

# A Smooth Model for the String Group

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## Abstract

We construct a model for the string group as an infinite-dimensional Lie group. In a second step we extend this model by a contractible Lie group to a Lie 2-group model. To this end we need to establish some facts on the homotopy theory of Lie 2-groups. Moreover, we provide an explicit comparison of string structures for the two models.

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## 1 Introduction

String structures and the string group play an important role in algebraic topology [He08b, Lu09, BN09], string theory [Ki87, FM06] and geometry [Wi88, St96]. The group *String* is defined to be a 3-connected cover of the spin group or, more generally of any simple simply

connected compact Lie group  $G$  [ST04]. This definition fixes only its homotopy type and makes abstract homotopy theoretic constructions possible. But for geometric applications these models are not very well suited, one is rather interested in concrete models that carry, for instance, topological or even Lie-group structures.

There is a direct cohomological argument showing that  $String_G$  cannot be a finite  $CW$ -complex or a finite-dimensional manifold (see Corollary 3.3), so the best thing one can hope for is a topological group or an infinite-dimensional Lie group. There have been various constructions of models of  $String_G$  as  $A_\infty$ -spaces or topological groups, but the question whether an infinite-dimensional Lie group model is also possible has been open so far. One of the main contributions of the present paper is to give an affirmative answer to this question and provide an explicit Lie group model, based on a topological construction of Stolz [St96].

Something that is not directly apparent from the setting of the problem is that string group models as Lie 2-groups are something more natural to expect when taking the perspective of string theory or higher homotopy theory into account. However, the notion of a Lie 2-group model deserves a thorough clarification itself. We discuss this notion carefully by establishing the relevant homotopy theoretic facts about infinite-dimensional Lie 2-groups and promote our Lie group model  $String_G$  to such a Lie 2-group model  $STRING_G$ .

Before we outline our construction let us briefly summarize the existing ones. One model for  $String_G$  can be obtained from pulling back the path fibration  $PK(\mathbb{Z}, 3) \rightarrow K(\mathbb{Z}, 3)$  along a characteristic map  $u: G \rightarrow K(\mathbb{Z}, 3)$ . This is a standard construction of the Whitehead tower and leads to a model of  $String_G$  as a space. Since this construction also works for a characteristic map  $BG \rightarrow K(\mathbb{Z}, 4)$ , each 3-connected cover is homotopy equivalent to a loop space and thus admits an  $A_\infty$ -structure. Taking a functorial construction of the Whitehead tower one even obtains a model as a topological group. Unfortunately, these models are not very tractable.

There are more geometric constructions of  $String_G$ , for instance the one by Stolz in [St96]. The model given there has as an input the basic principal  $PU(\mathcal{H})$ -bundle  $P$  over  $G$ , where  $\mathcal{H}$  is a separable Hilbert space. Stolz then defines a model for  $String_G$  as a topological group together with a homomorphism  $String_G \rightarrow G$  whose kernel is the group of continuous gauge transformations of the bundle  $P$ . Our constructions will be based on this idea. In [ST04] Stolz and Teichner construct a model for  $String_G$  as an extension of  $G$  by  $PU(\mathcal{H})$ . It is a natural idea to equip this model with a smooth structure. But this does not work since this extension is constructed as a pushout along a positive energy representation of the loop group of  $G$  which is not smooth.

We now come to Lie 2-group models. One construction has been given by Henriques [He08a], based on work of Getzler [Ge09]. Its basic idea is to apply a general integration procedure for  $L_\infty$ -algebras to the string Lie 2-algebra. To make this construction work one has to weaken the naive notion of a Lie 2-group and besides that work in the category of Banach spaces. Similarly, the model of Schommer-Pries [SP10] realizes  $String_G$  as a stacky Lie 2-group, but it has the advantage of being finite-dimensional. This model is constructed from a cocycle in Segal's Cohomology for  $G$  [Se70].

A common thing about the above Lie 2-group models is that they are not strict, i.e., not associative on the nose but only up to an additional coherence. This complication is not present in the strict 2-group model of Baez, Crans, Schreiber and Stevenson from [BCSS07]. It is constructed from a crossed module  $\widehat{\Omega G} \rightarrow P_e G$ , built out of the level one Kac-Moody central extension  $\widehat{\Omega G}$  of the loop group of  $G$  and its path space  $P_e G$ . The price to pay is

that the model is infinite dimensional, but the strictness makes the corresponding bundle theory more tractable [NW11].

Summarizing, quite some effort has been made in constructing models for  $String_G$  that are as close as possible to finite-dimensional Lie groups. However, one of the most natural questions, namely whether there exists an infinite-dimensional *Lie group* model for  $String_G$  is still open. We answer this question by the following result.

Let  $P \rightarrow G$  be a basic smooth principal  $PU(\mathcal{H})$ -bundle, i.e.,  $[P] \in [G, BPU(\mathcal{H})] \cong H^3(G, \mathbb{Z}) = \mathbb{Z}$  is a generator. In Section 2 we review the fact that  $\mathcal{G}au(P)$  is a Lie group modeled on the infinite-dimensional space of vertical vector fields on  $P$ . The main result of Section 3 is then

**Theorem** (Theorem 3.6). *Let  $G$  be a simple, simply connected and compact Lie group, then there exists a smooth string group model  $String_G$  turning*

$$\mathcal{G}au(P) \rightarrow String_G \rightarrow G$$

*into an extension of Lie groups. It is uniquely determined up to isomorphism by this property.*

From now on  $String_G$  will always refer to this particular model. The proof of the theorem is based on [St96] and [Wo07]. We also show that  $String_G$  is metrizable and Fréchet. This metrizability makes the homotopy theory that we use in the sequel work due to results of Palais [Pa66].

In Section 4 we introduce the concept of Lie 2-group models culminating in Definition 4.10. An important construction in this context is the geometric realization that produces topological groups from Lie 2-groups. We show that geometric realization is well-behaved under mild technical conditions, such as metrizability.

In Section 5 we then construct a central extension  $U(1) \rightarrow \widehat{\mathcal{G}au}(P) \rightarrow \mathcal{G}au(P)$  with contractible  $\widehat{\mathcal{G}au}(P)$ . We define an action of  $String_G$  on  $\widehat{\mathcal{G}au}(P)$  such that  $\widehat{\mathcal{G}au}(P) \rightarrow String_G$  is a smooth crossed module. Crossed modules are a source for Lie 2-groups (Example 4.3) and in that way we obtain a Lie 2-group  $STRING_G$ .

**Theorem** (Theorem 5.6).  *$STRING_G$  is a Lie 2-group model in the sense of Definition 4.10.*

The proof of this theorem relies on a comparison of the model  $String_G$  with the geometric realization of  $STRING_G$ . Moreover, this direct comparison allows to derive a comparison between the corresponding bundle theories and string structures, see Section 6. This explicit comparison is a distinct feature of our 2-group model that is not available for the other 2-group models.

In an appendix we have collected some elementary facts about infinite dimensional manifolds and Lie groups. A second appendix gives a useful characterization of smooth weak equivalences between Lie 2-groups.

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## 2 Preliminaries on gauge groups

Throughout this paper Lie groups are permitted to be infinite-dimensional. More precisely, a Lie group is a group, together with the structure of a locally convex manifold such that the group operations are smooth, see Appendix A. The term topological group throughout refers to a group in compactly generated spaces.

In this section the setting will be as follows:

- $M$  is a compact manifold.
- $K$  is a metrizable Banach–Lie group (or equivalently a paracompact Banach–Lie group).
- $P$  is a smooth principal  $K$ -bundle over  $M$ .

Note that if  $P$  is only a continuous principal bundle, then we always find a smooth principal bundle which is equivalent to it [MW09].

**Definition 2.1.** The group  $\text{Aut}(P)$  denotes the group of  $K$ -equivariant diffeomorphisms  $f: P \rightarrow P$ . Identifying  $M$  with  $P/K$  we have a natural homomorphism

$$Q: \text{Aut}(P) \rightarrow \text{Diff}(M), \quad Q(f)([p]) = [f(p)]$$

and we define the *gauge group* by  $\mathcal{Gau}(P) := \ker(Q)$ .

It will be convenient to identify  $\mathcal{Gau}(P)$  with  $C^\infty(P, K)^K$ , the smooth  $K$ -equivariant maps  $P \rightarrow K$ , via

$$C^\infty(P, K)^K \ni f \mapsto (p \mapsto p \cdot f(p)) \in \mathcal{Gau}(P).$$

If  $P$  is topologically trivial, then the left hand side  $C^\infty(P, K)^K$  is isomorphic to  $C^\infty(M, K)$ . In [Wo07] it is shown that in a certain sense this remains valid if  $P$  is only locally trivial

**Proposition 2.2.** *The group  $\mathcal{Gau}(P) \cong C^\infty(P, K)^K$  admits the structure of a Fréchet Lie-group modeled on the gauge algebra  $\mathfrak{gau}(P) := C^\infty(P, \mathfrak{k})^K$  of smooth equivariant maps  $P \rightarrow \mathfrak{k}$ . If  $\exp: \mathfrak{k} \rightarrow K$  is the exponential function of  $K$ , then*

$$\exp_*: C^\infty(P, \mathfrak{k})^K \rightarrow C^\infty(P, K)^K, \quad \xi \mapsto \exp \circ \xi \tag{1}$$

*is an exponential function and a local diffeomorphism.*

*Proof.* The proof of this proposition can be found in [Wo07, Theorem 1.11 and Lemma 1.14(c)]. We will therefore only sketch the arguments that become important in the sequel.

Let  $N$  be a manifold with boundary (the boundary might be empty) modeled on a locally convex space. The space  $C^\infty(N, K)$  can be given a topology by pulling back the compact open topology along

$$C^\infty(N, K) \rightarrow \prod_{i=0}^{\infty} C^0(T^i N, T^i K).$$

We refer to this topology as the  $C^\infty$ -topology. This also applies to the Lie algebra  $\mathfrak{k}$  of  $K$  and induces a locally convex vector space topology on  $C^\infty(N, \mathfrak{k})$ . Moreover,  $C^\infty(N, \mathfrak{k})$  is a Fréchet space if  $N$  is finite-dimensional [GL02]. If we now restrict to the case where  $N$  is compact and if  $\varphi: U \subset K \rightarrow W \subset \mathfrak{k}$  is a chart satisfying  $\varphi(e) = 0$ , then  $C^\infty(N, W)$  is in particular open in  $C^\infty(N, \mathfrak{k})$  and thus

$$\varphi_*: C^\infty(N, U) \rightarrow C^\infty(N, W), \quad \gamma \mapsto \varphi \circ \gamma \quad (2)$$

defines a manifold structure on  $C^\infty(N, U)$ . It can be shown that the (point-wise) group structures are compatible with this smooth structure and that it may be extended to a Lie group structure on  $C^\infty(N, K)$ . Details of this construction can be found in [Wo06] and [GN11].

The aforementioned topologies also endow the subspaces  $C^\infty(P, K)^K$  and  $C^\infty(P, \mathfrak{k})^K$  with the structure of topological groups and  $C^\infty(P, \mathfrak{k})^K$  with the structure of a topological Lie algebra, both with respect to point-wise operations. The exponential function  $\exp: \mathfrak{k} \rightarrow K$  is  $K$ -equivariant and, by the inverse function theorem for Banach spaces, a local diffeomorphism. It thus defines in particular a map

$$\exp_*: C^\infty(P, \mathfrak{k})^K \rightarrow C^\infty(P, K)^K, \quad \xi \mapsto \exp \circ \xi$$

Like in the case of a compact manifold with boundary  $N$ , it can be shown that this map restricts to a bijection on some open subset of  $C^\infty(P, \mathfrak{k})^K$ , which then gives rise to a manifold structure around the identity in  $C^\infty(P, K)^K$  that can be enlarged to a Lie group structure. The details of this are spelled out in [Wo07, Propositions 1.4 and 1.8].  $\square$

**Lemma 2.3.** *The topology underlying  $\mathcal{G}au(P)$  is metrizable.*

*Proof.* We first note that  $C^\infty(N, K)$  is metrizable for finite-dimensional  $N$  since  $C^0(T^i N, T^i K)$  is so [Bo98a, X.3.3] and countable products of metrizable spaces are metrizable. From [Wo07, Proposition 1.8] it follows that  $\mathcal{G}au(P)$  is identified with a closed subspace of  $C^\infty(\coprod \overline{V}_i, K)$ , where  $V_i, \dots, V_n$  is a cover of  $M$  such that  $\overline{V}_i$  is a manifold with boundary and  $P|_{\overline{V}_i}$  is trivial. Since  $C^\infty(\coprod \overline{V}_i, K)$  is metrizable,  $\mathcal{G}au(P)$  is so as well.  $\square$

**Remark 2.4.** ([Wo07, Remark 1.18]) There also is a continuous version of the gauge group, namely the group of  $K$ -equivariant homeomorphisms  $P \rightarrow P$  covering the identity on  $M$ . This group will be denoted  $\mathcal{G}au^c(P)$ . As above, we have that  $\mathcal{G}au^c(P) \cong C(P, K)^K$  and since  $C(X, K)$  is a Lie group modeled on  $C(X, \mathfrak{k})$  for each compact topological space  $X$  (with respect to the compact-open topology, cf. [GN11]) the above proof carries over to show that  $\mathcal{G}au^c(P)$  is also a metrizable Lie group modeled on  $C(P, \mathfrak{k})^K$ .

Now [Wo07, Proposition 1.20] and Theorem A.5 imply

**Proposition 2.5.** *The canonical inclusion*

$$\mathcal{G}au(P) \hookrightarrow \mathcal{G}au^c(P). \quad (3)$$

*is a homotopy equivalence.*

In the sequel we will also need the following slight variation. Consider a central extension

$$Z \rightarrow \widehat{K} \rightarrow K$$

of Banach–Lie groups admitting smooth local sections. Similar to  $\mathcal{G}au(P) \cong C^\infty(P, K)^K$ , the groups  $C^\infty(G, Z)$  and  $C^\infty(P, \widehat{K})^K$  possess Lie group structures, modeled on  $C^\infty(G, \mathfrak{z})$  and  $C^\infty(P, \widehat{\mathfrak{k}})^K$  [NW09, Appendix A], [Wo07, Theorem 1.11]. As in Proposition 2.2, charts can be obtained from the exponential map

$$\exp_*: C^\infty(P, \widehat{\mathfrak{k}})^K \rightarrow C^\infty(P, \widehat{K})^K, \quad \xi \mapsto \exp \circ \xi.$$

Moreover this is a central extension, as we show in proposition 2.7.

**Lemma 2.6.** ([EG54]) *If  $F \rightarrow E \rightarrow B$  is a fiber bundle with  $F$  and  $B$  metrizable, then  $E$  is metrizable.*

**Proposition 2.7.** *Let  $Z \rightarrow \widehat{K} \xrightarrow{q} K$  be a central extension of Banach–Lie groups, admitting a local smooth section. Then the exact sequence of Fréchet–Lie groups*

$$C^\infty(M, Z) \rightarrow C^\infty(P, \widehat{K})^K \rightarrow C^\infty(P, K)^K \quad (4)$$

*admits a smooth local section. Moreover,  $C^\infty(M, \widehat{K})^K$  is metrizable if  $Z$  and  $K$  are so.*

*Proof.* We have to recall some facts on the construction of the Lie group structure from [NW09, Appendix A] and [Wo07, Proposition 1.11]. Let  $V_1, \dots, V_n$  be an open cover of  $G$  such that each  $\overline{V_i}$  is a manifold (with boundary) and such that there exist smooth sections  $\sigma_i: \overline{V_i} \rightarrow P$ . These give rise to smooth transition functions  $k_{ij}: \overline{V_i} \cap \overline{V_j} \rightarrow K$  and we have that

$$\gamma \mapsto \Sigma(\gamma) := (\gamma \circ \sigma_i)_{i=1, \dots, n}$$

induces an isomorphism

$$C^\infty(P, K)^K \cong \{(\gamma_i)_{i=1, \dots, n} \in \prod_{i=1}^n C^\infty(\overline{V_i}, K) \mid \gamma_i = k_{ij} \cdot \gamma_j \cdot k_{ji} \text{ on } \overline{V_i} \cap \overline{V_j}\}$$

If now  $\exp: \mathfrak{k} \rightarrow K$  restricts to a diffeomorphism  $\exp: W \rightarrow U$ , then we have that

$$\mathfrak{W} := \{(\gamma_i)_{i=1, \dots, n} \in \prod_{i=1}^n C^\infty(\overline{V_i}, W) \mid \gamma_i = k_{ij} \cdot \gamma_j \cdot k_{ji} \text{ on } \overline{V_i} \cap \overline{V_j}\}$$

maps under  $\Sigma^{-1}$  to an identity neighborhood  $\Sigma^{-1}(\mathfrak{W})$  on which  $\exp_*$  restricts to a diffeomorphism (cf. [Wo07, Proposition 1.11]). Note that we may also assume w.l.o.g. that there exists a smooth section  $\tau: U \rightarrow \widehat{K}$  of  $q$  satisfying  $\tau(1_K) = 1_{\widehat{K}}$ .

Next we choose a smooth partition of unity  $\lambda_i: V_i \rightarrow [0, 1]$ . For  $\gamma \in \Sigma^{-1}(\mathfrak{W})$  we then set

$$\Lambda_i(\gamma) := \exp_*\left(\sum_{j \leq i} \lambda_j \cdot \log_*(\gamma)\right) \cdot \exp_*\left(\sum_{j < i} \lambda_j \cdot \log_*(\gamma)\right)^{-1}$$

and note that we have

$$\gamma = \Lambda_n(\gamma) \cdot \Lambda_{n-1}(\gamma) \cdots \Lambda_1(\gamma).$$

Moreover,  $\lambda_i(\pi(p)) = 0$  implies  $\Lambda_i(\gamma)(p) = 1$  and thus  $\text{supp}(\Lambda_i(\gamma)) \subset V_i$ . Moreover, we have  $\Sigma(\Lambda_i(\gamma))_i \in C^\infty(\overline{V}_i, W)$  by the definition of  $\mathfrak{W}$ .

We now use all the data that we collected so far to define lifts of each  $\Lambda_i(\gamma)$ . To this end we first introduce functions  $k_i: P|_{\overline{V}_i} \rightarrow K$ , defined by  $p = \sigma_i(\pi(p)).k_i(p)$ . Then the assignment

$$P|_{V_i} \ni p \mapsto k_i(p) \cdot \tau(\Sigma(\Lambda_i(\gamma))_i(\pi(p))) \quad (5)$$

is smooth since  $\tau$  and  $\Sigma(\Lambda_i(\gamma))_i$  are so and equivariant since  $k_i$  is so. Moreover, (5) vanishes on a neighborhood of each point in  $\partial\overline{V}_i$  since  $\lambda_i$  and thus  $\tau \circ \Sigma(\Lambda_i(\gamma))_i$  do so. Consequently, we may extend (5) by  $e_{\widehat{K}}$  to all of  $P$ , defining a lift  $\Theta_i(\gamma)$  of  $\Lambda_i(\gamma)$ . Indeed, we have for  $p \in \pi^{-1}(V_i)$

$$\begin{aligned} q(\Theta_i(\gamma)(p)) &= q(k_i(p) \cdot \tau(\Sigma(\Lambda_i(\gamma))_i(\pi(p)))) = k_i(p) \cdot q(\tau(\Sigma(\Lambda_i(\gamma))_i(\pi(p)))) = \\ &= k_i(p) \cdot \Sigma(\Lambda_i(\gamma))_i(\pi(p)) = k_i(p) \cdot \Lambda_i(\gamma)(\sigma_i(\pi(p))) = \Lambda_i(\sigma_i(\pi(p)).k_i(p)) = \Lambda_i(\gamma)(p) \end{aligned}$$

and for  $p \notin \pi^{-1}(V_i)$  we have  $q(\Theta_i(\gamma)(p)) = q(e_{\widehat{K}}) = e_K = \Lambda_i(\gamma)(p)$ . Eventually,

$$\Theta(\gamma) := \Theta_n(\gamma) \cdot \Theta_{n-1}(\gamma) \cdots \Theta_1(\gamma)$$

defines a lift of  $\gamma$ , since we have

$$\begin{aligned} q_*(\Theta_n(\gamma) \cdot \Theta_{n-1}(\gamma) \cdots \Theta_1(\gamma)) &= q_*(\Theta_n(\gamma)) \cdot q_*(\Theta_{n-1}(\gamma)) \cdots q_*(\Theta_1(\gamma)) = \\ &= \Lambda_n(\gamma) \cdots \Lambda_{n-1}(\gamma) \cdots \Lambda_1(\gamma) = \gamma. \end{aligned}$$

Since  $\Theta_i(\gamma)$  is constructed in terms of push-forwards of smooth maps, it depends smoothly on  $\gamma$  and so does  $\Theta(\gamma)$ .

The previous argument shows in particular that (4) is a fiber bundle (cf. A.1). As in Lemma 2.3 one sees that  $C^\infty(M, Z)$  is metrizable if  $Z$  is so, and thus the last claim follows from Lemma 2.6.  $\square$

**Remark 2.8.** Note that all results of this section remain valid in more general situations. For instance, if we replace  $K$  by an arbitrary Lie group with exponential function that is a local diffeomorphism, then  $\mathcal{Gau}(P)$  is a Lie group, modeled on  $\mathfrak{gau}(P)$ . Moreover, (1) still defines an exponential function which itself is a local diffeomorphism. If, in addition,  $K$  is metrizable, then the proof of Lemma 2.3 shows that  $\mathcal{Gau}(P)$  is also metrizable.

Proposition 2.7 generalizes to the situation where  $Z \rightarrow \widehat{K} \rightarrow K$  is a central extension of Lie groups for which  $\widehat{K}$  and  $K$  have exponential functions that are local diffeomorphisms. Since its proof only uses the fact that  $\widehat{K} \rightarrow K$  has a smooth local section, (4) still admits a smooth local section in this case.

This shows in particular that the construction applies to the smooth principal bundle  $\Omega G \rightarrow PG \rightarrow G$ , where  $\Omega G$  denotes the group of smooth loops (as for instance in [BCSS07, Section 3]) and the universal central extension  $U(1) \rightarrow \widehat{\Omega G} \rightarrow \Omega G$ .

### 3 The string group as a smooth extension of $G$

In this section we want to give a smooth model for the string group. Our construction is mainly based on [St96, Section 5]. By smooth model of the string group we mean a smooth 3-connected cover of a compact Lie-group  $G$  which is a Lie group itself. We are mainly interested in the case  $G = Spin(n)$  but we define more generally:

**Definition 3.1.** Let  $G$  be a compact, simple and simply connected Lie group. A *smooth string group model* for  $G$  is a Lie group  $\widehat{G}$  together with a smooth homomorphism

$$\widehat{G} \xrightarrow{q} G$$

such that  $q$  is a Serre fibration,  $\pi_k(\widehat{G}) = 0$  for  $k \leq 3$  and that  $\pi_i(q)$  is an isomorphism for  $i > 3$ .

**Proposition 3.2** (Cartan [Ca36]). *Let  $G$  be a compact, simple and simply-connected Lie group. Then*

$$\pi_2(G) = 0 \quad \text{and} \quad \pi_3(G) \cong H^3(G, \mathbb{Z}) \cong \mathbb{Z}.$$

**Corollary 3.3.** *If  $\widehat{G} \xrightarrow{q} G$  is a smooth string group model, then*

1.  $\ker(q)$  is a  $K(\mathbb{Z}, 2)$  (i.e.,  $\pi_k(\ker(q)) \cong \mathbb{Z}$  for  $k = 2$  and vanishes for  $k \neq 2$ );
2.  $\widehat{G}$  cannot be finite-dimensional.

*Proof.* 1. This follows from the long exact homotopy sequence.

2. If  $\widehat{G}$  were finite-dimensional, then it would have  $\ker(q)$  as a closed Lie subgroup. But by 1. we have  $H^{2n}(\ker(q), \mathbb{Z}) \cong H^{2n}(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z}$ , a contradiction.  $\square$

Now we come to the construction of our string group model. Let  $\mathcal{H}$  be an infinite-dimensional separable Hilbert space. Then it is well known that the projective unitary group  $PU(\mathcal{H})$ , together with the norm topology is a  $K(\mathbb{Z}, 2)$  [Ku65], so that  $BPU(\mathcal{H})$  is a  $K(\mathbb{Z}, 3)$ . Thus isomorphism classes of  $PU(\mathcal{H})$ -bundles over a manifold  $M$  are in bijection with  $H^3(M, \mathbb{Z})$ .

Now there is a canonical generator  $1 \in H^3(G, \mathbb{Z})$ . Let  $P \rightarrow G$  be a principal  $PU(\mathcal{H})$ -bundle over  $G$  that represents this generator. Note that  $PU(\mathcal{H})$  is a Banach–Lie group (see [GN03] and references therein) which is paracompact by [Du66, Theorem VIII.2.4] and [Br72, Theorem I.3.1]. In particular, it is metrizable. We can choose  $P$  to be smooth [MW09] and apply the results from Section 2. Recall in particular the map

$$Q : \text{Aut}(P) \rightarrow \text{Diff}(G)$$

that sends a bundle automorphism to its underlying diffeomorphism of the base.

**Definition 3.4.** Let  $G$  be connected, simple and simply connected and  $P \rightarrow G$  represent the generator  $1 \in H^3(G, \mathbb{Z})$ . Then we set

$$\text{String}_G := \{f \in \text{Aut}(P) \mid Q(f) \in G \subset \text{Diff}(G)\}$$

where the inclusion  $G \hookrightarrow \text{Diff}(G)$  sends  $g$  to left multiplication with  $g$ . In other words:  $\text{String}_G$  is the group consisting of bundle automorphisms that cover left multiplication in  $G$ .

Note that there is also a continuous version of  $\text{String}_G$ , given by

$$\text{String}_G^c := \{f \in \text{Homeo}(P) \mid f \text{ is } K\text{-equivariant and } Q(f) \in G \subset \text{Diff}(G)\}.$$

The motivation for constructing a smooth model for the String group as in the present paper now comes from the following fact [St96]. For the sake of completeness we include (a part of) the proof here.

**Proposition 3.5** (Stolz). *The fibration  $Q : \text{String}_G^c \rightarrow G$  is a 3-connected cover of  $G$ , i.e.  $\pi_i(\text{String}_G^c) = 0$  for  $i \leq 3$  and  $\pi_i(Q)$  is an isomorphism for  $i > 3$ .*

*Proof.* Pick a point in the fiber  $p \in P$  over  $1 \in G$ . Let  $ev$  be the evaluation that sends a bundle automorphisms  $f$  to  $f(p)$ . Then we obtain a diagram

$$\begin{array}{ccccc} \mathcal{G}au^c(P) & \longrightarrow & \text{String}_G^c & \xrightarrow{Q} & G \\ \downarrow ev & & \downarrow ev & & \downarrow \text{id} \\ PU(\mathcal{H}) & \longrightarrow & P & \xrightarrow{\pi} & G \end{array}$$

Now [St96, Lemma 5.6] asserts that  $ev : \mathcal{G}au^c(P) \rightarrow PU(\mathcal{H})$  is a (weak) homotopy equivalence. The long exact homotopy sequence and the Five Lemma then show that then  $ev : \text{String}_G^c \rightarrow P$  is also a homotopy equivalence. Hence it remains to show that  $P \rightarrow G$  is a 3-connected cover. By definition of  $P$  its classifying map

$$p : G \longrightarrow BPU(\mathcal{H}) \simeq K(\mathbb{Z}, 3)$$

is a generator of  $H^3(G, \mathbb{Z})$ , hence it induces isomorphisms on the first three homotopy groups. Thus the pullback  $P \cong p^*EPU(\mathcal{H})$  of the contractible space  $EPU(\mathcal{H})$  kills exactly the first three homotopy groups, i.e.  $P$  is a 3-connected cover.  $\square$

In the rest of this section we want to prove the following modification and enhancement of the preceding proposition. For its formulation recall that an extension of Lie groups is a sequence of Lie groups  $A \rightarrow B \rightarrow C$  such that  $B$  is a smooth locally trivial principal  $A$ -bundle over  $C$  [Ne07].

**Theorem 3.6.**  *$\text{String}_G$  is a smooth string group model according to Definition 3.1. Moreover,  $\text{String}_G$  is metrizable and there exists a Fréchet–Lie group structure on  $\text{String}_G$ , unique up to isomorphism, such that*

$$\mathcal{G}au(P) \rightarrow \text{String}_G \rightarrow G \tag{6}$$

*is an extension of Lie groups.*

*Proof.* We first show existence of the Lie group structure. To this end we recall that there exists an extension of Fréchet–Lie groups

$$\mathcal{G}au(P) \rightarrow \text{Aut}(P)_0 \rightarrow \text{Diff}(G)_0, \tag{7}$$

where  $\text{Aut}(P)_0$  is the inverse image  $Q^{-1}(\text{Diff}(G)_0)$  of the the identity component  $\text{Diff}(M)_0$  [Wo07, Theorem 2.14]. The embedding  $G \hookrightarrow \text{Diff}(G)_0$  given by left translation gives by the exponential law [GN11] a smooth homomorphism of Lie groups since the multiplication map  $G \times G \rightarrow G$  is smooth. Pulling back (7) along this embedding then yields the extension (6). Moreover,  $\text{String}_G$  is metrizable by Lemma 2.3 and Lemma 2.6.

We now discuss the uniqueness assertion, so let  $\mathcal{G}au(P) \rightarrow H_i \xrightarrow{q_i} G$  for  $i = 1, 2$  be two extensions of Lie groups. The requirement for it to be a locally trivial smooth principal bundle is equivalent to the existence of a smooth local section of  $q_i$  and we thus obtain a derived extension of Lie algebras

$$\mathfrak{gau}(P) \rightarrow L(H_i) \xrightarrow{L(q_i)} \mathfrak{g}.$$

The differential of the local smooth section implements a linear continuous section of  $L(q_i)$  and thus we have a (non-abelian) extension of Lie algebras in the sense of [Ne06]. Now the equivalence classes of such extensions are parametrized by  $H^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{gau}(P)))$  [Ne06, Theorem II.7]. Since  $\mathfrak{gau}(P) = C^\infty(P, \mathfrak{pu}(H))^K$  we clearly have  $\mathfrak{z}(\mathfrak{gau}(P)) = C^\infty(P, \mathfrak{z}(\mathfrak{pu}(H)))^K$ , which is trivial since  $\mathfrak{z}(\mathfrak{pu}(H))$  is so. Consequently, we have a morphism

$$\begin{array}{ccccc} \mathfrak{gau}(P) & \longrightarrow & L(H_1) & \longrightarrow & \mathfrak{g} \\ \parallel & & \downarrow \varphi & & \parallel \\ \mathfrak{gau}(P) & \longrightarrow & L(H_2) & \longrightarrow & \mathfrak{g} \end{array}$$

of extensions of Lie algebras. The long exact homotopy sequence for the fibration  $\mathcal{G}au(P) \rightarrow H_i \xrightarrow{q} G$  shows that  $H_i$  is 1-connected, and so  $\varphi$  integrates to a morphism

$$\begin{array}{ccccc} \mathcal{G}au(P) & \longrightarrow & H_1 & \longrightarrow & G \\ \parallel & & \downarrow \Phi & & \parallel \\ \mathcal{G}au(P) & \longrightarrow & H_2 & \longrightarrow & G \end{array}$$

of Lie groups. Since  $\Phi$  makes this diagram commute it is automatically an isomorphism.

It remains to show that  $\mathcal{String}_G$  is a smooth model for the String group. We have the following commuting diagram

$$\begin{array}{ccccc} \mathcal{G}au(P) & \longrightarrow & \mathcal{String}_G & \longrightarrow & G \\ \downarrow & & \downarrow & & \parallel \\ \mathcal{G}au^c(P) & \longrightarrow & \mathcal{String}_G^c & \longrightarrow & G \end{array}.$$

By Proposition 2.5 the inclusion  $\mathcal{G}au(P) \hookrightarrow \mathcal{G}au^c(P)$  is a homotopy equivalence. Since, furthermore,  $\mathcal{String}_G \rightarrow G$  and  $\mathcal{String}_G^c \rightarrow G$  are bundles, they are in particular fibrations and we obtain long exact sequences of homotopy groups. Applying the Five Lemma we see that the maps  $\pi_n(\mathcal{String}_G) \rightarrow \pi_n(\mathcal{String}_G^c)$  are isomorphisms for all  $n$ . By Proposition 3.5 we know that  $\mathcal{String}_G^c$  is a 3-connected cover, hence also  $\mathcal{String}_G$ .  $\square$

**Remark 3.7.** Note that the proof of the uniqueness assertion only used the fact that the center of  $\mathfrak{gau}(P)$  is trivial. In fact, this shows that for an arbitrary (regular) Lie group  $H$  which is a  $K(\mathbb{Z}, 2)$  and has trivial  $\mathfrak{z}(L(H))$  there exists, up to isomorphism, at most one Lie group  $\widehat{H}$ , together with smooth maps  $H \rightarrow \widehat{H}$  and  $\widehat{H} \rightarrow G$  turning

$$H \rightarrow \widehat{H} \rightarrow G$$

into an extension of Lie groups. Moreover, the proof shows that the uniqueness is not only up to isomorphism of Lie groups, but even up to isomorphism of *extensions*.

## 4 2-groups and 2-group models

One of the main problems about string group models is that they are not very tightly determined. In fact, the underlying space is just determined up to weak homotopy equivalence.

This implies that the group structure can only be determined up to  $A_\infty$ -equivalence and the smooth structure is not determined at all. Part of this problem is that there is in general not a good control about the fiber of  $\text{String}_G \rightarrow G$ , only the underlying homotopy type is determined to be a  $K(\mathbb{Z}, 2)$ .

Some of the problems can be cured by using 2-group models. This setting allows to fix the fiber more tightly. In particular there is a nice model of  $K(\mathbb{Z}, 2)$  as a 2-group, see Example 4.3 below and weak equivalences of 2-groups are more restrictive than homotopy equivalences of their geometric realizations. We first want to recall quickly the definition and some elementary properties of 2-groups. We restrict our attention to strict Lie 2-groups in this paper which for simplicity we just call Lie 2-groups.

**Definition 4.1.** A (strict) Lie 2-group is a category  $\mathcal{G}$  such that the set of objects  $\mathcal{G}_0$  and the set of morphisms  $\mathcal{G}_1$  are Lie groups, all structure maps

$$s, t : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \quad i : \mathcal{G}_0 \rightarrow \mathcal{G}_1 \quad \text{and} \quad \circ : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \rightarrow \mathcal{G}_1$$

are Lie group homomorphisms and  $s, t$  are submersions<sup>1</sup>. In the case that  $\mathcal{G}_0$  and  $\mathcal{G}_1$  are metrizable, we call  $\mathcal{G}$  a metrizable Lie 2-group. A morphism between 2-groups is a functor  $f : \mathcal{G} \rightarrow \mathcal{G}'$  that is a Lie group homomorphism on the level of objects and on the level of morphisms.

One reason to consider 2-groups here is that they can serve as models for topological spaces by virtue of the following construction.

**Definition 4.2.** Let  $\mathcal{G}$  be a Lie 2-group. Then the nerve  $N\mathcal{G}$  of the category  $\mathcal{G}$  is a simplicial manifold by Proposition A.3. Using this we define the *geometric realization* of  $\mathcal{G}$  to be the geometric realization of the simplicial space  $N\mathcal{G}$ , i.e., the coend

$$\int^{[n] \in \Delta} (N\mathcal{G})_n \times \Delta[n] = \bigsqcup_n (N\mathcal{G})_n \times \Delta[n] / \sim.$$

Note that the coend is taken in the category of compactly generated spaces.

- Example 4.3.**
1. Consider the category  $\mathcal{B}U(1)$  with one object and automorphisms given by the group  $U(1)$ . This is clearly a Lie 2-group. The geometric realization  $|\mathcal{B}U(1)|$  is the classifying space  $BU(1)$ , hence a  $K(\mathbb{Z}, 2)$ . The 2-group  $\mathcal{B}A$  exists moreover for each abelian Lie group  $A$ .
  2. If  $G$  is an arbitrary Lie group, then it gives rise to a 2-group by considering it as category with only identity morphisms. More precisely, in this case  $\mathcal{G}_0 = \mathcal{G}_1 = G$  and all structure maps are the identity.
  3. Let  $K \xrightarrow{\partial} L$  be a smooth crossed module of groups ([Ne07, Definition 3.1]). Then we can form a Lie 2-group  $\mathcal{G}$  using the Lie groups  $\mathcal{G}_0 := L$  and  $\mathcal{G}_1 := K \rtimes L$  together with the smooth maps  $s(k, l) = l$ ,  $t(k, l) = \partial(k)l$ ,  $i(l) = (1, l)$  and  $(k, l) \circ (k', l') = (kk', l)$ . Up to some technicalities, each Lie 2-group arises from a crossed module in this way.

**Lemma 4.4.** *If  $\mathcal{G}$  is a metrizable Lie 2-group, then*

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<sup>1</sup>Submersion in the sense that it is locally a projection, see Appendix A

1. all spaces  $N\mathcal{G}_n$  have the homotopy type of a CW complex;
2. the nerve  $N\mathcal{G}$  is good, i.e. all degeneracies are closed cofibrations;
3. the nerve  $N\mathcal{G}$  is proper, i.e. Reedy cofibrant as a simplicial space (with respect to the Strøm model structure);
4. the canonical map from the fat geometric realization  $\|N\mathcal{G}\|$  to the ordinary geometric realization  $|\mathcal{G}|$  is a homotopy equivalence;
5. the geometric realization  $|\mathcal{G}|$  has the homotopy type of a CW-complex.

*Proof.* 1) First note that all the spaces  $(N\mathcal{G})_n$  are subspaces of  $(\mathcal{G}_1)^n$  and thus are metrizable. Hence by Theorem A.5 they have the homotopy type of a CW-complex.

2) Again using the fact that all  $(N\mathcal{G})_n$  are metrizable and [Pa66, Theorem 7] we see that they are well-pointed in the sense that the basepoint inclusion is a closed cofibration. A statement of Roberts and Stevenson [RS11, Proposition 18] then shows that  $N\mathcal{G}$  is good, i.e., degeneracy maps are closed cofibrations. We roughly sketch a variant of their argument here: By the fact that  $\mathcal{G}$  is a 2-group we can write the nerve as

$$\cdots \rightrightarrows \ker(s) \times \ker(s) \times \mathcal{G}_0 \rightrightarrows \ker(s) \times \mathcal{G}_0 \rightrightarrows \mathcal{G}_0$$

where the decomposition is a decomposition on the level of topological spaces. Hence to show that the degeneracies are closed cofibrations it suffices to show that  $\ker(s)$  is well-pointed. But it is a retract of  $\mathcal{G}_1 = \mathcal{G}_0 \times \ker s$  hence well pointed by the fact that  $\mathcal{G}_1$  is well pointed.

3) Now we know that  $N\mathcal{G}$  is good and in this case [Le82, Corollary 2.4(b)] implies that  $N\mathcal{G}$  is also proper.

4) By [Se74, Proposition A1] (resp [tD74, Proposition 1]) the fat and the ordinary geometric realizations are homotopy equivalent.

5) Since all the spaces  $(N\mathcal{G})_n$  have the homotopy type of a CW-complex, also the fat geometric realization has the homotopy type of a CW complex [Se74, Proposition A1]. Thus also the ordinary realization by 4).  $\square$

**Proposition 4.5.** *If  $\mathcal{G}$  and  $\mathcal{G}'$  are metrizable Lie 2-groups and  $f : \mathcal{G} \rightarrow \mathcal{G}'$  is a homomorphism that is a weak homotopy equivalence on objects and morphisms, then*

$$|f| : |\mathcal{G}| \rightarrow |\mathcal{G}'|$$

*is a homotopy equivalence.*

*Proof.* First note that  $Nf : N\mathcal{G} \rightarrow N\mathcal{G}'$  is a levelwise weak homotopy equivalence. For the first two layers this is the assumption and for the rest it follows again from the product structure of the nerves given in the proof of Lemma 4.4 and the fact that  $Nf$  is also a product map. Then using [Ma74, Proposition A4] and the fact that  $N\mathcal{G}$  and  $N\mathcal{G}'$  are proper we conclude that also  $|f| : |\mathcal{G}| \rightarrow |\mathcal{G}'|$  is a weak homotopy equivalence. But since the geometric realizations have the homotopy type of a CW-complex, Whitehead's theorem shows that  $|f|$  is an honest homotopy equivalence.  $\square$

For smooth groupoids there is a notion of weak equivalence which is inspired by equivalence of the associated stacks, see e.g. [Me03, Definition 58 and Proposition 60]. We adopt this for 2-groups.

**Definition 4.6.** A morphism  $f : \mathcal{G} \rightarrow \mathcal{G}'$  of Lie 2-groups is called *smooth weak equivalence* if the following conditions are satisfied:

1. it is smoothly essentially surjective: the map

$$s \circ \text{pr}_2 : \mathcal{G}_0 \times_t \mathcal{G}'_1 \rightarrow \mathcal{G}'_0$$

is a surjective submersion.

2. it is smoothly fully faithful: the diagram

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{f_1} & \mathcal{G}'_1 \\ s \times t \downarrow & & \downarrow s \times t \\ \mathcal{G}_0 \times \mathcal{G}_0 & \xrightarrow{f_0 \times f_0} & \mathcal{G}'_0 \times \mathcal{G}'_0 \end{array}$$

is a pullback diagram.

**Proposition 4.7.** *Let  $f : \mathcal{G} \rightarrow \mathcal{G}'$  be a smooth weak equivalence between metrizable 2-groups. Then  $|f| : |\mathcal{G}| \rightarrow |\mathcal{G}'|$  is a homotopy equivalence.*

*Proof.* A smooth weak equivalence between 2-groups is in particular a topological weak equivalence of the underlying topological groupoids. But then Theorem 6.3 and Theorem 8.2. of [No10] together imply that the induced morphism  $\|f\| : \|\mathcal{G}\| \rightarrow \|\mathcal{G}'\|$  between the fat geometric realizations is a weak equivalence. Again by the fact the fat realizations are homotopy equivalent to the geometric realizations this completes the proof.  $\square$

**Definition 4.8.** If  $\mathcal{G}$  is a Lie 2-group, then we denote by  $\pi_0\mathcal{G}$  the group of isomorphism classes of objects in  $\mathcal{G}$  and by  $\pi_1\mathcal{G}$  the group of automorphisms of  $1 \in \mathcal{G}_0$ . Note that  $\pi_1\mathcal{G}$  is abelian. We call  $\mathcal{G}$  *smoothly separable* if  $\pi_1\mathcal{G}$  is a split Lie subgroup<sup>2</sup> of  $\mathcal{G}_1$  and  $\pi_0\mathcal{G}$  carries a Lie group structure such that  $\mathcal{G}_0 \rightarrow \pi_0\mathcal{G}$  is a submersion.

**Proposition 4.9.** *1. A morphism between smoothly separable Lie 2-groups is a smooth weak equivalence if and only if it induces Lie group isomorphisms on  $\pi_0$  and  $\pi_1$ .*

2. For a metrizable, smoothly separable Lie 2-group  $\mathcal{G}$  the sequence

$$|\mathcal{B}\pi_1\mathcal{G}| \rightarrow |\mathcal{G}| \rightarrow \pi_0\mathcal{G}$$

*is a fiber sequence of topological groups. Moreover, the right hand map is a fiber bundle and the left map is a homotopy equivalence to its fiber.*

*Proof.* The first claim will be proved in Appendix B. We thus show the second. Let us first consider the morphism  $q : \mathcal{G} \rightarrow \pi_0\mathcal{G}$  of 2-groups where  $\pi_0\mathcal{G}$  is considered as a 2-group with only identity morphisms. Let  $\mathcal{K}$  be the levelwise kernel of this map, i.e.,  $\mathcal{K}_0 = \ker(q_0)$  and  $\mathcal{K}_1 = \ker(q_1)$ . Since  $q_1 = q_0 \circ s$  it is a submersion,  $\mathcal{K}_0$  and  $\mathcal{K}_1$  are Lie subgroups and  $\mathcal{K}$  is a metrizable Lie 2-group. Then  $N\mathcal{K} \rightarrow N\mathcal{G} \rightarrow N\pi_0\mathcal{G}$  is an exact sequence of simplicial groups. It is easy to see that the geometric realization of this sequence is also exact, e.g., by using the fact that geometric realization preserves pullbacks [Ma74, Corollary 11.6]. Hence we have an exact sequence of topological groups.

$$|\mathcal{K}| \rightarrow |\mathcal{G}| \rightarrow \pi_0\mathcal{G}$$

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<sup>2</sup>Split Lie subgroup in the sense of Definition A.2

Moreover the right hand map is a  $|\mathcal{K}|$ -bundle since by the definition of smooth separability it admits local sections. Thus it only remains to show that  $|\mathcal{B}\pi_1\mathcal{G}| \simeq |\mathcal{K}|$ . Now the inclusion  $\mathcal{B}\pi_1\mathcal{G} \rightarrow \mathcal{K}$  is a smooth weak equivalence, which we can either see using the first part of the Proposition or by a direct argument. Then Proposition 4.7 shows that the realization is a homotopy equivalence.  $\square$

**Definition 4.10.** Let  $G$  be a compact simple and simply connected Lie group. A *smooth 2-group model* for the string group is a smooth 2-group  $\mathcal{G}$  which is smoothly separable together with isomorphisms

$$\pi_0\mathcal{G} \xrightarrow{\sim} G \quad \text{and} \quad \pi_1\mathcal{G} \xrightarrow{\sim} U(1)$$

such that  $|\mathcal{G}| \rightarrow G$  is a 3-connected cover.

**Remark 4.11.** • Note that for a smooth 2-group model the geometric realization  $|\mathcal{G}|$  with the canonical map  $|\mathcal{G}| \rightarrow G$  is automatically a topological group model for the string group.

- For a 2-group  $\mathcal{G}$  with isomorphisms  $\pi_0\mathcal{G} \xrightarrow{\sim} G$  and  $\pi_1\mathcal{G} \xrightarrow{\sim} U(1)$  we already know from Proposition 4.9 that  $|\mathcal{G}| \rightarrow G$  is a fibration with fiber  $|\mathcal{B}U(1)| \simeq K(\mathbb{Z}, 2)$ . Hence the condition that  $|\mathcal{G}| \rightarrow G$  is a 3-connected cover only ensures that it has the right level, i.e. the connecting homomorphism in the long exact homotopy sequence

$$\mathbb{Z} = \pi_3(G) \rightarrow \pi_2(K(\mathbb{Z}, 2)) = \mathbb{Z}$$

is an isomorphism.

- Considering  $\text{String}_G$  as a category with only identity morphisms we obtain a 2-group as in Example 4.3. However, in this case  $\pi_1\text{String}_G$  is trivial. So it is not a 2-group model as defined above, although its geometric realization is a topological group model.

## 5 The string group as a 2-group

The previous remark shows that Lie 2-group models have more structure than topological or Lie group models for the string group. In this section we promote our Lie group model from Section 3 to such a Lie 2-group model. Therefore the setting will be as in Section 3:  $G$  is a compact simple, simply-connected Lie group and  $P \rightarrow G$  is a smooth  $PU(\mathcal{H})$  bundle that represents the generator  $1 \in H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ .

Clearly we have the central extension  $U(1) \rightarrow U(\mathcal{H}) \rightarrow PU(\mathcal{H})$ . Furthermore  $PU(\mathcal{H})$  acts by conjugation on  $U(\mathcal{H})$ . Using these maps we obtain a sequence

$$C^\infty(G, U(1)) \rightarrow C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})} \rightarrow \mathcal{Gau}(P), \quad (8)$$

which is a central extension of Fréchet–Lie groups by Proposition 2.7.

For the next proposition note that each smooth function  $f \in C^\infty(G, U(1))$  is a quotient of a smooth function  $\hat{f} \in C^\infty(G, \mathbb{R})$  by the fact the  $G$  is simply connected. If we identify  $U(1)$  with  $\mathbb{R}/\mathbb{Z}$  we may thus identify  $C^\infty(G, U(1))$  with  $C^\infty(G, \mathbb{R})/\mathbb{Z}$ .

**Lemma 5.1.** *If  $\mu$  is the Haar measure on  $G$ , then the map*

$$I_G : C^\infty(G, U(1)) \rightarrow U(1), \quad I_G [\hat{f}] := \left[ \int_G \hat{f} d\mu \right]$$

*is a smooth group homomorphism. This map  $I_G$  is invariant under the right action of  $G$  on  $C^\infty(G, U(1))$  which is given by left multiplication in the argument.*

*Proof.* We denote by  $dI_G : C^\infty(G, \mathbb{R})$  the map on Lie algebras that is given by  $dI_G(f) := \int_G f d\mu$ . First note that  $dI_G$  is linear and continuous in the topology of uniform convergence since we have  $|\int_G f d\mu| \leq \int_G |f| d\mu$ . It thus is also continuous in the finer  $C^\infty$ -topology and in particular smooth. Furthermore it is invariant under left multiplication with  $G$ . Moreover,  $dI_G$  factors since it maps  $\mathbb{Z} \subset C^\infty(G, \mathbb{R})$  to  $\mathbb{Z} \subset \mathbb{R}$ .  $\square$

Now we can use the group homomorphism  $I_G$  to turn the smooth extension (8) into a  $U(1)$  extension:

**Definition 5.2.** We define

$$\widehat{\mathcal{G}au}(P) := C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})} \times U(1) / \sim,$$

where we identify  $(\varphi \cdot \mu, \lambda) \sim (\varphi, I_G(\mu) \cdot \lambda)$  for  $\mu \in C^\infty(G, U(1))$ .

**Proposition 5.3.** *The sequence*

$$U(1) \rightarrow \widehat{\mathcal{G}au}(P) \rightarrow \mathcal{G}au(P) \tag{9}$$

*is a central extension of metrizable Fréchet Lie groups and the space  $\widehat{\mathcal{G}au}(P)$  is contractible.*

*Proof.* By definition of  $\widehat{\mathcal{G}au}(P)$  it is just the association of the bundle  $C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})} \rightarrow \mathcal{G}au(P)$  along the homomorphism  $I_G : C^\infty(G, U(1)) \rightarrow U(1)$ . Hence it is a smooth manifold and a central extension of  $\mathcal{G}au(P)$ . More precisely we may take a locally smooth  $C^\infty(G, U(1))$ -valued cocycle describing the central extension (8). Composing this with  $I_G$  yields then a locally smooth cocycle representing the central extension (9) (cf. [Ne02, Proposition 4.2]). Since the modeling space is the product of the modeling space of the fiber and the base it is in particular Fréchet. In addition,  $\widehat{\mathcal{G}au}(P)$  is metrizable by Lemma 2.3 and Lemma 2.6.

Now we come to the second part of the claim. In order to show that  $\widehat{\mathcal{G}au}(P)$  is weakly contractible we first define another space  $\widetilde{\mathcal{G}au}(P)$  using the homomorphism  $ev : C^\infty(G, U(1)) \rightarrow U(1)$  instead of  $I_G$ . More precisely,

$$\widetilde{\mathcal{G}au}(P) := C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})} \times U(1) / \sim_{ev}$$

where we identify  $(\varphi \cdot \mu, \lambda) \sim_{ev} (\varphi, \mu(1) \cdot \lambda)$  for  $\mu \in C^\infty(G, U(1))$ . Note that  $ev$  is smooth since arbitrary point evaluations are so. Thus  $\widetilde{\mathcal{G}au}(P)$  is a  $U(1)$  central extension of  $\mathcal{G}au(P)$  as well as also metrizable by Lemma 2.6.

We claim that the  $\widehat{\mathcal{G}au}(P)$  and  $\widetilde{\mathcal{G}au}(P)$  are homeomorphic as spaces (not as groups). Therefore we first show that the homomorphisms  $ev$  and  $I_G$  are homotopic as group homomorphisms, i.e. there is a homotopy

$$H : C^\infty(G, U(1)) \times [0, 1] \rightarrow U(1)$$

such that each  $H_t := H(-, t)$  is a Lie group homomorphism,  $H_0 = ev$  and  $H_1 = I_G$ . We first define the smooth map

$$dH: C^\infty(G, \mathbb{R}) \times [0, 1] \rightarrow \mathbb{R}, \quad (f, t) \mapsto t \cdot f(1) + (1 - t) \cdot \int_G f \, d\mu$$

Since each  $dH_t$  maps  $\mathbb{Z}$  into  $\mathbb{Z}$  it in particular induces a smooth group homomorphism  $H_t$  via the identification  $C^\infty(G, U(1)) \cong C^\infty(G, \mathbb{R})/\mathbb{Z}$ . Now we can use  $H_t$  to define a  $U(1)$ -bundle  $E$  over  $\mathcal{G}au(P) \times [0, 1]$  by

$$E := C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})} \times U(1) \times [0, 1] / \sim_H$$

where we identify  $(\varphi \cdot \mu, \lambda, t) \sim_H (\varphi, H(\mu, t) \cdot \lambda, t)$ . Obviously  $E|_{\mathcal{G}au(P) \times 0} \cong \widetilde{\mathcal{G}au}(P)$  and  $E|_{\mathcal{G}au(P) \times 1} \cong \widehat{\mathcal{G}au}(P)$ . Thus  $\widetilde{\mathcal{G}au}(P)$  and  $\widehat{\mathcal{G}au}(P)$  are isomorphic as continuous bundles [tD08, Theorem 14.3.2].

Since we now know that  $\widehat{\mathcal{G}au}(P) \cong \widetilde{\mathcal{G}au}(P)$ , it is sufficient to show that  $\widetilde{\mathcal{G}au}$  is contractible. To this end we first pick a point  $p \in P$  in the fiber over  $1 \in G$ . Evaluation at  $p$  yields a group homomorphism

$$ev: \mathcal{G}au(P) = C^\infty(P, PU(\mathcal{H}))^{PU(\mathcal{H})} \rightarrow PU(\mathcal{H}).$$

which is a weak homotopy equivalence by [St96, Lemma 5.6] and Proposition 2.5. We now define another Lie group homomorphism  $\Phi: \widetilde{\mathcal{G}au}(P) \rightarrow U(\mathcal{H})$  by  $\Phi([\varphi, \lambda]) := \lambda \cdot \varphi(p)$ . By definition of  $\widetilde{\mathcal{G}au}(P)$  this is well defined and the diagram

$$\begin{array}{ccccc} U(1) & \longrightarrow & \widetilde{\mathcal{G}au}(P) & \longrightarrow & \mathcal{G}au(P) \\ \parallel & & \downarrow \Phi & & \downarrow ev \\ U(1) & \longrightarrow & U(\mathcal{H}) & \longrightarrow & PU(\mathcal{H}) \end{array}$$

commutes. Since  $ev$  is a weak homotopy equivalence it follows from the long exact homotopy sequence and the Five Lemma that also  $\Phi$  is a weak homotopy equivalence. Therefore the weak contractibility of  $\widetilde{\mathcal{G}au}(P)$  is implied by the weak contractibility of  $U(\mathcal{H})$ . This also implies contractibility of  $\widehat{\mathcal{G}au}(P)$  by Theorem A.5.  $\square$

Combining the two sequences (6) and (9) we obtain an exact sequence

$$1 \rightarrow U(1) \rightarrow \widehat{\mathcal{G}au}(P) \xrightarrow{\partial} String_G \rightarrow G \rightarrow 1 \quad (10)$$

of Fréchet Lie groups, where  $\partial$  is the composition  $\widehat{\mathcal{G}au}(P) \rightarrow \mathcal{G}au(P) \rightarrow String_G$ . We furthermore define a smooth right action of  $String_G$  on  $\widehat{\mathcal{G}au}(P)$  by:

$$[\varphi, \lambda]^f := [\varphi \circ f, \lambda] \quad \text{for } f \in String_G \subset \text{Aut}(P). \quad (11)$$

**Proposition 5.4.** *The action is well defined. Together with the morphism  $\partial: \widehat{\mathcal{G}au}(P) \rightarrow String_G$  this forms a smooth crossed module.*

*Proof.* The action is well-defined since for  $\varphi \in C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})}$ ,  $\mu \in C^\infty(G, U(1))$  and  $f \in \text{String}_G$  we have

$$[(\varphi \cdot \mu) \circ f, \lambda] = [(\varphi \circ f) \cdot (\mu \circ Q(f)), \lambda] = [\varphi \circ f, I_G(\mu \circ Q(f)) \cdot \lambda] = [\varphi \circ f, I_G(\mu) \cdot \lambda]$$

where the last equality holds by the fact that  $I_G$  is invariant under left multiplication as shown in Lemma 5.1.

The action of  $\text{Aut}(P)$  on  $\mathcal{G}au(P) \cong C^\infty(P, PU(\mathcal{H}))^{PU(\mathcal{H})}$ , given by  $\varphi^f := \varphi \circ f$  is the conjugation action of  $\mathcal{G}au(P)$  on itself [Wo07, Remark 2.8]. This shows that  $\partial$  is equivariant and that (10) and (11) define indeed a crossed module. It thus remains to show that the action map  $\widehat{\mathcal{G}au}(P) \times \text{String}_G \rightarrow \widehat{\mathcal{G}au}(P)$  is smooth. Since  $\text{String}_G$  acts by diffeomorphisms it suffices to show that the restriction of the action map  $U \times \widehat{\mathcal{G}au}(P) \rightarrow \widehat{\mathcal{G}au}(P)$  for  $U$  some identity neighborhood in  $\text{String}_G$  is smooth. By Theorem 3.6 we find some  $U$  which is diffeomorphic to  $\mathcal{G}au(P) \times O$  for some open  $O \subset G$  with  $1_G \in O$ . Writing out the induced map  $\widehat{\mathcal{G}au}(P) \times \mathcal{G}au(P) \times O \rightarrow \widehat{\mathcal{G}au}(P)$  in local coordinates one sees that the smoothness of this map is implied from the smoothness of the action of  $\mathcal{G}au(P)$  on  $C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})}$  and the smoothness of the natural action  $C^\infty(G, U(H)) \times \text{Diff}(G) \rightarrow C^\infty(G, U(H))$ ,  $(\varphi, f) \mapsto \varphi \circ f$  [GN11].  $\square$

**Definition 5.5.** Let  $G$  be a compact simple and simply connected Lie group. Then we define  $\text{STRING}_G$  to be the metrizable Fréchet Lie 2-group associated to the crossed module  $(\widehat{\mathcal{G}au}(P) \xrightarrow{\partial} \text{String}_G)$  according to example 4.3.

In more detail we have

$$(\text{STRING}_G)_0 := \text{String}_G \quad \text{and} \quad (\text{STRING}_G)_1 := \widehat{\mathcal{G}au}(P) \rtimes \text{String}_G$$

with structure maps given by

$$s(g, f) = f \quad t(g, f) = \partial(g)h \quad i(f) = (1, f) \quad \text{and} \quad (g, f) \circ (g', f') = (gg', f).$$

From the sequence (10) we obtain isomorphisms

$$\pi_0 \text{STRING}_G = \text{coker}(\partial) \xrightarrow{\sim} G \quad \text{and} \quad \pi_1 \text{STRING}_G = \ker(\partial) \xrightarrow{\sim} U(1). \quad (12)$$

Moreover we can consider the Lie group  $\text{String}_G$  from Definition 3.4 also as a 2-group which has only identity morphisms, see Example 4.3. Then there is clearly an inclusion  $\text{String}_G \rightarrow \text{STRING}_G$  of 2-groups.

**Theorem 5.6.** *The 2-group  $\text{STRING}_G$  together with the isomorphisms (12) is a smooth 2-group model for the string group (in the sense of Definition 4.10). The inclusion  $\text{String}_G \rightarrow \text{STRING}_G$  induces a homotopy equivalence*

$$\text{String}_G \rightarrow |\text{STRING}_G|$$

*Proof.* We first want to show that the map  $\text{String}_G = |\text{String}_G| \rightarrow |\text{STRING}_G|$  is a homotopy equivalence. Therefore note that the inclusion functor  $\text{String}_G \rightarrow \text{STRING}_G$  is given by the identity on the level of objects and by the canonical inclusion

$$\text{String}_G \rightarrow \widehat{\mathcal{G}au} \rtimes \text{String}_G$$

on the level of morphisms. Both of these maps are homotopy equivalences, the identity for trivial reasons and the inclusion by the fact that  $\widehat{\mathcal{G}au}$  is contractible as shown in Proposition 5.3. Since, furthermore, both Lie-2-groups are metrizable we can apply Proposition 4.5 and conclude that the geometric realization of the functor is a homotopy equivalence.

It only remains to show that  $|\mathbf{STRING}_G| \rightarrow G$  is a 3-connected cover. The homotopy equivalence  $String_G \simeq |\mathbf{STRING}_G|$  clearly commutes with the projection to  $G$ . Thus the claim is a consequence of the fact that  $String_G$  is a smooth String group model (in particular a 3-connected cover) as shown in Theorem 3.6.  $\square$

**Remark 5.7.** From Remark 2.8 we obtain a crossed module  $\widetilde{\mathcal{G}au}(P_e G) \rightarrow \mathcal{P}String_G$ , where  $\mathcal{P}String_G$  is the restriction of the Lie group extension

$$\mathcal{G}au(P) \rightarrow \mathrm{Aut}(P)_0 \rightarrow \mathrm{Diff}(G)_0 \quad (13)$$

from [Wo07, Theorem 2.14] to  $G \subset \mathrm{Diff}(G)_0$  and  $\mathcal{P}String_G \subset \mathrm{Aut}(P_e G)$  acts canonically  $\widetilde{\mathcal{G}au}(P_e G) := C^\infty(P_e G, \widehat{\Omega G})^{\Omega G}$ . As in Definition 5.2 we then define  $\widehat{\mathcal{G}au}(P_e G)$  to be associated to  $\mathcal{G}au(P_e G)$  along the homomorphism  $I_G$ . This furnishes another crossed module

$$\widehat{\mathcal{G}au}(P_e G) \rightarrow \mathcal{P}String_G,$$

where the action of  $\mathcal{P}String_G \subset \mathrm{Aut}(P_e G)$  is defined in the same way as in (11).

## 6 Comparison of string structures

One reason for the importance of Lie 2-groups is that they allow for a bundle theory analogous to bundles for Lie groups. These 2-bundles play for example a role in mathematical physics. In particular in supersymmetric sigma models, which are used to describe fermionic string theories, they serve as target space background data [FM06, Wa09, Bu09]. For a precise definition of 2-bundles we refer the reader to [NW11] or [Wo09]. We mainly need the following facts about smooth 2-bundles here

1. For a Lie 2-group  $\mathcal{G}$  and a finite dimensional manifold  $M$  all 2-bundles form a bicategory  $2\text{-}\mathcal{B}un_{\mathcal{G}}(M)$  [NW11, Definition 6.1.5].
2. For a smoothly separable, metrizable Lie 2-group  $\mathcal{G}$  isomorphism classes of  $\mathcal{G}$ -2-bundles are in bijection with non-abelian cohomology  $\check{H}^1(M, \mathcal{G})$  and with isomorphism classes of continuous  $|\mathcal{G}|$ -bundles [NW11, Theorem 4.6, Theorem 5.3.2 and Theorem 7.1].
3. For a Lie group  $G$  considered as a Lie 2-group (as in example 4.3) the definition of 2-bundles reduces to that of 1-bundles. More precisely we have an equivalence of bicategories  $\mathcal{B}un_G(M) \rightarrow 2\text{-}\mathcal{B}un_G(M)$  where  $\mathcal{B}un_G(M)$  is considered as a bicategory with only identity 2-morphisms [NW11, Example 5.1.8]. Moreover non-abelian cohomology  $\check{H}^1(M, G)$  reduces in this case to the ordinary Čech-cohomology.
4. For a morphism of  $\mathcal{G} \rightarrow \mathcal{G}'$  of Lie 2-groups we have an induced functor  $2\text{-}\mathcal{B}un_{\mathcal{G}}(M) \rightarrow 2\text{-}\mathcal{B}un_{\mathcal{G}'}(M)$  and an induced morphism  $\check{H}^1(M, \mathcal{G}) \rightarrow \check{H}^1(M, \mathcal{G}')$ . For a smooth weak equivalence between metrizable, smoothly separable 2-groups the induced functor is an equivalence of bicategories. [NW11, Theorem 6.2.2].

**Proposition 6.1.** *The inclusion  $\text{String}_G \rightarrow \text{STRING}_G$  induces a functor*

$$\mathcal{B}un_{\text{String}_G}(M) \rightarrow 2\text{-}\mathcal{B}un_{\text{STRING}_G}(M)$$

*which on isomorphism classes is given by the induced map*

$$\check{H}^1(M, \text{String}_G) \rightarrow \check{H}^1(M, \text{STRING}_G)$$

*for each finite dimensional manifold  $M$ . This map is a bijection.*

*Proof.* This follows essentially from the fact that the geometric realizations of the functor  $\text{String}_G \rightarrow |\text{STRING}_G|$  is a homotopy equivalence as shown in Theorem 5.6. Then one knows that the induced map between isomorphism classes of continuous  $\text{String}_G$ -bundles and  $|\text{STRING}_G|$ -bundles is an isomorphism. Then the claim follows by the facts given above.  $\square$

The importance of the last proposition is that it allows to directly compare  $\text{String}_G$ -structures and  $\text{STRING}_G$ -structures. We mainly built the 2-group model  $\text{STRING}_G$  in order to have such a comparison available. Now one can use the  $\text{STRING}_G$  2-group and compare it in the world of Lie 2-groups to other smooth 2-group models and so obtain an overall comparison. We will make precise what this means in detail:

**Definition 6.2.** A morphism between 2-group models  $\mathcal{G}$  and  $\mathcal{G}'$  is a smooth homomorphism  $f : \mathcal{G} \rightarrow \mathcal{G}'$  such that the diagrams

$$\begin{array}{ccc} \pi_0 \mathcal{G} & \xrightarrow{\pi_0 f} & \pi_0 \mathcal{G}' \\ & \searrow \sim & \swarrow \sim \\ & G & \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_1 \mathcal{G} & \xrightarrow{\pi_1 f} & \pi_1 \mathcal{G}' \\ & \searrow \sim & \swarrow \sim \\ & U(1) & \end{array}$$

commute.

**Proposition 6.3.** *Let  $f : \mathcal{G} \rightarrow \mathcal{G}'$  be a morphism between metrizable, smoothly separable smooth 2-group models.*

1. *Then  $f$  is automatically a smooth weak equivalence of 2-groups.*
2. *The geometric realization  $|f| : |\mathcal{G}| \rightarrow |\mathcal{G}'|$  is a homotopy equivalence of topological groups. Furthermore it commutes with the projection to  $G$  and the inclusion of  $|BU(1)|$  (see proposition 4.9).*
3. *For a manifold  $M$  the induced functor*

$$f_* : 2\text{-}\mathcal{B}un_{\mathcal{G}}(M) \rightarrow 2\text{-}\mathcal{B}un_{\mathcal{G}'}(M).$$

*is an equivalence of bicategories.*

*Proof.* The first assertion follows from the characterization of weak equivalences given in Proposition 4.9 and the second from Proposition 4.7. The last statement is then implied by fact 4 mentioned above.  $\square$

This shows that from such a morphism between 2-group models we can directly derive comparisons between the bundle theories. Of course one should allow spans of such morphisms. An interesting thing would be to give directly such a span connecting our model  $\text{STRING}_G$  to the model given in [BCSS07]. There are cohomological reasons to expect that such a span should exist [WW].

## A Locally convex manifolds and Lie groups

In this section we provide the necessary information to clarify the differential geometric background. If  $X, Y$  are locally convex vector spaces and  $U \subset X$  is open, then  $f: U \rightarrow Y$  is called *continuously differentiable* if for each  $v \in X$  the limit

$$df(x).v := \lim_{h \rightarrow 0} \frac{1}{h}(f(x + hv) - f(x)) \quad (14)$$

exists and the map  $U \times X \rightarrow Y$ ,  $(x, v) \mapsto df(x).v$  is continuous. It is called *smooth* if the iterated derivatives  $d^n f: U \times X^n \rightarrow Y$  exist and are also continuous. Concepts like manifolds and tangent bundles carry over to this setting of differential calculus, in particular the notion of Lie groups and their associated Lie algebras [GN11]. Moreover, manifolds in this sense are in particular topological manifolds in the sense of [Pa66].

If  $M, N$  are manifolds and  $f: M \rightarrow N$  is smooth, then we call  $f$  an *immersion* if for each  $m \in M$  there exist charts around  $m$  and  $f(m)$  such that the corresponding coordinate representation of  $f$  is an inclusion of the modeling space of  $M$  as a direct summand into the modeling space of  $N$ . Analogously,  $f$  is called *submersion* if for each  $m \in M$  the corresponding coordinate representation is a projection onto a direct summand (cf. [La99, §II.2], [Ha82, Definition 4.4.8]).

If  $G$  is a Lie group, then a closed subgroup  $H \subset G$  is called *Lie subgroup* if it is also a submanifold. This is not automatically the case in infinite dimensions (cf. [Bo98b, Exercise III.8.2]). Moreover, if  $H$  is a closed Lie subgroup, then it need not be immersed as the example of a non-complemented subspace in a Banach space shows.

**Lemma A.1.** *If  $H \subset G$  is a closed subgroup and  $G/H$  carries an arbitrary Lie group structure such that  $G \rightarrow G/H$  is smooth, then the following are equivalent.*

1.  $G \rightarrow G/H$  admits smooth local sections around each point.
2.  $G \rightarrow G/H$  is a locally trivial bundle.
3.  $G \rightarrow G/H$  is a submersion.

*In any of these cases  $H$  is an immersed Lie subgroup and  $G/H$  carries the quotient topology.*

*Proof.* If  $G \rightarrow G/H$  admits local sections, then

$$q^{-1}(U) \ni g \mapsto (q(g), g \cdot \sigma(q(g))^{-1}) \in U \times H$$

defines a local trivialization of  $G \rightarrow G/H$ . This shows equivalence of the first two statements and with this aid one sees also the equivalence with the last statement. From the second it follows in particular that  $H \hookrightarrow G$  is an immersion. Since submersions are open, and since surjective open maps are quotient maps, the topology on  $G/H$  has to be the quotient topology.  $\square$

**Definition A.2.** (cf. [Ne07, Definition 2.1]) A *split Lie subgroup* of a Lie group is a closed subgroup that fulfills one of the three equivalent conditions of the preceding lemma.

Note that each immersed Lie subgroup of a Banach–Lie group is split by [Bo98b, Proposition III.1.10]. This implies in particular that each closed subgroup of a finite-dimensional Lie group is split by [Bo98b, Theorem III.8.2]. Also note that if  $H$  is closed and normal and  $G/H$  carries a Lie group structure such that  $G \rightarrow G/H$  is smooth, then a single local smooth section can be moved around with the group multiplication to yield a local smooth section around each point.

**Proposition A.3.** *If  $X, Y, Z$  are manifolds,  $f: X \rightarrow Z$  is smooth and  $g: Y \rightarrow Z$  is a submersion then the fiber product  $X \times_Z Y$  exists in the category of smooth manifolds and the projection*

$$X \times_Z Y \rightarrow X$$

*is a submersion. Moreover the identity is a submersion and the composition of submersions is again a submersion. That means submersions form a Grothendieck pretopology (see [Me03, Definition 5]) on the category of smooth manifolds*

*Proof.* This is a slight generalization of [Ha82, 4.4.10]. The proof of [La99, Proposition II.2.6], showing that the first statement is a local one and of [La99, Proposition II.2.7], showing this for a projection carry over literally to our more general setting. Moreover, the question of being a submersion is also local, so [La99, Proposition II.2.7] shows that  $X \times_Z Y \rightarrow X$  is one.  $\square$

**Corollary A.4.** *The fibers of a submersion are submanifolds.*

A manifold is called metrizable if the underlying topology is so. Note that metrizable is equivalent to paracompact and locally metrizable [Pa66, Theorem 1]. Thus a Fréchet manifold is metrizable if and only if it is paracompact. Moreover, we have the following

**Theorem A.5.** *A metrizable manifold has the homotopy type of a CW-complex. In particular, weak homotopy equivalences between metrizable manifolds are homotopy equivalences.*

*Proof.* By [Pa66, Theorem 14] a metrizable manifold is dominated by CW-complex. By a theorem of Whitehead this implies that it has the homotopy type of a CW-complex (cf. [Ha02, Prop. A.11]).  $\square$

## B A characterization of smooth weak equivalences

In this section we will exclusively be concerned with smoothly separable Lie 2-groups. Recall that for a smoothly separable Lie 2-group  $\mathcal{G}$  we require among other things that  $\pi_1 \mathcal{G}$  is a split Lie subgroup. Our main goal here is to prove part 1 of Proposition 4.9. This will be done in several steps.

**Lemma B.1.** *Let  $\mathcal{G}$  be a smoothly separable Lie 2-group. Then the map  $s \times t: \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times_{\pi_0 \mathcal{G}} \mathcal{G}_0$  is a surjective submersion.*

*Proof.* By definition the map  $s \times t$  is a surjective map onto the submanifold  $\mathcal{G}_0 \times_{\pi_0 \mathcal{G}} \mathcal{G}_0$  of  $\mathcal{G}_0 \times \mathcal{G}_0$

It admits local sections because its kernel  $\pi_1 \mathcal{G}$  is a split Lie subgroup. By Lemma A.1 this implies that it is a submersion.  $\square$

**Proposition B.2.** *Let  $f : \mathcal{G} \rightarrow \mathcal{G}'$  be a morphism of smoothly separable Lie 2-groups inducing an isomorphism on  $\pi_1$ . Then  $f$  is smoothly fully faithful, i.e.,*

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{f} & \mathcal{G}'_1 \\ s \times t \downarrow & & \downarrow s \times t \\ \mathcal{G}_0 \times \mathcal{G}_0 & \xrightarrow{f \times f} & \mathcal{G}'_0 \times \mathcal{G}'_0 \end{array}$$

is a pullback diagram of Lie groups.

*Proof.* It is clear that this is a pullback diagram of groups by the general theory of 2-groups. Let  $\mathcal{H}$  be a Lie group and consider the diagram

$$\begin{array}{ccccc} \mathcal{H} & & & & \\ & \searrow h & & \searrow a & \\ & & \mathcal{G}_1 & \xrightarrow{f} & \mathcal{G}'_1 \\ & \searrow b & \downarrow s \times t & & \downarrow s \times t \\ & & \mathcal{G}_0 \times \mathcal{G}_0 & \xrightarrow{f \times f} & \mathcal{G}'_0 \times \mathcal{G}'_0 \end{array}$$

where  $a, b$  are morphisms of Lie groups. We have to show that the unique map  $h : \mathcal{H} \rightarrow \mathcal{G}_1$  supplied by the pullback of groups is also smooth. By Lemma B.1 there exists a smooth local section  $\gamma : U \rightarrow \mathcal{G}_1$  of  $s \times t$ , defined on an identity neighbourhood  $U \subset \mathcal{G}_0 \times_{\pi_0} \mathcal{G}_0$ . Since  $b$  maps to  $\mathcal{G}_0 \times_{\pi_0} \mathcal{G}_0$ ,  $V := b^{-1}(U)$  is an open identity neighborhood in  $\mathcal{H}$ .

We now observe that

$$h' : V \rightarrow \mathcal{G}_1, \quad x \mapsto \gamma(b(x)) \cdot (\pi_1 f_1)^{-1}(f_1(\gamma(b(x))))^{-1} \cdot a(x)$$

is smooth since  $f_1(\gamma(b(x)))^{-1} \cdot a(x) \in \pi_1 \mathcal{G}'$  and  $f_1$  restricts to a diffeomorphism  $\pi_1 \mathcal{G} \rightarrow \pi_1 \mathcal{G}'$ . It satisfies  $f_1 \circ h' = a|_V$ , and we also have  $(s \times t) \circ h' = b$  since  $\gamma$  is a section of  $s \times t$ . Thus  $h$  coincides with  $h'$  on  $V$ , showing that  $h$  is a smooth homomorphism of Lie groups.  $\square$

**Proposition B.3.** *Let  $f : \mathcal{G} \rightarrow \mathcal{G}'$  be a morphism of smoothly separable Lie 2-groups inducing an isomorphism on  $\pi_0$ . Then  $f$  is smoothly essentially surjective, i.e., the morphism*

$$s \circ \text{pr}_2 : \mathcal{G}_0 \times_{f_0} \mathcal{G}'_1 \rightarrow \mathcal{G}'_0$$

is a smooth submersion.

*Proof.* Surjectivity is clear because  $f$  is surjective on  $\pi_0$ . To see that  $s \circ \text{pr}_2$  is a submersion we will construct a local smooth section. Since the map  $p : \mathcal{G}_0 \rightarrow \pi_0 \mathcal{G}$  is a submersion there exists a local section  $\sigma : U \rightarrow \mathcal{G}_0$  of  $p$ .

For brevity let us denote the “roundtrip” map, restricted to  $V := p'^{-1}(\pi_0 f(U))$  as  $R = f_0 \circ \sigma \circ (\pi_0 f)^{-1} \circ p'$ . For  $x \in V$  we then have  $x \cong R(x)$  and thus  $(x, R(x)) \in \mathcal{G}'_0 \times_{\pi_0 \mathcal{G}} \mathcal{G}'_0$ . Now there exists a local smooth section  $\tau : W \rightarrow \mathcal{G}'_1$  of  $s' \times t'$  for  $W \subset V \times_{\pi_0 \mathcal{G}'} V$  open. Then

$$\begin{aligned} S : (\text{id}_{\mathcal{G}'_0}|_V \times R)^{-1}(W) &\rightarrow \mathcal{G}_0 \times_{f_0} \mathcal{G}'_1 \\ x &\mapsto (\sigma((\pi_0 f)^{-1}(p'(x))), \tau(x, R(x))) \end{aligned}$$

is the required section since we have

$$f_0(\sigma((\pi_0 f)^{-1}(p'(x)))) = R(x) = t(\tau(x, R(x)))$$

and  $s(\tau(x)) = x$ .  $\square$

**Corollary B.4.** *If  $f : \mathcal{G} \rightarrow \mathcal{G}'$  is a morphism of smoothly separable Lie 2-groups inducing isomorphisms on  $\pi_0$  and  $\pi_1$  then  $f$  is a weak equivalence.*

The converse of the first part of Proposition 4.9 also holds:

**Proposition B.5.** *A smooth weak equivalence  $f : \mathcal{G} \rightarrow \mathcal{G}'$  of smoothly separable Lie 2-groups induces isomorphisms on  $\pi_0$  and  $\pi_1$ .*

*Proof.* Since  $f$  is in particular an equivalence of the underlying categories in the set-theoretic sense, it is clear that its induced morphisms  $\pi_0 f : \pi_0 \mathcal{G} \rightarrow \pi_0 \mathcal{G}'$  and  $\pi_1 f : \pi_1 \mathcal{G} \rightarrow \pi_1 \mathcal{G}'$  are group isomorphisms. From the diagram

$$\begin{array}{ccc}
 \mathcal{G}_0 \times_t \mathcal{G}'_1 & \xrightarrow{\text{pr}_2} & \mathcal{G}'_1 \\
 \downarrow & & \downarrow s \\
 \mathcal{G}_0 & \xrightarrow{f_0} & \mathcal{G}'_0 \\
 \downarrow p & & \downarrow p' \\
 \pi_0 \mathcal{G} & \xrightarrow{\pi_0 f} & \pi_0 \mathcal{G}'_0
 \end{array} \tag{15}$$

we see that  $\pi_0 f$  is smooth since we can pick a local section  $\sigma : \pi_0 \mathcal{G} \rightarrow \mathcal{G}_0$  of the submersion  $p : \mathcal{G}_0 \rightarrow \pi_0 \mathcal{G}$ , which shows that locally

$$\pi_0 f = p' \circ f_0 \circ \sigma.$$

To see that  $(\pi_0 f)^{-1}$  is smooth as well we choose a local section  $\sigma' : \pi_0 \mathcal{G}' \rightarrow \mathcal{G}'_0$ . Since we know that  $s \circ \text{pr}_2 : \mathcal{G}_0 \times_t \mathcal{G}'_1 \rightarrow \mathcal{G}'_0$  is a submersion, we can also choose a section  $\tau$  for that map, and composing  $\tau \circ \sigma'$  with the projection to  $\mathcal{G}_0$  and finally to  $\pi_0 \mathcal{G}$  coincides with  $(\pi_0 f)^{-1}$  which is therefore smooth.

To see that  $\pi_1 f$  is a diffeomorphism we use the fact that the diagram of part 2 of the definition of a smooth weak equivalence is a pullback diagram. This implies in particular that the restriction of  $f_1$  to the fiber over  $(1, 1)$ , which is the submanifold  $\pi_1 \mathcal{G}$ , is a smooth bijective map. That its inverse is also smooth follows from the universal property of the pullback: there exists a unique smooth map  $H : \pi_1 \mathcal{G}' \rightarrow \pi_1 \mathcal{G}$  that makes the diagram

$$\begin{array}{ccc}
 \pi_1 \mathcal{G}' & & \\
 \downarrow \text{id} & \searrow H & \downarrow f_1 \\
 \pi_1 \mathcal{G} & \xrightarrow{f_1} & \pi_1 \mathcal{G}' \\
 \downarrow s \times t & & \downarrow s \times t \\
 (1, 1) & \xrightarrow{f_0 \times f_0} & (1, 1)
 \end{array}$$

commute, so  $f_1 \circ H = \text{id}_{\pi_1 \mathcal{G}'}$  which means that  $H$  is the inverse of  $f_1$  on  $\pi_1 \mathcal{G}'$ , which thus is smooth.  $\square$

This concludes the proof of the first part of Proposition 4.9.

## References

- [BCSS07] Baez, J. C., A. S. Crans, D. Stevenson, and U. Schreiber, *From loop groups to 2-groups*, Homology, Homotopy Appl. **9**(2) (2007), 101–135, <http://arxiv.org/abs/math/0504123>.
- [BN09] Bunke, U. and N. Naumann, *Secondary Invariants for String Bordism and  $tmf$* , 2009, <http://arxiv.org/abs/0912.4875>.
- [Bo98a] Bourbaki, N., “General topology,” Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998.
- [Bo98b] —, “Lie groups and Lie algebras,” Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998.
- [Br72] Bredon, G. E., “Introduction to compact transformation groups,” Academic Press, New York, 1972, pure and Applied Mathematics, Vol. 46.
- [Bu09] Bunke, U., *String Structures and Trivialisations of a Pfaffian Line Bundle*, 2009, <http://arxiv.org/abs/0909.0846>.
- [Ca36] Cartan, E., *La topologie des groupes de Lie*, Actual. Sci. et Industr. **358**.
- [Du66] Dugundji, J., “Topology,” Allyn and Bacon Inc., Boston, Mass., 1966.
- [EG54] Etter, D. O. and J. S. Griffin, Jr., *On the metrizability of the bundle space*, Proc. Amer. Math. Soc. **5** (1954), 466–467.
- [FM06] Freed, D. and G. Moore, *Setting the quantum integrand of M-theory*, Comm. Math. Phys. **263**(1) (2006), 89–132.
- [Ge09] Getzler, E., *Lie theory for nilpotent  $L_\infty$ -algebras*, Ann. of Math. (2) **170**(1) (2009), 271–301, URL <http://dx.doi.org/10.4007/annals.2009.170.271>.
- [Gl02] Glöckner, H., *Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups*, J. Funct. Anal. **194**(2) (2002), 347–409.
- [GN03] Glöckner, H. and K.-H. Neeb, *Banach-Lie quotients, enlargability, and universal complexifications*, J. Reine Angew. Math. **560** (2003), 1–28.
- [GN11] —, “Infinite-dimensional Lie groups,” volume I, Basic Theory and Main Examples of *Graduate Texts in Mathematics*, Springer-Verlag, 2011.
- [Ha82] Hamilton, R. S., *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7**(1) (1982), 65–222.
- [Ha02] Hatcher, A., “Algebraic topology,” Cambridge University Press, Cambridge, 2002.
- [He08a] Henriques, A., *Integrating  $L_\infty$ -algebras*, Compos. Math. **144**(4) (2008), 1017–1045, <http://arxiv.org/abs/math/0603563>.
- [He08b] —, *Lecture by Michael Hopkins: the string orientation of  $tmf$* , 2008, <http://arxiv.org/abs/0805.0743>.

- [Ki87] Killingback, T. P., *World-sheet anomalies and loop geometry*, Nuclear Phys. B **288**(3-4) (1987), 578–588.
- [Ku65] Kuiper, N. H., *The homotopy type of the unitary group of Hilbert space*, Topology **3** (1965), 19–30.
- [La99] Lang, S., “Fundamentals of differential geometry,” volume 191 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1999.
- [Le82] Lewis, L. G., Jr., *When is the natural map  $X \rightarrow \Omega\Sigma X$  a cofibration?*, Trans. Amer. Math. Soc. **273**(1) (1982), 147–155, URL <http://dx.doi.org/10.2307/1999197>.
- [Lu09] Lurie, J., *A survey of elliptic cohomology*, in “Algebraic topology,” volume 4 of *Abel Symp.*, 219–277, Springer, Berlin, 2009, URL [http://dx.doi.org/10.1007/978-3-642-01200-6\\_9](http://dx.doi.org/10.1007/978-3-642-01200-6_9).
- [Ma74] May, J. P.,  *$E_\infty$  spaces, group completions, and permutative categories*, in “New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972),” 61–93. London Math. Soc. Lecture Note Ser., No. 11, Cambridge Univ. Press, London, 1974.
- [Me03] Metzler, D., *Topological and Smooth Stacks*, 2003, <http://arxiv.org/abs/math/0306176>.
- [MW09] Müller, C. and C. Wockel, *Equivalences of Smooth and Continuous Principal Bundles with Infinite-Dimensional Structure Group*, Adv. Geom. **9**(4) (2009), 605–626, <http://arxiv.org/abs/math/0604142>.
- [Ne02] Neeb, K.-H., *Central extensions of infinite-dimensional Lie groups*, Ann. Inst. Fourier (Grenoble) **52**(5) (2002), 1365–1442.
- [Ne06] —, *Non-abelian extensions of topological Lie algebras*, Comm. Algebra **34**(3) (2006), 991–1041, <http://arxiv.org/abs/math/0411256>.
- [Ne07] —, *Non-abelian extensions of infinite-dimensional Lie groups*, Ann. Inst. Fourier (Grenoble) **57**(1) (2007), 209–271, <http://arxiv.org/abs/math/0504295>.
- [No10] Noohi, B., *Homotopy types of topological stacks*, 2010, <http://arxiv.org/abs/0808.3799>.
- [NW09] Neeb, K.-H. and C. Wockel, *Central extensions of groups of sections*, Ann. Glob. Anal. Geom. **36**(4) (2009), 381–418, URL <http://dx.doi.org/10.1007/s10455-009-9168-6>, <http://arxiv.org/abs/0711.3437>.
- [NW11] Nikolaus, T. and K. Waldorf, *Four Equivalent Versions of Non-Abelian Gerbes*, 2011, <http://arxiv.org/abs/1103.4815>.
- [Pa66] Palais, R. S., *Homotopy theory of infinite dimensional manifolds*, Topology **5** (1966), 1–16.

- [RS11] Roberts, D. M. and D. Stevenson, *Simplicial principal bundles in parametrized spaces*, 2011, unpublished Draft.
- [Se70] Segal, G., *Cohomology of topological groups*, in “Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69),” 377–387, Academic Press, London, 1970.
- [Se74] —, *Categories and cohomology theories*, *Topology* **13**(3) (1974), 293–312.
- [SP10] Schommer-Pries, C., *Central Extensions of Smooth 2-Groups and a Finite-Dimensional String 2-Group*, 2010, <http://arxiv.org/abs/0911.2483>.
- [St96] Stolz, S., *A conjecture concerning positive Ricci curvature and the Witten genus*, *Math. Ann.* **304**(4) (1996), 785–800.
- [ST04] Stolz, S. and P. Teichner, *What is an elliptic object?*, in “Topology, geometry and quantum field theory,” volume 308 of *London Math. Soc. Lecture Note Ser.*, 247–343, Cambridge Univ. Press, Cambridge, 2004, <http://math.berkeley.edu/~teichner/Papers/Oxford.pdf>.
- [tD74] tom Dieck, T., *On the homotopy type of classifying spaces*, *Manuscripta Math.* **11** (1974), 41–49.
- [tD08] —, “Algebraic topology,” EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008.
- [Wa09] Waldorf, K., *String Connections and Chern-Simons Theory*, 2009, <http://arxiv.org/abs/0906.0117>.
- [Wi88] Witten, E., *The index of the Dirac operator in loop space*, in “Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986),” volume 1326 of *Lecture Notes in Math.*, 161–181, Springer, Berlin, 1988.
- [Wo06] Wockel, C., *Smooth extensions and spaces of smooth and holomorphic mappings*, *J. Geom. Symmetry Phys.* **5** (2006), 118–126, <http://arxiv.org/abs/math/0511064>.
- [Wo07] —, *Lie group structures on symmetry groups of principal bundles*, *J. Funct. Anal.* **251**(1) (2007), 254–288, <http://arxiv.org/abs/math/0612522>.
- [Wo09] —, *Principal 2-bundles and their gauge 2-groups*, accepted for publication in *Forum Math.* 46 pp., <http://arxiv.org/abs/0803.3692>.
- [WW] Wagemann, F. and C. Wockel, *Continuous and smooth group cohomology: locally continuous and smooth vs. simplicial cohomology*, (in preparation).