# Connectivity and tree structure in finite graphs

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**Abstract**

Considering systems of separations in a graph that separate every pair of a given set of vertex sets that are themselves not separated by these separations, we determine conditions under which such a separation system contains a nested subsystem that still separates those sets and is invariant under the automorphisms of the graph.

As an application, we show that the $k$-blocks – the maximal vertex sets that cannot be separated by at most $k$ vertices – of a $k$-connected graph $G$ live in distinct parts of a suitable tree-decomposition of $G$ of adhesion $k$, whose decomposition tree is invariant under the automorphisms of $G$. This extends a recent result of Dunwoody and Krön and, like theirs, generalizes a similar theorem of Tutte for $k = 2$.

Under mild additional assumptions, which are necessary, our decompositions can be combined into one overall tree-decomposition that distinguishes, for all $k$ simultaneously, all the $k$-blocks of a finite graph.

## 1 Introduction

Ever since graph connectivity began to be systematically studied, from about 1960 onwards, it has been an equally attractive and elusive quest to 'decompose a $k$-connected graph into its $(k+1)$-connected components'. The idea was modelled on the well-known block-cutvertex tree, which for $k = 1$ displays the global structure of a connected graph 'up to 2-connectedness'. For general $k$, the precise meaning of what those '$(k+1)$-connected components' should be varied, and came to be considered as part of the problem. But the aim was clear: it should allow for a decomposition of the graph into those 'components', so that their relative structure would display some more global structure of the graph.

While originally, perhaps, these 'components' were thought of as subgraphs, it soon became clear that, for larger $k$, they would have to be defined differently. For $k = 2$, Tutte [8] found a decomposition which, in modern terms,[1] would

[1]Readers not acquainted with the terminology of graph minor theory can skip the details of this example without loss. The main point is that those 'torsos' are not subgraphs, but subgraphs plus some additional edges reflecting the additional connectivity that the rest of the graph provides for their vertices.

be described as a tree-decomposition of adhesion 2 whose torsos are either 3-connected or cycles.

For general $k$, Robertson and Seymour [7] re-interpreted those '$(k+1)$-connected components' in a radically new (but interesting) way as 'tangles of order $k$'. They showed, as a cornerstone of their theory on graph minors, that every finite graph admits a tree-decomposition that separates all its maximal tangles, regardless of their order, in that they inhabit different parts of the decomposition. Note that this solves the modified problem for all $k$ simultaneously, a feature we shall achieve also for the original problem.

More recently still, Dunwoody and Krön [3], taking their lead directly from Tutte (and from Dunwoody's earlier work on tree-structure induced by edge-cuts [4]), followed up Tutte's observation that his result for $k=2$ can alternatively be described as a tree-like decomposition of a graph $G$ into vertex sets that are '2-inseparable': such that no set of at most 2 vertices can separate any two vertices of that set in $G$. Note that such '$k$-inseparable' sets of vertices differ markedly from $k$-connected subgraphs, in that their connectivity resides not on the set itself but in the ambient graph. For example, joining $r > k$ isolated vertices pairwise by $k+1$ independent paths of length 2, all disjoint, makes this set into a '$k$-block', a maximal $k$-inseparable set of vertices. This then plays an important structural (hub-like) role for the connectivity of the graph, but it is still independent.

External connectivity of a set of vertices in the ambient graph had been considered before in the context of tree-decompositions and tangles [2, 6]. But it was Dunwoody and Krön who realized that $k$-inseparability can serve to extend Tutte's result to $k > 2$: they showed that the $k$-blocks of a finite $k$-connected graph can, in principle, be separated canonically in a tree-like way [3]. We shall re-prove this in a stronger form,[2] and cast the 'tree-like way' in the standard form of tree-decompositions: an application of our results says that every $k$-connected graph has a canonical tree-decomposition of adhesion $k$ such that distinct $k$-blocks are contained in different parts. For graphs $G$ whose $k$-blocks have size at least $3k/2$, this can even be done for all $k$ simultaneously, in one overall tree-decomposition. This will be another application of our results.

Our paper is independent of the results stated in [3].[3] Our approach will be as follows. We first develop a more general theory of separation systems to deal with the following abstract problem. Let $\mathcal{S}$ be a set of separations in a graph, and let $\mathcal{I}$ be a collection of $\mathcal{S}$-*inseparable* sets of vertices, sets which, for every separation $(A, B) \in \mathcal{S}$, lie entirely in $A$ or entirely in $B$. Under what condition does $\mathcal{S}$ have a *nested* subsystem $\mathcal{N}$ that still separates all the sets in $\mathcal{I}$? In a further step we show how such nested separation systems $\mathcal{N}$ can be captured by tree-decompositions.[4]

[2] See Section 6 for a brief discussion of the differences.

[3] The starting point for this paper was that, despite some effort, we were unable to verify some of the results claimed in [3]. In some cases, but not all, this was mended in later versions.

[4] It is easy to see tree-decompositions give rise to nested separation systems. The converse is less clear.

Our construction of $\mathcal{N}$ and its associated tree-decomposition will be canonical in that they depend only on $\mathcal{S}$ and $\mathcal{I}$. In particular, if these are invariant under the automorphisms of the graph, then so will be $\mathcal{N}$, and the automorphism group of the graph will act also on the associated decomposition tree.

The gain from having an abstract theory of how to extract nested subsystems from a given separation system is its flexibility. For example, we shall use it in [5] to prove that every finite graph has a canonical (in the sense above) tree-decomposition separating all its maximal tangles. This improves on the result of Robertson and Seymour [7] mentioned earlier, in that their decomposition is not canonical in our sense: it depends on an assumed vertex enumeration to break ties when choosing which of two crossing separations should be picked for the nested subsystem. The choices made by our decomposition will depend only on the structure of the graph. In particular, it will be invariant under its automorphisms.

To state our main results precisely, let us define their terms more formally. In addition to the terminology explained in [1] we say that a set $X$ of vertices in a graph $G$ is *$k$-inseparable* in $G$ if $|X| > k$ and no set $S$ of at most $k$ vertices separates two vertices of $X \setminus S$ in $G$. A maximal $k$-inseparable set of vertices is a *$k$-block*,[5] or simply a *block*. The smallest $k$ for which a block is a $k$-block is the *rank* of that block; the largest such $k$ is its *order*. The *adhesion* of a tree-decomposition $(\mathcal{T}, \mathcal{V})$ is the maximum size of the overlaps $V_t \cap V_{t'}$ of adjacent parts, i.e. for edges $tt'$ of $\mathcal{T}$.

**Theorem 1.** *Given any integer $k \geq 0$, every $k$-connected finite graph $G$ has a tree-decomposition $(\mathcal{T}, \mathcal{V})$ such that*

(i) *every $k$-inseparable set of vertices is contained in a unique part of $(\mathcal{T}, \mathcal{V})$, and distinct $k$-blocks are contained in different parts;*

(ii) *$(\mathcal{T}, \mathcal{V})$ has adhesion $k$;*

(iii) $\mathrm{Aut}(G)$ *acts on $\mathcal{T}$ as a group of automorphisms.*

In Section 6 we shall prove this theorem in a slightly stronger form.

If all the blocks of a graph $G$ (of any connectivity) are *robust*, a weak[6] (and necessary) additional constraint we define in Section 6, we shall be able to find a tree-decomposition of $G$ that distinguishes not only all its $\kappa(G)$-blocks pairwise (as does Theorem 1), but every non-nested pair of blocks, regardless of their rank. Some blocks (especially those of large order) will reside in a single part of this decomposition, while others (of smaller order) will inhabit a subtree consisting of several parts. For any fixed $k$ however we can contract those subtrees to single nodes, to reobtain the tree-decomposition from Theorem 1 in

[5] Belonging to the same $k$-block is not an equivalence relation on $V(G)$, but almost: distinct $k$-blocks can be separated by $k$ or fewer vertices. A long cycle has exactly one $k$-block for $k \in \{0, 1\}$ and no $k$-block for $k \geq 2$. A large grid has a unique $k$-block for $k \in \{0, 1\}$, five 2-blocks (each of the corner vertices with its neighbours, plus the set of non-corner vertices), and one 3-block (the set of its inner vertices). It has no $k$-block for $k \geq 4$.

[6] For example, all $k$-blocks that are complete or have at least $3k/2$ vertices will be 'robust'.

which the $k$-blocks (for this fixed $k$) inhabit distinct single parts. As $k$ grows, we thus have a sequence $(\mathcal{T}_k, \mathcal{V}_k)_{k\in\mathbb{N}}$ of tree-decompositions, each refining the previous, that gives rise to our overall tree-decomposition in the last step of the sequence.

Formally, let us write $(\mathcal{T}_m, \mathcal{V}_m) \preccurlyeq (\mathcal{T}_n, \mathcal{V}_n)$ for tree-decompositions $(\mathcal{T}_m, \mathcal{V}_m)$ and $(\mathcal{T}_n, \mathcal{V}_n)$ if the decomposition tree $\mathcal{T}_m$ of the first is a minor of the decomposition tree $\mathcal{T}_n$ of the second, and a part $V_t \in \mathcal{V}_m$ of the first decomposition is the union of those parts $V_{t'}$ of the second whose nodes $t'$ were contracted to the node $t$ of $\mathcal{T}_m$.

**Theorem 2.** *For every finite graph $G$ there is a sequence $(\mathcal{T}_k, \mathcal{V}_k)_{k\in\mathbb{N}}$ of tree-decompositions such that, for all $k$,*

(i) *every $k$-inseparable set is contained in a unique part of $(\mathcal{T}_k, \mathcal{V}_k)$, and distinct robust $k$-blocks are contained in different parts;*

(ii) *$(\mathcal{T}_k, \mathcal{V}_k)$ has adhesion at most $k$;*

(iii) *$(\mathcal{T}_k, \mathcal{V}_k) \preccurlyeq (\mathcal{T}_{k+1}, \mathcal{V}_{k+1})$;*

(iv) $\mathrm{Aut}(G)$ *acts on $\mathcal{T}_k$ as a group of automorphisms.*

In Section 6 we shall prove this theorem in a slightly stronger form.

This paper is organized as follows. In Section 2 we collect together some properties of pairs of separations, either crossing or nested. In Section 3 we define a structure tree $\mathcal{T}$ associated canonically with a nested set of separations of a graph $G$. In Section 4 we construct a tree-decomposition of $G$ modelled on $\mathcal{T}$, and study its parts. In Section 5 we find conditions under which, given a set $\mathcal{S}$ of separations and a collection $\mathcal{I}$ of $\mathcal{S}$-inseparable set of vertices, there is a nested subsystem of $\mathcal{S}$ that still separates all the sets in $\mathcal{I}$. In Section 6, finally, we apply all this to the special case of $k$-separations and $k$-blocks and derive our main results, including slightly stronger versions of Theorems 1 and 2.

# 2 Separations

Let $G = (V, E)$ be a finite graph. A *separation* of $G$ is an ordered pair $(A, B)$ such that $A, B \subsetneq V$ and $G[A] \cup G[B] = G$. The *order* of a separation $(A, B)$ is the cardinality of its *separator* $A \cap B$; the sets $A, B$ are its *sides*. A separation of order $k$ is a *$k$-separation*.

A separation $(A, B)$ *separates* a set $I \subseteq V$ if $I$ meets both $A \setminus B$ and $B \setminus A$. Two sets $I_0, I_1$ are *weakly separated* by a separation $(A, B)$ if $I_i \subseteq A$ and $I_{1-i} \subseteq B$ for an $i \in \{0, 1\}$. They are *properly separated*, or simply *separated*, by $(A, B)$ if in addition neither $I_0$ nor $I_1$ is contained in $A \cap B$.

Given a set $\mathcal{S}$ of separations, we call a set of vertices $\mathcal{S}$-*inseparable* if no separation in $\mathcal{S}$ separates it. A maximal $\mathcal{S}$-inseparable set of vertices is an $\mathcal{S}$-*block*, or simply a *block* if $\mathcal{S}$ is fixed in the context.

**Lemma 2.1.** *Distinct $\mathcal{S}$-blocks $b_1, b_2$ are separated by some $(A, B) \in \mathcal{S}$.*

*Proof.* Since $b_1$ and $b_2$ are maximal $\mathcal{S}$-inseparable sets, $b := b_1 \cup b_2$ can be separated by some $(A, B) \in \mathcal{S}$. Then $b \setminus B \neq \emptyset \neq b \setminus A$, but being $\mathcal{S}$-inseparable, $b_1$ and $b_2$ are each contained in $A$ or $B$. Hence $(A, B)$ separates $b_1$ from $b_2$. $\square$

A set of vertices is *small* with respect to $\mathcal{S}$ if it is contained in the separator of some separation in $\mathcal{S}$. If $\mathcal{S}$ is given from the context, we simply call such a set *small*. Note that if two sets are weakly but not properly separated by some separation in $\mathcal{S}$ then at least one of them is small.

Let us look at how different separations of $G$ can relate to each other. The set of all separations of $G$ is partially ordered by

$$(A, B) \leq (C, D) \;:\Leftrightarrow\; A \subseteq C \text{ and } B \supseteq D. \tag{1}$$

Indeed, reflexivity, antisymmetry and transitivity follow easily from the corresponding properties of set inclusion on $\mathcal{P}(V)$. Note that changing the order in each pair reverses the relation:

$$(A, B) \leq (C, D) \;\Leftrightarrow\; (B, A) \geq (D, C). \tag{2}$$

Let $(C, D)$ be any separation.

> *No separation $(A, B)$ is $\leq$-comparable with both $(C, D)$ and $(D, C)$. In particular, $(C, D) \not\leq (D, C)$.* (3)

Indeed, if $(A, B) \leq (C, D)$ and also $(A, B) \leq (D, C)$, then $A \subseteq C \subseteq B$, a contradiction. By (2), the other cases all reduce to this case by changing notation: just swap $(A, B)$ with $(B, A)$ or $(C, D)$ or $(D, C)$.

The way in which two separations relate to each other can be illustrated by a *cross-diagram* as in Figure 1. In view of such diagrams, we introduce the following terms for any set $\{(A, B), (C, D)\}$ of two separations, not necessarily distinct. The set $A \cap B \cap C \cap D$ is their *centre*, and $A \cap C$, $A \cap D$, $B \cap C$, $B \cap D$ are their *corners*. The corners $A \cap C$ and $B \cap D$ are *opposite*, as are the corners

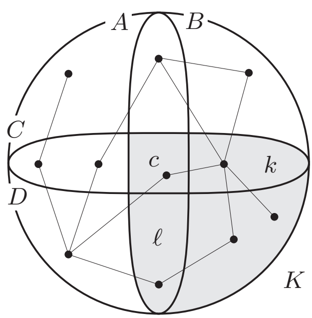


*Figure 1:* The cross-diagram $\{(A, B), (C, D)\}$ with centre $c$ and a corner $K$ and its links $k, \ell$.

$A \cap D$ and $B \cap C$. Two corners that are not opposite are *adjacent.* The *link* between two adjacent corners is their intersection minus the centre. A corner minus its links and the centre is the *interior* of that corner; the rest – its two links and the centre – are its *boundary.* We shall write $\partial K$ for the boundary of a corner $K$.

A corner whose interior is non-empty forms a separation of $G$ together with the union of the other three corners. We call these separations *corner separations.* For example, $(A \cap C, B \cup D)$ (in this order) is the corner separation for the corner $A \cap C$ in $\{(A, B), (C, D)\}$.

The four corner separations of a cross-diagram compare with the two separations forming it, and with the inverses of each other, in the obvious way:

*Any two separations* $(A, B)$, $(C, D)$ *satisfy* $(A \cap C, B \cup D) \leq (A, B)$. (4)

*If* $(I, J)$ *and* $(K, L)$ *are distinct corner separations of the same cross-diagram, then* $(I, J) \leq (L, K)$. (5)

Inspection of the cross-diagram for $(A, B)$ and $(C, D)$ shows that $(A, B) \leq (C, D)$ if and only if the corner $A \cap D$ has an empty interior and empty links, i.e., the entire corner $A \cap D$ is contained in the centre:

$$(A, B) \leq (C, D) \;\Leftrightarrow\; A \cap D \subseteq B \cap C. \tag{6}$$

Another consequence of $(A, B) \leq (C, D)$ is that $A \cap B \subseteq C$ and $C \cap D \subseteq B$. So both separators live entirely on one side of the other separation.

A separation $(A, B)$ is *tight* if every vertex of $A \cap B$ has a neighbour in $A \setminus B$ and another neighbour in $B \setminus A$. For tight separations, one can establish that $(A, B) \leq (C, D)$ by checking only one of the two inclusions in (1):

*If* $(A, B)$ *and* $(C, D)$ *are separations such that* $A \subseteq C$ *and* $(C, D)$ *is tight, then* $(A, B) \leq (C, D)$. (7)

Indeed, suppose $D \not\subseteq B$. Then as $A \subseteq C$, there is a vertex $x \in (C \cap D) \setminus B$. As $(C, D)$ is tight, $x$ has a neighbour $y \in D \setminus C$, but since $x \in A \setminus B$ we see that $y \in A$. So $A \setminus C \neq \emptyset$, contradicting our assumption.

Let us call $(A, B)$ and $(C, D)$ *nested*, and write $(A, B) \parallel (C, D)$, if $(A, B)$ is comparable with $(C, D)$ or with $(D, C)$ under $\leq$. (By (3), it cannot be comparable with both.) By (2), this is a symmetrical relation. For example, we saw in (4) and (5) that the corner separations of a cross-diagram are nested with the two separations forming it, as well as with each other.

Separations $(A, B)$ and $(C, D)$ that are not nested are said to *cross*; we then write $(A, B) \nparallel (C, D)$.

Nestedness is invariant under 'flipping' a separation: if $(A, B) \parallel (C, D)$ then also $(A, B) \parallel (D, C)$, by definition of $\parallel$, but also $(B, A) \parallel (C, D)$ by (2). Thus although nestedness is defined on the separations of $G$, we may think of it as a symmetrical relation on the unordered pairs $\{A, B\}$ such that $(A, B)$ is a separation.

By (6), nested separations have a simple description in terms of cross-diagrams:

*Two separations are nested if and only if one of their four corners has an empty interior and empty links.* (8)

In particular:

*Neither of two nested separations separates the separator of the other.* (9)

The converse of (9) fails only if there is a corner with a non-empty interior whose links are both empty.

Although nestedness is reflexive and symmetric, it is not in general transitive. However when transitivity fails, we can still say something:

**Lemma 2.2.** *If $(A, B) \parallel (C, D)$ and $(C, D) \parallel (E, F)$ but $(A, B) \nparallel (E, F)$, then $(C, D)$ is nested with every corner separation of $\{(A, B), (E, F)\}$, and for one corner separation $(I, J)$ we have either $(C, D) \leq (I, J)$ or $(D, C) \leq (I, J)$.*

*Proof.* Changing notation as necessary, we may assume that $(A, B) \leq (C, D)$, and that $(C, D)$ is comparable with $(E, F)$.[7] If $(C, D) \leq (E, F)$ we have $(A, B) \leq (E, F)$, contrary to our assumption. Hence $(C, D) \geq (E, F)$, or equivalently by (2), $(D, C) \leq (F, E)$. As also $(D, C) \leq (B, A)$, we thus have $D \subseteq F \cap B$ and $V(G) \supsetneq C \supseteq E \cup A$. Hence $(F \cap B, E \cup A)$ is a separation, and

$$(D, C) \leq (F \cap B, E \cup A) \underset{(5)}{\leq} (L, K)$$

for each of the other three corner separations $(K, L)$ of $\{(A, B), (E, F)\}$. □

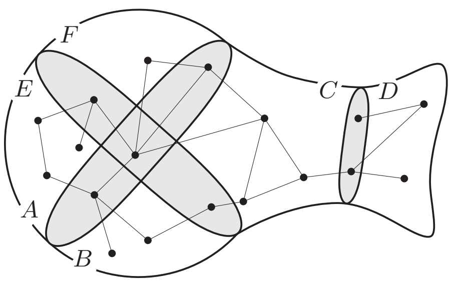


*Figure 2:* Separations as in Lemma 2.2

Figure 2 shows an example of three separations witnessing the non-transitivity of nestedness. Note also that there are two ways in which three separations can be pairwise nested. One is that they or their inverses form a chain under $\leq$. But there is also another way, which will be important later; this is illustrated in Figure 3.

[7] Note that such change of notation will not affect the set of corner separations of the cross-diagram of $(A, B)$ and $(E, F)$, nor the nestedness (or not) of $(C, D)$ with those corner separations.

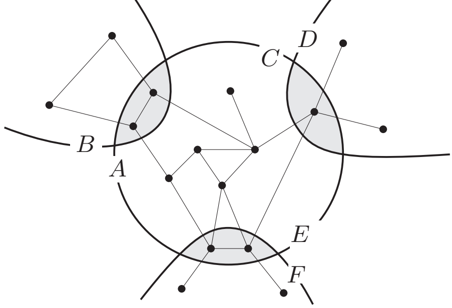


*Figure 3:* Three nested separations not coming from a $\leq$-chain

We need one more lemma.

**Lemma 2.3.** *Let $\mathcal{N}$ be a set of separations of $G$ that are pairwise nested. Let $(A, B)$ and $(C, D)$ be two further separations, each nested with all the separations in $\mathcal{N}$. Assume that $(A, B)$ separates an $\mathcal{N}$-block $b$, and that $(C, D)$ separates an $\mathcal{N}$-block $b' \neq b$. Then $(A, B) \parallel (C, D)$. Moreover, $A \cap B \subseteq b$ and $C \cap D \subseteq b'$.*

*Proof.* By Lemma 2.1, there is a separation $(E, F) \in \mathcal{N}$ with $b \subseteq E$ and $b' \subseteq F$. Suppose $(A, B) \nparallel (C, D)$. By Lemma 2.2 we may assume that

$$(E, F) \leq (A \cap C, B \cup D).$$

But then $b \subseteq E \subseteq A \cap C \subseteq A$, contradicting the fact that $(A, B)$ separates $b$. Hence $(A, B) \parallel (C, D)$, as claimed.

If $A \cap B \nsubseteq b$, then there is a $(K, L) \in \mathcal{N}$ which separates $b \cup (A \cap B)$. We may assume that $b \subseteq L$ and that $A \cap B \nsubseteq L$. The latter implies that $(K, L) \nleq (A, B)$ and $(K, L) \nleq (B, A)$. So $(K, L) \parallel (A, B)$ implies that either $(L, K) \leq (A, B)$ or $(L, K) \leq (B, A)$. Thus $b \subseteq L \subseteq A$ or $b \subseteq L \subseteq B$, a contradiction to the fact that $(A, B)$ separates $b$. Similarly we obtain $C \cap D \subseteq b'$. $\square$

# 3 Nested separation systems and tree structure

A set $\mathcal{S}$ of separations is *symmetric* if $(A, B) \in \mathcal{S}$ implies $(B, A) \in \mathcal{S}$, and *nested* if every two separations in $\mathcal{S}$ are nested. Any symmetric set of separations is a *separation system*. Throughout this section and the next, we consider a fixed nested separation system $\mathcal{N}$ of our graph $G$.

Our aim in this section will be to describe $\mathcal{N}$ by way of a *structure tree* $\mathcal{T} = \mathcal{T}(\mathcal{N})$, whose edges will correspond to the separations in $\mathcal{N}$. Its nodes[8] will correspond to subgraphs of $G$. Every automorphism of $G$ that leaves $\mathcal{N}$ invariant will also act on $\mathcal{T}$. Although our notion of a separation system differs from that of Dunwoody and Krön [3, 4], the main ideas of how to describe a nested system by a structure tree can already be found there.

[8] While our graphs $G$ have *vertices*, structure trees will have *nodes*.

Our main task in the construction of $\mathcal{T}$ will be to define its nodes. They will be the equivalence classes of the following equivalence relation $\sim$ on $\mathcal{N}$, induced by the ordering $\leq$ from (1):

$$(A,B) \sim (C,D) \;:\Leftrightarrow \begin{cases} (A,B) = (C,D) \text{ or} \\ (B,A) \text{ is a predecessor of } (C,D) \text{ in } (\mathcal{N}, \leq). \end{cases} \tag{10}$$

(Recall that, in a partial order $(P, \leq)$, an element $x \in P$ is a *predecessor* of an element $z \in P$ if $x < z$ but there is no $y \in P$ with $x < y < z$.)

Before we prove that this is indeed an equivalence relation, it may help to look at an example: the set of vertices in the centre of Figure 3 will be the node of $\mathcal{T}$ represented by each of the equivalent nested separations $(A,B)$, $(C,D)$ and $(E,F)$.

**Lemma 3.1.** *The relation $\sim$ is an equivalence relation on $\mathcal{N}$.*

*Proof.* Reflexivity holds by definition, and symmetry follows from (2). To show transitivity assume that $(A,B) \sim (C,D)$ and $(C,D) \sim (E,F)$, and that all these separations are distinct. Thus,

(i) $(B,A)$ is a predecessor of $(C,D)$;

(ii) $(D,C)$ is a predecessor of $(E,F)$.

And by (2) also

(iii) $(D,C)$ is a predecessor of $(A,B)$;

(iv) $(F,E)$ is a predecessor of $(C,D)$.

By (ii) and (iii), $(A,B)$ is incomparable with $(E,F)$. Hence, since $\mathcal{N}$ is nested, $(B,A)$ is comparable with $(E,F)$. If $(E,F) \leq (B,A)$ then by (i) and (ii), $(D,C) \leq (C,D)$, which contradicts (3). Thus $(B,A) < (E,F)$, as desired.

Suppose there is a separation $(X,Y) \in \mathcal{N}$ with $(B,A) < (X,Y) < (E,F)$. As $\mathcal{N}$ is nested, $(X,Y)$ is comparable with either $(C,D)$ or $(D,C)$. By (i) and (ii), $(X,Y) \not< (C,D)$ and $(D,C) \not< (X,Y)$. Now if $(C,D) \leq (X,Y) < (E,F)$ then by (iv), $(C,D)$ is comparable to both $(E,F)$ and $(F,E)$, contradicting (3). Finally, if $(D,C) \geq (X,Y) > (B,A)$, then by (iii), $(D,C)$ is comparable to both $(B,A)$ and $(A,B)$, again contradicting (3). We have thus shown that $(B,A)$ is a predecessor of $(E,F)$, implying that $(A,B) \sim (E,F)$ as claimed. □

Note that, by (3), the definition of equivalence implies:

**Lemma 3.2.** *Distinct equivalent separations are incomparable under $\leq$.* □

We can now define the nodes of $\mathcal{T} = \mathcal{T}(\mathcal{N})$ as planned, as the equivalence classes of $\sim$:

$$V(\mathcal{T}) := \{[(A,B)] : (A,B) \in \mathcal{N}\}.$$

Having defined the nodes of $\mathcal{T}$, let us define its edges. For every separation $(A,B) \in \mathcal{N}$ we shall have one edge, joining the nodes represented by $(A,B)$

and $(B, A)$, respectively. To facilitate notation later, we formally give $\mathcal{T}$ the abstract edge set

$$E(\mathcal{T}) := \{\{(A, B), (B, A)\} \mid (A, B) \in \mathcal{N}\}$$

and declare an edge $e$ to be incident with a node $t \in V(\mathcal{T})$ whenever $e \cap t \neq \emptyset$ (so that the edge $\{(A, B), (B, A)\}$ of $\mathcal{T}$ joins its nodes $[(A, B)]$ and $[(B, A)]$). We have thus, so far, defined a multigraph $\mathcal{T}$.

As $(A, B) \not\sim (B, A)$ by definition of $\sim$, our multigraph $\mathcal{T}$ has no loops. Whenever an edge $e$ is incident with a node $t$, the non-empty set $e \cap t$ that witnesses this is a singleton set containing one separation. We denote this separation by $(e \cap t)$. Every separation $(A, B) \in \mathcal{N}$ occurs as such an $(e \cap t)$, with $t = [(A, B)]$ and $e = \{(A, B), (B, A)\}$. Thus,

*Every node $t$ of $\mathcal{T}$ is the set of all the separations $(e \cap t)$ such that $e$ is incident with $t$. In particular, $t$ has degree $|t|$ in $\mathcal{T}$.* (11)

Our next aim is to show that $\mathcal{T}$ is a tree.

**Lemma 3.3.** *Let $W = t_1 e_1 t_2 e_2 t_3$ be a walk in $\mathcal{T}$ with $e_1 \neq e_2$. Then $(e_1 \cap t_1)$ is a predecessor of $(e_2 \cap t_2)$.*

*Proof.* Let $(e_1 \cap t_1) = (A, B)$ and $(e_2 \cap t_2) = (C, D)$. Then $(B, A) = (e_1 \cap t_2)$ and $(B, A) \sim (C, D)$. Since $e_1 \neq e_2$ we have $(B, A) \neq (C, D)$. Thus, $(A, B)$ is a predecessor of $(C, D)$ by definition of $\sim$. □

And conversely:

**Lemma 3.4.** *Let $(E_0, F_0), \ldots, (E_k, F_k)$ be separations in $\mathcal{N}$ such that each $(E_{i-1}, F_{i-1})$ is a predecessor of $(E_i, F_i)$ in $(\mathcal{N}, \leq)$. Then $[(E_0, F_0)], \ldots, [(E_k, F_k)]$ are the nodes of a walk in $\mathcal{T}$, in this order.*

*Proof.* By definition of $\sim$, we know that $(F_{i-1}, E_{i-1}) \sim (E_i, F_i)$. Hence for all $i = 1, \ldots, k$, the edge $\{(E_{i-1}, F_{i-1}), (F_{i-1}, E_{i-1})\}$ of $\mathcal{T}$ joins the node $[(E_{i-1}, F_{i-1})]$ to the node $[(E_i, F_i)] = [(F_{i-1}, E_{i-1})]$. □

**Theorem 3.5.** *The multigraph $\mathcal{T}(\mathcal{N})$ is a tree.*

*Proof.* We have seen that $\mathcal{T}$ is loopless. Suppose that $\mathcal{T}$ contains a cycle $t_1 e_1 \cdots t_{k-1} e_{k-1} t_k$, with $t_1 = t_k$ and $k > 2$. Applying Lemma 3.3 $(k-1)$ times yields

$$(A, B) := (e_1 \cap t_1) < \ldots < (e_{k-1} \cap t_{k-1}) < (e_1 \cap t_k) = (A, B),$$

a contradiction. Thus, $\mathcal{T}$ is acyclic; in particular, it has no parallel edges.

It remains to show that $\mathcal{T}$ contains a path between any two given nodes $[(A, B)]$ and $[(C, D)]$. As $\mathcal{N}$ is nested, we know that $(A, B)$ is comparable with either $(C, D)$ or $(D, C)$. Since $[(C, D)]$ and $[(D, C)]$ are adjacent, it suffices to construct a walk between $[(A, B)]$ and one of them. Swapping the names for $C$ and $D$ if necessary, we may thus assume that $(A, B)$ is comparable with $(C, D)$.

Reversing the direction of our walk if necessary, we may further assume that $(A, B) < (C, D)$. Since our graph $G$ is finite, there is a chain

$$(A, B) = (E_0, F_0) < \cdots < (E_k, F_k) = (C, D)$$

such that $(E_{i-1}, F_{i-1})$ is a predecessor of $(E_i, F_i)$, for every $i = 1, \ldots, k$. By Lemma 3.4, $\mathcal{T}$ contains the desired path from $[(A, B)]$ to $[(C, D)]$. □

**Corollary 3.6.** *If $\mathcal{N}$ is invariant under a group $\Gamma \leq \mathrm{Aut}(G)$ of automorphisms of $G$, then $\Gamma$ also acts on $\mathcal{T}$ as a group of automorphisms.*

*Proof.* Any automorphism $\alpha$ of $G$ maps separations to separations, and preserves their partial ordering defined in (1). If both $\alpha$ and $\alpha^{-1}$ map separations from $\mathcal{N}$ to separations in $\mathcal{N}$, then $\alpha$ also preserves the equivalence of separations under $\sim$. Hence $\Gamma$, as stated, acts on the nodes of $\mathcal{T}$ and preserves their adjacencies and non-adjacencies. □

# 4 From structure trees to tree-decompositions

Throughout this section, $\mathcal{N}$ continues to be an arbitrary nested separation system of our graph $G$. Our aim now is to show that $G$ has a tree-decomposition, in the sense of Robertson and Seymour, with the structure tree $\mathcal{T} = \mathcal{T}(\mathcal{N})$ defined in Section 3 as its decomposition tree. The separations of $G$ associated with the edges of this decomposition tree[9] will be precisely the separations in $\mathcal{N}$ identified by those edges in the original definition of $\mathcal{T}$.

Recall that a *tree-decomposition* of $G$ is a pair $(T, \mathcal{V})$ of a tree $T$ and a family $\mathcal{V} = (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V(G)$, one for every node of $T$, such that:

(T1) $V(G) = \bigcup_{t \in T} V_t$;

(T2) for every edge $e \in G$ there exists a $t \in T$ such that both ends of $e$ lie in $V_t$;

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_2$ lies on the $t_1$–$t_3$ path in $T$.

To define our desired tree-decomposition $(\mathcal{T}, \mathcal{V})$, we thus have to define the family $\mathcal{V} = (V_t)_{t \in V(\mathcal{T})}$ of its parts: with every node $t$ of $\mathcal{T}$ we have to associate a set $V_t$ of vertices of $G$. We define these as follows:

$$V_t := \bigcap \{\, A \mid (A, B) \in t \,\} \tag{12}$$

**Example 1.** Assume that $G$ is connected, and consider as $\mathcal{N}$ the nested set of all 1-separations $(A, B)$ and $(B, A)$ such that $A \setminus B$ is connected in $G$. Then $\mathcal{T}$ is very similar to the block-cutvertex tree of $G$: its nodes will be the blocks in the usual sense (maximal 2-connected subgraphs or bridges) plus those cutvertices that lie in at least three blocks.

In Figure 4, this separation system $\mathcal{N}$ contains all the 1-separations of $G$. The separation $(A, B)$ defined by the cutvertex $s$, with $A := U \cup V \cup W$ and

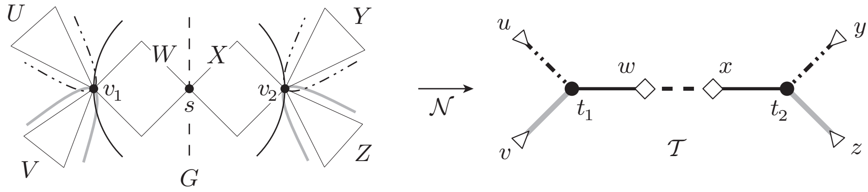


*Figure 4:* $\mathcal{T}$ has an edge for every separation in $\mathcal{N}$. Its nodes correspond to the blocks and some of the cutvertices of $G$.

$B := X \cup Y \cup Z$ say, defines the edge $\{(A, B), (B, A)\}$ of $\mathcal{T}$ joining its nodes $w = [(A, B)]$ and $x = [(B, A)]$.

In Figure 5 we can add to $\mathcal{N}$ one of the two crossing 1-separations not in $\mathcal{N}$ (together with its inverse), to obtain a set $\mathcal{N}'$ of separations that is still nested. For example, let

$$\mathcal{N}' := \mathcal{N} \cup \{(A, B), (B, A)\}$$

with $A := X_1 \cup X_2$ and $B := X_3 \cup X_4$. This causes the central node $t$ of $\mathcal{T}$ to split into two nodes $a = [(A, B)]$ and $b = [(B, A)]$ joined by the new edge $\{(A, B), (B, A)\}$. However the new nodes $a, b$ still define the same part of the tree-decomposition of $G$ as $t$ did before: $V_a = V_b = V_t = \{v\}$.

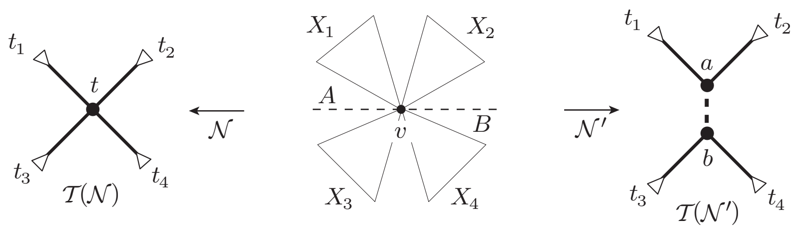


*Figure 5:* $\mathcal{T}' = \mathcal{T}(\mathcal{N}')$ has distinct nodes $a, b$ whose parts in the tree-decomposition $(\mathcal{T}', \mathcal{V})$ coincide: $V_a = \{v\} = V_b$.

Before we prove that $(\mathcal{T}, \mathcal{V})$ is indeed a tree-decomposition, let us collect some information about its parts $V_t$, the vertex sets defined in (12).

**Lemma 4.1.** *Every $V_t$ is $\mathcal{N}$-inseparable.*

*Proof.* Let us show that a given separation $(C, D) \in \mathcal{N}$ does not separate $V_t$. Pick $(A, B) \in t$. Since $\mathcal{N}$ is nested, and swapping the names of $C$ and $D$ if necessary, we may assume that $(A, B)$ is $\leq$-comparable with $(C, D)$. If $(A, B) \leq (C, D)$ then $V_t \subseteq A \subseteq C$, so $(C, D)$ does not separate $V_t$. If $(C, D) < (A, B)$, there is a $\leq$-predecessor $(E, F)$ of $(A, B)$ with $(C, D) \leq (E, F)$. Then $(F, E) \sim (A, B)$ and hence $V_t \subseteq F \subseteq D$, so again $(C, D)$ does not separate $V_t$. $\square$

[9] as in the theory of tree-decompositions, see e.g. [1, Lemma 12.3.1]

The sets $V_t$ will come in two types: they can be

- $\mathcal{N}$-blocks (that is, maximal $\mathcal{N}$-inseparable sets of vertices), or
- 'hubs' (defined below).

Nodes $t \in \mathcal{T}$ such that $V_t$ is an $\mathcal{N}$-block are *block nodes*. A node $t \in \mathcal{T}$ such that $V_t = A \cap B$ for some $(A, B) \in t$ is a *hub node* (and $V_t$ a *hub*).

In Example 1, the $\mathcal{N}$-blocks were the (usual) blocks of $G$; the hubs were singleton sets consisting of a cutvertex. Example 2 will show that $t$ can be a hub node and a block node at the same time. Every hub is a subset of a block: by (9), hubs are $\mathcal{N}$-inseparable, so they extend to maximal $\mathcal{N}$-inseparable sets.

Hubs can contain each other properly (Example 2 below). But a hub $V_t$ cannot be properly contained in a separator $A \cap B$ of any $(A, B) \in t$. Let us prove this without assuming that $V_t$ is a hub:

**Lemma 4.2.** *Whenever $(A, B) \in t \in \mathcal{T}$, we have $A \cap B \subseteq V_t$. In particular, if $V_t \subseteq A \cap B$, then $V_t = A \cap B$ is a hub with hub node $t$.*

*Proof.* Consider any vertex $v \in (A \cap B) \setminus V_t$. By definition of $V_t$, there exists a separation $(C, D) \in t$ such that $v \notin C$. This contradicts the fact that $B \subseteq C$ since $(A, B) \sim (C, D)$. $\square$

**Lemma 4.3.** *Every node of $\mathcal{T}$ is either a block node or a hub node.*

*Proof.* Suppose $t \in \mathcal{T}$ is not a hub node; we show that $t$ is a block node. By Lemma 4.1, $V_t$ is $\mathcal{N}$-inseparable. We show that $V_t$ is maximal in $V(G)$ with this property: that for every vertex $x \notin V_t$ the set $V_t \cup \{x\}$ is not $\mathcal{N}$-inseparable.

By definition of $V_t$, any vertex $x \notin V_t$ lies in $B \setminus A$ for some $(A, B) \in t$. Since $t$ is not a hub node, Lemma 4.2 implies that $V_t \nsubseteq A \cap B$. As $V_t \subseteq A$, this means that $V_t$ has a vertex in $A \setminus B$. Hence $(A, B)$ separates $V_t \cup \{x\}$, as desired. $\square$

Conversely, all the $\mathcal{N}$-blocks of $G$ will be parts of our tree-decomposition:

**Lemma 4.4.** *Every $\mathcal{N}$-block is the set $V_t$ for a node $t$ of $\mathcal{T}$.*

*Proof.* Consider an arbitrary $\mathcal{N}$-block $b$.

Suppose first that $b$ is small. Then there exists a separation $(A, B) \in \mathcal{N}$ with $b \subseteq A \cap B$. As $\mathcal{N}$ is nested, $A \cap B$ is $\mathcal{N}$-inseparable by (9), so in fact $b = A \cap B$ by the maximality of $b$. We show that $b = V_t$ for $t = [(A, B)]$. By Lemma 4.2, it suffices to show that $V_t \subseteq b = A \cap B$. As $V_t \subseteq A$ by definition of $V_t$, we only need to show that $V_t \subseteq B$. Suppose there is an $x \in V_t \setminus B$. As $x \notin A \cap B = b$, the maximality of $b$ implies that there exists a separation $(E, F) \in \mathcal{N}$ such that

$$F \nsupseteq b \subseteq E \text{ and } x \in F \setminus E \qquad (*)$$

(compare the proof of Lemma 2.1). By $(*)$, all corners of the cross-diagram $\{(A, B), (E, F)\}$ other than $B \cap F$ contain vertices not in the centre. Hence by (8), the only way in which $(A,B)$ and $(E,F)$ can be nested is that $B \cap F$ does lie in the centre, i.e. that $(B,A) \le (E,F)$. Since $(B,A) \ne (E,F)$, by $(*)$ and $b = A \cap B$, this means that $(B,A)$ has a successor $(C,D) \le (E,F)$. But then $(C,D) \sim (A,B)$ and $x \notin E \supseteq C \supseteq V_t$, a contradiction.

Suppose now that $b$ is not small. We shall prove that $b = V_t$ for $t = t(b)$, where $t(b)$ is defined as the set of separations $(A,B)$ that are minimal with $b \subseteq A$. Let us show first that $t(b)$ is indeed an equivalence class, i.e., that the separations in $t(b)$ are equivalent to each other but not to any other separation in $\mathcal{N}$.

Given distinct $(A,B),(C,D) \in t(b)$, let us show that $(A,B) \sim (C,D)$. Since both $(A,B)$ and $(C,D)$ are minimal as in the definition of $t(b)$, they are incomparable. But as elements of $\mathcal{N}$ they are nested, so $(A,B)$ is comparable with $(D,C)$. If $(A,B) \le (D,C)$ then $b \subseteq A \cap C \subseteq D \cap C$, which contradicts our assumption that $b$ is not small. Hence $(D,C) < (A,B)$. To show that $(D,C)$ is a predecessor of $(A,B)$, suppose there exists a separation $(E,F) \in \mathcal{N}$ such that $(D,C) < (E,F) < (A,B)$. This contradicts the minimality either of $(A,B)$, if $b \subseteq E$, or of $(C,D)$, if $b \subseteq F$. Thus, $(C,D) \sim (A,B)$ as desired.

Conversely, we have to show that every $(E,F) \in \mathcal{N}$ equivalent to some $(A,B) \in t(b)$ also lies in $t(b)$. As $(E,F) \sim (A,B)$, we may assume that $(F,E) < (A,B)$. Then $b \not\subseteq F$ by the minimality of $(A,B)$ as an element of $t(b)$, so $b \subseteq E$. To show that $(E,F)$ is minimal with this property, suppose that $b \subseteq X$ also for some $(X,Y) \in \mathcal{N}$ with $(X,Y) < (E,F)$. Then $(X,Y)$ is incomparable with $(A,B)$: by Lemma 3.2 we cannot have $(A,B) \le (X,Y) < (E,F)$, and we cannot have $(X,Y) < (A,B)$ by the minimality of $(A,B)$ as an element of $t(b)$. But $(X,Y) \parallel (A,B)$, so $(X,Y)$ must be comparable with $(B,A)$. Yet if $(X,Y) \le (B,A)$, then $b \subseteq X \cap A \subseteq B \cap A$, contradicting our assumption that $b$ is not small, while $(B,A) < (X,Y) < (E,F)$ is impossible, since $(B,A)$ is a predecessor of $(E,F)$.

Hence $t(b)$ is indeed an equivalence class, i.e., $t(b) \in V(\mathcal{T})$. By definition of $t(b)$, we have $b \subseteq \bigcap \{\, A \mid (A,B) \in t(b) \,\} = V_{t(b)}$. The converse inclusion follows from the maximality of $b$ as an $\mathcal{N}$-inseparable set. □

We have seen so far that the parts $V_t$ of our intended tree-decomposition associated with $\mathcal{N}$ are all the $\mathcal{N}$-blocks of $G$, plus some hubs. Our last lemma on this topic shows what has earned them their name:

**Lemma 4.5.** *A hub node $t$ has degree at least 3 in $\mathcal{T}$, unless it has the form $t = \{(A,B),(C,D)\}$ with $A \supsetneq D$ and $B = C$ (in which case it has degree 2).*

*Proof.* Let $(A,B) \in t$ be such that $V_t = A \cap B$. As $(A,B) \in t$ but $V_t \ne A$, we have $d(t) = |t| \ge 2$; cf. (11). Suppose that $d(t) = 2$, say $t = \{(A,B),(C,D)\}$. Then $B \subseteq C$ by definition of $\sim$, and $B \supseteq V_t \supseteq C$ by definition of $V_t$. So $B = C$. As $(A,B)$ and $(C,D)$ are equivalent but not equal, this implies $D \subsetneq A$. □

Figure 6 shows that the exceptional situation from Lemma 4.5 can indeed occur. In the example, we have $\mathcal{N} = \{(A,B),(B,A),(C,D),(D,C)\}$ with $B = C$ and $D \subsetneq A$. The structure tree $\mathcal{T}$ is a path between two block nodes $\{(D,C)\}$

and $\{(B, A)\}$ with a central hub node $t = \{(A, B), (C, D)\}$, whose set $V_t = A \cap B$ is not a block since it is properly contained in the $\mathcal{N}$-inseparable set $B = C$.

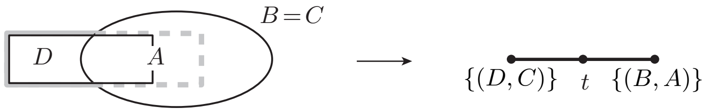


*Figure 6:* A hub node $t = \{(A, B), (C, D)\}$ of degree 2

Our last example answers some further questions about the possible relationships between blocks and hubs that will naturally come to mind:

**Example 2.** Consider the vertex sets $X_1, \dots, X_4$ shown on the left of Figure 7. Let $A$ be a superset of $X_1 \cup X_2$ and $B$ a superset of $X_3 \cup X_4$, so that $A \cap B \not\subseteq X_1 \cup \dots \cup X_4$ and different $X_i$ do not meet outside $A \cap B$. Let $\mathcal{N}$ consist of $(A, B)$, $(B, A)$, and $(X_1, Y_1), \dots, (X_4, Y_4)$ and their inverses $(Y_i, X_i)$, where $Y_i := (A \cap B) \cup \bigcup_{j \neq i} X_j$. The structure tree $\mathcal{T} = \mathcal{T}(\mathcal{N})$ has four block nodes $t_1, \dots, t_4$, with $t_i = [(X_i, Y_i)]$ and $V_{t_i} = X_i$, and two central hub nodes

$$a = \{(A, B), (Y_1, X_1), (Y_2, X_2)\} \quad \text{and} \quad b = \{(B, A), (Y_3, X_3), (Y_4, X_4)\}$$

joined by the edge $\{(A, B), (B, A)\}$. The hubs corresponding to $a$ and $b$ coincide: they are $V_a = A \cap B = V_b$, which is also a block.

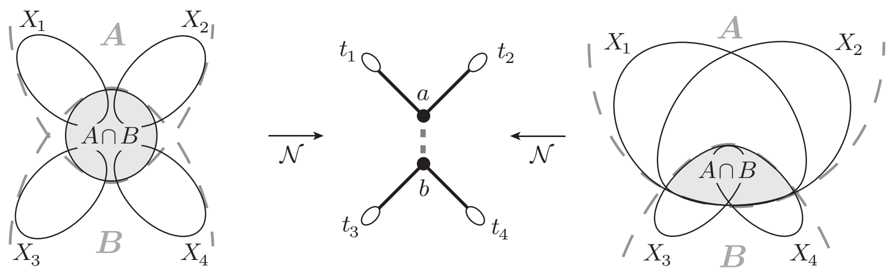


*Figure 7:* The two nested separation systems of Example 2, and their structure trees

Let us now modify this example by enlarging $X_1$ and $X_2$ so that they meet outside $A \cap B$ and each contain $A \cap B$. Thus, $A = X_1 \cup X_2$. Let us also shrink $B$ a little, down to $B = X_3 \cup X_4$ (Fig. 7, right). The structure tree $\mathcal{T}$ remains unchanged by these modifications, but the corresponding sets $V_t$ have changed:

$$V_b = A \cap B \subsetneq X_1 \cap X_2 = X_1 \cap Y_1 = X_2 \cap Y_2 = V_a,$$

and neither of them is a block, because both are properly contained in $X_1$, which is also $\mathcal{N}$-inseparable.

Our next lemma shows that deleting a separation from our nested system $\mathcal{N}$ corresponds to contracting an edge in the structure tree $\mathcal{T}(\mathcal{N})$. For a separation $(A, B)$ that belongs to different systems, we write $[(A, B)]_{\mathcal{N}}$ to indicate in which system $\mathcal{N}$ we are taking the equivalence class.

**Lemma 4.6.** *Given $(A,B) \in \mathcal{N}$, the tree $\mathcal{T}' := \mathcal{T}(\mathcal{N}')$ for*

$$\mathcal{N}' = \mathcal{N} \setminus \{(A,B),(B,A)\}$$

*arises from $\mathcal{T} = \mathcal{T}(\mathcal{N})$ by contracting the edge $e = \{(A,B),(B,A)\}$. The contracted node $z$ of $\mathcal{T}'$ satisfies $z = x \cup y \setminus e$ and $V_z = V_x \cup V_y$, where $x = [(A,B)]_{\mathcal{N}}$ and $y = [(B,A)]_{\mathcal{N}}$, and $V(\mathcal{T}') \setminus \{z\} = V(\mathcal{T}) \setminus \{x,y\}$.*[10]

*Proof.* To see that $V(\mathcal{T}') \setminus \{z\} = V(\mathcal{T}) \setminus \{x,y\}$ and $z = x \cup y \setminus e$, we have to show for all $(C,D) \in \mathcal{N}'$ that $[(C,D)]_{\mathcal{N}} = [(C,D)]_{\mathcal{N}'}$ unless $[(C,D)]_{\mathcal{N}} \in \{x,y\}$, in which case $[(C,D)]_{\mathcal{N}'} = x \cup y \setminus e$. In other words, we have to show:

> *Two separations $(C,D),(E,F) \in \mathcal{N}'$ are equivalent in $\mathcal{N}'$ if and only if they are equivalent in $\mathcal{N}$ or are both in $x \cup y \setminus e$.* $(*)$

Our further claim that $\mathcal{T}' = \mathcal{T}/e$, i.e. that the node-edge incidences in $\mathcal{T}'$ arise from those in $\mathcal{T}$ as defined for graph minors, will follow immediately from the definition of these incidences in $\mathcal{T}$ and $\mathcal{T}'$.

Let us prove the backward implication of $(*)$ first. As $\mathcal{N}' \subseteq \mathcal{N}$, predecessors in $(\mathcal{N}, \leq)$ are still predecessors in $\mathcal{N}'$, and hence $(C,D) \sim_{\mathcal{N}} (E,F)$ implies $(C,D) \sim_{\mathcal{N}'} (E,F)$. Moreover if $(C,D) \in x$ and $(E,F) \in y$ then, in $\mathcal{N}$, $(D,C)$ is a predecessor of $(A,B)$ and $(A,B)$ is a predecessor of $(E,F)$. In $\mathcal{N}'$, then, $(D,C)$ is a predecessor of $(E,F)$, since by Lemma 3.4 and Theorem 3.5 there is no separation $(A',B') \neq (A,B)$ in $\mathcal{N}$ that is both a successor of $(D,C)$ and a predecessor of $(E,F)$. Hence $(C,D) \sim_{\mathcal{N}'} (E,F)$.

For the forward implication in $(*)$ note that if $(D,C)$ is a predecessor of $(E,F)$ in $\mathcal{N}'$ but not in $\mathcal{N}$, then in $\mathcal{N}$ we have a sequence of predecessors $(D,C) < (A,B) < (E,F)$ or $(D,C) < (B,A) < (E,F)$. Then one of $(C,D)$ and $(E,F)$ lies in $x$ and the other in $y$, as desired.

It remains to show that $V_z = V_x \cup V_y$. Consider the sets

$$x' := x \setminus \{(A,B)\} \quad \text{and} \quad y' := y \setminus \{(B,A)\};$$

then $z = y' \cup x'$. Since all $(E,F) \in x'$ are equivalent to $(A,B)$ but not equal to it, we have $(B,A) \leq (E,F)$ for all those separations. That is,

$$B \subseteq \bigcap_{(E,F) \in x'} E = V_{x'}. \tag{13}$$

By definition of $V_x$ we have $V_x = V_{x'} \cap A$. Hence (13) yields $V_{x'} = V_x \cup (B \setminus A)$, and since $A \cap B \subseteq V_x$ by Lemma 4.2, we have $V_{x'} = V_x \cup B$. An analogous argument yields

$$V_{y'} = \bigcap_{(E,F) \in y'} E = V_y \cup A.$$

Hence,

[10] The last identity says more than that there exists a canonical bijection between $V(\mathcal{T}') \setminus \{z\}$ and $V(\mathcal{T}) \setminus \{x,y\}$: it says that the nodes of $\mathcal{T} - \{x,y\}$ and $\mathcal{T}' - z$ are the same also as sets of separations.

$$
\begin{aligned}
V_z &= \bigcap_{(E,F)\,\in\, z} E \\
&= V_{x'} \cap V_{y'} \\
&= (V_x \cup B) \cap (V_y \cup A) \\
&= (V_x \cap V_y) \cup (V_x \cap A) \cup (V_y \cap B) \cup (B \cap A) \\
&= (V_x \cap V_y) \cup V_x \cup V_y \cup (B \cap A) \\
&= V_x \cup V_y .
\end{aligned}
$$

□

Every edge $e$ of $\mathcal{T}$ separates $\mathcal{T}$ into two components. The vertex sets $V_t$ for the nodes $t$ in these components induce a corresponding separation of $G$, as in [1, Lemma 12.3.1]. This is the separation that defined $e$:

**Lemma 4.7.** *Given any separation $(A, B) \in \mathcal{N}$, consider the corresponding edge $e = \{(A, B), (B, A)\}$ of $\mathcal{T} = \mathcal{T}(\mathcal{N})$. Let $\mathcal{T}_A$ denote the component of $\mathcal{T} - e$ that contains the node $[(A, B)]$, and let $\mathcal{T}_B$ be the other component. Then $\bigcup_{t\in\mathcal{T}_A} V_t = A$ and $\bigcup_{t\in\mathcal{T}_B} V_t = B$.*

*Proof.* We apply induction on $|E(\mathcal{T})|$. If $\mathcal{T}$ consists of a single edge, the assertion is immediate from the definition of $\mathcal{T}$. Assume now that $|E(\mathcal{T})| > 1$. In particular, there is an edge $e^* = xy \neq e$.

Consider $\mathcal{N}' := \mathcal{N} \smallsetminus e^*$, and let $\mathcal{T}' := \mathcal{T}(\mathcal{N}')$. Then $\mathcal{T}' = \mathcal{T}/e^*$, by Lemma 4.6. Let $z$ be the node of $\mathcal{T}'$ contracted from $e^*$. Define $\mathcal{T}'_A$ as the component of $\mathcal{T}' - e$ that contains the node $[(A, B)]$, and let $\mathcal{T}'_B$ be the other component. We may assume $e^* \in \mathcal{T}_A$. Then

$$V(\mathcal{T}_A) \smallsetminus \{x, y\} = V(\mathcal{T}'_A) \smallsetminus \{z\} \text{ and } V(\mathcal{T}_B) = V(\mathcal{T}'_B).$$

As $V_z = V_x \cup V_y$ by Lemma 4.6, we can use the induction hypothesis to deduce that

$$\bigcup_{t\in\mathcal{T}_A} V_t = \bigcup_{t\in\mathcal{T}'_A} V_t = A \quad \text{and} \quad \bigcup_{t\in\mathcal{T}_B} V_t = \bigcup_{t\in\mathcal{T}'_B} V_t = B,$$

as claimed. □

Let us summarize some of our findings from this section. Recall that $\mathcal{N}$ is an arbitrary nested separation system of an arbitrary finite graph $G$. Let $\mathcal{T} := \mathcal{T}(\mathcal{N})$ be the structure tree associated with $\mathcal{N}$ as in Section 3, and let $\mathcal{V} := (V_t)_{t\in\mathcal{T}}$ be defined by (12). Let us call the separations of $G$ that correspond as in [1, Lemma 12.3.1] to the edges of the decomposition tree of a tree-decomposition of $G$ the separations *induced by* this tree-decomposition.

**Theorem 4.8.** *The pair $(\mathcal{T}, \mathcal{V})$ is a tree-decomposition of $G$.*

(i) *Every $\mathcal{N}$-block is a part of the decomposition.*

(ii) *Every part of the decomposition is either an $\mathcal{N}$-block or a hub.*

(iii) *The separations of $G$ induced by the decomposition are precisely those in $\mathcal{N}$.*

(iv) *Every $\mathcal{N}' \subseteq \mathcal{N}$ satsfies $(\mathcal{T}', \mathcal{V}') \preccurlyeq (\mathcal{T}, \mathcal{V})$ for $\mathcal{T}' = \mathcal{T}(\mathcal{N}')$ and $\mathcal{V}' = V(\mathcal{T}')$.*[11]

*Proof.* Of the three axioms for a tree-decomposition, (T1) and (T2) follow from Lemma 4.4, because single vertices and edges form $\mathcal{N}$-inseparable vertex sets, which extend to $\mathcal{N}$-blocks. For the proof of (T3), let $e = \{(A, B), (B, A)\}$ be an edge at $t_2$ on the $t_1$–$t_3$ path in $\mathcal{T}$. Since $e$ separates $t_1$ from $t_3$ in $\mathcal{T}$, Lemmas 4.7 and 4.2 imply that $V_{t_1} \cap V_{t_3} \subseteq A \cap B \subseteq V_{t_2}$.

Statement (i) is Lemma 4.4. Assertion (ii) is Lemma 4.3. Assertion (iii) follows from Lemma 4.7 and the definition of the edges of $\mathcal{T}$. Statement (iv) follows by repeated application of Lemma 4.6. □

# 5 Extracting nested separation systems

Our aim in this section will be to find inside a given separation system $\mathcal{S}$ a nested subsystem $\mathcal{N}$ that can still distinguish the elements of some given set $\mathcal{I}$ of $\mathcal{S}$-inseparable sets of vertices. As we saw in Sections 3 and 4, such a nested subsystem will then define a tree-decomposition of $G$, and the sets from $\mathcal{I}$ will come to lie in different parts of that decomposition.

This cannot be done for all choices of $\mathcal{S}$ and $\mathcal{I}$. Indeed, consider the following example of where such a nested subsystem does not exist. Let $G$ be the $3 \times 3$-grid, let $\mathcal{S}$ consist of the two 3-separations cutting along the horizontal and the vertical symmetry axis, and let $\mathcal{I}$ consist of the four corners of the resulting cross-diagram. Each of these is $\mathcal{S}$-inseparable, and any two of them can be separated by a separation in $\mathcal{S}$. But since the two separations in $\mathcal{S}$ cross, any nested subsystem contains at most one of them, and thus fails to separate some sets from $\mathcal{I}$.

However, we shall prove that the desired nested subsystem does exist if $\mathcal{S}$ and $\mathcal{I}$ satisfy the following condition. Given a separation system $\mathcal{S}$ and a set $\mathcal{I}$ of $\mathcal{S}$-inseparable sets, let us say that $\mathcal{S}$ *separates $\mathcal{I}$ well* if the following holds for every pair of crossing – that is, not nested – separations $(A, B), (C, D) \in \mathcal{S}$:

> *For all $I_1, I_2 \in \mathcal{I}$ with $I_1 \subseteq A \cap C$ and $I_2 \subseteq B \cap D$ there is an $(E, F) \in \mathcal{S}$ such that $I_1 \subseteq E \subseteq A \cap C$ and $F \supseteq B \cup D$.*

Note that such a separation satisfies both $(E, F) \leq (A, B)$ and $(E, F) \leq (C, D)$.

In our grid example, $\mathcal{S}$ did not separate $\mathcal{I}$ well, but we can mend this by adding to $\mathcal{S}$ the four corner separations. And as soon as we do that, there is a nested subsystem that separates all four corners – for example, the set of the four corner separations.

More abstractly, the idea behind the notion of $\mathcal{S}$ separating $\mathcal{I}$ well is as follows. In the process of extracting $\mathcal{N}$ from $\mathcal{S}$ we may be faced with a pair of crossing separations $(A, B)$ and $(C, D)$ in $\mathcal{S}$ that both separate two given sets $I_1, I_2 \in \mathcal{I}$, and wonder which of them to pick for $\mathcal{N}$. (Obviously we cannot

[11] See the Introduction for the definition of $(\mathcal{T}', \mathcal{V}') \preccurlyeq (\mathcal{T}, \mathcal{V})$.

choose both.) If $\mathcal{S}$ separates $\mathcal{I}$ well, however, we can avoid this dilemma by choosing $(E,F)$ instead: this also separates $I_1$ from $I_2$, and since it is nested with both $(A,B)$ and $(C,D)$ it will not prevent us from choosing either of these later too, if desired.

Let us call a separation $(E,F) \in \mathcal{S}$ *extremal* in $\mathcal{S}$ if for all $(C,D) \in \mathcal{S}$ we have either $(E,F) \leq (C,D)$ or $(E,F) \leq (D,C)$. In particular, extremal separations are nested with all other separations in $\mathcal{S}$. Being extremal implies being $\leq$-minimal in $\mathcal{S}$; if $\mathcal{S}$ is nested, extremality and $\leq$-minimality are equivalent. If $(E,F) \in \mathcal{S}$ is extremal, then $E$ is an $\mathcal{S}$-block; we call it an *extremal block* in $\mathcal{S}$.

A separation system, even a nested one, typically contains many extremal separations. For example, given a tree-decomposition of $G$ with decomposition tree $\mathcal{T}$, the separations corresponding to the edges of $\mathcal{T}$ that are incident with a leaf of $\mathcal{T}$ are extremal in the (nested) set of all the separations of $G$ corresponding to edges of $\mathcal{T}$.[12]

Our next lemma shows that separating a set $\mathcal{I}$ of $\mathcal{S}$-inseparable sets well is enough to guarantee the existence of an extremal separation among those that separate sets from $\mathcal{I}$. Call a separation *$\mathcal{I}$-relevant* if it weakly separates some two sets in $\mathcal{I}$. If all the separations in $\mathcal{S}$ are $\mathcal{I}$-relevant, we call $\mathcal{S}$ itself *$\mathcal{I}$-relevant.*

**Lemma 5.1.** *Let $\mathcal{R}$ be a separation system that is $\mathcal{I}$-relevant for some set $\mathcal{I}$ of $\mathcal{R}$-inseparable sets. If $\mathcal{R}$ separates $\mathcal{I}$ well, then every $\leq$-minimal $(A,B) \in \mathcal{R}$ is extremal in $\mathcal{R}$. In particular, if $\mathcal{R} \neq \emptyset$ then $\mathcal{R}$ contains an extremal separation.*

*Proof.* Consider a $\leq$-minimal separation $(A,B) \in \mathcal{R}$, and let $(C,D) \in \mathcal{R}$ be given. If $(A,B)$ and $(C,D)$ are nested, then the minimality of $(A,B)$ implies that $(A,B) \leq (C,D)$ or $(A,B) \leq (D,C)$, as desired. So let us assume that $(A,B)$ and $(C,D)$ cross.

As $(A,B)$ and $(C,D)$ are $\mathcal{I}$-relevant and the sets in $\mathcal{I}$ are $\mathcal{R}$-inseparable, we can find opposite corners of the cross-diagram $\{(A,B),(C,D)\}$ that each contains a set from $\mathcal{I}$. Renaming $(C,D)$ as $(D,C)$ if necessary, we may assume that these sets lie in $A \cap C$ and $B \cap D$, say $I_1 \subseteq A \cap C$ and $I_2 \subseteq B \cap D$. As $\mathcal{R}$ separates $\mathcal{I}$ well, there exists $(E,F) \in \mathcal{R}$ such that $I_1 \subseteq E \subseteq A \cap C$ and $F \supseteq B \cup D$, and hence $(E,F) \leq (A,B)$ as well as $(E,F) \leq (C,D)$. By the minimality of $(A,B)$, this yields $(A,B) = (E,F) \leq (C,D)$ as desired. $\square$

Let us say that a set $\mathcal{S}$ of separations *distinguishes* two given $\mathcal{S}$-inseparable sets $I_1, I_2$ (or *distinguishes* them *properly*) if it contains a separation that separates them. If it contains a separation that separates them weakly, it *weakly distinguishes* $I_1$ from $I_2$. We then also call $I_1$ and $I_2$ *(weakly) distinguishable* by $\mathcal{S}$, or *(weakly) $\mathcal{S}$-distinguishable.*

Here is our main result for this section:

[12] More precisely, every such edge of $\mathcal{T}$ corresponds to an inverse pair of separations of which, usually, only one is extremal: the separation $(A,B)$ for which $A$ is the part $V_t$ with $t$ a leaf of $\mathcal{T}$. The separation $(B,A)$ will not be extremal, unless $\mathcal{T} = K^2$.

**Theorem 5.2.** *Let $\mathcal{S}$ be any separation system that separates some set $\mathcal{I}$ of $\mathcal{S}$-inseparable sets of vertices well. Then $\mathcal{S}$ has a nested $\mathcal{I}$-relevant subsystem $\mathcal{N}(\mathcal{S},\mathcal{I}) \subseteq \mathcal{S}$ that weakly distinguishes all weakly $\mathcal{S}$-distinguishable sets in $\mathcal{I}$.*

*Proof.* If $\mathcal{I}$ has no two weakly distinguishable elements, let $\mathcal{N}(\mathcal{S},\mathcal{I})$ be empty. Otherwise let $\mathcal{R} \subseteq \mathcal{S}$ be the subsystem of all $\mathcal{I}$-relevant separations in $\mathcal{S}$. Then $\mathcal{R} \neq \emptyset$, and $\mathcal{R}$ separates $\mathcal{I}$ well. Let $\mathcal{E} \subseteq \mathcal{R}$ be the subset of those separations that are extremal in $\mathcal{R}$, and put

$$\overline{\mathcal{E}} := \{(A,B) \mid (A,B) \text{ or } (B,A) \text{ is in } \mathcal{E}\}.$$

By Lemma 5.1 we have $\overline{\mathcal{E}} \neq \emptyset$, and by definition of extremality all separations in $\overline{\mathcal{E}}$ are nested with all separations in $\mathcal{R}$. In particular, $\overline{\mathcal{E}}$ is nested.

Let

$$\mathcal{I}_{\mathcal{E}} := \{I \in \mathcal{I} \mid \exists (E,F) \in \mathcal{E} : I \subseteq E\}.$$

This is non-empty, since $\mathcal{E} \subseteq \mathcal{R}$ is non-empty and $\mathcal{I}$-relevant. Let us prove that $\mathcal{E}$ weakly distinguishes all pairs of weakly distinguishable elements $I_1, I_2 \in \mathcal{I}$ with $I_1 \in \mathcal{I}_{\mathcal{E}}$. Pick $(A,B) \in \mathcal{R}$ with $I_1 \subseteq A$ and $I_2 \subseteq B$. Since $I_1 \in \mathcal{I}_{\mathcal{E}}$, there is an $(E,F) \in \mathcal{E}$ such that $I_1 \subseteq E$. By the extremality of $(E,F)$ we have either $(E,F) \leq (A,B)$, in which case $I_1 \subseteq E$ and $I_2 \subseteq B \subseteq F$, or we have $(E,F) \leq (B,A)$, in which case $I_1 \subseteq E \cap A \subseteq E \cap F$. In both cases $I_1$ and $I_2$ are weakly separated by $(E,F)$.

As $\mathcal{I}' := \mathcal{I} \smallsetminus \mathcal{I}_{\mathcal{E}}$ is a set of $\mathcal{S}$-inseparable sets with fewer elements than $\mathcal{I}$, induction gives us a nested $\mathcal{I}'$-relevant subsystem $\mathcal{N}(\mathcal{S},\mathcal{I}')$ of $\mathcal{S}$ that weakly distinguishes all weakly distinguishable elements of $\mathcal{I}'$. Then

$$\mathcal{N}(\mathcal{S},\mathcal{I}) := \overline{\mathcal{E}} \cup \mathcal{N}(\mathcal{S},\mathcal{I}')$$

is $\mathcal{I}$-relevant and weakly distinguishes all weakly distinguishable elements of $\mathcal{I}$. As $\mathcal{I}' \subseteq \mathcal{I}$, and thus $\mathcal{N}(\mathcal{S},\mathcal{I}') \subseteq \mathcal{R}$, the separations in $\overline{\mathcal{E}}$ are nested with those in $\mathcal{N}(\mathcal{S},\mathcal{I}')$. Hence, $\mathcal{N}(\mathcal{S},\mathcal{I})$ too is nested. □

An important feature of the proof of Theorem 5.2 is that the subset $\mathcal{N}(\mathcal{S},\mathcal{I})$ it constructs is *canonical*, given $\mathcal{S}$ and $\mathcal{I}$: there are no choices made anywhere in the proof. We may thus think of $\mathcal{N}$ as a recursively defined operator that assigns to every pair $(\mathcal{S},\mathcal{I})$ as given in the theorem a certain nested subsystem $\mathcal{N}(\mathcal{S},\mathcal{I})$ of $\mathcal{S}$. This subsystem $\mathcal{N}(\mathcal{S},\mathcal{I})$ is canonical also in the structural sense that it is invariant under any automorphisms of $G$ that leave $\mathcal{S}$ and $\mathcal{I}$ invariant.

To make this more precise, we need some notation. Every automorphism $\alpha$ of $G$ acts also on (the set of) its vertex sets $U \subseteq V(G)$, on the collections $\mathcal{X}$ of such vertex sets, on the separations $(A,B)$ of $G$, and on the sets $\mathcal{S}$ of such separations. We write $U^\alpha$, $\mathcal{X}^\alpha$, $(A,B)^\alpha$ and $\mathcal{S}^\alpha$ and so on for their images under $\alpha$.

**Corollary 5.3.** *Let $\mathcal{S}$ and $\mathcal{I}$ be as in Theorem 5.2, and let $\mathcal{N}(\mathcal{S},\mathcal{I})$ be the nested subsystem of $\mathcal{S}$ constructed in the proof. Then for every automorphism $\alpha$ of $G$ we have $\mathcal{N}(\mathcal{S}^\alpha,\mathcal{I}^\alpha) = \mathcal{N}(\mathcal{S},\mathcal{I})^\alpha$. In particular, if $\mathcal{S}$ and $\mathcal{I}$ are invariant under the action of a group $\Gamma$ of automorphisms of $G$, then so is $\mathcal{N}(\mathcal{S},\mathcal{I})$.*

*Proof.* The proof of the first assertion is immediate from the construction of $\mathcal{N}(\mathcal{S},\mathcal{I})$ from $\mathcal{S}$ and $\mathcal{I}$. The second assertion follows, as

$$\mathcal{N}(\mathcal{S},\mathcal{I})^\alpha = \mathcal{N}(\mathcal{S}^\alpha,\mathcal{I}^\alpha) = \mathcal{N}(\mathcal{S},\mathcal{I})$$

for every $\alpha \in \Gamma$. □

# 6 Separating the $k$-blocks of a graph

In this section we apply the theory developed so far to our original problem, to 'decompose a graph into its $(k+1)$-connected components'. We shall answer this question in several ways. We first find, for $k \geq 0$ arbitrary but fixed, in any $k$-connected graph $G$ a nested set of separations that distinguishes all its $k$-blocks, its maximal $k$-inseparable sets of vertices. We then turn this into a tree-decomposition such that different $k$-blocks of $G$ lie in different parts of the decomposition.

In a second tack we show that, if we restrict our attention to 'robust' $k$-blocks, we can unify both results over all $k$. Indeed, we shall be able to find one nested set of separations that distinguishes every two robust blocks of $G$, regardless of their rank (as long as neither contains the other). And we can turn this into a sequence of tree-decompositions, one for each $k$, that refine one another as $k$ increases and are such that different robust $k$-blocks of $G$ lie in different parts of the $k$th decomposition.

All our decompositions will be canonical, in that they depend only on the structure of $G$. More precisely, our nested sets of separations will be invariant under the automorphism group $\mathrm{Aut}(G)$ of $G$, which will act also on the decomposition trees associated with our tree-decompositions of $G$.

A separation system $\mathcal{S}$ will be said to *$k$-distinguish* a set $\mathcal{I}$ of $\mathcal{S}$-inseparable sets if every two sets in $\mathcal{I}$ are distinguished by a $k$-separation in $\mathcal{S}$. Recall that a separation $(A,B)$ is *tight* if every vertex in $A \cap B$ has a neighbour in $A \setminus B$ and another in $B \setminus A$. Let us call a separation system *tight* if all its separations are tight.

**Theorem 6.1.** *Let $G$ be a finite graph that has more than one $k$-block for some $k \in \mathbb{N}$, and let $k$ be minimal with this property. Then $G$ has as a tight, nested, and* $\mathrm{Aut}(G)$*-invariant separation system $\mathcal{N}$ that $k$-distinguishes its $k$-blocks.*

*Proof.* Let $\mathcal{S}$ be the set of all separations of order at most $k$ that separate two $k$-blocks.[13] These are tight separations of order exactly $k$: two $k$-blocks separated by $\ell < k$ vertices could be extended to $\ell$-blocks separated by those $\ell$ vertices, contradicting the minimality of $k$. Clearly, $\mathcal{S}$ is symmetric. Distinct $k$-blocks are separated by a separation of order at most $k$ (by Lemma 2.1 applied to the set of all these separations), which lies in $\mathcal{S}$ by definition; so $\mathcal{S}$ ($k$-)distinguishes all $k$-blocks. By Theorem 5.2 and Corollary 5.3 it suffices to prove that $\mathcal{S}$

[13] These need not be all the separations of order at most $k$. For example, consider for $k = 2$ a complete graph and subdivide every edge at least twice. See also Footnote 15.

separates the set of $k$-blocks well: the weak separations it provides will be proper separations, since $k$-blocks have size at least $k+1$.

So consider two crossing separations $(A,B),(C,D) \in \mathcal{S}$, with $k$-blocks $I_1 \subseteq A \cap C$ and $I_2 \subseteq B \cap D$. By the minimality of $k$, the $k$-blocks $I_1$ and $I_2$ cannot be separated by fewer than $k$ vertices (as earlier), so $|\partial(A\cap C)| \geq k$ and $|\partial(B \cap D)| \geq k$. But

$$|\partial(A \cap C)| + |\partial(B \cap D)| = |A \cap B| + |C \cap D| = 2k,$$

so both these inequalities hold with equality. In particular, $|\partial(A \cap C)| = k$, so the corner separation $(A \cap C, B \cup D)$ lies in $\mathcal{S}$ as desired. □

Theorem 6.1 extends a theorem of Dunwoody and Krön [3], who use a tailor-made notion of 'cut systems' to prove a similar result for a weaker notion of nestedness [3, Theorem 9.5]. Their notion, in which 'nested' separations need not have an empty corner (only a corner not containing a $k$-block), is particularly suited to infinite graphs: unlike finite graphs,[14] infinite graphs can satisfy the premise of Theorem 6.1 when they are transitive, and for transitive graphs the weaker notion of nestedness coincides with the usual notion.

We now come to the first main result of this paper, a slight strenthgening of Theorem 1 from the Introduction. Note that, although our theory of separation systems is essential to its proof, the theorem itself can be stated just in terms of tree-decompositions.

**Theorem 6.2.** *Let $G$ be a finite graph. Let $k$ be the least integer such that $G$ has more than one $k$-block; if no such $k$ exists, let $k \in \mathbb{N}$ be arbitrary. Then $G$ has a tree-decomposition $(\mathcal{T}, \mathcal{V})$ such that*

(i) *every $k$-inseparable set of vertices is contained in a unique part of $(\mathcal{T}, \mathcal{V})$, and distinct $k$-blocks are contained in different parts;*

(ii) *$(\mathcal{T}, \mathcal{V})$ has adhesion $k$;*

(iii) $\mathrm{Aut}(G)$ *acts on $\mathcal{T}$ as a group of automorphisms.*

If $G$ has more than one $\kappa(G)$-block, as is usually the case, the assertion of Theorem 6.2 becomes that of Theorem 1. If $G$ has only one $\kappa(G)$-block but distinct $k$-blocks for some $k > \kappa(G)$, which can happen,[15] the theorem is a little stronger: the trivial decomposition into just one part then satisfies (i)–(iii) for $k = \kappa(G)$, but the theorem also yields a non-trivial tree-decomposition satisfying (i)–(iii) for some larger $k$. If $k$ does not exist as stated and is taken to be arbitrary, the decomposition offered by the theorem will be trivial.

[14] By Lemma 5.1 and the proof of Theorem 6.1, finite connected transitive graphs cannot have more than one $k$-block for any $k$.

[15] Take a $10 \times 10$ grid, make the left and the right column complete, and add a new vertex to each of them to create two $K^{11}$s. The resulting graph $G$ has $\kappa(G) = 3$, but $k = 10$.

*Proof of Theorem 6.2.* If $G$ has hat most one $k$-block for every $k \in \mathbb{N}$, the trivial tree-decomposition with only one part is as desired. Assume now that $G$ has distinct $k$-blocks for some $k$, choose $k$ minimal with this property, and let $\mathcal{N}$ be the nested separation system from Theorem 6.1 that distinguishes all these $k$-blocks. Let $(\mathcal{T}, \mathcal{V})$ be the tree-decomposition associated with $\mathcal{N}$ as in Section 4, i.e. with $\mathcal{T} = \mathcal{T}(\mathcal{N})$ and $\mathcal{V} = \mathcal{V}(\mathcal{N})$. Then (iii) holds by Corollary 3.6.

For (ii), recall that the adhesion of a tree-decomposition is the maximum size of an overlap $V_t \cap V_{t'}$ of two parts associated with an edge $e = tt'$ of the decomposition tree. If $e = \{(A, B), (B, A)\} \in \mathcal{T}$, then by Lemma 4.7 and [1, Lemma 12.3.1] this overlap $V_t \cap V_{t'}$ is precisely $A \cap B$. Since the separations in $\mathcal{N}$ all have order $k$, this proves (ii).

For (i), note that every $k$-inseparable set $I$ is $\mathcal{N}$-inseparable. It therefore extends to an $\mathcal{N}$-block, which by Lemma 4.4 is a part of $(\mathcal{T}, \mathcal{V})$. By (ii) and [1, Lemma 12.3.1], distinct parts of $(\mathcal{T}, \mathcal{V})$ are separated by a separation of order $k$, so as $|I| > k$ this part is unique. Finally, suppose that two $k$-blocks $I_1, I_2$ lie in the same part $V_t$. Since $\mathcal{N}$, by definition, distinguishes all $k$-blocks, this means that some separation in $\mathcal{N}$ separates $V_t$. But that is not the case; cf. Lemma 4.1. $\square$

Let us now turn to the question of how our decompositions can be unified over all $k$. For example, can we find one nested set $\mathcal{N}$ of separations of $G$ such that, for every $k \geq 0$, every two $k$-blocks are separated by a separation in $\mathcal{N}$ of order at most $k$? The idea would be to find such a separation system recursively: once we have a system that distinguishes all the $k$-blocks for some $k$ in this way, we could try to refine it to a larger system that does the same even for $k + 1$.

Figure 8 shows that this need not always be possible, indeed that no nested system that works for all $k$ need exist. The graph depicted arises from the disjoint union of a $K^{(k/2)-1}$, two $K^{k/2}$, a $K^{(k/2)+2}$ and two $K^{9k}$, by joining the $K^{(k/2)-1}$ completely to the two $K^{k/2}$, the $K^{(k/2)+2}$ completely to the two $K^{9k}$, the first $K^{k/2}$ completely to the first $K^{9k}$, and the second $K^{k/2}$ completely to the second $K^{9k}$. The horizontal $k$-separator consisting of the two $K^{k/2}$ defines the only separation of order at most $k$ that distinguishes the two $k$-blocks consisting of the top five complete graphs versus the bottom three. On the other hand, the

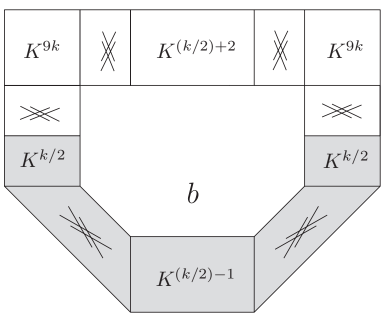


*Figure 8:* A horizontal $k$-separation needed to distinguish two $k$-blocks, crossed by a vertical $(k+1)$-separation needed to distinguish two $(k+1)$-blocks.

vertical $(k+1)$-separator consisting of the $K^{(k/2)-1}$ and the $K^{(k/2)+2}$ defines the only separation of order at most $(k+1)$ that distinguishes the two $(k+1)$-blocks consisting, respectively, of the left $K^{k/2}$ and $K^{9k}$ and the $K^{(k/2)+2}$, and of the right $K^{k/2}$ and $K^{9k}$ and the $K^{(k/2)+2}$. Hence any separation system that distinguishes all $k$-blocks as well as all $(k+1)$-blocks must contain both separations. Since the two separations cross, such a system cannot be nested.

In view of this example it may be surprising that we can find a separation system that distinguishes, for all $k \geq 0$ simultaneously, all *large* $k$-blocks of $G$, those with at least $\lfloor \frac{3}{2}k \rfloor$ vertices. The example of Figure 8 shows that this value is best possible: here, all blocks are large except for the $k$-block $b$ consisting of the two $K^{k/2}$ and the $K^{(k/2)-1}$, which has size $\frac{3}{2}k - 1$.

Indeed, we shall prove something considerably stronger: that the only obstruction to the existence of a unified tree-decomposition is a $k$-block that is not only not large but positioned exactly like $b$ in Figure 8, inside the union of a $k$-separator and a larger separator crossing it.

Let us call a set $U$ of vertices *$k$-robust* if $U$ is $k$-inseparable and, for every $k$-separation $(C, D)$ with $U \subseteq D$ and every separation $(A, B) \nparallel (C, D)$ with

$$|\partial(A \cap D)| < k > |\partial(B \cap D)| \,, \tag{14}$$

we have either $U \subseteq A$ or $U \subseteq B$. By $U \subseteq D$ and (14), the only way in which this can fail is that $|A \cap B| > k$ and $U$ is contained in the union $T$ of the boundaries of $A \cap D$ and $B \cap D$ (Fig. 9): exactly the situation of $b$ in Figure 8.

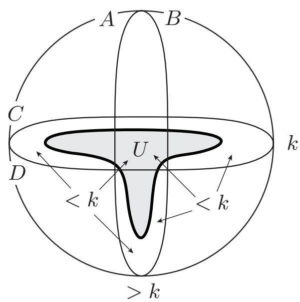


*Figure 9:* The shaded set $U$ is $k$-inseparable but not $k$-robust.

Our next lemma says that large $k$-blocks are $k$-robust. But there are more kinds of $k$-robust subsets than these: the vertex set of any $K^{k+1}$ subgraph, for example, is $k$-robust.

**Lemma 6.3.** *Large $k$-blocks are $k$-robust.*

*Proof.* By the remark following the definition of '$k$-robust', it suffices to show that the set $T = \partial(A \cap D) \cup \partial(B \cap D)$ in Figure 9 has size at most $\frac{3}{2}k - 1$. Let $\ell := |(A \cap B) \setminus C|$ be the size of the common link of the corners $A \cap D$ and $B \cap D$. Then by $|C \cap D| = k$ and (14),

$$|T| \;\leq\; \min\{\, k + \ell \,,\, 2(k-1) - \ell \,\} \leq \tfrac{3}{2}k - 1,$$

as is easy to check. $\square$

For the remainder of this paper, a *block* of $G$ is again a subset of $V(G)$ that is a $k$-block for some $k$. A block is *robust* if it is a $k$-robust $k$-block[16] for some $k$.

It is not difficult to find examples of robust $k$-blocks that are not $k$-robust, but only $\ell$-robust (and $\ell$-blocks) for some $\ell < k$. A $k$-block that is $k'$-robust for $k' > k$, however, is also $k$-robust. More generally:

**Lemma 6.4.** *Let $k \le k'$ be integers.*

(i) *Every $k$-inseparable set $I$ containing a $k'$-robust set $I'$ is $k$-robust.*

(ii) *Every $k$-block $b$ that contains a robust $k'$-block $b'$ is robust.*

*Proof.* (i) Suppose that $I$ is not $k$-robust, and let this be witnessed by a $k$-separation $(C, D)$ crossed by a separation $(A, B)$. Put $S := C \cap D$ and $L := (A \cap B) \setminus C$. Then $I \subseteq S \cup L$, as remarked after the definition of '$k$-robust'.

Extend $S$ into $L$ to a $k'$-set $S'$ that is properly contained in $S \cup L$ (which is large enough, since it contains $I' \subseteq I$), and put $C' := C \cup S'$. Then $(C', D)$ is a $k'$-separation with separator $S'$ and corners $D \cap A$ and $D \cap B$, whose boundaries by assumption have size $< k$. As $I'$ is $k'$-robust, it lies in one of these corners, say $I' \subseteq A \cap D$. Since

$$|I'| > k > |\partial(A \cap D)|\,,$$

this implies that $I'$ has a vertex in the interior of the corner $A \cap D$. As $I' \subseteq I$, this contradicts the fact that $I \subseteq S \cup L$.

(ii) Since $b'$ is robust, it is an $\ell$-robust $\ell$-block for some $\ell$. If $\ell \ge k$, then $b$ is $k$-robust by (i). Then $b$ is a $k$-robust $k$-block, and hence robust. If $\ell < k$ then $b' = b$, by the maximality of $b'$ as an $\ell$-inseparable set. Then $b$, like $b'$, is an $\ell$-robust $\ell$-block, and hence robust. $\square$

Recall that the smallest $k$ for which a block $b$ is a $k$-block is its *rank*; let us denote this by $r(b)$. By Lemma 6.4 (i), a block $b$ is robust if and only if it is $r(b)$-robust. Two blocks are *incomparable* if neither is a subset of the other. A set $\mathcal{S}$ of separations *distinguishes* two blocks $b_1, b_2$ if these are separated by a separation in $\mathcal{S}$ of order at most $\min\{r(b_1), r(b_2)\}$.[17]

We next prove that every finite graph $G$ has a nested set of separations that distinguishes every pair of incomparable robust blocks. We shall then turn this into a sequence of tree-decompositions, one for each $k \in \mathbb{N}$, that refine one another as $k$ increases. The separators of the $k$th decomposition will distinguish every robust block of rank up to $k$ from all robust blocks incomparable with it, and every robust block of order at least $k$ will lie in a unique part. Incomparable robust blocks of both rank at most $k$ and order at least $k$ – these are precisely the robust $k$-blocks – will thus lie in different parts.

[16]It is important to require this. If we called an $n$-block 'robust' as soon as it was $k$-robust for some $k$, without having to be an $k$-block too, the block $b$ in Figure 8 would become 'robust', and the example would disprove Theorems 6.5 and 6.6.

[17]This definition extends our earlier definition of when $\mathcal{S}$ distinguishes two $\mathcal{S}$-inseparable sets, since we no longer require that $b_1$ and $b_2$ be $\mathcal{S}$-inseparable for the entire $\mathcal{S}$. If they are, e.g. if $\mathcal{S}$ contains all separations up to the minimum of the *orders* of $b_1$ and $b_2$, our new definition can be more restrictive: we now want to separate $b_1$ from $b_2$ by a separation whose order is at most the minimum of their *ranks*, so not all separations in $\mathcal{S}$ may qualify.

**Theorem 6.5.** *Every finite graph $G$ has a tight, nested, and* $\mathrm{Aut}(G)$*-invariant separation system that distinguishes every two incomparable robust blocks.*

*Proof.* Recursively for all integers $k \geq 0$ we shall construct a sequence of separation systems $\mathcal{N}_k$ with the following properties:

(i) $\mathcal{N}_k$ is tight, nested, and $\mathrm{Aut}(G)$-invariant;

(ii) $\mathcal{N}_{k-1} \subseteq \mathcal{N}_k$ (put $\mathcal{N}_{-1} := \emptyset$);

(iii) every separation in $\mathcal{N}_k \setminus \mathcal{N}_{k-1}$ has order $k$;

(iv) $\mathcal{N}_k$ distinguishes every two incomparable robust blocks $b_1, b_2$ such that $\min\{r(b_1), r(b_2)\} \leq k$.

(v) every separation in $\mathcal{N}_k \setminus \mathcal{N}_{k-1}$ separates some robust $k$-blocks that are not distinguished by $\mathcal{N}_{k-1}$.

Then, clearly, $\mathcal{N} := \bigcup_k \mathcal{N}_k$ will satisfy the assertions of the theorem. It remains to construct the separation systems $\mathcal{N}_k$.

As $\mathcal{N}_0$ we take the set of all separations that have one component of $G$ on one side and the union of all the other components on the other side. In particular, if $G$ is connected we let $\mathcal{N}_0$ be empty. Clearly, $\mathcal{N}_0$ satisfies (i)–(v).

For $k > 0$ we assume inductively that we already have separation systems $\mathcal{N}_{k'}$ satisfying (i)–(v) for $k' = 0, \dots, k-1$. Then

$$\text{For all } \ell \leq k, \text{ robust } \ell\text{-blocks } b_1, b_2 \text{ of } G \text{ that are not distinguished by } \mathcal{N}_{\ell-1} \text{ cannot be separated in } G \text{ by fewer than } \ell \text{ vertices.} \tag{15}$$

For suppose they can. Then $b_1$ and $b_2$ extend to distinct $(\ell-1)$-blocks. By Lemma 6.4 (ii) these are again robust, so by hypothesis (iv) for $\ell - 1$ they are distinguished by $\mathcal{N}_{\ell-1}$. But then $\mathcal{N}_{\ell-1}$ also distinguishes $b_1$ from $b_2$, by hypothesis (iii) for $\ell - 1$, a contradiction.

By hypothesis (iii), every $k$-block is $\mathcal{N}_{k-1}$-inseparable, so it extends to some $\mathcal{N}_{k-1}$-block; let $\mathcal{B}$ denote the set of those $\mathcal{N}_{k-1}$-blocks that contain more than one robust $k$-block. For each $b \in \mathcal{B}$ let $\mathcal{I}_b$ be the set of all robust $k$-blocks contained in $b$. Let $\mathcal{S}_b$ denote the set of all those $k$-separations of $G$ that separate some two elements of $I_b$ and are nested with all the separations in $\mathcal{N}_{k-1}$.

Clearly $\mathcal{S}_b$ is symmetric, so it is a separation system of $G$. By (15) for $\ell = k$, the separations in $\mathcal{S}_b$ are tight. Our aim is to apply Theorem 5.2 to extract from $\mathcal{S}_b$ a nested subsystem $\mathcal{N}_b$ that we can add to $\mathcal{N}_{k-1}$.

Before we verify the premise of Theorem 5.2, let us prove that it will be useful:

(∗) *$\mathcal{S}_b$ distinguishes every two elements of $\mathcal{I}_b$.*

For a proof of (∗) it suffices to find for any two blocks $I_1, I_2 \in \mathcal{I}_b$ a separation in $\mathcal{S}_b$ that separates them: since $I_1, I_2$ are proper subsets of $b$ they have rank $k$, not smaller.

So let distinct $I_1, I_2 \in \mathcal{I}_b$ be given. By Lemma 2.1, there is a separation $(A, B)$ of order at most $k$ such that $I_1 \subseteq A$ and $I_2 \subseteq B$. Choose $(A, B)$ so that it is nested with as many separations in $\mathcal{N}_{k-1}$ as possible. We prove that $(A, B) \in \mathcal{S}_b$, by showing that $(A, B)$ has order exactly $k$ and is nested with every separation $(C, D) \in \mathcal{N}_{k-1}$. Let $(C, D) \in \mathcal{N}_{k-1}$ be given.

Being elements of $\mathcal{I}_b$, the sets $I_1$ and $I_2$ cannot be separated by fewer than $k$ vertices, by (15). Hence $(A, B)$ has order exactly $k$, and $(C, D)$ does not separate $I_1$ from $I_2$, by (iii). Since $I_1$ is $k$-inseparable it lies on one side of $(C, D)$, say in $C$, so $I_1 \subseteq A \cap C$. As $(C, D)$ does not separate $I_1$ from $I_2$, we then have $I_2 \subseteq B \cap C$.

Let $\ell < k$ be such that $(C, D) \in \mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1}$. By hypothesis (v) for $\ell$, there are robust $\ell$-blocks $J_1 \subseteq C$ and $J_2 \subseteq D$ that are not distinguished by $\mathcal{N}_{\ell-1}$. By (15),

$$J_1 \text{ and } J_2 \text{ are not separated in } G \text{ by fewer than } \ell \text{ vertices.} \tag{16}$$

Being robust, each of $J_1, J_2$ is an $m$-robust $m$-block for some $m$. These $m$ must be at least $\ell$, since $J_1$ and $J_2$ are properly contained in $J_1 \cup J_2$, which by (16) is $(\ell - 1)$-inseparable. By Lemma 6.4 (i), then,

$$J_1 \text{ and } J_2 \text{ are } \ell\text{-robust.} \tag{17}$$

Let us show that we may assume the following:

> The corner separations of the corners $A \cap C$ and $B \cap C$ are nested with every separation $(C', D') \in \mathcal{N}_{k-1}$ that $(A, B)$ is nested with. (18)

Since $(C, D)$ and $(C', D')$ are both elements of $\mathcal{N}_{k-1}$, they are nested with each other. Thus,

$$(A, B) \parallel (C', D') \parallel (C, D).$$

Unless $(A, B)$ is nested with $(C, D)$ (in which case our proof of $(*)$ is complete), this implies by Lemma 2.2 that $(C', D')$ is nested with all the corner separations of the cross-diagram for $(A, B)$ and $(C, D)$, especially with those of the corners $A \cap C$ and $B \cap C$ that contain $I_1$ and $I_2$.

Since the corner separations of $A \cap C$ and $B \cap C$ are nested with the separation $(C, D) \in \mathcal{N}_{k-1}$ that $(A, B)$ is not nested with (as we assume), (18) and the choice of $(A, B)$ imply that

$$|\partial(A \cap C)| \geq k + 1 \quad \text{and} \quad |\partial(B \cap C)| \geq k + 1.$$

Since the sizes of the boundaries of two opposite corners sum to

$$|A \cap B| + |C \cap D| = k + \ell,$$

this means that the boundaries of the corners $A \cap D$ and $B \cap D$ have sizes $< \ell$. Since $J_2$ is $\ell$-robust, by (17), we have $J_2 \subseteq A \cap D$ or $J_2 \subseteq B \cap D$, say the former. But as $J_1 \subseteq C \subseteq B \cup C$, this contradicts (16).

Let us now verify the premise of Theorem 5.2:

$(**)$ *$\mathcal{S}_b$ separates $\mathcal{I}_b$ well.*

Consider a pair $(A,B),(C,D) \in \mathcal{S}_b$ of crossing separations with sets $I_1, I_2 \in \mathcal{I}_b$ such that $I_1 \subseteq A \cap C$ and $I_2 \subseteq B \cap D$. We shall prove that $(A \cap C, B \cup D) \in \mathcal{S}_b$.

By (15) and $I_1, I_2 \in \mathcal{I}_b$, the boundaries of the corners $A \cap C$ and $B \cap D$ have size at least $k$. Since their sizes sum to $|A \cap B| + |C \cap D| = 2k$, they each have size exactly $k$. So all that remains to show is that $(A \cap C, B \cup D)$ is nested with every separation $(C', D') \in \mathcal{N}_{k-1}$. But this follows once more by Lemma 2.2, because $(A,B),(C,D) \in \mathcal{S}_b$ implies that $(A,B)$ and $(C,D)$ are both nested with $(C', D') \in \mathcal{N}_{k-1}$. This completes the proof of $(**)$.

By $(*)$ and $(**)$, Theorem 5.2 implies that $\mathcal{S}_b$ has a nested $\mathcal{I}_b$-relevant subsystem $\mathcal{N}_b := \mathcal{N}(\mathcal{S}_b, \mathcal{I}_b)$ that weakly distinguishes all the sets in $\mathcal{I}_b$. But these are $k$-inseparable and hence of size $> k$, so they cannot lie inside a $k$-separator. So $\mathcal{N}_b$ even distinguishes the sets in $\mathcal{I}_b$ properly. Let

$$\mathcal{N}_{\mathcal{B}} := \bigcup_{b \in \mathcal{B}} \mathcal{N}_b \quad \text{and} \quad \mathcal{N}_k := \mathcal{N}_{k-1} \cup \mathcal{N}_{\mathcal{B}}.$$

Let us verify the inductive statements (i)–(v) for $k$. We noted earlier that every $\mathcal{S}_b$ is tight, hence so is every $\mathcal{N}_b$. The separations in each $\mathcal{N}_b$ are nested with each other and with $\mathcal{N}_{k-1}$. Separations from different sets $\mathcal{N}_b$ are nested by Lemma 2.3. So the entire set $\mathcal{N}_k$ is nested. Since $\mathcal{N}_{k-1}$ is $\mathrm{Aut}(G)$-invariant, by hypothesis (i), so is $\mathcal{B}$. For every automorphism $\alpha$ and every $b \in \mathcal{B}$ we then have $\mathcal{I}_{b^\alpha} = (\mathcal{I}_b)^\alpha$ and $\mathcal{S}_{b^\alpha} = (\mathcal{S}_b)^\alpha$, so Corollary 5.3 yields $(\mathcal{N}_b)^\alpha = \mathcal{N}_{b^\alpha}$. Thus, $\mathcal{N}_{\mathcal{B}}$ is $\mathrm{Aut}(G)$-invariant too, completing the proof of (i). Assertions (ii) and (iii) hold by definition of $\mathcal{N}_k$. Assertion (v) holds, because each $\mathcal{N}_b$ is $\mathcal{I}_b$-relevant.

To verify (iv), consider incomparable robust blocks $b_1, b_2$ of ranks $k_1 \le k$ and $k_2 \ge k_1$, respectively. By the induction hypothesis, we may assume that $k_1 = k$. Extend $b_2$ to a $k$-block $b_2'$; this is robust by Lemma 6.4 (ii), and distinct from $b_1$ since $b_2 \not\subseteq b_1$. Unless $b_1, b_2'$ are already distinguished by $\mathcal{N}_{k-1}$, they are robust $k$-blocks in the same $\mathcal{N}_{k-1}$-block $b$, and hence are distinguished by $\mathcal{N}_b \subseteq \mathcal{N}$. As $|b_2| > k$, the $k$-separator in $\mathcal{N}_b$ witnessing this also separates $b_1$ from $b_2$. $\square$

We now come to the second main result of this paper, a slight strengthening of Theorem 2 from the Introduction. See there for how the theorem may be read.

**Theorem 6.6.** *For every finite graph $G$ there is a sequence $(\mathcal{T}_k, \mathcal{V}_k)_{k \in \mathbb{N}}$ of tree-decompositions such that, for all $k$,*

(i) *every $k$-inseparable set is contained in a unique part of $(\mathcal{T}_k, \mathcal{V}_k)$;*

(ii) *any two incomparable robust blocks $b_1, b_2$ with $\min\{r(b_1), r(b_2)\} = k$ lie in different parts of $(\mathcal{T}_k, \mathcal{V}_k)$;*

(iii) *$(\mathcal{T}_k, \mathcal{V}_k)$ has adhesion at most $k$;*

(iv) *$(\mathcal{T}_k, \mathcal{V}_k) \preccurlyeq (\mathcal{T}_{k+1}, \mathcal{V}_{k+1})$;*

(v) *$\mathrm{Aut}(G)$ acts on $\mathcal{T}_k$ as a group of automorphisms.*

*Proof.* Consider the nested separation system $\mathcal{N}$ given by Theorem 6.5. As in the proof of that theorem, let $\mathcal{N}_k$ be the subsystem of $\mathcal{N}$ consisting of its separations of order at most $k$. By Theorem 6.5, both $\mathcal{N}$and $\mathcal{N}_k$ are $\mathrm{Aut}(G)$-invariant, for every $k$.

Let $(\mathcal{T}_k, \mathcal{V}_k)$ be the tree-decomposition associated with $\mathcal{N}_k$ as in Section 4. Then (v) holds by Corollary 3.6, (iii) and (iv) by Theorem 4.8 (iii) and (iv). By (iii) and [1, Lemma 12.3.1], any $k$-inseparable set is contained in a unique part of $(\mathcal{T}_k, \mathcal{V}_k)$, giving (i). By (iv) of Theorem 6.5, blocks $b_1, b_2$ as in (ii) are distinguished by $\mathcal{N}_k$. By (i) and Lemma 4.7, this implies that such blocks lie in different parts of $(\mathcal{T}_k, \mathcal{V}_k)$, giving (ii). □

By Theorem 6.6 (iii), a part of the decomposition $(\mathcal{T}_k, \mathcal{V}_k)$ that contains a block of order $\geq k$ cannot be contained in an adjacent part, and therefore in no other part. Induction along the structure of $\mathcal{T}_k$ thus shows that $(\mathcal{T}_k, \mathcal{V}_k)$ has at most $|G|$ parts containing blocks of order $\geq k$. By (i), every such block lies in a part, and by (ii) no part contains two incomparable robust such blocks. So in particular, $G$ has at most $|G|$ robust $k$-blocks. Applying the same reasoning inductively with Theorem 6.6 (iv) as $k$ grows yields the following:

**Corollary 6.7.** *Every set of pairwise incomparable robust blocks in $G$ has at most $|G|$ elements. In particular, for every $k$ there are at most $|G|$ robust $k$-blocks.* □


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