DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 77/20
March 1977

On Bunch Lengthening and the Particle Distribution in Electron Storage Rings

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On Bunch Lengthening and the Particle Distribution in Electron Storage Rings

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Introduction

In the present theories of bunch lengthening [1)-7)] beyond the limit of a pure potential well distortion the bunch length is described by that r.m.s. width in energy which makes the bunch stable. Whereas protons have an infinite set of stable solutions as a result of the Vlasov equation, electrons must obey the Fokker-Planck equation which leads to a Gaussian distribution in energy for low currents. The theories so far assume also a Gaussian distribution for the widened high current case except Chao and Gareyte[3)] who used a waterbag model for scaling purposes. A. Sessler and P. Channell[2)] in their turbulence model achieve their exploded Gaussian distribution by an additional diffusion term due to strong turbulence of ad hoc character. This might be sufficient for scaling.

In electron machines, however, one of the crucial parameters is the quantum life time. Therefore it is of major interest to obtain information about the tail of the distribution in case of energy widening. To this end experiments at DORIS were started for investigating the tail region by measuring the life time with scrapers. The first experiments already indicated that the tail is less widened than the core of the bunch. In this note we present a model from which the distribution function can be deduced. Thus apart from bunch lengthening also quantum life time and higher order mode losses can be calculated.

Presentation of the Model

Consider a Fokker-Planck equation of the general form:

$$\boxed{\frac{\partial}{\partial t} w = \underline{V} w + \underline{D} w} \qquad (1)$$

w is the density function. The Vlasov operator $\underline{V}$ describes the interaction with the beam surroundings and $\underline{D}$, the Fokker-Planck operator, accounts for radiation effects.

As can be seen from Renieri[5)] or Ruggiero[6)] the stationary solution of eq. (1) with

$$\boxed{\frac{\partial}{\partial t} w = 0 \quad \text{stationary}} \tag{2}$$

describes only bunch lengthening due to potential well distortion. In case of widening, however, there can only be a large time range balance around a distribution $\langle w \rangle_{tp} = \rho$ distinct from the solution of eq.(2) and characterized by

$$\boxed{\left\langle \frac{\partial}{\partial t} w \right\rangle_{t_p} = 0 \quad \text{balance}} \tag{3}$$

The time range t_p is expected within the interval

$$\frac{1}{\Omega_s} < t_p < \tau_r = \text{radiation damping time.}$$

For the balance we assume a mechanism such that the operator $\underline{V}$ of eq. (1) generates an unstable growth limited by non-linearities and filamentation which makes a change of w

$$\boxed{w = \rho \xRightarrow{\text{instability}} \rho + \delta w_i} \tag{4a}$$

whereas the operator $\underline{D}$ generates a change

$$\boxed{w = \rho \xRightarrow{\text{radiation}} \rho - \delta w_r} \tag{4b}$$

We finally shall determine ρ by the requirement

$$\boxed{\langle \delta w_i \rangle_{t_p} = \langle \delta w_r \rangle_{t_p}} \tag{4c}$$

Putting $w = \rho + f$ we write eq. (1) using also eq. (3)

$$\boxed{0 = \langle \underline{V}(\rho + f) + \underline{D} f \rangle_{t_p} + \langle \underline{D} \rho \rangle_{t_p}} \tag{5}$$

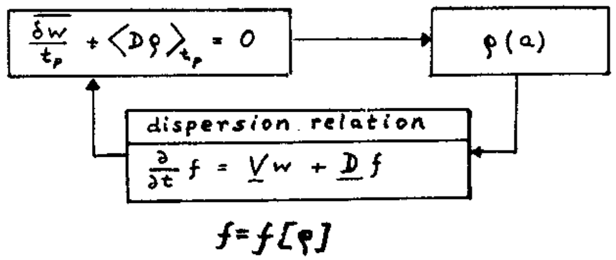
Since the first term of the r.h.s. of eq. (5) is the change of w according to

$$\boxed{\frac{\partial}{\partial t} f = \underline{V} w + \underline{D} f} \tag{6}$$

within a time range t_p, eq.(5) is equivalent to

$$\boxed{0 = \frac{\overline{\delta w}}{t_p} + \langle \underline{D} \rho \rangle_{t_p}} \tag{7}$$

where $\frac{\overline{\delta w}}{t_p}$ is the average change of w within t_p due to eq.(6). It will turn out later that eq.(7) has the form of a stationary Fokker-Planck equation modified by an additional diffusion term whose energy dependence is unambiguously defined. The diffusion term is of collective nature and determined by the stability criterion in a self consistent manner. A recipe to find the solution $\rho(a)$ of the Fokker-Planck eq.(7) with a modified diffusion is sketched in the flow diagram.



Flow diagram
of the model
Fig. 1

The Vlasov-Fokker-Planck Equation

The basic equation determining $\frac{\overline{\delta w}}{t_p}$ is eq. (6).
The Vlasov part being non-linear in f describes unstable growth and self-stabilization, the Fokker-Planck part governs the radiation effects. Since the essential part of eq.(6) is the Vlasov part $\underline{V}w$ we drop $\underline{D}f$. Introducing polar coordinates for the synchrotron motion we put

$$\phi = r \sin\psi$$

where ϕ is the azimuth taken in rf radians. Then eq.(6) reads

$$\boxed{\left(\frac{\partial}{\partial t} + \Omega_s \frac{\partial}{\partial \psi}\right) f = \left\{ F([f]; r, \psi) \left(\cos\psi \frac{\partial}{\partial r} - \frac{\sin\psi}{r} \frac{\partial}{\partial \psi} \right) \right\} w} \tag{8}$$

F denotes the force generated by the interaction.

Since F is a linear functional of f we can write

$$\boxed{\mathcal{F} = \lambda\, \mathcal{F}\left([\hat{f}];\, r, \psi\right)} \tag{9}$$

where $\hat{f}$ is a normalized f and λ the norm. Ω_s is the unperturbed angular synchrotron frequency. Since we are not interested in the angular modulation of the distribution but in the blow-up of the smooth function we average over ψ and obtain with $H = r^2$ explicitely

$$\boxed{\frac{\overline{\overline{\delta w}}}{t_p} + \frac{1}{\tau_r}\frac{d}{dH}\left[H\left(\rho + H_o \frac{d}{dH}\rho\right)\right] = 0} \tag{10}$$

$\sqrt{H_o/2}$ is the low current Gauss parameter.

$\overline{\overline{\delta w}}$ is taken as the symbol for δw averaged over both, the angle ψ and the finite time t_p. In order to proceed with the evaluation of $\frac{\overline{\overline{\delta w}}}{t_p}$ from eq.(8) we make the following assumptions:

i.) The nonlinear eq.(8) can be solved by iteration

ii.) We take only the first non-vanishing approximation contributing to $\frac{\overline{\overline{\delta w}}}{t_p}$. $\quad f = f^{(1)} + f^{(2)} + \ldots$

iii.) We neglect mode coupling according to arguments given in ref.(7).

Calculation of $\frac{\overline{\overline{\delta w}}}{t_p}$ (Radial Material Transport)

For first approximation of eq.(8) we factorize the time dependent $\hat{f}$ so that $\bar{f}$ is a normalized and time independent state. This yields

$$\boxed{\left(\frac{\partial}{\partial t} + \Omega_s \frac{\partial}{\partial \psi}\right) f^{(1)} = \lambda g(t) \sin(\omega t + \chi)\left\{\mathcal{F}([\bar{f}];\phi)\cos\psi \frac{\partial}{\partial r}\right\}\rho} \tag{11}$$

The strength of the forces scales with λ and they oscillate with the collective frequency ω, χ being an arbitrary phase. The slowly varying function g(t)

describes the growth and self stabilization. In the second approximation we obtain according to ii.)

$$\boxed{\left(\frac{\partial}{\partial t} + \Omega_s \frac{\partial}{\partial \psi}\right) f^{(2)} = \lambda g(t) \sin(\omega t + \chi)\left\{\mathcal{F}([\bar{f}];\phi)\left(\cos\psi\frac{\partial}{\partial r} - \frac{1}{r}\sin\psi\frac{\partial}{\partial \psi}\right)\right\} f^{(1)}} \tag{12}$$

The function $\mathcal{F}([\bar{f}];\phi)$ can be represented by

$$\boxed{\mathcal{F}([\bar{f}],\phi) = \int dp\; a([\bar{f}],p)\, G(p)\, e^{i\frac{p}{h}\phi}} \tag{13}$$

with $\quad h =$ the harmonic number

$G(p)$ = the interaction function

$a([\bar{f}];p)$ the Fourier transform of $\bar{f}$ with respect to ϕ and is of the form $\int r\,dr \int d\psi\; \bar{f}(r,\psi)\, \exp\left(-i\frac{p}{h} r \sin\psi\right)$

According to iii.) we only project F to the fixed angular mode μ with the Bessel function relations:

$$\boxed{\begin{aligned}
&\int e^{i\frac{p}{h}\phi} \cos\psi\; e^{-i\mu\psi}\, d\psi = \frac{h\mu}{pr}\, J_\mu\left(\frac{p}{h}r\right)\\
&\int e^{i\frac{p}{h}\phi} \sin\psi\; e^{-i\mu\psi}\, d\psi = -i\,\frac{h}{p}\frac{\partial}{\partial r} J_\mu\left(\frac{p}{h}r\right)\\
&J_\mu(-z) = (-1)^\mu\, J_\mu(z) = J_{-\mu}(z)
\end{aligned}} \tag{14}$$

For odd μ we assume $a(p) = a(-p)$ for simplicity (this has no influence on the final result); it then follows from eq.(13) with eq.(14)

$$\frac{\partial}{\partial t} + \Omega_s \frac{\partial}{\partial \psi} = I_\mu \qquad \text{where}$$

$$\boxed{I_\mu = \lambda g(t)\sin(\omega t+\chi)\left\{\mu\frac{G_\mu(r)}{r}\cos(\mu\psi)\frac{\partial}{\partial r} - \frac{\frac{\partial}{\partial r}G_\mu(r)}{r}\sin(\mu\psi)\frac{\partial}{\partial \psi}\right\}} \tag{15a}$$

$$\boxed{G_\mu(r) = G_\mu([\bar{f}];r) = h\int_0^\infty \frac{dp}{p}\, a([\bar{f}];p)\, J_\mu\left(\frac{p}{h}r\right) G(p)\,; \quad \mu = \text{odd}} \tag{15b}$$

A similar relation can be derived for even modes.

From eqs.(11), (13), (14) and (15) we obtain

$$\boxed{\left(\frac{\partial}{\partial t}+\Omega_s\frac{\partial}{\partial\psi}\right)f^{(1)} = \lambda g(t)\sin(\omega t+\chi)\mu\,\frac{G_\mu(r)}{r}\cos(\mu\psi)\frac{\partial\rho}{\partial r}} \qquad (16)$$

The solution of eq.(16) yields

$$\boxed{f^{(1)}=\frac{G_\mu(r)}{r}\left(h(t)\sin(\mu\psi)+\frac{1}{\Omega_s}\frac{dh(t)}{dt}\cos(\mu\psi)\right)\frac{\partial\rho}{\partial r}} \qquad (17a)$$

with

$$\boxed{h(t)=\frac{1}{\Omega_s}\int_0^t d\tau\,\lambda g(\tau)\sin(\omega\tau+\chi)\sin(\mu\Omega_s(t-\tau))} \qquad (17b)$$

$f^{(1)}$ averaged over ψ vanishes. Thus we proceed to $f^{(2)}$ but shall only retain contributions to $f^{(2)}$ which are independent of ψ. We obtain

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}f^{(2)} &= \frac{1}{\Omega_s}\lambda g(t)\sin(\omega t+\chi)\frac{dh(t)}{dt}\,\cdot\\ &\cdot\left\{\mu\frac{G_\mu(r)}{r}\frac{\partial}{\partial r}\left(\frac{G_\mu(r)}{r}\frac{\partial\rho}{\partial r}\right)+\mu\frac{G_\mu(r)}{r^2}\left(\frac{\partial}{\partial r}G_\mu(r)\right)\frac{\partial}{\partial r}\rho\right\}\end{aligned}} \qquad (18)$$

After averaging eq.(18) over the time period t_p we obtain the radial material transport which is:

$$\boxed{\frac{\overline{\overline{\delta w}}}{t_p}=\frac{a_\mu}{4\tau_r}\left\{\frac{G_\mu(r)}{r}\frac{\partial}{\partial r}\left(\frac{G_\mu(r)}{r}\frac{\partial}{\partial r}\rho\right)+\frac{G_\mu(r)}{r^2}\left(\frac{\partial}{\partial r}G_\mu(r)\right)\frac{\partial\rho}{\partial r}\right\}} \qquad (19)$$

where

$$\frac{a_\mu}{4\tau_r}=\left\langle\frac{1}{\Omega_s}\lambda g(t)\sin(\omega t+\chi)\frac{dh(t)}{dt}\right\rangle_{t_p}$$

This can also be written in $H = r^2$ coordinates

$$\boxed{\begin{aligned}\frac{\overline{\overline{\delta w}}}{t_p}&=\frac{a_\mu}{\tau_r}\frac{d}{dH}\left[g_\mu([\bar f];H)\frac{d\rho}{dH}\right]\\ g_\mu([\bar f];H)&=G_\mu^2([\bar f];\sqrt{H})\end{aligned}} \qquad (20)$$

Since $\frac{\overline{\overline{\delta w}}}{t_p}$ is a gradient the particle number is conserved as expected because $G(H) \to 0$ for $H \to 0$.

With eq.(20) eq.(10) assumes the form

$$\boxed{\frac{d}{dH}\left[H\left\{\rho+\left(H_o+\frac{a_\mu g_\mu([\bar f];H)}{H}\right)\frac{d\rho}{dH}\right\}\right]=0} \qquad (21)$$

The constant B of the integral of eq.(21):

$$\boxed{H\rho+\left(HH_o+a_\mu g_\mu([\bar f];H)\right)\frac{d\rho}{dH}=B} \qquad (22)$$

must vanish for finite solutions.

Denoting $\frac{d}{dH}\rho = \rho'$ eq.(22) becomes

$$\boxed{\begin{aligned}\rho'&=-\frac{1}{H_o+\frac{1}{H}a_\mu g_\mu([\bar f];H)}\rho\\ \bar f&=\bar f[\rho]\end{aligned}} \qquad (23)$$

<u>Discussion of the Result</u>

We interpret eq.(23) that there is an extra diffusion term $\Delta H_o = \frac{1}{H}a_\mu g_\mu([\bar f];H)$ which adds to the low current H_o of single particle diffusion type. In the strong turbulence theory that term was assumed to be constant leading to a Gaussian explosion. In our model it follows from eq.(15) that ΔH_o decreases stronger than H^{-1} for large H so that clearly the widening of the tail is less than the widening of the core.

Let $H_1 = H_o + \Delta H_{o\,max}$. Then the least strong decay anywhere goes as

$$\boxed{\begin{array}{c} \rho \sim \exp(-H/H_1) \\ \hline \text{least strong decay} \end{array}} \tag{24}$$

The asymptotic decay of ρ beyond H_1 is considerably stronger since in turn the behaviour of f is governed by the asymptotic behaviour of ρ according to eqs.(17) and (18).

We therefore can even conclude that for smooth impedance functions G(p) the self-consistent solution $\rho(H)$ decreases almost as strongly as the unperturbed distribution in the tail. Since H_o and H_1 define the unperturbed and widened r.m.s. bunch lengths respectively this means, that the widening affects only the bunch core region ($H < H_1$) whereas for $H > H_1$ there is no significant widening.

When solving eq.(23) by iteration according to the flow diagram of Fig.1 we expect a fairly rapid convergence when starting with a sound guess for $\Delta H_o(H)$. We are still looking into this problem which must be treated such that the stability criterion is satisfied. For the moment we may accept as a reasonable guess

$$\rho' = \begin{cases} -\frac{1}{H_1}\rho & \text{for } H < H_1 \\ -\frac{1}{H_o}\rho & \text{for } H \gg H_1 \end{cases} \tag{25}$$

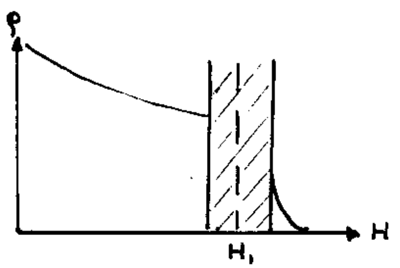


Fig. 2

In the shaded region of Fig.2 the gradient of $\ln\rho$ decreases continuously from $-\frac{1}{H_1}$, to $-\frac{1}{H_o}$.

Guess for quantum life time

If the width of the shaded area around H_1 in Fig.2 is small we may put for a crude demonstration

$$\boxed{\rho = \frac{1}{N}\begin{cases} \exp(-x/R^2) & \text{for } x = H/H_o < R^2 \\ \exp(R^2-1)\exp(-x) & \text{for } x > R^2 \end{cases}} \tag{26}$$

where N is the normalization constant for

$$\boxed{\int_0^\infty \rho\, dx = 1 \qquad ; \; N = \frac{R^2(e-1)+1}{e}} \tag{27}$$

R is nearly the bunch lengthening factor.

The life time in the presence of widening T_R is related to the natural life time T_o by

$$\boxed{\frac{T_R}{T_o} = \frac{\rho_o(x_L)}{\rho_R(x_L)} \; ; \; x_L = \frac{H_L}{H_o}} \tag{28}$$

where ρ_o is the natural distribution and ρ_R the widened distribution. H_L denotes the maximum available H. We find from eqs.(26), (27), (28)

$$\boxed{\frac{T_R}{T_o} = \frac{R^2}{\exp(R^2-1)}} \tag{29}$$

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