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Abstract

A classical result of Halin states that if a graph G contains n disjoint rays for every $n \in \mathbb{N}$, then G contains infinitely many disjoint rays. The question how this generalizes to other graphs than rays leads to the notion of ubiquity: a graph A is *ubiquitous* with respect to a relation $\leq$ between graphs (such as the subgraph relation or the minor relation) if $nA \leq G$ for all $n \in \mathbb{N}$ implies $\aleph_0 A \leq G$, where nA denotes the disjoint union of n copies of A (for $n \in \mathbb{N}$ or $n = \aleph_0$). A connected graph is *tree-like* if all its blocks are finite. The main result of the present paper establishes a link between the concepts of ubiquity and well-quasi-ordering, thus offering the possibility to apply well-quasi-ordering results (such as the graph minor theorem or Nash-Williams' tree theorem) to ubiquity problems. Several corollaries are derived showing that wide classes of locally finite tree-like graphs are ubiquitous with respect to the minor or topological minor relation.

Keywords:
Infinite graphs, Ubiquity, Well-quasi-ordering, Minors, Topological Minors, Trees, Tree-like graphs

1. Introduction

A basic result in infinite graph theory which is due to Halin [5] states that if a graph G contains n disjoint rays for every $n \in \mathbb{N}$, then G contains infinitely many disjoint rays. For a proof of Halin's theorem differing from Halin's original argument, see the textbook of Diestel [4]. The question how this result generalizes to other graphs than rays leads to the notion of ubiquity: following the terminology of Diestel [4], a graph A is *ubiquitous* with respect to a relation $\leq$ between graphs (or $\leq$-*ubiquitous* for short) if $nA \leq G$ for all $n \in \mathbb{N}$ implies $\aleph_0 A \leq G$, where nA denotes the disjoint union of n copies of A (for $n \in \mathbb{N}$ or $n = \aleph_0$). Typical relations $\leq$ to be considered in this context are the subgraph relation ('H is isomorphic to a subgraph of G'), the topological minor relation, and the minor relation. Halin's result, and its extension to double rays [6], can then be reformulated as follows.

Theorem A (Halin [5, 6])**.** *Each ray or double ray is ubiquitous with respect to the subgraph relation.*

We write $H \preccurlyeq G$ to indicate that H is a minor of G and $H \subseteq G$ means that H is a topological minor of G. One easily finds that each finite graph is ubiquitous with respect to any of the three mentioned graph relations. Also non-ubiquitous graphs with respect to each of these standard graph relations exist: examples for the subgraph relation and the topological minor relation are displayed in Figures 1 and 2. Proofs that these graphs possess the claimed properties can be obtained by slightly modifying the constructions described in [1]; for other examples, see [7, 13].

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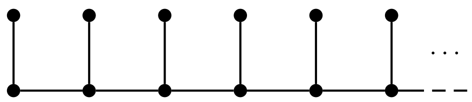

Fig. 1: A non-ubiquitous graph with respect to the subgraph relation.

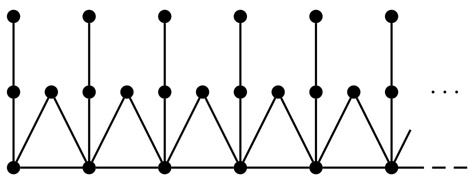

Fig. 2: A non-ubiquitous graph with respect to the topological minor relation.

Examples for non-ubiquitous graphs with respect to the minor relation were constructed in [3]: those examples are more complex and it is an open problem whether countable examples of such graphs exist (cf. [3]). On the positive side, the following result has been obtained.

Theorem B (Andreae [2])**.** *Every locally finite tree is ubiquitous with respect to the topological minor relation.*

As a direct consequence of Theorem B, one obtains that each 'k-star' is ubiquitous with respect to the subgraph relation, where a *k-star* is a graph that results from k disjoint rays by identifying their initial vertices ($k \in \mathbb{N}$). Thus, in particular, Theorem B implies Theorem A. The mentioned results on the subgraph relation and the topological minor relation can roughly be summarized as follows:

- some locally finite trees are ubiquitous with respect to the subgraph relation, but others do not share this property (cf. Figure 1);
- all locally finite trees are ubiquitous with respect to the topological minor relation, but still there exist simple structured 'ray-like' graphs which are not $\subseteq$-ubiquitous (cf. Figure 2).

These results suggest to go one step further and consider minors rather than topological minors. In [3] the author of the present paper set up the following conjecture which has been named *ubiquity conjecture* (cf. Diestel [4]):

Ubiquity conjecture (Andreae [3])**.** *Every locally finite connected graph is ubiquitous with respect to the minor relation.*

Call a graph *tree-like* if it is connected and all its blocks are finite. Then, of course, every tree is a special kind of tree-like graph and the graph displayed in Figure 2 is an example of a tree-like graph which is not a tree. In order to establish partial results on the ubiquity conjecture, we focus on locally finite tree-like graphs. An interesting feature of the proof of Theorem B given in [2] is that it makes use of Nash-Williams' tree theorem [9]. This suggests that, in order to obtain results on $\preccurlyeq$-ubiquity, one should bring into play the well-quasi-ordering results for minors due to Robertson and Seymour [11] and Thomas [12].

The central results of the present paper are the Theorems 1 and 2 presented in Section 3. Theorem 1 deals with the ubiquity of locally finite tree-like graphs with respect to both the minor and the topological minor relation. Generally speaking, the theorem provides a structural condition which is sufficient for a locally finite tree-like graph to be $\preccurlyeq$-ubiquitous; Theorem 1 also includes a similar result for $\subseteq$-ubiquity and for a class of graphs called 'strongly tree-like' (for a definition of this class, cf. Section 2). Theorem 2 is a direct consequence of Theorem 1 and an additional lemma (Lemma 1). In the case of the minor relation, the content of Theorem 2 can roughly be summarized by saying that a locally finite tree-like graph A is

$\preccurlyeq$-ubiquitous if certain sets of infinite rooted subgraphs of A are well-quasi-ordered by minors; in addition, Theorem 2 provides a similar result for topological minors and for the aforementioned class of strongly tree-like graphs. We shall derive from Theorem 2 a series of corollaries which we state now. (For the notion of an end of an infinite graph, cf. Diestel [4].)

Corollary 1. *A locally finite tree-like graph is $\preccurlyeq$-ubiquitous if its number of ends is finite.*

Corollary 2. *A locally finite tree-like graph is $\preccurlyeq$-ubiquitous if there exists a finite planar graph which is not a minor of it.*

Corollary 1 is derived from Theorem 2 of the present paper by making use of the graph minor theorem of Robertson and Seymour [11], while Corollary 2 follows from Theorem 2 by application of the main result of Thomas [12]. By making use of a result of Robertson and Seymour [10], one easily obtains that Corollary 2 can be reformulated in terms of bounded tree-width: a locally finite tree-like graph A is $\preccurlyeq$-ubiquitous if there exists a $k \in \mathbb{N}$ such that all blocks of A have tree-width less than k.

As a direct consequence of Theorem 2 and Nash-Williams' tree-theorem [9], one obtains the above Theorem B. This yields another proof of Theorem B with similar methods as in [2] but in a more general context. In fact we shall obtain the following result which is an extended version of Theorem B.

Corollary 3. *For a locally finite connected graph A assume that all of its blocks are complete graphs. Then A is $\preccurlyeq$-ubiquitous. Further, if all blocks of A are complete graphs K_2, then A is $\subseteq$-ubiquitous.*

The proof of Corollary 3 is based on Theorem 2 of the present paper and a labelled version of Nash-Williams' tree-theorem which is due to Laver [8]. The following conjecture has been proposed by Thomas [12].

Thomas' conjecture. *The class of countable graphs is well-quasi-ordered by minors.*

As a further consequence of Theorem 2, it is shown that the truth of Thomas' conjecture would imply that all locally finite tree-like graphs are $\preccurlyeq$-ubiquitous (cf. Corollary 4 in Section 3). Actually, as pointed out in Section 3, this would already follow from the truth of a considerably relaxed version of Thomas' conjecture.

The paper is organized as follows. In the remainder of the introduction, we shall collect some definitions and notations. For graph-theoretic terminology used but not explained here, the reader is referred to the textbook of Diestel [4]. Section 2 is devoted to Lemma 1 and its proof; also additional terminology is introduced which will play a central role in the subsequent sections. In Section 3, Theorem 1 is presented without proof and it is shown how Theorem 2 and its corollaries follow from Theorem 1. The remaining sections are devoted to the proof of Theorem 1.

The graphs considered in this paper are undirected and without loops or multiple edges. Given a graph G, we denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively. A graph for which some vertex has been exhibited as its *root* is called a *rooted graph* (or *rooted subgraph* if it is addressed as a subgraph of some graph). If in a graph G there exists a vertex which is incident with just one edge of G, then this edge is called a *pendant edge* of G. Given sets A and B of vertices, we call a path $P = (x_0, \dots, x_k)$ an *A, B-path* if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$. We write *a, B-path* rather than $\{a\}, B$-path and in a similar way use *A, b-path* and *a, b-path*. For graphs R and S we write *R, S-path* rather than $V(R), V(S)$-path. For a ray P, let a be its unique vertex of degree one and let b be an arbitrary vertex of P. Then P is called an *a-ray* and a is the *initial vertex* of P; moreover, the uniquely determined b-ray contained in P is the *b-tail* of P and the a, b-path contained in P is an *initial segment* of P. A *block* of a graph is a maximal connected subgraph without a cutvertex. A set X is *well-quasi-ordered* (wqo) by $\leq$, if $\leq$ is a reflexive and transitive relation on X and if for every infinite sequence $x_1,\ x_2,\ \dots$ of elements of X there are indices i, j such that $i < j$ and $x_i \leq x_j$. The symbol $\mathbb{N}$ denotes the set of positive integers.

A *$\subseteq$-embedding* of a graph G into a graph H is an injective mapping φ defined on $V(G) \cup E(G)$ which assigns to each $v \in V(G)$ a vertex of H and to each edge $vw \in E(G)$ a $\varphi(v), \varphi(w)$-path of H such that, for every pair e_1, e_2 of distinct edges of G and every $u \in V(G)$, neither $\varphi(u)$ nor a vertex of $\varphi(e_2)$ occurs as an inner vertex of $\varphi(e_1)$.

A $\preccurlyeq$-*embedding* of G into H is a mapping φ which assigns to each $v \in V(G)$ a connected subgraph $\varphi(v)$ of H and to each $e \in E(G)$ an edge of H such that $\varphi(v) \cap \varphi(w) = \emptyset$ whenever v, w are distinct vertices of G and such that $\varphi(e)$ is an edge between $\varphi(v)$ and $\varphi(w)$ if $e = vw$.

Let φ be a $\leq$-embedding of G into H with $\leq$ denoting one of the relations $\subseteq$ or $\preccurlyeq$. Then $\varphi(G)$ denotes the subgraph of H formed by all $\varphi(x)$ with $x \in V(G) \cup E(G)$; the so-defined graph $\varphi(G)$ is called a $\leq$-*copy* of G in H. For φ as before and $G' \subseteq G$, $\varphi(G')$ is defined similarly. If φ is a $\preccurlyeq$-embedding of some locally finite graph into an arbitrary graph, then all graphs $\varphi(v)$ can and without further mention will be assumed to be finite.

2. Good representations

Let A be a locally finite infinite tree-like graph. Then there obviously exists an infinite sequence $A_0, A_1, \ldots$ of finite connected nontrivial and pairwise distinct subgraphs of A together with a corresponding sequence of (not necessarily distinct) vertices $r_n \in A_n$ $(n = 0, 1, \ldots)$ such that

(i) $(A_0 \cup \ldots \cup A_{n-1}) \cap A_n = r_n$ $(n = 1, 2, \ldots)$ and

(ii) $A = \bigcup_{n=0}^{\infty} A_n$.

Indeed, one can choose the graphs $A_0, A_1, \ldots$ as the blocks of A, but each A_n may as well be the union of two or more blocks. We call a family $\big(A_n, r_n\big)$ $(n = 0, 1, \ldots)$ of rooted subgraphs of A with the above properties a *decomposition* of A; r_0 is the *root* of the decomposition and the rooted graphs $\big(A_n, r_n\big)$ are its *parts*. If no confusion is possible, we also refer to the unrooted graphs A_n as the parts of the decomposition. Typically, whenever a locally finite infinite tree-like graph A is given, some fixed decomposition of it, denoted by $\mathcal{D}(A)$, is given too; then the root r_0 of $\mathcal{D}(A)$ is denoted by r_A and sometimes A is tacitly considered as a rooted graph with root r_A. If it is clear from the context to which decomposition $\mathcal{D}(A)$ we refer, then the parts of $\mathcal{D}(A)$ are also called *parts of* A.

For A as above, let $\mathcal{D}(A)$ be a fixed decomposition with parts and roots denoted as before. We define a corresponding rooted tree T, called *decomposition tree*, as follows. The vertex set of T is the union $R \cup \mathcal{P}$ where $R = \big\{v \in V(A) : v = r_n \text{ for some } n \geq 0\big\}$ and $\mathcal{P} = \big\{A_n : n \geq 0\big\}$; an edge between vertices x and y of T is drawn whenever $x \in R$, $y \in \mathcal{P}$ and x is contained in y. The vertex $r_A = r_0$ is the root of T. By $\leq_T$, we denote the *tree-order* of T, i.e., $a \leq_T b$ whenever a, b are vertices of T such that a is on the uniquely determined r_A, b-path of T. For each $v \in R$, we define the *branch* $B(A, v)$ of A as the subgraph of A which is formed by all graphs A_n with $v \leq_T A_n$. A branch $B(A, v)$ will generally be considered as a rooted subgraph of A with root v. For an infinite branch $B(A, v)$, let $P \subseteq B(A, v)$ be a v-ray. A vertex v' of P is an *essential vertex* of P if $v' = v$ or v' is incident with two edges of P belonging to distinct parts of A.

Let C be a connected subgraph of A with $A_0 \subseteq C$ and such that C is the union of a finite number of parts of A; moreover, let $B(A, v_1), \ldots, B(A, v_k)$ be disjoint branches of A. We say that $C, B(A, v_j)$ $(j = 1, \ldots, k)$ is a *representation* of A if $C \cap B(A, v_j) = v_j$ $(j = 1, \ldots, k)$ and if A is the union of C with the branches $B(A, v_j)$ $(j = 1, \ldots, k)$. When considering a representation of A, we sometimes want to explicitly mention the underlying decomposition $\mathcal{D}(A)$; we then say that the representation is *based on* $\mathcal{D}(A)$.

Let $\leq_r$ be a relation defined on the class of rooted graphs. A representation $C, B(A, v_j)$ $(j = 1, \ldots, k)$ of A is *good with respect to* $\leq_r$ if, for each $j \in \big\{1, \ldots, k\big\}$, there exists a v_j-ray $P_j \subseteq B(A, v_j)$ such that $B(A, v_j) \leq_r B(A, w)$ for infinitely many essential vertices w of P_j $(j = 1, \ldots, k)$. For an r_A-ray $P \subseteq A$, we denote by $\mathcal{B}\big(\mathcal{D}(A), P\big)$ the set of branches $B(A, v)$ with v being an essential vertex of P.

Lemma 1. *For a locally finite infinite tree-like graph A, let $\mathcal{D}(A)$ be a decomposition of A with root r_A. Let $\leq_r$ be a relation on the class of rooted graphs and assume that, for each r_A-ray P of A, the set $\mathcal{B}(\mathcal{D}(A), P)$ is wqo by $\leq_r$. Then A possesses a representation based on $\mathcal{D}(A)$ which is good with respect to $\leq_r$.*

Proof. Let (A_n, r_n) $(n = 0, 1, \ldots)$ denote the parts of $\mathcal{D}(A)$ and, as above, let $R = \{v \in V(A) : v = r_n \text{ for some } n \geq 0\}$. Denote by T the corresponding decomposition tree and recall that $r_A = r_0$. Let R' be the set of vertices $v \in R \setminus \{r_A\}$ for which there exists a v-ray $P \subseteq B(A, v)$ such that $B(A, v) \leq_r B(A, w)$ for infinitely many essential vertices w of P. Define the subgraph C of A as the union of those A_n $(n \geq 0)$ for which the corresponding r_A, r_n-path of T contains no vertex of R'. Then (trivially) $A_0 \subseteq C$ and one easily finds that C is connected. We claim that C is finite.

For a contradiction, suppose that C is infinite. Then, by a well-known theorem of König, there exists an r_A-ray $Q \subseteq C$. Let $w_0 = r_A, w_1, w_2, \ldots$ be the roots of the branches contained in $\mathcal{B}(\mathcal{D}(A), Q)$, in the order in which they appear on Q. By assumption, $\mathcal{B}(\mathcal{D}(A), Q)$ is wqo by $\leq_r$, from which one readily obtains that there exists an $n_0 \geq 1$ such that $B(A, w_{n_0}) \leq_r B(A, w_n)$ for infinitely many $n \geq n_0$. Hence, by considering the w_{n_0}-tail P of Q, one obtains $w_{n_0} \in R'$ and thus we have $w_{n_0+1} \notin C$, which is a contradiction.

Let $v_1, \ldots, v_k$ be the vertices of $V(C) \cap R'$. It then follows from the definitions of R' and C that $C, B(A, v_j)$ $(j = 1, \ldots, k)$ is a representation of A based on $\mathcal{D}(A)$ which is good with respect to $\leq_r$. $\square$

For both the minor relation $\preccurlyeq$ and the topological minor relation $\Subset$ we need variants $\preccurlyeq_r$ and $\Subset_r$ for rooted graphs which are defined as follows. Let $\leq$ be either $\preccurlyeq$ or $\Subset$ and further let G and H be graphs for which some vertices r_G and r_H have been exhibited as roots of G and H, respectively. Then $G \leq_r H$ means that there exists a $\leq$-embedding $\varphi : G \to H$, together with an $r_H, \varphi(r_G)$-path $Q \subseteq H$, such that Q has exactly one vertex in common with $\varphi(G)$. A $\leq$-embedding with this additional property is called a *$\leq_r$-embedding.* Obviously, for the case that $\leq$ is the minor relation, $G \leq_r H$ is equivalent to the existence of a $\leq$-embedding $\varphi : G \to H$ with $r_H \in \varphi(r_G)$. The relations $\preccurlyeq_r$ and $\Subset_r$ are called the *rooted minor relation* and the *rooted topological minor relation*, respectively. Instead of saying 'with respect to the minor relation' or 'with respect to the rooted minor relation', we frequently just say 'with respect to minors', and a similar remark applies where topological minors are concerned.

A graph is *strongly tree-like* if it is connected and all its bridgeless connected subgraphs are finite. Obviously, each strongly tree-like graph is tree-like, but the converse does not hold (cf. Figure 2). For a locally finite infinite tree-like graph A, let (A_n, r_n) $(n = 0, 1, \ldots)$ be a decomposition of A for which the following additional condition (iii) holds:

(iii) The degree of r_n in $A_0 \cup \ldots \cup A_{n-1}$ is 1 $(n = 1, 2, \ldots)$.

Then the decomposition (A_n, r_n) $(n = 0, 1, \ldots)$ is called a *strong decomposition* of A. Obviously, if A possesses a strong decomposition, then A is strongly tree-like. But also the converse is true: assuming that A is a locally finite infinite strongly tree-like graph, a decomposition of A meeting condition (iii) can easily be obtained by considering the maximal bridgeless connected subgraphs of A and the bridges connecting them: let A_0 emerge from an arbitrary maximal bridgeless connected subgraph of A by adding to it all bridges emanating from it (plus incident vertices), and construct $A_1, A_2, \ldots$ in a similar way by adding to appropriate maximal bridgeless connected subgraphs all but one of the emanating bridges. (We leave the details to the reader.) A representation $C, B(A, v_j)$ $(j = 1, \ldots, k)$ of A based on a strong decomposition of A is called a *strong representation.* Note that if $C, B(A, v_j)$ $(j = 1, \ldots, k)$ is a strong representation of A, then it follows from the definitions that v_j has degree 1 in C $(j = 1, \ldots, k)$.

3. Main results and corollaries

Theorem 1. *Let A be a locally finite infinite tree-like graph.*

(i) If A possesses a good representation with respect to minors, then A is $\preccurlyeq$-ubiquitous.

(ii) If A possesses a strong representation which is good with respect to topological minors, then A is $\Subset$-ubiquitous.

The proof of Theorem 1 is postponed to the subsequent sections. The next theorem is an immediate consequence of Theorem 1 and Lemma 1.

Theorem 2. *Let A be a locally finite infinite tree-like graph.*

(i) If A possesses a decomposition $\mathcal{D}(A)$ with root r_A such that, for each r_A-ray P of A, the set $\mathcal{B}(\mathcal{D}(A), P)$ is wqo by $\preccurlyeq_r$, then A is $\preccurlyeq$-ubiquitous.

(ii) If A possesses a strong decomposition $\mathcal{D}(A)$ with root r_A such that, for each r_A-ray P of A, the set $\mathcal{B}(\mathcal{D}(A), P)$ is wqo by $\subseteq_r$, then A is $\subseteq$-ubiquitous.

We next derive several corollaries from Theorem 2.

Corollary 1. *A locally finite tree-like graph is $\preccurlyeq$-ubiquitous if its number of ends is finite.*

Proof. Let A be a locally finite tree-like graph with a finite number of ends. It may be assumed that A is infinite. Let $\mathcal{D}(A)$ be an arbitrary decomposition of A with root r_A and let P be an r_A-ray of A. By Theorem 2, we are done if we can show that $\mathcal{B}(\mathcal{D}(A), P)$ is wqo by $\preccurlyeq_r$. To this end, let $a_0 = r_A, a_1, a_2, \ldots$ be the essential vertices of P, in the order in which they appear on P. Since A has just a finite number of ends, there exists an $n_0 \geq 0$ such that the branch $B(A, a_{n_0})$ possesses only one end. Let $a'_i := a_{n_0+i}$ $(i = 0, 1, \ldots)$. It follows that there are finite connected graphs $A'_0, A'_1, \ldots$ having the following properties (cf. Figure 3):

(i) $a'_0 \in A'_0$;

(ii) $A'_{i-1} \cap A'_i = a'_i$ $(i = 1, 2, \ldots)$;

(iii) $B(A, a'_i) = \bigcup\limits_{j=i}^{\infty} A'_j$ $(i = 0, 1, \ldots)$.

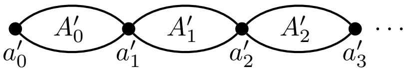

Fig. 3: A one-ended subgraph of A.

Call a graph *birooted* if an ordered pair of distinct vertices has been exhibited as its pair of roots. For birooted graphs (G, r_1, r_2) and (H, s_1, s_2), we write $(G, r_1, r_2) \preccurlyeq_2 (H, s_1, s_2)$ to indicate that there exists a $\preccurlyeq$-embedding $\varphi : G \to H$ with $s_i \in \varphi(r_i)$ $(i = 1, 2)$. We shall use a 'birooted version' of the graph minor theorem of Robertson and Seymour [11] stating that the class of finite birooted graphs is wqo by $\preccurlyeq_2$. This version of the graph minor theorem is well-known. It can be derived from statement (10.4) of [11] in a similar way as (10.5) is derived there. (Leaving all details to the reader, we just remark that in the context of (10.4) and (10.5) of [11] it is useful to think of a birooted graph (G, r_1, r_2) as an 'edge-labelled graph with loops' resulting from G by adding loops ℓ_1 and ℓ_2 at r_1 and r_2, respectively, with ℓ_1 labelled 1, ℓ_2 labelled 2, and all other edges labelled 0.)

We now consider the birooted graphs (A'_i, a'_i, a'_{i+1}) $(i = 0, 1, \ldots)$ and obtain as an easy consequence of the mentioned birooted version of the graph minor theorem that there exists an $i_0 \geq 0$ such that, for each $i \geq i_0$, there are infinitely many indices $j \geq i$ such that $(A'_i, a'_i, a'_{i+1}) \preccurlyeq_2 (A'_j, a'_j, a'_{j+1})$. From this one obtains $B(A, a'_i) \preccurlyeq_r B(A, a'_k)$ for each pair i, k with $i_0 \leq i < k$, which implies that $\mathcal{B}(\mathcal{D}(A), P)$ is wqo by $\preccurlyeq_r$. $\square$

Corollary 2. *A locally finite tree-like graph A is $\preccurlyeq$-ubiquitous if there exists a finite planar graph which is not a minor of A.*

Proof. We may assume that A is infinite. Let $\mathcal{D}(A)$ be a decomposition of A with root r_A and, further, let P be an r_A-ray of A. By Theorem 2, it is sufficient to show that $\mathcal{B}(\mathcal{D}(A), P)$ is wqo by $\preccurlyeq_r$, but this easily follows from the results of Thomas [12]. (Cf. the remark after this proof.) $\square$

Remark. Denoting by Forb(H) the class of graphs which do not contain a given graph H as a minor, the unrooted version of Thomas' theorem [12] states that Forb(H) is wqo by $\preccurlyeq$ provided that H is a finite planar graph. As an easy consequence of this result, one obtains the following rooted version of it which meets the requirements of the above proof.

(1) For H finite and planar, let $(G_1, r_1), (G_2, r_2), \ldots$ be a sequence of rooted graphs with $G_i \in \mathrm{Forb}(H)$ $(i = 1, 2, \ldots)$. Then there exist indices $i < j$ with $(G_i, r_i) \preccurlyeq_r (G_j, r_j)$.

We show how (1) can be derived from the corresponding statement for unrooted graphs: w.l.o.g we may assume $p_1 \leq p_2 \leq \ldots$ where p_i denotes the number of pendant edges of G_i which are incident with the root r_i $(i = 1, 2, \ldots)$. For an infinite cardinal α with $\alpha > |G_i|$ $(i = 1, 2, \ldots)$, let G_i^+ be a graph that results from G_i by adding α new pendant edges to G_i, all incident with r_i $(i = 1, 2, \ldots)$. Let $H^+ \supseteq H$ be a finite planar graph without vertices of degree ≤ 1. Then $G_i \in \mathrm{Forb}(H)$ implies $G_i^+ \in \mathrm{Forb}(H^+)$ and thus there exists a $\preccurlyeq$-embedding $h : G_i^+ \to G_j^+$ for some i, j with $i < j$. From the choice of α, one obtains $r_j \in h(r_i)$. Moreover, one easily concludes from $p_i \leq p_j$ that $h(G_i) \subseteq G_j$ may be assumed. Hence $(G_i, r_i) \preccurlyeq_r (G_j, r_j)$.

Corollary 3. *For a locally finite connected graph A assume that all of its blocks are complete graphs. Then A is $\preccurlyeq$-ubiquitous. Further, if all blocks of A are complete graphs K_2, then A is $\subseteq$-ubiquitous.*

Proof. The statement that A is $\subseteq$-ubiquitous if A is a tree immediately follows from Theorem 2 and Nash-Williams' tree-theorem. The statement on $\preccurlyeq$-ubiquity is an immediate consequence of Theorem 2 and the following statement (2).

(2) The class $\mathcal{B}$ of connected rooted graphs for which all blocks are complete is wqo by $\preccurlyeq_r$.

We sketch a proof of (2) leaving some of the details to the reader. Given a non-trivial rooted graph $(H, r_H) \in \mathcal{B}$, associate with (H, r_H) a rooted labelled tree T as follows. Define $V(T)$ as the union of $V(H)$ with the set of blocks of H and draw an edge between two vertices of T whenever one is a block and the other one is a vertex of H contained in this block. Label a vertex of T with 1 if it is a block of H and with 0, otherwise; take r_H as the root of T. Consider a sequence (H_n, r_n) $(n = 1, 2, \ldots)$ of non-trivial rooted graphs contained in $\mathcal{B}$ and let T_n be the rooted labelled tree associated with (H_n, r_n) $(n = 1, 2, \ldots)$. Then it follows from a labelled version of Nash-Williams' tree-theorem which is due to Laver [8] that there are indices $i < j$ for which there exists a $\subseteq_r$-embedding $h : T_i \to T_j$ preserving the labels, i.e, 0-vertices are mapped onto 0-vertices and 1-vertices onto 1-vertices. From this one readily obtains $(H_i, r_i) \preccurlyeq_r (H_j, r_j)$. (Leaving all further details to the reader, we just mention that a $\subseteq_r$-embedding $h : T_i \to T_j$ induces a $\preccurlyeq_r$-embedding $\widetilde{h} : H_i \to H_j$ which maps each vertex v of H_i onto a 'star with center $h(v)$', i.e., a graph consisting of paths emanating from $h(v)$ and having only $h(v)$ in common.) $\square$

The above corollaries demonstrate how Theorem 2 can be used, in conjunction with appropriate wqo-results, to obtain ubiquity results for certain classes of locally finite tree-like graphs. The following conjecture is due to Thomas [12].

Thomas' conjecture. *The class of countable graphs is wqo by minors.*

We now show that the truth of Thomas' conjecture would imply that all locally finite tree-like graphs are $\preccurlyeq$-ubiquitous. Actually, we shall find that the truth of a considerably relaxed version of Thomas' conjecture would also do.

Call a graph *almost locally finite* if all its vertices have finite degree, except for exactly one vertex which has countable degree and for which all but a finite number of neighbours have degree 1. Let $\mathcal{C}$ be the class of almost locally finite tree-like graphs.

Corollary 4. *If $\mathcal{C}$ is wqo by $\preccurlyeq$, then every locally finite tree-like graph is $\preccurlyeq$-ubiquitous.*

Proof. This is a direct consequence of Theorem 2 and the following statement which is easily seen to be true.

(3) If $\mathcal{C}$ is wqo by $\preccurlyeq$, then the class of locally finite infinite rooted tree-like graphs is wqo by $\preccurlyeq_r$. $\square$

4. Thick subsets

In this section we will prove a series of lemmas preparing the proof of Theorem 1. With the exception of Lemma 6, these lemmas deal with families of sets rather than graphs.

Definition. For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be a set with $|\mathcal{A}_n| = n$; put $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. A subset $\mathcal{A}'$ of $\mathcal{A}$ is *thick* with respect to the family of sets $\mathcal{A}_n$ ($n \in \mathbb{N}$) if, for each $k \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $|\mathcal{A}' \cap \mathcal{A}_n| \geq k$. A set $\mathcal{A}' \subseteq \mathcal{A}$ is *thin* with respect to $\mathcal{A}_n$ ($n \in \mathbb{N}$) if it is not thick with respect to the given family $\mathcal{A}_n$ ($n \in \mathbb{N}$).

In the following, we shall frequently use the notions 'thick' and 'thin' without specifying to which family of sets we refer. In such cases there will always be under consideration a family of sets denoted by $\mathcal{A}_n$ ($n \in \mathbb{N}$) and 'thick' is just a shorthand for 'thick with respect to the family $\mathcal{A}_n$ ($n \in \mathbb{N}$)'. A similar remark applies to 'thin'.

The next two lemmas are immediate consequences of the definitions. They will frequently be used without being mentioned explicitly.

Lemma 2. *For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be a set with $|\mathcal{A}_n| = n$ and put $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$.*

(i) The union of a finite number of thin subsets of $\mathcal{A}$ is again a thin subset of $\mathcal{A}$.

(ii) If $\mathcal{A}'$ is a thick and $\mathcal{A}''$ is a thin subset of $\mathcal{A}$, then $\mathcal{A}' \setminus \mathcal{A}''$ is a thick subset of $\mathcal{A}$.

Lemma 3. *For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be a set of n disjoint sets and let F be some finite set. Put $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. Then $\mathcal{A}' = \{A \in \mathcal{A} : A \cap F \neq \emptyset\}$ is a thin subset of $\mathcal{A}$.*

Lemma 4. *For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be a set with $|\mathcal{A}_n| = n$. Then there exist disjoint sets $\mathcal{A}'_1, \mathcal{A}'_2, \ldots$ such that $\mathcal{A}'_k \subseteq \mathcal{A}_{n_k}$ for some $n_k \in \mathbb{N}$ and $|\mathcal{A}'_k| = k$ ($k = 1, 2, \ldots$).*

Proof. Put $n_1 := 1$ and $\mathcal{A}'_1 := \mathcal{A}_1$. For some $k \in \mathbb{N}$, assume that positive integers n_j and disjoint sets $\mathcal{A}'_j$ have already been defined such that $\mathcal{A}'_j \subseteq \mathcal{A}_{n_j}$ and $|\mathcal{A}'_j| = j$ ($j = 1, \ldots, k$). Then the union $\mathcal{A}'_1 \cup \ldots \cup \mathcal{A}'_k$ is a finite set. Thus we can find an $n_{k+1} \in \mathbb{N}$ such that there exists a subset $\mathcal{A}'_{k+1}$ of $\mathcal{A}_{n_{k+1}}$ with $\mathcal{A}'_{k+1} \cap (\mathcal{A}'_1 \cup \ldots \cup \mathcal{A}'_k) = \emptyset$ and $|\mathcal{A}'_{k+1}| = k + 1$. This inductively defines $n_1, n_2, \ldots$ and $\mathcal{A}'_1, \mathcal{A}'_2, \ldots$ as desired. $\square$

The next lemma is more complex; it will be used to obtain the subsequent Lemma 6 which is a cornerstone in the proof of Theorem 1.

Lemma 5. *For each $n \in \mathbb{N}$, let $A^{(n,1)}, \ldots, A^{(n,n)}$ be n distinct sets and put $\mathcal{A}_n = \{A^{(n,m)} : m = 1, \ldots, n\}$. Assume $\mathcal{A}_{n_1} \cap \mathcal{A}_{n_2} = \emptyset$ for all $n_1, n_2 \in \mathbb{N}$ ($n_1 \neq n_2$) and let $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. Further, for each $A^{(n,m)} \in \mathcal{A}$, assume that $B_i^{(n,m)}$ ($i \in I^{(n,m)}$) is a family of subsets of $A^{(n,m)}$, where $I^{(n,m)}$ is some index set. Then there exists a thick subset $\mathcal{A}'$ of $\mathcal{A}$ for which either (a) or (b) holds.*

(a) For all $A^{(n,m)} \in \mathcal{A}'$ and all $i \in I^{(n,m)}$ there exists a thin subset $\mathcal{E}$ of $\mathcal{A}$ such that $B_i^{(n,m)} \cap A^{(q,p)} \neq \emptyset$ for all $A^{(q,p)} \in \mathcal{A}' \setminus \mathcal{E}$.

(b) *For all* $A^{(n,m)} \in \mathcal{A}'$ *there exists an* $i \in I^{(n,m)}$ *and a thin subset* $\mathcal{E}$ *of* $\mathcal{A}$ *such that* $B_i^{(n,m)} \cap A^{(q,p)} = \emptyset$ *for all* $A^{(q,p)} \in \mathcal{A}' \setminus \mathcal{E}$.

Proof. We assume that there exists no thick subset $\mathcal{A}'$ of $\mathcal{A}$ for which (a) holds and show that, under this hypothesis, there is one for which (b) holds. Thus by assumption the following holds.

(4) For all thick subsets $\mathcal{A}'$ of $\mathcal{A}$ there exists an $A^{(n,m)} \in \mathcal{A}'$ and a corresponding $i \in I^{(n,m)}$ such that, for every thin subset $\mathcal{E}$ of $\mathcal{A}$, there exists an $A^{(q,p)} \in \mathcal{A}' \setminus \mathcal{E}$ such that $B_i^{(n,m)} \cap A^{(q,p)} = \emptyset$.

We claim that, for each $k \in \mathbb{N}$, the following statement (5) is a consequence of (4).

(5) For all thick subsets $\mathcal{A}'$ of $\mathcal{A}$ there exist (for an appropriate $n \in \mathbb{N}$) k distinct $A^{(n,m_j)} \in \mathcal{A}'$ $(j = 1, \ldots, k)$, together with corresponding indices $i_j \in I^{(n,m_j)}$ $(j = 1, \ldots, k)$ and together with a thick $\mathcal{A}'' \subseteq \mathcal{A}'$, such that $B_{i_j}^{(n,m_j)} \cap A^{(q,p)} = \emptyset$ for all $A^{(q,p)} \in \mathcal{A}''$ and all $j \in \{1, \ldots, k\}$.

The proof of (5) is carried out by induction on k. For $k = 1$, statement (5) readily follows from (4): for given $\mathcal{A}'$, let $A^{(n,m)}$ and i be as in (4) and put $\mathcal{A}'' = \{A^{(q,p)} \in \mathcal{A}' : B_i^{(n,m)} \cap A^{(q,p)} = \emptyset\}$. Then $\mathcal{A}''$ is not a thin subset of $\mathcal{A}$ since, otherwise, choosing $\mathcal{E} = \mathcal{A}''$ would result in a contraction to (4). Hence $\mathcal{A}''$ is thick, which shows (5) for $k = 1$.

We now assume that (5) holds for some fixed $k \in \mathbb{N}$. For a contradiction, suppose that (5) does not hold for $k+1$. Let $\mathcal{A}^{(1)}$ be a thick subset of $\mathcal{A}$ showing that (5) is not valid for $k+1$, i.e., one cannot find $k+1$ members of $\mathcal{A}^{(1)}$ having the properties addressed in (5).

Let $\mathcal{A}_*^{(1)}$ be the set of those $A^{(n,m)} \in \mathcal{A}^{(1)}$ for which $|\mathcal{A}_n \cap \mathcal{A}^{(1)}| \geq k+1$. Because $\mathcal{A}^{(1)}$ is thick, the same holds for $\mathcal{A}_*^{(1)}$. Hence, by the induction hypothesis, we can find k distinct $A^{(n_1,m_j)} \in \mathcal{A}_*^{(1)}$ $(j = 1, \ldots, k)$, together with corresponding $i_j \in I^{(n_1,m_j)}$ $(j = 1, \ldots, k)$ and a thick $\mathcal{A}^{(2)} \subseteq \mathcal{A}_*^{(1)}$, such that $B_{i_j}^{(n_1,m_j)} \cap A^{(q,p)} = \emptyset$ for all $A^{(q,p)} \in \mathcal{A}^{(2)}$ and all $j \in \{1, \ldots, k\}$. By the definition of $\mathcal{A}_*^{(1)}$, we can pick some $A^{(n_1,m_{k+1})} \in \mathcal{A}_{n_1} \cap \mathcal{A}^{(1)}$ such that $A^{(n_1,m_{k+1})} \neq A^{(n_1,m_j)}$ $(j = 1, \ldots, k)$. It follows that, for each $i \in I^{(n_1,m_{k+1})}$, there exists a thin subset $\mathcal{E}$ of $\mathcal{A}$ such that $B_i^{(n_1,m_{k+1})} \cap A^{(q,p)} \neq \emptyset$ for all $A^{(q,p)} \in \mathcal{A}^{(2)} \setminus \mathcal{E}$ since, otherwise, we could pick $i_{k+1} \in I^{(n_1,m_{k+1})}$ such that $\{A^{(q,p)} \in \mathcal{A}^{(2)} : B_{i_{k+1}}^{(n_1,m_{k+1})} \cap A^{(q,p)} = \emptyset\}$ is thick, in contradiction to our supposition that (5) does not hold for $\mathcal{A}^{(1)}$ and $k+1$. We define

$$\mathcal{B}_1 := \Big\{A^{(n_1,m_{k+1})}\Big\}.$$

Next, put $\mathcal{A}_*^{(2)} := \{A^{(n,m)} \in \mathcal{A}^{(2)} : |\mathcal{A}_n \cap \mathcal{A}^{(2)}| \geq k+2\}$ and note that $\mathcal{A}_*^{(2)}$ is thick. Application of the induction hypothesis to $\mathcal{A}_*^{(2)}$ yields k distinct $A^{(n_2,\ell_j)} \in \mathcal{A}_*^{(2)}$ $(j = 1, \ldots, k)$, together with an element of $I^{(n_2,\ell_j)}$ for each $j \in \{1, \ldots, k\}$ and a thick $\mathcal{A}^{(3)} \subseteq \mathcal{A}_*^{(2)}$ according to (5). By the definition of $\mathcal{A}_*^{(2)}$, we can pick distinct sets $A^{(n_2,\ell_{k+1})}, A^{(n_2,\ell_{k+2})} \in \mathcal{A}_{n_2} \cap \mathcal{A}^{(2)}$ such that $A^{(n_2,\ell_{k+r})} \neq A^{(n_2,\ell_j)}$ for $r = 1, 2$ and $j = 1, \ldots, k$. Similar as before it follows that, for each $r \in \{1, 2\}$ and each $i \in I^{(n_2,\ell_{k+r})}$, there exists a thin $\mathcal{E}$ such that $B_i^{(n_2,\ell_{k+r})} \cap A^{(q,p)} \neq \emptyset$ for all $A^{(q,p)} \in \mathcal{A}^{(3)} \setminus \mathcal{E}$. We put

$$\mathcal{B}_2 := \Big\{A^{(n_2,\ell_{k+r})} : r = 1, 2\Big\}.$$

We next define $\mathcal{A}_*^{(3)} := \{A^{(n,m)} \in \mathcal{A}^{(3)} : |\mathcal{A}_n \cap \mathcal{A}^{(3)}| \geq k+3\}$ and proceed as before to obtain a 3-element set $\mathcal{B}_3$ defined in a similar way as $\mathcal{B}_1$ and $\mathcal{B}_2$. Going on like that, one inductively obtains for every $q \in \mathbb{N}$ a q-element set $\mathcal{B}_q \subseteq \mathcal{A}_{n_q}$ (for some $n_q \in \mathbb{N}$) such that $\mathcal{B} = \bigcup_{q=1}^{\infty} \mathcal{B}_q$ is a thick subset of $\mathcal{A}$ which meets (a). This contradicts our assumption that no such thick subset exists. Hence (5).

With the aid of (5), it is easy to find a thick subset of $\mathcal{A}$ which meets (b). To this end, we apply (5) to the case $\mathcal{A}' = \mathcal{A}$ and $k = 1$. This results into a set $A^{(n_1,m_1)}$, together with an element of $I^{(n_1,m_1)}$ and a thick $\mathcal{A}^{(1)}$ according to (5). Next, we apply (5) to the case $\mathcal{A}' = \mathcal{A}^{(1)}$ and $k = 2$ to obtain $A^{(n_2,\ell_j)} \in \mathcal{A}^{(1)}$ $(j = 1, 2)$, together with elements i_j of $I^{(n_2,\ell_j)}$ $(j = 1, 2)$ and a thick $\mathcal{A}^{(2)} \subseteq \mathcal{A}^{(1)}$ according to (5). Going on in this way, one inductively obtains for every $q \in \mathbb{N}$ a q-element set $\mathcal{C}_q \subseteq \mathcal{A}_{n_q}$ (for some $n_q \in \mathbb{N}$) such that $\mathcal{C} = \bigcup\limits_{q=1}^{\infty} \mathcal{C}_q$ is a thick subset of $\mathcal{A}$ which meets (b). □

Given for each $n \in \mathbb{N}$ a set $\mathcal{A}_n = \{A^{(n,m)} : m = 1, \ldots, n\}$ of n disjoint graphs, we shall frequently say that one of the preceding lemmas is applied to the sets $\mathcal{A}_n$ $(n \in \mathbb{N})$, by which we mean that the lemma is applied to the corresponding sets $\{V(A^{(n,m)}) : m = 1, \ldots, n\}$ $(n \in \mathbb{N})$. In a similar way, notions like 'thick' and 'thin' are used when sets $\mathcal{A}_n$ $(n \in \mathbb{N})$ of graphs are considered.

Lemma 6. *For a locally finite infinite tree-like graph A, assume that $\mathcal{D}(A)$ is a decomposition of A and let $C, B(A, v_j)$ $(j = 1, \ldots, k)$ be a representation of A based on $\mathcal{D}(A)$. Assume further that each branch $B(A, v_j)$ is infinite and let $P_j \subseteq B(A, v_j)$ be a v_j-ray $(j = 1, \ldots, k)$. Let $\leq$ be either the minor or the topological minor relation. For a graph G, assume that for every $n \in \mathbb{N}$ there exist n disjoint subgraphs $A^{(n,m)}$ $(m = 1, \ldots, n)$ of G such that $A^{(n,m)} = \varphi^{(n,m)}(A)$ for a $\leq$-embedding $\varphi^{(n,m)} : A \to G$ $(n, m \in \mathbb{N}, n \geq m)$. Then one can choose the graphs $A^{(n,m)}$ and the corresponding $\leq$-embeddings $\varphi^{(n,m)}$ such that, for each $j \in \{1, \ldots, k\}$, either (a.j) or (b.j) holds (with $\mathcal{A} = \bigcup\limits_{n=1}^{\infty} \mathcal{A}_n$ for $\mathcal{A}_n = \{A^{(n,m)} : m = 1, \ldots, n\}$).*

(a.j) For all $n, m \in \mathbb{N}$ with $n \geq m$ and all essential vertices v of P_j there exists a thin subset $\mathcal{E}$ of $\mathcal{A}$ such that $\varphi^{(n,m)}\big(B(A, v)\big) \cap A^{(q,p)} \neq \emptyset$ for all $A^{(q,p)} \in \mathcal{A} \setminus \mathcal{E}$.

(b.j) For all $n, m \in \mathbb{N}$ with $n \geq m$ there exists an essential vertex v of P_j and a thin subset $\mathcal{E}$ of $\mathcal{A}$ such that $\varphi^{(n,m)}\big(B(A, v)\big) \cap A^{(q,p)} = \emptyset$ for all $A^{(q,p)} \in \mathcal{A} \setminus \mathcal{E}$.

Proof. We start with arbitrarily choosing sets $\mathcal{A}_n = \{A^{(n,m)} : m = 1, \ldots, n\}$ of n disjoint subgraphs of G for which there exist $\leq$-embeddings $\varphi^{(n,m)} : A \to G$ with $\varphi^{(n,m)}(A) = A^{(n,m)}$ $(n, m \in \mathbb{N}, n \geq m)$. By Lemma 4, we may assume that the sets $\mathcal{A}_n$ $(n = 1, 2, \ldots)$ are pairwise disjoint; we put $\mathcal{A} = \bigcup\limits_{n=1}^{\infty} \mathcal{A}_n$.

Now, Lemma 6 can be proved in an obvious way by iterated application of Lemma 5. The first step is to apply Lemma 5 to the just chosen sets $\mathcal{A}_n$ $(n \in \mathbb{N})$ with $I^{(n,m)} = \{v : v \text{ is an essential vertex of } P_1\}$ for all $n, m \in \mathbb{N}$ $(n \geq m)$ and with the vertex sets of the graphs $\varphi^{(n,m)}\big(B(A, v)\big)$ in the role of the sets $B_i^{(n,m)}$. This yields a thick subset $\mathcal{A}'$ of $\mathcal{A}$ for which either (a') or (b') holds.

(a') For all $A^{(n,m)} \in \mathcal{A}'$ and all essential vertices v of P_1 there exists a thin subset $\mathcal{E}$ of $\mathcal{A}$ such that $\varphi^{(n,m)}\big(B(A, v)\big) \cap A^{(q,p)} \neq \emptyset$ for all $A^{(q,p)} \in \mathcal{A}' \setminus \mathcal{E}$.

(b') For all $A^{(n,m)} \in \mathcal{A}'$ there exists an essential vertex v of P_1 and a thin subset $\mathcal{E}$ of $\mathcal{A}$ such that $\varphi^{(n,m)}\big(B(A, v)\big) \cap A^{(q,p)} = \emptyset$ for all $A^{(q,p)} \in \mathcal{A}' \setminus \mathcal{E}$.

For each $n \in \mathbb{N}$ and a suitable q_n, we pick exactly n members of $\mathcal{A}' \cap \mathcal{A}_{q_n}$ and, for simplicity, denote the so chosen graphs with $A^{(n,m)}$ $(m = 1, \ldots, n)$. Accordingly, we redefine $\mathcal{A}_n$ as the set of these graphs $A^{(n,m)}$ and, in a similar manner, also $\varphi^{(n,m)}$ and $\mathcal{A}$ are redefined. One readily finds that, after these redefinitions, either (a.1) or (b.1) holds. (Note that if $\mathcal{E}$ is a set as in (a') or (b'), then (after the redefinition of $\mathcal{A}$) the intersection of $\mathcal{E}$ with the new $\mathcal{A}$ is a thin set with respect to the new family $\mathcal{A}_n$ $(n \in \mathbb{N})$.) Iteration of the procedure for $P_2, \ldots, P_k$ yields the lemma. □

5. Proof of Theorem 1

Let A be a locally finite infinite tree-like graph. We prove both parts of Theorem 1 simultaneously. To this end, let $\leq$ be either the minor relation $\preccurlyeq$ or the topological minor relation $\subseteq$. Let (A_n, r_n) $(n = 0, 1, \ldots)$ be a decomposition of A and, if $\leq$ is the topological minor relation, assume that the decomposition is strong. As before, we put $r_A := r_0$. Let further $C, B(A, v_j)$ $(j = 1, \dots, k)$ be a representation of A based on (A_n, r_n) $(n = 0, 1, \dots)$ which is good with respect to $\leq_r$. (For a definition of the relation $\leq_r$, see Section 2.) Thus there exists a v_j-ray $P_j \subseteq B(A, v_j)$ such that $B(A, v_j) \leq_r B(A, v)$ for infinitely many essential vertices v of P_j $(j = 1, \dots, k)$. Hence, by the transitivity of $\leq_r$, we have

(6) $B(A, v_j) \leq_r B(A, v)$ for all essential vertices v of P_j $(j = 1, \dots, k)$.

In order to show the $\leq$-ubiquity of A, let G be a graph for which $nA \leq G$ for all $n \in \mathbb{N}$. Application of Lemma 6 yields $A^{(n,m)}$, $\varphi^{(n,m)}$, $\mathcal{A}_n$ and $\mathcal{A}$ as described there (for $n, m \in \mathbb{N}, n \geq m$). Recall that we assume $\varphi^{(n,m)}(v)$ to be finite for all n, m and all $v \in V(A)$ (cf. the remark at the end of the introduction). Let (a.j) and (b.j) be as in Lemma 6. We put

$$J_1 := \Big\{1 \leq j \leq k : j \text{ meets (a.j)}\Big\}$$

and

$$J_2 := \Big\{1 \leq j \leq k : j \text{ meets (b.j)}\Big\}.$$

Note that $J_1 \cap J_2 = \emptyset$ and $J_1 \cup J_2 = \big\{1, \dots, k\big\}$. We start with considering the case $J_1 = \emptyset$ and claim that, under this hypothesis, the following holds.

(7) For all $n, m \in \mathbb{N}$ with $n \geq m$ there exists a subgraph $\widehat{A}^{(n,m)}$ of $A^{(n,m)}$, together with a $\leq$-embedding $\widehat{\varphi}^{(n,m)} : A \to G$ with $\widehat{\varphi}^{(n,m)}(A) = \widehat{A}^{(n,m)}$ and together with a thin subset $\mathcal{F}^{(n,m)}$ of $\mathcal{A}$, such that $\widehat{A}^{(n,m)} \cap A^{(q,p)} = \emptyset$ for all $A^{(q,p)} \in \mathcal{A} \setminus \mathcal{F}^{(n,m)}$.

The assumption $J_1 = \emptyset$ means that for all $n, m \in \mathbb{N}$ with $n \geq m$ and all $j \in \big\{1, \dots, k\big\}$ there exists an essential vertex $v_j^{(n,m)}$ of P_j and a thin subset $\mathcal{E}_j^{(n,m)}$ of $\mathcal{A}$ such that $\varphi^{(n,m)}\big(B(A, v_j^{(n,m)})\big) \cap A^{(q,p)} = \emptyset$ for all $A^{(q,p)} \in \mathcal{A} \setminus \mathcal{E}_j^{(n,m)}$. By (6) we have $B(A, v_j) \leq_r B(A, v_j^{(n,m)})$ from which one concludes (by considering a corresponding $\leq$-embedding $B(A, v_j) \to B(A, v_j^{(n,m)})$ and composing it with $\varphi^{(n,m)}$) that there exists a $\leq$-embedding $\tau_j^{(n,m)} : B(A, v_j) \to \varphi^{(n,m)}\big(B(A, v_j^{(n,m)})\big)$ together with a $\varphi^{(n,m)}(v_j^{(n,m)}), \tau_j^{(n,m)}(v_j)$-path $Q_j^{(n,m)} \subseteq \varphi^{(n,m)}\big(B(A, v_j^{(n,m)})\big)$ such that $Q_j^{(n,m)}$ has exactly one vertex in common with $\tau_j^{(n,m)}\big(B(A, v_j)\big)$ (for all $n, m \in \mathbb{N}$ with $n \geq m$ and all $j \in \big\{1, \dots, k\big\}$). Moreover, by appropriately extending $Q_j^{(n,m)}$, one obtains a $\varphi^{(n,m)}(v_j), \tau_j^{(n,m)}(v_j)$-path $P_j^{(n,m)} \subseteq \varphi^{(n,m)}\big(B(A, v_j)\big)$ which has exactly one vertex in common with $\tau_j^{(n,m)}\big(B(A, v_j)\big)$ $(n, m \in \mathbb{N},\ n \geq m,\ j \in \big\{1, \dots, k\big\})$. For all n, m with $n \geq m$, let

$$\widehat{A}^{(n,m)} := \varphi^{(n,m)}(C) \cup \bigcup_{j=1}^{k} \Big(\tau_j^{(n,m)}\big(B(A, v_j)\big) \cup P_j^{(n,m)}\Big).$$

A corresponding $\leq$-embedding $\widehat{\varphi}^{(n,m)} : A \to G$ with $\widehat{\varphi}^{(n,m)}(A) = \widehat{A}^{(n,m)}$ is defined as follows. If $\leq$ is $\preccurlyeq$, then $\widehat{\varphi}^{(n,m)}(v_j)$ is defined as $\varphi^{(n,m)}(v_j) \cup P_j^{(n,m)} \cup \tau_j^{(n,m)}(v_j)$. If $\leq$ is $\subseteq$, note that (because the representation $C, B(A, v_j)$ $(j = 1, \dots, k)$ is strong) each v_j has degree 1 in C and thus we can define $\widehat{\varphi}^{(n,m)}(v_j) := \tau_j^{(n,m)}(v_j)$ and $\widehat{\varphi}^{(n,m)}(e_j) := \varphi^{(n,m)}(e_j) \cup P_j^{(n,m)}$, where e_j is the uniquely determined edge of C which is incident with v_j $(j = 1, \dots, k)$. For all other vertices and edges of A, $\widehat{\varphi}^{(n,m)}$ is defined such that, in the obvious way, $\widehat{\varphi}$ either coincides with $\varphi^{(n,m)}$ or with $\tau_j^{(n,m)}$ for some $j \in \big\{1, \dots, k\big\}$. From the construction of the graphs $\widehat{A}^{(n,m)}$ and with the aid of the Lemmas 2(i) and 3, one readily obtains (7).

By application of (7), one obtains infinitely many disjoint $\leq$-copies of A in G in the following (obvious) way. Put $A^{(1)} := \widehat{A}^{(n_1,m_1)}$ for arbitrary $n_1, m_1 \in \mathbb{N}$, $n_1 \geq m_1$. If disjoint graphs $A^{(p)} = \widehat{A}^{(n_p,m_p)}$ $(p = 1, \dots, q)$

have been defined, then pick some $A^{(n_{q+1},m_{q+1})}$ which is not in $\mathcal{F}^{(n_p,m_p)}$ for all $p \in \{1,\ldots,q\}$ and put $A^{(q+1)} := \widehat{A}^{(n_{q+1},m_{q+1})}$. This defines $\leq$-copies $A^{(1)}, A^{(2)}, \ldots$ of A as desired, which settles the case $J_1 = \emptyset$.

Let $J_1 \neq \emptyset$. We denote by T the decomposition tree associated with the decomposition (A_n, r_n) $(n = 0, 1, \ldots)$ of A (cf. Section 2). Denote by $a_1, a_2, \ldots$ those vertices of $\bigcup_{j \in J_1} B(A, v_j)$ which are contained in the set $\{v \in V(A) : v = r_n \text{ for some } n \geq 0\}$, where $a_{n_1} \neq a_{n_2}$ for $n_1 \neq n_2$. If $a_n \in V\big(B(A, v_j)\big)$, then denote by $d(a_n)$ the distance in the tree T between a_n and v_j $(n = 1, 2, \ldots)$. We may assume that the order of the vertices $a_1, a_2, \ldots$ has been chosen such that

(8) $d(a_{n_1}) \leq d(a_{n_2})$ whenever $n_1 \leq n_2$.

We now inductively define subgraphs $S_1, S_2, \ldots$ of A as follows. Let $S_1 := C \cup \bigcup_{j \in J_2} B(A, v_j)$. For $n \geq 1$, if S_n has already been defined, let S_{n+1} be the union of S_n with those parts A_i of A having a_n as its root. Then clearly $S_1 \subseteq S_2 \subseteq \ldots$ and $\bigcup_{n=1}^{\infty} S_n = A$. Also note that, as a consequence of (8), we have $a_n \in V(S_n)$ $(n = 1, 2, \ldots)$ and thus all graphs S_n are connected. Put

$$V_n := \Big\{a_i \in V(S_n) : i \geq n\Big\} \ (n = 1, 2, \ldots).$$

It then follows from the definitions that V_n is finite and that, for each $a_i \in V_n$, there is just one part of A which contains a_i and which is a subgraph of S_n $(n = 1, 2, \ldots)$.

For $n \in \mathbb{N}$, a subgraph D of G is of *type* n if it is locally finite and if there exists a $\leq$-embedding $\psi : S_n \to D$, together with a family $D(v)$ $(v \in V_n)$ of disjoint connected graphs, such that, firstly, D is the union of $\psi(S_n)$ with the graphs $D(v)$ and, secondly,

(9) $\psi(S_n) \cap D(v) = \psi(v)$ for all $v \in V_n$.

The graphs $D(v)$ are called *tentacles* of D (with respect to ψ). Note that, if $\widehat{\psi} : A \to G$ is a $\leq$-embedding, then $\widehat{\psi}(A)$ is of type n for every $n \in \mathbb{N}$ with the restriction of $\widehat{\psi}$ to S_n in the role of a corresponding $\leq$-embedding $\psi : S_n \to \widehat{\psi}(A)$ and with the graphs $\widehat{\psi}\big(B(A, v)\big)$ $(v \in V_n)$ as tentacles.

Let D be of type n with a corresponding $\leq$-embedding $\psi : S_n \to D$ and tentacles $D(v)$ $(v \in V_n)$. A thick subset $\mathcal{A}'$ of $\mathcal{A}$ is *compatible* with D and ψ if it meets the conditions (10) and (11).

(10) $\psi(S_n) \cap A' = \emptyset$ for all $A' \in \mathcal{A}'$.

(11) For each $v \in V_n$, either (11.1) or (11.2) holds:

- (11.1) $D(v) \cap A' \neq \emptyset$ for all $A' \in \mathcal{A}'$.
- (11.2) $D(v) \cap A' = \emptyset$ for all $A' \in \mathcal{A}'$ and, in addition, there exists a $\leq$-embedding $\varphi : B(A, v) \to D(v)$ with $\varphi(v) = \psi(v)$.

For $n \in \mathbb{N}$, let $\psi : S_n \to G$ and $\psi' : S_{n+1} \to G$ be $\leq$-embeddings such that $\psi(S_n)$ is a subgraph of $\psi'(S_{n+1})$ and such that ψ' coincides with ψ on $S_n - a_n$. Then ψ' is called an *extension* of ψ.

Let $\mathcal{D}_0 := \emptyset$. For some $n \geq 0$, assume that we have defined a set $\mathcal{D}_n$ of n disjoint subgraphs of G. If $n \geq 1$ let $\mathcal{D}_n = \{D^{(n,m)} : m = 1, \ldots, n\}$ and assume that the graphs $D^{(n,m)}$ are of type n with corresponding $\leq$-embeddings $\psi^{(n,m)} : S_n \to D^{(n,m)}$ $(m = 1, \ldots, n)$; for $n \geq 1$ also assume that there exists a thick subset $\mathcal{A}^{(n)}$ of $\mathcal{A}$ which is compatible with $D^{(n,m)}$ and $\psi^{(n,m)}$ for all $m = 1, \ldots, n$.

Claim. Then there exists a set $\mathcal{D}_{n+1} := \{D^{(n+1,m)} : m = 1, \ldots, n+1\}$ of $n+1$ disjoint subgraphs of G such that each $D^{(n+1,m)}$ is of type $n+1$ and such that, for corresponding $\leq$-embeddings $\psi^{(n+1,m)} : S_{n+1} \to D^{(n+1,m)}$ $(m = 1, \ldots, n+1)$, the following statements (12) and (13) hold.

(12) If $n \geq 1$, then $\psi^{(n+1,m)}$ is an extension of $\psi^{(n,m)}$ $(m = 1, \ldots, n)$.

(13) There exists a thick subset $\mathcal{A}^{(n+1)}$ of $\mathcal{A}$ which is compatible with $D^{(n+1,m)}$ and $\psi^{(n+1,m)}$ for $m = 1, \ldots, n+1$.

We show that the validity of this claim implies $\aleph_0 A \leq G$. Indeed, assuming the truth of the claim, one inductively obtains, for each $n \in \mathbb{N}$, a set $\mathcal{D}_n = \{D^{(n,m)} : m = 1, \ldots, n\}$ of n disjoint subgraphs of G which are of type n, together with corresponding $\leq$-embeddings $\psi^{(n,m)} : S_n \to D^{(n,m)}$ $(m = 1, \ldots, n)$, such that $\psi^{(n+1,m)}$ is an extension of $\psi^{(n,m)}$ for all $n, m \in \mathbb{N}$, $n \geq m$. Put

$$A^{(m)} := \bigcup_{n=m}^{\infty} \psi^{(n,m)}(S_n) \ (m = 1, 2, \ldots).$$

Then, obviously, the graphs $A^{(m)}$ $(m = 1, 2, \ldots)$ are disjoint. Put $S_n^- := S_n - V_n$ $(n = 1, 2, \ldots)$. Then $S_1^- \subseteq S_2^- \subseteq \ldots$ and $A = \bigcup_{n=1}^{\infty} S_n^-$. Given $m \in \mathbb{N}$ it follows from (12) that, for each $n \geq m$, all $\leq$-embeddings $\psi^{(n,m)}, \psi^{(n+1,m)}, \ldots$ coincide on S_n^-. From this one immediately obtains that each $A^{(m)}$ is a $\leq$-copy of A. Hence $\aleph_0 A \leq G$.

Thus, in order to finish the proof of Theorem 1, we have to prove the above claim. To this end, fix $n \geq 0$ and let $\mathcal{D}_n$ be as above; if $n \geq 1$, then also let $D^{(n,m)}, \psi^{(n,m)}$ $(m = 1, \ldots, n)$ and $\mathcal{A}^{(n)}$ be as in the paragraph before the claim. For the remainder of the proof, we shall mostly be concerned with the case $n \geq 1$ and thus, until stated otherwise, we assume $n \geq 1$. Let $D^{(n,m)}(v)$ denote the tentacles of $D^{(n,m)}$ with respect to $\psi^{(n,m)}$ $(v \in V_n, m = 1, \ldots, n)$. Since $\mathcal{A}^{(n)}$ is compatible with $D^{(n,m)}$ and $\psi^{(n,m)}$ $(m = 1, \ldots, n)$, there are two kinds of tentacles $D^{(n,m)}(v)$ (cf. (11)): let $\mathcal{D}^{(1)}$ be the set of those $D^{(n,m)}(v)$ for which $D^{(n,m)}(v) \cap A' \neq \emptyset$ for all $A' \in \mathcal{A}^{(n)}$ and let $\mathcal{D}^{(2)}$ be the remainder of the $D^{(n,m)}(v)$. For notational simplicity, we put

$$\mathcal{D}^{(1)} = \Big\{D_1, \ldots, D_s\Big\} \quad \text{and} \quad \mathcal{D}^{(2)} = \Big\{D_{s+1}, \ldots, D_t\Big\}.$$

Let $q := s^2 + 1$. Since $\mathcal{A}^{(n)}$ is a thick subset of $\mathcal{A}$, there exists an $n_q \in \mathbb{N}$ with $\left|\mathcal{A}_{n_q} \cap \mathcal{A}^{(n)}\right| \geq q$. Pick exactly q members of $\mathcal{A}_{n_q} \cap \mathcal{A}^{(n)}$, say, $A^{(n_q,p)}$ $(p = 1, \ldots, q)$. Let

$$R := \bigcup_{m=1}^{n} \psi^{(n,m)}(S_n) \cup \bigcup_{r=1}^{t-s} D_{s+r}.$$

Since $\mathcal{A}^{(n)}$ is compatible with $D^{(n,m)}$ and $\psi^{(n,m)}$ $(m = 1, \ldots, n)$, and by the definition of $\mathcal{D}^{(2)}$, we have

(14) $R \cap A^{(n_q,p)} = \emptyset$ $(p = 1, \ldots, q)$.

Moreover, by the definition of $\mathcal{D}^{(1)}$, we have $D_r \cap A^{(n_q,p)} \neq \emptyset$ for $r = 1, \ldots, s$ and $p = 1, \ldots, q$. For each $r \in \{1, \ldots, s\}$, choose a finite connected subgraph D_r^* of D_r, together with a set $I_r \subseteq \{p : 1 \leq p \leq q\}$, such that $|I_r| = s$ and such that

(15) $D_r \cap R \subseteq D_r^*$ $(r = 1, \ldots, s)$,

(16) $D_r^* \cap A^{(n_q,p)} \neq \emptyset \Leftrightarrow p \in I_r$ $(r = 1, \ldots, s)$.

One easily finds that these choices of D_r^* and I_r are possible. (To verify this, note in particular that $D_r \cap R$ is finite and connected (cf. (9)) and that (by (14)) $D_r \cap R$ is not met by any of the graphs $A^{(n_q,p)}$ for $p = 1, \ldots, q$.) We put

$$F := \bigcup_{r=1}^{s} D_r^*.$$

From $q = s^2 + 1$, one concludes that there exists an element $p_0 \in \{1, \ldots, q\}$ such that $A^{(n_q,p_0)} \cap F = \emptyset$. We shall define $D^{(n+1,n+1)}$ as a subgraph of $A^{(n_q,p_0)}$, together with a corresponding $\leq$-embedding $\psi^{(n+1,n+1)} : S_{n+1} \to D^{(n+1,n+1)}$. In order to cover the case $n = 0$ by these definitions, we put $q = 1$ and $n_q = p_0 = 1$ if $n = 0$.

It follows from (b.j) (cf. Lemma 6) that, for each $j \in J_2$, there exists an essential vertex w_j of P_j and a thin subset $\mathcal{E}_j$ of $\mathcal{A}$ such that

(17) $\varphi^{(n_q,p_0)}\big(B(A, w_j)\big) \cap A' = \emptyset$ for all $A' \in \mathcal{A} \setminus \mathcal{E}_j$ $(j \in J_2)$.

By (6), $B(A, v_j) \leq_r B(A, w_j)$. From this one concludes (similar as in the proof of (7)) that there exists a $\leq$-embedding

$$\tau_j : B(A, v_j) \to \varphi^{(n_q,p_0)}\big(B(A, w_j)\big)$$

together with a $\varphi^{(n_q,p_0)}(v_j), \tau_j(v_j)$-path $P'_j \subseteq \varphi^{(n_q,p_0)}\big(B(A, v_j)\big)$ which has exactly one vertex in common with $\tau_j\big(B(A, v_j)\big)$ $(j \in J_2)$. We define

$$D^{(n+1,n+1)} := \varphi^{(n_q,p_0)}\Big(C \cup \bigcup_{j \in J_1} B(A, v_j)\Big) \cup \bigcup_{j \in J_2} \Big(\tau_j\big(B(A, v_j)\big) \cup P'_j\Big).$$

A $\leq$-embedding $\widehat{\varphi} : A \to G$ with $\widehat{\varphi}(A) = D^{(n+1,n+1)}$ can be defined in the obvious way, similar to the definition of $\widehat{\varphi}^{(n,m)}$ in the proof of (7). In particular, this can be done such that, firstly, $\widehat{\varphi}$ coincides with $\varphi^{(n_q,p_0)}$ on all branches $B(A, v_j)$ with $j \in J_1$ and, secondly, $\widehat{\varphi}$ coincides with τ_j for all vertices and edges of $B(A, v_j) - v_j$ $(j \in J_2)$. Hence, by the remarks in the paragraph after (9), $D^{(n+1,n+1)}$ is of type $n + 1$ with a corresponding $\leq$-embedding $\psi^{(n+1,n+1)} : S_{n+1} \to D^{(n+1,n+1)}$ defined as the restriction of $\widehat{\varphi}$ to S_{n+1} and with the graphs $\widehat{\varphi}\big(B(A, v)\big) = \varphi^{(n_q,p_0)}\big(B(A, v)\big)$ as the tentacles $D^{(n+1,n+1)}(v)$ of $D^{(n+1,n+1)}$ with respect to $\psi^{(n+1,n+1)}$ (for $v \in V_{n+1}$).

We now return to assuming $n \geq 1$ (until further notice). Let $j_n \in J_1$ be the index for which $a_n \in B(A, v_{j_n})$. For each $p \in \bigcup_{r=1}^{s} I_r$, pick a ray $U_p \subseteq \varphi^{(n_q,p)}(P_{j_n})$. Further, for each $r \in \{1, \ldots, s\}$ and each $p \in I_r$, pick a vertex $u_{r,p} \in D^*_r \cap A^{(n_q,p)}$ (cf. (16)) and let $U_{r,p} \subseteq A^{(n_q,p)}$ be a $u_{r,p}$-ray having a tail in common with U_p. Put

$$H := F \cup \bigcup_{r=1}^{s} \bigcup_{p \in I_r} U_{r,p}.$$

We claim that

(18) in H there are s disjoint rays $Q_1, \ldots, Q_s$ such that Q_r starts in $D_r \cap R$ and such that Q_r has only its initial vertex in common with R $(r = 1, \ldots, s)$.

For the proof of (18), recall that (by (15) and because $D^*_r \subseteq D_r$) $D^*_r \cap R = D_r \cap R$. Let H' be the graph that results from H by contracting each $D^*_r \cap R$ into a single vertex x_r $(r = 1, \ldots, s)$. Note that, because of (14), none of the rays $U_{r,p}$ meets any of the just contracted graphs and thus, if two of these rays are disjoint in H, they are still disjoint in H'.

For each $p \in \bigcup_{r=1}^{s} I_r$ pick a vertex $y_p \in U_p$ such that, firstly, the y_p-tail U'_p of U_p does not meet the finite graph F and, secondly, $U'_p \subseteq U_{r,p}$ for each r with $p \in I_r$. Let $X = \{x_1, \ldots, x_s\}$ and denote by Y the set of the just defined vertices y_p, i.e., $Y = \{y_p : p \in \bigcup_{r=1}^{s} I_r\}$. In H', Y cannot be separated from X by deletion of less than s vertices: this follows from the fact that the graphs $D^*_1, \ldots, D^*_s$ are disjoint, in conjunction with the fact that (for every $r \in \{1, \ldots, s\}$) the rays $U_{r,p}$ $(p \in I_r)$ are a family of s disjoint graphs. Hence, by Menger's Theorem, there are s disjoint X, Y-paths $W_r \subseteq H'$ $(r = 1, \ldots, s)$. Note that, by the choice of the vertices y_p, one can extend the paths W_r to obtain s disjoint rays $Q'_r \subseteq H'$ starting in X $(r = 1, \dots, s)$. From this the claimed statement (18) immediately follows.

Since the rays $U_{r,p}$ and U_p have a common tail, it follows from (18) that, for each $r \in \{1, \dots, s\}$, there exists a $p_r \in \{1, \dots, q\} \setminus \{p_0\}$ such that Q_r and U_{p_r} have a common tail; moreover, the rays $Q_1, \dots, Q_s$ are disjoint and thus we have $p_{r_1} \neq p_{r_2}$ for $r_1 \neq r_2$. Without loss of generality, assume $p_r = r$ $(r = 1, \dots, s)$ and $p_0 > s$. Then Q_r and U_r have a common tail and thus, because $U_r \subseteq \varphi^{(n_q,r)}(P_{j_n})$, there exists a vertex $\widetilde{v}_r \in Q_r$ such that the $\widetilde{v}_r$-tail $\widetilde{Q}_r$ of Q_r is contained in $\varphi^{(n_q,r)}(P_{j_n})$ $(r = 1, \dots, s)$. Let $\overline{Q}_r$ be the initial segment of Q_r ending in $\widetilde{v}_r$ $(r = 1, \dots, s)$. Pick an essential vertex u of P_{j_n} such that, for all $r \in \{1, \dots, s\}$, $\varphi^{(n_q,r)}\big(B(A,u)\big)$ does not meet $\overline{Q}_1 \cup \ldots \cup \overline{Q}_s$; this is possible since $\overline{Q}_1 \cup \ldots \cup \overline{Q}_s$ is finite. From this, together with the disjointness of the graphs $\varphi^{(n_q,r)}(A)$ $(r = 1, \dots, s)$ and the disjointness of the rays $Q_1, \dots, Q_s$, one concludes that the graphs $Q_r \cup \varphi^{(n_q,r)}\big(B(A,u)\big)$ $(r = 1, \dots, s)$ are disjoint.

Let P'_{j_n} be the u-tail of P_{j_n}. Then $P'_{j_n} \subseteq B(A,u)$ and thus $\varphi^{(n_q,r)}(P'_{j_n}) \subseteq \varphi^{(n_q,r)}\big(B(A,u)\big)$ $(r = 1, \dots, s)$. Moreover, $\widetilde{v}_r \notin \varphi^{(n_q,r)}\big(B(A,u)\big)$ by the choice of u and thus $\widetilde{v}_r \notin \varphi^{(n_q,r)}(P'_{j_n})$ $(r = 1, \dots, s)$. From this, together with the fact that $\widetilde{v}_r \in \varphi^{(n_q,r)}(P_{j_n})$, one easily finds that, for each $r \in \{1, \dots, s\}$, there exists an initial segment Q^*_r of Q_r such that, firstly, Q^*_r ends in $\varphi^{(n_q,r)}(u)$ and, secondly, Q^*_r has just one vertex in common with $\varphi^{(n_q,r)}\big(B(A,u)\big)$. For $r = 1, \dots, s$ let $Z_r := (D_r \cap R) \cup Q^*_r \cup \varphi^{(n_q,r)}\big(B(A,u)\big)$. Then (by construction)

(19) the graphs $Z_r = (D_r \cap R) \cup Q^*_r \cup \varphi^{(n_q,r)}\big(B(A,u)\big)$ $(r = 1, \dots, s)$ are disjoint.

It follows from $a_n \in V_n$ that, for each $m \in \{1, \dots, n\}$, there exists an $r_m \in \{1, \dots, t\}$ such that $D_{r_m} = D^{(n,m)}(a_n)$. For the purpose of defining the graphs $D^{(n+1,m)}$ $(m = 1, \dots, n)$, we shall replace some of the tentacles D_r by appropriately chosen graphs $\widehat{D}_r$ and, in order to accomplish this for $r = r_m$, we consider two cases. Let $m \in \{1, \dots, n\}$.

Case 1: $1 \leq r_m \leq s$.

Since u is an essential vertex of P_{j_n}, we can apply (6) to find $B(A, v_{j_n}) \leq_r B(A,u)$. Moreover, since $a_n \in B(A, v_{j_n})$, we have $B(A, a_n) \leq_r B(A, v_{j_n})$. Hence $B(A, a_n) \leq_r B(A,u)$. From this, together with the definition of Z_{r_m} and the fact that $D_{r_m} \cap R = \psi^{(n,m)}(a_n)$, one finds that there exists a $\leq$-embedding $\sigma_m : B(A, a_n) \to Z_{r_m}$ such that $\sigma_m\big(B(A,a_n)\big) \subseteq \varphi^{(n_q,r_m)}\big(B(A,u)\big)$ and such that there exists a $\psi^{(n,m)}(a_n), \sigma_m(a_n)$-path $Q'_m \subseteq Z_{r_m}$ which has just one vertex in common with $\sigma_m\big(B(A,a_n)\big)$.

Case 2: $s + 1 \leq r_m \leq t$.

Because $\mathcal{A}^{(n)}$ is compatible with $D^{(n,m)}$ and $\psi^{(n,m)}$, one obtains (cf. (11.2)) that there exists a $\leq$-embedding $\sigma_m : B(A, a_n) \to D^{(n,m)}(a_n)$ with $\sigma_m(a_n) = \psi^{(n,m)}(a_n)$. There also exists a $\psi^{(n,m)}(a_n), \sigma_m(a_n)$-path $Q'_m \subseteq D^{(n,m)}(a_n)$ which has just one vertex in common with $\sigma_m\big(B(A,a_n)\big)$: just choose Q'_m as a trivial path.

Thus, in either case, we have defined a $\leq$-embedding σ_m of $B(A, a_n)$ together with a corresponding path Q'_m, i.e., σ_m and Q'_m are defined for all $m \in \{1, \dots, n\}$. For all $r \in \{1, \dots, t\}$, we now define a graph $\widehat{D}_r$ as follows:

$$\begin{aligned} \widehat{D}_{r_m} &:= \psi^{(n,m)}(a_n) \cup Q'_m \cup \sigma_m\big(B(A,a_n)\big) \ (m = 1, \dots, n) \\ \widehat{D}_r &:= \begin{cases} Z_r & \text{, if } r \leq s \text{ and } r \neq r_m \ (m = 1, \dots, n) \\ D_r & \text{, if } s+1 \leq r \text{ and } r \neq r_m \ (m = 1, \dots, n). \end{cases} \end{aligned}$$

Let $R_m := \{r : 1 \leq r \leq t, D_r \subseteq D^{(n,m)}\}$ $(m = 1, \dots, n)$; we define

$$D^{(n+1,m)} := \psi^{(n,m)}(S_n) \cup \bigcup_{r \in R_m} \widehat{D}_r \ (m = 1, \dots, n).$$

By construction (in particular, cf. (19)), the graphs $D^{(n+1,m)}$ $(m = 1, \dots, n+1)$ are disjoint. For each $m \in \{1, \dots, n\}$ define a $\leq$-embedding $\psi^{(n+1,m)} : S_{n+1} \to D^{(n+1,m)}$ as follows. If $\leq$ means $\preccurlyeq$, then let

$$\psi^{(n+1,m)}(a_n) := \psi^{(n,m)}(a_n) \cup Q'_m \cup \sigma_m(a_n).$$

If $\leq$ means $\subseteq$, then (A_n, r_n) $(n = 0, 1, \dots)$ is a strong decomposition of A and thus a_n has degree 1 in S_n; in this case, we define

$$\psi^{(n+1,m)}(a_n) := \sigma_m(a_n) \quad \text{and} \quad \psi^{(n+1,m)}(e_n) := \psi^{(n,m)}(e_n) \cup Q'_m$$

with e_n denoting the unique edge of S_n which is incident to a_n.

For all other vertices and edges of S_{n+1}, define $\psi^{(n+1,m)}$ such that $\psi^{(n+1,m)}$ coincides either with $\psi^{(n,m)}$ or with σ_m, depending on whether the considered element of $V(S_{n+1}) \cup E(S_{n+1})$ is in S_n or not. It then follows from the construction that $\psi^{(n+1,m)}$ is a $\leq$-embedding $S_{n+1} \to D^{(n+1,m)}$ $(m = 1, \dots, n)$. Moreover, $\psi^{(n+1,m)}$ is an extension of $\psi^{(n,m)}$ $(m = 1, \dots, n)$ and one immediately obtains from the construction that (20) holds.

(20) Let $m \in \{1, \dots, n\}$. Then the graph $D^{(n+1,m)}$ is of type $n+1$ and $\psi^{(n+1,m)} : S_{n+1} \to D^{(n+1,m)}$ is a corresponding $\leq$-embedding; the tentacles of $D^{(n+1,m)}$ with respect to $\psi^{(n+1,m)}$ are

(i) the graphs $\widehat{D}_r$ with $r \in R_m \setminus \{r_m\}$ and

(ii) the graphs $\sigma_m\big(B(A, v)\big)$ with $v \in V\big(B(A, a_n)\big) \cap V_{n+1}$.

Further, for $v \in V_{n+1}$, if $D^{(n+1,m)}(v)$ is a tentacle of $D^{(n+1,m)}$ with respect to $\psi^{(n+1,m)}$, then $\psi^{(n+1,m)}(v) = \psi^{(n,m)}(v)$ if $D^{(n+1,m)}(v)$ is of type (i), and $\psi^{(n+1,m)}(v) = \sigma_m(v)$, otherwise.

For the remainder of the proof, let $n \geq 0$. It remains to show (13). One concludes from Lemma 2(i), Lemma 3 and from the construction of $D^{(n+1,n+1)}$ and $\psi^{(n+1,n+1)}$ that there exists a thin subset $\mathcal{F}$ of $\mathcal{A}$ such that $\psi^{(n+1,n+1)}(S_{n+1}) \cap A' = \emptyset$ for all $A' \in \mathcal{A} \setminus \mathcal{F}$. (Leaving the details to the reader, we just mention that (17) is the main observation for this.) Moreover, if $n \geq 1$, then $\psi^{(n,m)}(S_n) \cap A' = \emptyset$ for all $A' \in \mathcal{A}^{(n)}$; there are also just a finite number of vertices of $\psi^{(n+1,m)}(S_{n+1})$ which are not in $\psi^{(n,m)}(S_n)$ $(m = 1, \dots, n)$. Hence one can apply Lemma 3 to find that there exists a thin subset $\mathcal{F}'$ of $\mathcal{A}$ such that $\psi^{(n+1,m)}(S_{n+1}) \cap A' = \emptyset$ for all $A' \in \mathcal{A}^{(n)} \setminus \mathcal{F}'$ and all $m \in \{1, \dots, n\}$. If $n = 0$, we put $\mathcal{F}' := \emptyset$ and $\mathcal{A}^{(n)} := \mathcal{A}$.

Let $\widetilde{\mathcal{A}} := \mathcal{A}^{(n)} \setminus (\mathcal{F} \cup \mathcal{F}')$ $(n \geq 0)$. Then, by Lemma 2, $\widetilde{\mathcal{A}}$ is a thick subset of $\mathcal{A}$ and, by the choice of $\mathcal{F}$ and $\mathcal{F}'$, we have

$$\psi^{(n+1,m)}(S_{n+1}) \cap A' = \emptyset \text{ for all } A' \in \widetilde{\mathcal{A}} \ (m = 1, \dots, n+1).$$

As before, we denote by $D^{(n+1,m)}(v)$ the tentacles of $D^{(n+1,m)}$ with respect to $\psi^{(n+1,m)}$ (for $m \in \{1, \dots, n+1\}$ and $v \in V_{n+1}$). It remains to show that there exists a thick $\mathcal{A}^{(n+1)} \subseteq \widetilde{\mathcal{A}}$ such that, for each $v \in V_{n+1}$ and all $m \in \{1, \dots, n+1\}$, either (21) or (22) holds:

(21) $D^{(n+1,m)}(v) \cap A' \neq \emptyset$ for all $A' \in \mathcal{A}^{(n+1)}$;

(22) $D^{(n+1,m)}(v) \cap A' = \emptyset$ for all $A' \in \mathcal{A}^{(n+1)}$ and there exists a $\leq$-embedding $\varphi : B(A, v) \to D^{(n+1,m)}(v)$ with $\varphi(v) = \psi^{(n+1,m)}(v)$.

Let $v \in V_{n+1}$ and $m \in \{1, \dots, n+1\}$. For showing the existence of $\mathcal{A}^{(n+1)}$ with the above properties, we consider three cases; each case is dealing with a certain type of graph $D^{(n+1,m)}(v)$. (For the graphs $D^{(n+1,n+1)}(v)$, cf. the paragraph after the definition of $D^{(n+1,n+1)}$.)

Case 1: $1 \leq m \leq n$ and $D^{(n+1,m)}(v) = \widehat{D}_r$ for some r with $s+1 \leq r \leq t$ and $r \neq r_m$.

From the hypothesis of this case one obtains $D^{(n+1,m)}(v) = D^{(n,m)}(v)$ and $D^{(n+1,m)}(v) \cap A' = \emptyset$ for all $A' \in \mathcal{A}^{(n)}$. Further, $\psi^{(n+1,m)}(v) = \psi^{(n,m)}(v)$ (cf. (20)) and thus (since $\mathcal{A}^{(n)}$ is compatible with $D^{(n,m)}$ and $\psi^{(n,m)}$) there exists a $\leq$-embedding $\varphi : B(A,v) \to D^{(n+1,m)}(v)$ with $\varphi(v) = \psi^{(n+1,m)}(v)$.

Case 2: $1 \leq m \leq n$ and $D^{(n+1,m)}(v) = \widehat{D}_r$ for some r with $1 \leq r \leq s$ and $r \neq r_m$.

In this case, $D^{(n+1,m)}(v) = Z_r$ and one obtains from the definition of Z_r that $D^{(n+1,m)}(v)$ contains $\varphi^{(n_q,r)}\big(B(A,u)\big)$ as a subgraph. Consequently, since u is an essential vertex of P_{j_n} and because $j_n \in J_1$, we have $D^{(n+1,m)}(v) \cap A' \neq \emptyset$ for all $A' \in \mathcal{A}$, except for the members of some thin subset of $\mathcal{A}$.

As a consequence of the preceding discussion of the cases 1 and 2, one finds that there exits a thick $\widehat{\mathcal{A}} \subseteq \widetilde{A}$ such that, for each $D^{(n+1,m)}(v)$ addressed in these cases, either (21') or (22') holds, where (21') and (22') are the statements obtained from (21) and (22) when $\mathcal{A}^{(n+1)}$ is replaced by $\widehat{\mathcal{A}}$. It remains to consider the following case (cf. (20)).

Case 3: $m = n+1$ or $D^{(n+1,m)}(v) = \sigma_m\big(B(A,v)\big)$ for $m \in \big\{1,\dots,n\big\}$ and $v \in B(A,a_n)$.

Under the hypothesis of this case, it follows that

(23) there exists a $\leq$-embedding $\varphi : B(A,v) \to D^{(n+1,m)}(v)$ with $\varphi(v) = \psi^{(n+1,m)}(v)$.

Indeed, if $m = n+1$, then $D^{(n+1,m)}(v) = \varphi^{(n_q,p_0)}\big(B(A,v)\big)$ and $\psi^{(n+1,m)}(v) = \varphi^{(n_q,p_0)}(v)$ (cf. the paragraph after the definition of $D^{(n+1,n+1)}$). Hence, choosing φ as the restriction of $\varphi^{(n_q,p_0)}$ to $B(A,v)$ yields φ as required. Further, if $D^{(n+1,m)}(v) = \sigma_m\big(B(A,v)\big)$ for $m \in \big\{1,\dots,n\big\}$ and $v \in B(A,a_n)$, then $\psi^{(n+1,m)}(v) = \sigma_m(v)$ by (20), and thus we can choose φ as the restriction of σ_m to $B(A,v)$ to obtain φ as desired.

We now denote the tentacles addressed in case 3 by $T_1,\dots,T_\lambda$ (in arbitrary order). For T_1 there exists a thick $\mathcal{B}^{(1)} \subseteq \widehat{\mathcal{A}}$ such that either $T_1 \cap A' \neq \emptyset$ for all $A' \in \mathcal{B}^{(1)}$ or $T_1 \cap A' = \emptyset$ for all $A' \in \mathcal{B}^{(1)}$. Further, there is a thick $\mathcal{B}^{(2)} \subseteq \mathcal{B}^{(1)}$ such that either $T_2 \cap A' \neq \emptyset$ for all $A' \in \mathcal{B}^{(2)}$ or $T_2 \cap A' = \emptyset$ for all $A' \in \mathcal{B}^{(2)}$. Continuing in this way, one obtains a sequence $\widehat{\mathcal{A}} \supseteq \mathcal{B}^{(1)} \supseteq \ldots \supseteq \mathcal{B}^{(\lambda)}$ of thick subsets of $\mathcal{A}$ such that, for each $i \in \big\{1,\dots,\lambda\big\}$, either $T_i \cap A' \neq \emptyset$ for all $A' \in \mathcal{B}^{(i)}$ or $T_i \cap A' = \emptyset$ for all $A' \in \mathcal{B}^{(i)}$. Hence, putting $\mathcal{A}^{(n+1)} := \mathcal{B}^{(\lambda)}$, it follows from (23) that we have obtained $\mathcal{A}^{(n+1)}$ as required. $\square$

References

[1] T. Andreae, Bemerkungen zu einem Problem aus der Theorie der unendlichen Graphen, Abh. Math. Sem. Univ. Hamburg 46 (1978) 91–95.
[2] T. Andreae, Über eine Eigenschaft lokalfiniter, unendlicher Bäume, J. Combinatorial Theory, Ser. B 27 (1979) 202–215.
[3] T. Andreae, On disjoint configurations in infinite graphs, J. Graph Theory 39 (2002) 222–229.
[4] R. Diestel, Graph Theory, fourth edition, Springer, Heidelberg, 2010.
[5] R. Halin, Über die Maximalzahl fremder unendlicher Wege in Graphen, Mathematische Nachrichten 30 (1965) 63–85.
[6] R. Halin, Die Maximalzahl fremder zweiseitig unendlicher Wege in Graphen, Mathematische Nachrichten 44 (1970) 119–127.
[7] J. Lake, A problem concerning infinite graphs, Discrete Math. 14 (1976) 343–345.
[8] R. Laver, On Fraissé's order type conjecture, Ann. of Math. 93 (1971) 89–111.
[9] C.S.J.A. Nash-Williams, On well-quasi-ordering infinite trees, Proc. Cambridge Philos. Soc. 61 (1965) 697–720.
[10] N. Robertson, P. Seymour, Graph Minors. V. Excluding a planar graph, J. Combinatorial Theory, Ser. B 41 (1986) 92–114.
[11] N. Robertson, P. Seymour, Graph Minors. XX. Wagner's conjecture, J. Combinatorial Theory, Ser. B 92 (2004) 325–357.
[12] R. Thomas, Well-quasi-ordering infinite graphs with forbidden finite planar minors, Trans. Amer. Math. Soc. 312 (1989) 279–313.
[13] D. Woodall, A note on a problem of Halin's, J. Combinatorial Theory, Ser. B 21 (1976) 132–134.