 2 generations in the fundamental representation, is studied by Deshpande et al. [6]. These authors, however, do not study the horizontal subgroup problem.

The aim of this paper is to show the impossibility of the above mentioned breaking of $\underline{G}$ into $SU_V(2) \times U(1) \times SU_H(2)$ or the embedding of a horizontal symmetry together with a Salam-Weinberg group in a higher simple group like $\underline{G}$. The direct result of this statement is that: if we want to explain the generation problem (mass, Cabbibo angles) through a horizontal symmetry, then we must be satisfied with a semi-simple group like $SU(M) \times SU(N)$.

It is known that in order to have a product subgroup as the result of spontanously symmetry breaking of some simple unitary group, one must introduce at least one Higgs field in the adjoint representation [7]. This follows from the fact that the mass terms of gauge bosons associated with the SU(N) symmetry consist of those neutral components of the adjoint Higgs field which results from commutations between the generators of SU(N) and the components of the Higgs field in the adjoint representation. The necessary condition for such a breaking is that one must find some (at least one!) operators F_Y, which commute with all generators of $SU_H(n) \times SU_V(m) \times U(1)$ to leave their gauge bosons massless, but with nonzero commutators with other generators of SU(N) to give masses to their gauge bosons. These are obviously very strong conditions. If these operators exist, then one must choose some suitable nonvanishing vacuum expectation value (VEV) for the neutral components of the adjoint Higgs field which result from those commutators to give masses to the desired gauge bosons while the gauge bosons associated with $SU_H(n) \times SU_V(m) \times U(1)$ symmetry remain massless. This can break the SU(N) symmetry in the required way:

$$SU(N) \longrightarrow SU_H(n) \times SU_V(m) \times U(1)$$

or make possible the desired embedding. But because of such strong group theoretical conditions (commutations) one can show group theoretically that such a breaking (or embedding) is not possible. To do this it is enough to show this impossibility for the case $n = m = 2$. Then the generalization for $n > 2;\ m > 2;\ n + m \leq N$ is straightforward because the existence of a $SU(n > 2)$ group which satisfies the stated conditions depends directly on the existence of some SU(2) subgroup which must obey the same conditions.

We demonstrate in this paper the impossibility for the cases $N = 4;\ n = m = 2$ and $N = 6;\ n = m = 2$.
These cases are sufficient for $N \leqslant 6$ because for $N < 4$ there is no subgroup of the kind SU(2) X SU(2) and for N = 5 such a breaking goes through the SU(4) group. The generalization for $N \leqslant 8$ has the same structure as $4 \leqslant N \leqslant 6$. It seems that in general one can not embed a horizontal subgroup together with a vertical one: $SU_H(n) \times SU_V(m)$ in a higher simple group SU(N).

We write the general group theoretical conditions for such a breaking for SU(N) in section II and give a general construction for the N x N matrix as a general model for the generators of U(N); N:even. This set of matrices contains both unitary and hermitean matrices and is very suitable for introducing the commutation conditions mentioned above to see the impossibility in a direct way. In section III we treat the case N = 4 and N = 6; n = m = 2. In section IV we make some remarks about the connection between our work and the question of grand unified theories which contain more than one family.

II-Group theoretical symmetry breaking

By introducing the adjoint representation for the Higgs field we find that the mass terms of the SU(N) gauge bosons have the form [8]

$$\frac{1}{2} g^2 W_i^+ M_{ij} W_j = \frac{g^2}{2} f_{kil} W_i^+ \varphi_l^+ f_{kjm} W_j \varphi_m = \left(\frac{ig}{\sqrt{2}} W_i^+ [F^i, \varphi_k^+]\right)\left(-\frac{ig}{\sqrt{2}} W_j [F^j, \varphi_k]\right); \quad (1)$$

$$[F_i, F_j] = i f_{ijk} F_k \quad , \quad \{F_i\} \in SU(N) \quad , \quad i = 1, \cdots, (N^2-1) \quad . \quad (2)$$

Then the breaking $SU(N) \supset SU_H(n) \times SU_V(m) \times U(1)$ is possible if and only if

$$\{F_\alpha\} \in SU_H(n) \quad , \quad \alpha = 1, \cdots, (n^2-1) \quad (3)$$

$$\{F_\beta\} \in SU_V(m) \quad , \quad \beta = 1, \cdots, (m^2-1) \quad (4)$$

$$[F_\alpha, F_\beta] = 0 \quad (5)$$

and one can find some operators (at least one) F_γ such that:

for $\{F_\gamma\} \subset \{F_\lambda\}$, $\lambda = 1, \cdots, [(N^2-1)-(n^2-1)-(m^2-1)]$

the following conditions are obeyed:

$$[F_\alpha, F_\gamma] = 0 \tag{6}$$

$$[F_\beta, F_\gamma] = 0 \tag{7}$$

$$[F_\lambda, F_\gamma] \neq 0 \quad \text{but} \quad [F_\lambda, F_\gamma] \propto F_n \quad [9] \tag{8}$$

If such operators (F_γ) exist, then one can choose some suitable VEV

for $\{\varphi_n\}$ related to $\{F_n\}$

$$\langle \varphi_n \rangle \propto \langle [F_\lambda, F_\gamma] \rangle \longleftrightarrow [F_\lambda, F_\gamma] \propto F_n \tag{9}$$

to give masses to the gauge bosons W_λ associated with F_λ. The gauge bosons W_α and W_β associated with F_α and F_β remain massless because of conditions (6) and (7). This breaks the SU(N) symmetry down to $SU(n)_H$ X $SU_V(m)$. But we show in the next section that such strong conditions (6) — (8) make this breaking impossible.

The general matrix form which contain unitary and hermitean matrices, can be built by Pauli-matrices and the identity operator (σ_i, i = o, 1, 2, 3). This means that one can construct all the generators of U(N) by some linear combination of such general matrices. Then one can enforce conditions to form different desired matrices (generators). We show this for the example of U(4) and U(6) groups. The fact is that by introducing such a general matrix form one has the smallest preconditions and therefore greatest possibility to check the effects of the desired conditions.

III - SU(4) and SU(6)

The general matrix form for 4 x 4 matrices which contains all generators of U(4):SU(4) XU(1) has the form

$$\{U\} : \begin{pmatrix} a\sigma_i & b\sigma_j \\ c\sigma_k & d\sigma_l \end{pmatrix} \quad , \{i,j,k,l\} : \{0,\cdots,3\} \tag{10}$$

with a, b, c, d real numbers. This means that one can construct all generators of U(4) by some suitable linear combinations of $\{U\}$. In other words, our desired operators F_γ are of course some subset of $\{U\}$. Now to distinguish between the horizontal (interfamily) and vertical (Salam-Weinberg) symmetries, take the electric charge basis for the fermions to be $(e_1, e_2; e_1, e_2)$. This is then in term of quarks some fundamental basis like (u, d; c, s). Then the convenient horizontal SU(2) symmetry consists of three generators $H_\alpha \in SU_H(2)$ $(\{H_\alpha\} \subset \{U\})$ $(\alpha = 1, 2, 3)$.

$$H_1 : \begin{pmatrix} & \sigma_0 \\ \sigma_0 & \end{pmatrix} \quad , H_2 : \begin{pmatrix} & -i\sigma_0 \\ i\sigma_0 & \end{pmatrix} \quad , H_3 : \begin{pmatrix} \sigma_0 & \\ & -\sigma_0 \end{pmatrix} \tag{11}$$

and the vertical symmetry $SU_V(2)$ whose generators W_β obey the product subgroup condition: $SU_H(2) \times SU_V(2)$ consists of

$$W_\beta : \begin{pmatrix} \sigma_\beta & \\ & \sigma_\beta \end{pmatrix} \quad , \quad \beta = 1,2,3, \ (\sigma_\beta\text{: Pauli matrices}) \tag{12}$$

$$W_\beta \updownarrow \begin{pmatrix} u \\ d \end{pmatrix} \xleftrightarrow{H_\alpha} \begin{pmatrix} c \\ s \end{pmatrix} \updownarrow W_\beta \tag{13}$$

Now to find some F_γ we force initially the conditions (6) and (7) on the general set U :

$$[H_\alpha, U] = 0 \tag{14}$$

$$[W_\beta, U] = 0 \tag{15}$$

and then of course we need the condition (8). Then we have

$$[H_1, U] = 0 \rightarrow \begin{cases} K = j & c = b \\ \ell = i & d = a \end{cases} \rightarrow \left(\{U\} \rightarrow \{U'\} \right) \tag{16}$$

$$[H_2, U'] = 0 \rightarrow b = c = 0 \rightarrow \left(\{U'\} \rightarrow \{U''\} \right) \tag{17}$$

$$[H_3, U''] = 0 \qquad \text{no other restriction.} \tag{18}$$

That is, $\{U\}$ after condition (6) must have the form:

$$\{U\} : a \begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix} \tag{19}$$

It is now obvious that the second conditon

$$(2) \quad [W_\beta, U] = 0, \tag{20}$$

cannot be satisfied by such $\{U\}$, because apart from i = 0 (the identity operator) all of the other ones are some multiples of the same W_β operators. This means that one cannot find even one such operator (in realistic examples generally one needs more than one!) [10] which can satisfy even the conditions (6) and (7) together. (Where we have three conditions (6) - (8).) In other words, one cannot break SU(4) into $SU(2)_H \times \underset{S-W}{SU(2)}$ or conversely embed such a product subgroup into SU(4).

The same impossibility can be shown for N = 6, n = m = 2. The general form for 6 x 6 matrices consisting of hermitean and unitary ones is

$$\{U\} : \begin{pmatrix} \sigma_i' & \sigma_j' & \sigma_K' \\ \sigma_\ell' & \sigma_m' & \sigma_n' \\ \sigma_x' & \sigma_y' & \sigma_z' \end{pmatrix}, \{i, j, K, \ell, m, n, x, y, z\} : \{0, \cdots, 3\} \tag{21}$$

$\{\sigma'\}$ are any multiples of one of $\sigma_o, \sigma_1, \sigma_2, \sigma_3$.

All generators of U(6) can be constructed from some linear combinations of $\{U\}$. The electric charge basis to distinguish between horizontal and vertical symmetries is $(e_1, e_2; e_1, e_2; e_1, e_2)$ or (u, d; c, s; t, b) and this shows that the $H_\alpha \in SU_H(2)$ are of the form:

$$H_1 : \begin{pmatrix} & \sigma_0 & \\ \sigma_0 & & \\ & & \end{pmatrix} , \quad H_2 : \begin{pmatrix} & & \\ & & \sigma_0 \\ & \sigma_0 & \end{pmatrix} , \quad H_3 : \begin{pmatrix} & & -i\sigma_0 \\ & & \\ i\sigma_0 & & \end{pmatrix} \qquad (22)$$

$$\begin{pmatrix} u \\ d \end{pmatrix} \xleftrightarrow{H_1} \begin{pmatrix} c \\ s \end{pmatrix} , \quad \begin{pmatrix} c \\ s \end{pmatrix} \xleftrightarrow{H_2} \begin{pmatrix} t \\ b \end{pmatrix} , \quad \begin{pmatrix} u \\ d \end{pmatrix} \xleftrightarrow{H_3} \begin{pmatrix} t \\ b \end{pmatrix} \qquad (23)$$

Then the $W_\beta \in SU_V(2)$ which must obey the $SU_H(2) \times SU_V(2)$ conditions are

$$W_\beta : \begin{pmatrix} \sigma_\beta & & \\ & \sigma_\beta & \\ & & \sigma_\beta \end{pmatrix} , \quad \beta = 1, 2, 3 \qquad (24)$$

The first two commutation conditions which must be enforced on the $\{U\}$ to find those generators F_γ, which are necessary for the desired symmetry breaking are [11] :

$$[H_\alpha, U] = 0 \qquad (25)$$

$$[W_\beta, U] = 0 \qquad (26)$$

(1) implies that.

$$[H_1, U] = 0 \rightarrow \left\{ \begin{matrix} \ell = j, \ m = i \\ \sigma_x' = \sigma_y' = \sigma_k' = \sigma_n' = 0 \end{matrix} \right\} \rightarrow (\{U\} \rightarrow \{U'\}) \tag{27}$$

$$[H_2, U'] = 0 \rightarrow (z = i, \ \sigma_j' = 0) \rightarrow (\{U'\} \rightarrow \{U''\}) \tag{28}$$

$$[H_3, U''] = 0 \qquad \text{no other restriction} \tag{29}$$

In other words $\{U''\}$ are some multiples of the $\{W_\beta\}$ [12].
This shows that because there is no operator (we need generally more than one) which can satisfy even two ((1) and (2)) of our three conditions, one cannot break SU(6) into $SU(2)_H \times SU_V(2)$ or, conversely embed such a subgroup into SU(6).

The same method can be employed to demonstrate that also the SU(8) case is not possible,which we have done explicitly but do not show in detail here for sake of brevity.

IV - Conclusions

It seems that in a formulation of the weak electromagnetic interactions of fermions, where the quarks and leptons are labeled independently in the fundamental basis (representation) of a SU(N) local gauge symmetry group, it is not possible to embed a continous horizontal (interfamily) symmetry and the vertical symmetry together into a higher simple symmetry group : SU(N).

This fact suggest that the simplest possibility for a grand unified symmetry [5] which contains the horizontal one and explains the generation problem, is a semi-simple symmetry group like SU(M)XSU(N) which can break in such a way that in the first step one has

$$SU(M) \times SU(N) \longrightarrow SU_H(n) \times SU(N)$$

with

$$SU(N) \supset SU_C(3) \times SU(2)_V \times U(1)$$

and in the next steps

$$SU_H(n) \times SU(N) \longrightarrow SU_H(n) \times SU_C(3) \times SU(2)_V \times U(1)$$

$$\longrightarrow SU_C(3) \times SU(2)_V \times U(1)$$

$$\longrightarrow SU_C(3) \times U(1)_{e.m.}$$

Generally our result and the statement summarized in the last paragraph is in accord with the conclusion reached by A. Davidson [13] that only semi-simple groups like SU(5) X SU(5) [14] are suitable for the generations containing grand unified theories, although these authors had other reasons to reject a simple symmetry group like SU(N) as a unifying group incorporating several families.

Acknowledgements: I am grateful to Prof. G. Kramer and Prof. D. McKay for helpful discussions and for reading the manuscript. Furthermore I would like to thank DAAD for a fellowship.

References and footnotes

[1] The muon, electron mass difference or generations of quarks and leptons.

[2] In the sense of spontaneously broken symmetry.

[3] For example, S. Barr and A. Zee, Phys. Rev. D17 (1978), 1854; F. Wilczek and A. Zee, Phys. Rev. Lett. 42 (1979), 421; T. Machara and T. Yanagida, Prog. Theor. Phys. 61 (1979), 1434; A. Davidson, M. Koca and K.C. Wali, Phys. Rev. Lett. 43 (1979), 92.

[4] V:vertical: S-W: Salam-Weinberg; normally n = 2 or 3.

[5] Georgi-Glashow and related theories; H. Georgi and S.L. Glashow, Phys. Rev. Lett. 32 (1974), 438.

[6] N.G. Deshpande et al., Phys. Rev. D19 (1979), 2686,

[7] For example, L.F.Li, Phys. Rev. D9 (1974), 1723; F. Buccella et al., Geneva Preprint UGVA-DPT 1979/11-221; H. Ruegg, SLAC-PUB-2518, May 1980 (T/E)

[8] N.G. Deshpande et al., Phys. Rev. D19 (1979), 2686,

[9] n: neutral in the sense of electric charge transformations;

[10] For example see Ref. [8], where the gauge boson mass pattern in SU(4) x U(1) is discussed.

[11] Obviously if these are obeyed then the condition (3) must be obeyed by the same operators.

[12] And of course W_o as the 6 x 6 intentity operator.

[13] A. Davidson, Phys. Lett. 90B (1980), 87

[14] A. Davidson and K.C. Wali, SU-4217-166 (1980)