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Dynamical Charges in the Quantized, Renormalized Massive Thirring Model

by

M. Lüscher
II. Institut für Theoretische Physik der Universität Hamburg

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I. Introduction

Since Coleman [2] proved the equivalence of the quantized sine-Gordon equation and the charge zero sector of the massive Thirring model many workers in the field speculated that this model should have some outstanding features. Among them is the complete integrability of the classical version of the model (with anticommuting fields) and the existence of infinitely many conserved currents for both the quantized and the classical theory. Such currents have now been obtained for the classical case independently by Berg, Karowski and Thun [3] and by Flume, Mitter and Papanicolaou [4]. Furthermore, Michailov [5] claimed recently that the $\mathbb{C}$-number massive Thirring model can be solved by an inverse scattering method. An infinite set of conserved quantities then arises as a byproduct of this method.

In this paper we are going to construct a set of infinitely many conserved, local charges for the quantized massive Thirring model put on a (space-) lattice. By a local charge I mean a quantity Q that can be written in the form

$$(1) \qquad Q = \int_{-\infty}^{\infty} dx \; \rho(t,x) \quad + \quad \text{const.}$$

where the charge density $\rho(t,x)$ is a local, translationally co-variant (composite) field operator. In our case ρ will turn out to be a polynomial of the Thirring field Ψ and its derivative

The lattice theory has recently been elaborated by Luther [1]. By using known results for the Baxter model he succeeded in calculating the spectrum of low-lying states in the quantized, renormalized massive Thirring model exactly.

In sec.II Luther's work is briefly rewiewed. With the help of Baxter's transfermatrix we then (sec.III) construct an infinite set of local charges for the XYZ-spin-chain, a model that is close related to the lattice Thirring model [1]. The exact values of these charges on the one particle states are given in the subseque section and these results are then used (sec.V) to discuss various scattering processes. The paper ends with some concluding remarks and an appendix, where it is proven that the charges constructed in the text are local.

II. From the massive Thirring model to the eight-vertex model [1]

In this section we establish the connection between the Thirring model and the eight-vertex model (the Baxter model). The Hamiltonian of the massive Thirring model on a one dimensional lattice with spacing a and N = 2r sites is given by:

$$(2) \qquad H = \sum_{n=-r+1}^{r} \Big\{ \frac{i}{2a} v(G) (\phi_n^+ \phi_{n+1} - \phi_{n+1}^+ \phi_n) + (-1)^n \frac{m_0}{2} (\phi_n^+ \phi_{n+1}^+ + \phi_{n+1} \phi_n) - \frac{G}{2a} (\phi_n^+ \phi_n - \tfrac{1}{2})(\phi_{n+1}^+ \phi_{n+1} - \tfrac{1}{2}) \Big\} \; - \; E_0$$

The ϕ_n's are fermion operators: $\{\phi_n, \phi_m^+\} = \delta_{nm}$ and $\phi_{r+1} = \phi_{-r+1}$. G is a renormalized coupling constant and v(G) is the finite renormalization constant needed to make the speed of light equal to unity. Up to first order in G we have $v(G) = 1 + \frac{G}{\pi}$ [1]. Finally m_0 denotes the bare mass and a number E_0 is included to make the ground state energy vanish.

The leading short distance behaviour of the massive Thirring model is the same as the one of a massless Thirring model with coupling constant g [2] [6]. The relation between the two coupling constants G and g is:

$$(3) \qquad G = -\frac{4\epsilon}{\pi} \operatorname{ctg} \epsilon \qquad \text{where} \quad \frac{\epsilon}{\pi} = (\tfrac{1}{2} + \tfrac{g}{\pi})/(1 + \tfrac{g}{\pi}) \; ; \; 0 < \epsilon < \pi$$

Unlike G and v(G) the bare mass m_0 depends on the lattice constant a (i.e. on the cutoff):

$$(4) \qquad m_0 = \frac{8\epsilon \sin\epsilon}{\pi \cdot a} \left(\frac{a \cdot M}{4} \right)^{2\epsilon/\pi} \; ; \quad M \text{ is the physical soliton mass}$$

Finally we have

$$(5) \qquad v(G) = \frac{2\epsilon}{\pi \sin\epsilon}$$

The continuum limit of the lattice theory is taken by first letting $N \rightarrow \infty$ and then performing the limit $a \searrow 0$. Formally, the Thirring field operator becomes in this limit[3]:

$$(6) \qquad \begin{aligned} \chi_1(t, n\cdot a) &= (2a\, Z_2(a))^{-\frac{1}{2}}\ \phi_n(t) \qquad (n \text{ even}) \\ \chi_2(t, n\cdot a) &= (2a\, Z_2(a))^{-\frac{1}{2}}\ \phi_n(t) \qquad (n \text{ odd}) \end{aligned}$$

$Z_2(a)$ is a cutoff dependent wave function renormalization constant.

Luther has proven that the low-lying eigenvalues of H converge in the continuum limit (c.f. sec.IV). It is however not yet clear, wether the n-point functions of χ, χ^+ approach a nontrivial limit for $a \searrow 0$. If they do, we may expect the limiting n-point functions to be the Wightman distributior of the renormalized massive Thirring model.

Consider now the lattice theory that is defined by the Hamiltonian of eq.(2). With the help of a Jordan-Wigner-transformation we can relate this model to a Heisenberg spin-chain problem, the XYZ-model. Its Hamiltonian is given by:

$$(7) \qquad H_{XYZ} = -\frac{1}{2}\sum_{k=-r+1}^{r}\left\{ J_x\, \sigma^x_k \sigma^x_{k+1} + J_y\, \sigma^y_k \sigma^y_{k+1} + J_z\, \sigma^z_k \sigma^z_{k+1} \right\}$$

$\sigma^x_k, \sigma^y_k, \sigma^z_k$ denote the three Pauli-matrices at site k and periodic boundary conditions are implied.

The Jordan-Wigner-trick yields a representation of fermion operators in terms of spin operators:

$$(8) \qquad \phi^+_k = e^{i\frac{\pi}{4}(N+1)}\ \sigma^+_k \prod_{j=-r+1}^{r}\left[i\, \sigma^z_j \right] \quad ; \quad \sigma^+_k \doteq \frac{1}{2}(\sigma^x_k + i\sigma^y_k)$$

Under this transformation H becomes:

$$(9) \qquad H = H_{XYZ} + \frac{1}{2}(1 + (-1)^{r+F})\left\{ J_x\, \sigma^x_r \sigma^x_{-r+1} + J_y\, \sigma^y_r \sigma^y_{-r+1} \right\} + \text{const.}$$

with the identifications:

$$(10) \qquad J_x = \frac{v}{2a} + \frac{m_0}{2} \quad ; \quad J_y = \frac{v}{2a} - \frac{m_0}{2} \quad ; \quad J_z = \frac{G}{4a}$$

and the definition:

$$(11) \qquad (-1)^F = \prod_{k=-r+1}^{r} \sigma^z_k = e^{i\pi\sum_{k=-r+1}^{r}\phi^+_k\phi_k} \qquad [4]$$

Note that $(-1)^F$ commutes with H_{XYZ} and H. We thus see that in the sector where $(-1)^F = (-1)^{r+1}$ we have $H = H_{XYZ}$, but if $(-1)^F = (-1)^r$, we find H to be equal to another Hamiltonian $\tilde{H}_{XYZ}$. This operator is given by eq.(7) and anticyclic boundary conditions:

$$(12) \qquad \sigma^x_{r+1} = -\sigma^x_{-r+1} \quad ; \quad \sigma^y_{r+1} = -\sigma^y_{-r+1} \quad ; \quad \sigma^z_{r+1} = \sigma^z_{-r+1}$$

Unfortunately we cannot be cavalier about boundary conditions. For, as can be verified explicitely for $J_z = 0$ (e.g. [7]), the three Hamiltonians H_{XYZ}, $\tilde{H}_{XYZ}$ and H have different spectra. In view of the presence of topological solitons we should not be surprised by the fact that in the boson language the theory depends delicately on what boundary conditions are assumed.

If we introduce projection operators

$$(13) \qquad P_+ \doteq \frac{1}{2}(1 - (-1)^{r+F}) \quad ; \quad P_- \doteq \frac{1}{2}(1 + (-1)^{r+F})$$

we have $[H_{XYZ}, P_\pm] = [\tilde{H}_{XYZ}, P_\pm] = [H, P_\pm] = 0$ and eq.(9) becomes

$$(14) \qquad H = P_+ H_{XYZ} + P_- \tilde{H}_{XYZ} + \text{const.}$$

From experience with the free model ($J_z = 0$) and the exact results obtained by Baxter [8] and Johnson, Krinsky and McCoy [9] for $J_z \neq 0$ we may then infer that

a) P_+ resp. P_- is the projector on the subspace with an even resp. odd fermion number above physical ground state.

b) For large N the spectra of $P_+ H_{XYZ}$ and $P_- H_{XYZ}$ are the same. This is true for $\tilde{H}_{XYZ}$ aswell.

We now proceed to exhibit the connection of the XYZ-spin-chain with the eight-vertex-model. The reader interested in this model itself should consult the original articles of Baxter [8]. The central object in the discussion of the eight-vertex-model is the transfer matrix T. This is an operator acting in the same space as H_{XYZ}. It depends on three real parameters V, φ, ℓ with

$$(15) \qquad 0 < \ell < 1 \quad , \quad 0 < \varphi < K_\ell$$

K_ℓ is the complete elliptic integral of the first kind of modulus ℓ ([10] §8.112). Let

$$\sigma^1 = \sigma^x \; , \; \sigma^2 = \sigma^y \; , \; \sigma^3 = \sigma^z \; , \; \sigma^4 = 1$$

$$(16) \quad \begin{aligned} w_1 &= cn(V,\ell)/cn(\varphi,\ell) \; ; \quad w_2 = dn(V,\ell)/dn(\varphi,\ell) \\ w_3 &= 1 \qquad\qquad\qquad\qquad ; \quad w_4 = sn(V,\ell)/sn(\varphi,\ell) \end{aligned}$$

$$R(\alpha,\beta)^{\lambda}{}_{\lambda'} = \sum_{j=1}^{4} w_j \, \sigma^j_{\alpha,\beta} \, \sigma^j_{\lambda,\lambda'}$$

$sn(u,\ell)$, $cn(u,\ell)$ and $dn(u,\ell)$ denote the Jacobian elliptic functions of argument u and modulus ℓ [10]. For each pair α, β we look at $R(\alpha,\beta)$ as some operator acting in a two dimensional auxiliary space.

The transfermatrix T is now given by:

$$(17) \quad T^{\alpha}{}_{\beta} = Tr\{R(\alpha_{-r+1},\beta_{-r+1}) \cdot R(\alpha_{-r+2},\beta_{-r+2}) \cdot \ldots \cdot R(\alpha_r,\beta_r)\}$$

$$\alpha \doteq (\alpha_{-r+1}, \ldots, \alpha_r) \quad ; \quad \beta \doteq (\beta_{-r+1}, \ldots, \beta_r)$$

(the indices α_k, β_k refer to site k of the spin-chain). As has been shown by Baxter [8] H_{XYZ} is a logarithmic derivative of T:

$$(18) \quad H_{XYZ} = J_x \, sn(2\varphi,\ell) \, U\left[T^{-1} \frac{\partial}{\partial V} T\right]_{V=\varphi} U^{-1} + \text{const.}$$

if we identify

$$(19) \quad \ell^2 = (J_x^2 - J_y^2)/(J_x^2 - J_z^2) \quad ; \quad cn(2\varphi,\ell) = -J_z/J_x$$

U denotes a simple unitary transformation [5]:

$$U \sigma^x_k U^{-1} = -\sigma^z_k$$

$$(20) \quad U \sigma^y_k U^{-1} = \begin{cases} -\sigma^y_k & (k \text{ odd}) \\ \sigma^y_k & (k \text{ even}) \end{cases}$$

$$U \sigma^z_k U^{-1} = \begin{cases} -\sigma^x_k & (k \text{ odd}) \\ \sigma^x_k & (k \text{ even}) \end{cases}$$

Formula (18) holds also for $\tilde{H}_{XYZ}$ if we replace T by

$$(21) \quad \tilde{T}^{\alpha}{}_{\beta} = Tr\{R(\alpha_{-r+1},\beta_{-r+1}) \cdot R(\alpha_{-r+2},\beta_{-r+2}) \cdot \ldots \cdot R(\alpha_r,\beta_r) \cdot i\sigma^x\}$$

The transfer matrix T has a remarkable property that will be the clue to the construction of many conserved quantities. If we keep φ and ℓ fixed and let V vary, we find [8]:

$$(22) \quad [T(V), T(V')] = [T(V), T^{+}(V')] = 0 \qquad (\text{for all } V, V')$$

(the same is true for $\tilde{T}$ too). Thus, T(V) is a one parameter family of simultanously diagonalizable matrices. Of course, if we knew the eigenstates and eigenvalues of T(V) we could solve the XYZ-problem. In fact, this observation provides one of the cornerstones for the exact evaluation of eigenvalues of H_{XYZ} [8], [9].

III. Construction of local charges for the lattice theory

We first construct a set of charges for the XYZ-model with cyclic resp. anticyclic (eq.(12)) boundary conditions. Let

$$(23) \qquad G_n \doteq U\left[T^{-1}\frac{\partial^n}{\partial V^n}T\right]_{V=\gamma}U^{-1} \quad ; \quad n = 0,1,2,\dots$$

From eq.(22) it is clear that the real and imaginary parts of these operators commute with each other and also with H_{XYZ}. However in the continuum limit the G_n's do not become local charges (c.f. sec.I). This defect can be removed by taking cumulants [6]:

$$(24) \qquad C_n \doteq U\left[\frac{\partial^{n+1}}{\partial V^{n+1}}\ln\left(T^{-1}(\gamma)T(V)\right)\right]_{V=\gamma}U^{-1} \quad ; \quad n = 0,1,2,\dots$$

A proof that the C_n's are indeed local charges is included in an appendix. It is shown there that

$$C_n = \sum_{x=-r+1}^{r} C_n(x) \quad ; \quad C_n(x) \text{ a polynomial of } \sigma^i_y \text{ , } |x-y| \leq n+$$

Furthermore, $C_n(x')$ can be obtained from $C_n(x)$ just by applying a translation by $x' - x$ lattice units.

For example, we have:

$$i\,(J_x\,\mathrm{sn}(2\gamma,\ell))^2\,C_1 = \frac{1}{2}\sum_{k=-r+1}^{r}\vec{J}_k\cdot(\vec{\sigma}_{k+1}\times\vec{J}_{k+2})$$

$$\text{with} \quad \vec{J}_k \doteq (J_x\sigma^x_k\,,\,J_y\sigma^y_k\,,\,J_z\sigma^z_k)$$

Replacing T by $\tilde{T}$ (eq.(21)) we obtain a set of commuting charges $\tilde{C}_n$. They differ from the C_n's by boundary terms only.

We now combine C_n and $\tilde{C}_n$ in such a way that the new charges will be local in the fermion language:

$$(25) \qquad Q_n \doteq i^n\,(J_x\,\mathrm{sn}(2\gamma,\ell))^{n+1}\{P_+C_n + P_-\tilde{C}_n\} \quad - \text{ ground state expectation va}$$

Summarizing, we have found an infinite set of local, conserved charges for the fermion problem (2):

$$(26) \qquad Q_0 = H \quad ; \quad [Q_n, Q_m] = 0 \quad , \quad n,m = 0,1,2,\dots$$

IV. Evaluation of the charges on the one particle states

In this section we are going to use exact results for the largest and second largest eigenvalues of the transfer matrix T. In order to state these results we have to introduce some more of Baxter's parameters [7]:

$$(27) \qquad \tau = \pi K_\ell/K'_\ell \quad ; \quad \lambda = \pi\,\gamma/K'_\ell \quad ; \quad \mu = \pi\,\gamma/K_\ell$$

K'_ℓ denotes the complete elliptic integral of the first kind with modulus $\ell' = (1-\ell^2)^{1/2}$. We also define the new moduli k_1, k_2 by:

$$(28) \qquad \pi K'_1/K_1 = \lambda \quad ; \quad \pi K'_2/K_2 = 2\lambda$$

(K_1 is a shorthand for K_{k_1}).

Let $T_0(V)$ be the eigenvalue of $UT(V)U^{-1}$ on the ground state of H. Similarly, define $\tilde{T}_s(V)$, $\tilde{T}_{\bar{s}}(V)$ and $T_b(V)$ to be the eigenvalues of $U\tilde{T}(V)U^{-1}$ (resp. $UT(V)U^{-1}$) on soliton, anti-soliton and breather states respectively.

From the work of Johnson, Krinsky and McCoy [9] we then infer that for $N \to \infty$

$$(29) \qquad \begin{aligned} \frac{\tilde{T}_s(V)}{T_0(V)} &= \sqrt{k_2}\;\mathrm{sn}\left[\frac{K_2}{\pi}\left(\phi - i\frac{\pi V}{K'_\ell}\right); k_2\right] \quad ; \quad 0 \leq \phi \leq 2\pi \\ \frac{\tilde{T}_{\bar{s}}(V)}{T_0(V)} &= \sqrt{k_2}\;\mathrm{sn}\left[\frac{K_2}{\pi}\left(\phi - i\frac{\pi V}{K'_\ell}\right); k_2\right] \quad ; \quad -2\pi \leq \phi \leq 0 \end{aligned}$$

ϕ is a parameter related to the momentum of the particle (see below). Breather states (i.e. bound states of soliton-antisoliton pairs) occur only if $\mu > \pi/2$ i.e. by eqs.(10),(19),(27) if $G > 0$. They are labeled by ϕ and an internal quantum number $n = 1,2,3,\ldots;\ n < (\frac{\pi}{\mu}-1)^{-1}$. For the corresponding eigenvalues of T we have:

$$(30)\quad \frac{T_b(V)}{T_0(V)} = k_2\, \mathrm{sn}\left[\frac{K_2}{\pi}(\phi_+ - i\frac{\pi V}{K_\ell'});k_2\right]\cdot \mathrm{sn}\left[\frac{K_2}{\pi}(\phi_- - i\frac{\pi V}{K_\ell'});k_2\right]$$

$$\phi_\pm \doteq \phi \pm in(\tau-\lambda) \mp i\lambda \quad ; \quad 0 \leq \phi \leq 2\pi$$

To express the parameter ϕ in terms of the momentum p of the particle, we must find a translation operator $\exp i\,P\cdot 2a$ such that the field ϕ_n transforms as

$$(31)\quad e^{iP\cdot 2a}\,\phi_n\,e^{-iP\cdot 2a} = \phi_{n-2} \qquad \text{(cyclic boundary conditions)}$$

Note that we are using a staggered lattice (eq.(6)) so that only translations by an even number of lattice units are pure space translations (more about the staggered lattice can be found in ref. [11]). We now observe (c.f. appendix) that $T(\varphi)$ is a shift operator for the spin operators:

$$T(\varphi)\,\vec{\sigma}_k\,T(\varphi)^{-1} = \vec{\sigma}_{k-1}$$

It is then not hard to prove that the choice

$$(32)\quad e^{iP\cdot 2a} \doteq 4^{-N}\left\{P_+ U T^2(\varphi) U^{-1} + P_- U \tilde{T}^2(\varphi) U^{-1}\right\}$$

meets all requirements.

We know that $T_0(\varphi) = 2^N$. Hence, for large N the momentum p of the soliton is given by

$$(33)\quad e^{ip\cdot 2a} = \left(\frac{\tilde{T}_s(\varphi)}{T_0(\varphi)}\right)^2 \quad , \quad |p| \leq \frac{\pi}{2a}$$

For later calculational convenience let us define a scaled momentum q [8]:

$$(34)\quad e^{iq} \doteq \frac{\tilde{T}_s(\varphi)}{T_0(\varphi)} \quad ; \quad |q| \leq \pi$$

Also, we will henceforth adopt the convention that a Jacobian elliptic function of modulus k_1 resp. k_1' and argument u is written as snu resp. sn'u etc.. We then find

$$(35)\quad p = q/a \quad (\mathrm{mod}\ \tfrac{\pi}{a})$$

and with the help of [10] we get

$$(36)\quad \sin q = -\,\mathrm{cn}\frac{K_1\phi}{\pi} \quad ; \quad \cos q = \mathrm{sn}\frac{K_1\phi}{\pi}$$

The same formulae hold also for the antisoliton whereas for the breather modes eq.(36) must be replaced by

$$(37)\quad \begin{aligned} \sin q &= -2\,\mathrm{sn}\frac{K_1\phi}{\pi}\,\mathrm{cn}\frac{K_1\phi}{\pi}\,\mathrm{sn}'y\left[\mathrm{sn}^2\frac{K_1\phi}{\pi}(\mathrm{cn}'y)^2 + (\mathrm{sn}'y)^2\right]^{-1} \\ \cos q &= \left[\mathrm{sn}^2\frac{K_1\phi}{\pi} - \mathrm{cn}^2\frac{K_1\phi}{\pi}(\mathrm{sn}'y)^2\right]\cdot\left[\mathrm{sn}^2\frac{K_1\phi}{\pi}(\mathrm{cn}'y)^2 + (\mathrm{sn}'y)^2\right]^{-1} \end{aligned}$$

where $y \doteq \frac{K_1}{\pi}n(\tau-\lambda) = K_1'\,n(\frac{\pi}{\mu}-1)$.

Recalling eqs. (14) and (18) we now proceed to calculate the energy E of the one particle states [9]:

a) soliton, antisoliton

$$(38)\quad \begin{aligned} E &= J_x\,\mathrm{sn}(2\varphi;\ell)\,K_1/K_\ell'\ \mathrm{dn}\frac{K_1\phi}{\pi} = \\ &= J_x\,\mathrm{sn}(2\varphi;\ell)\,K_1/K_\ell'\ \sqrt{1-k_1^2\cos^2 q} \end{aligned}$$

b) breather modes

$$(39)\quad \begin{aligned} E &= J_x\,\mathrm{sn}(2\varphi;\ell)\,K_1/K_\ell'\left\{\mathrm{dn}\frac{K_1\phi_+}{\pi} + \mathrm{dn}\frac{K_1\phi_-}{\pi}\right\} = \\ &= J_x\,\mathrm{sn}(2\varphi;\ell)\,K_1/K_\ell'\ \frac{2}{\mathrm{sn}'y}\sqrt{1-(\mathrm{cn}'y\cos\tfrac{q}{2})^2}\sqrt{\sin^2\tfrac{q}{2} + (k_1'\,\mathrm{sn}'y\cos\tfrac{q}{2})^2} \end{aligned}$$

From eqs. (3),(4),(5),(10) and (19) we find that in the continuum limit $a \to 0$

(40) $\mu = \epsilon \; ; \; k_l' = a \cdot M \; ; \; \ell = 4\left(\frac{a \cdot M}{4}\right)^{\epsilon/\pi} \; ; \; \varphi = \frac{\epsilon}{2} \; ; \; J_x = \frac{\epsilon}{a \cdot \pi \sin\epsilon}$

and therefore

(41)
$$E = \sqrt{M^2 + p^2} \qquad \text{(soliton, antisoliton)}$$
$$E = \sqrt{M_b^2 + p^2} \; ; \; M_b = 2M \sin \frac{n\pi}{2}\left(\frac{\pi}{\epsilon} - 1\right) \qquad \text{(breather)}$$

Thus, in the continuum limit the lattice theory reproduces the correct relativistic dispersion law for the one particle states.

We are now well prepared to calculate Q_n (eq.(25)) for the one particle states in the lattice theory ($a > 0$). For the soliton we obtain:

(42) $$Q_n = i^n (J_x \, sn(2\varphi;\ell))^{n+1} \frac{\partial^{n+1}}{\partial V^{n+1}} \ln e^{-iq} \frac{\tilde{T}_s(V)}{T_o(V)} \Bigg|_{V=\varphi}$$

The factor $2^N T_o^{-1}(V)$ accounts for the subtraction of the ground state expectation value of $i^n (J_x \, sn(2\varphi;\ell))^{n+1} \{P_+ C_n + P_- \tilde{C}_n\}$. Inserting the explicit expression (29) for $\tilde{T}_s(V)/T_o(V)$ yields:

(43) $$Q_n = \left[\frac{\pi}{K_\ell'} J_x \, sn(2\varphi;\ell)\right]^n \frac{\partial^n}{\partial \phi^n} E$$

and with $\frac{dq}{d\phi} = \left[\frac{\pi}{K_\ell'} J_x \, sn(2\varphi;\ell)\right]^{-1} E$ we get the beautiful formula

(44) $$Q_n = \left(E \frac{\partial}{\partial q}\right)^n E$$

Eqs.(43) and (44) are valid for all three types of particles discussed if we use the corresponding energy expressions (38) resp. (39). Note that the soliton and the antisoliton carry the same charges. The electric (topological) charge is hence not a linear combination of Q_n's.

We conclude this section by remarking that one should not take the continuum limit of eq.(44) directly by replacing q by p and E(q) by the continuum expressions (41). The result would be trivial. Instead, one must take linear combinations of the Q_n's and divide them by appropriate powers of the lattice constant a. This obstacle does however not disturb the subsequent considerations, because we are going to discuss scattering processes on the (infinite volume) lattice. The results are then formulated in a way which is independent of the lattice constant and the continuum limit of these statements can hence be taken trivially.

V. Conservation laws for scattering processes

In this section we consider a general scattering process in the infinite volume lattice Thirring model involving solitons, antisolitons and breathers. The charges Q_k imply conservation laws that severly restrict the possible scatterings. These conservation laws emerge from the fact that a conserved charge like Q (eq.(1); the constant has to be choosen such that Q annihilates the vacuum state) acts additively on asymptotic states (this can be proven by arguments similar to those given in ref. [12]). This theorem applies also to a lattice theory like the one considered here[9].

Thus, given a scattering process with m incoming particles with scaled momenta $q_1,\dots,q_m$ and m' outgoing particles with momenta $q_1',\dots,q_{m'}'$ they satisfy the conservation laws

(45) $$Q_k(q_1) + \dots + Q_k(q_m) = Q_k(q_1') + \dots + Q_k(q_{m'}')$$

$$k = 0,1,2,\dots$$

Here the Q_k's are given by eq.(44) where $E(q)$ is the energy (38) resp. (39) of the corresponding particle.

To explore the full content of eq.(45) it is most convenient to define an operatorvalued generating function $G(z)$ for the charges Q_k:

$$(46)\qquad G(z) \doteq \left[J_x\,\mathrm{sn}(2\varphi;\ell)\,K_1/K_\ell'\right]^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, Q_k \left[\frac{\pi}{K_\ell'} J_x\,\mathrm{sn}(2\varphi;\ell)\right]^{-k} z^k$$

z is a complex number. Let $G_s(z,\phi)$, $G_{\bar{s}}(z,\phi)$ resp. $G_b(z,\phi,n)$ be the eigenvalue of $G(z)$ when applied to a soliton, antisoliton resp. breather state with quantum numbers ϕ resp. ϕ, n (c.f.(36),(37)). Upon inserting (43) into (46) we obtain:

$$(47)\qquad G_s(z,\phi) = G_{\bar{s}}(z,\phi) = \mathrm{dn}\,\frac{K_1}{\pi}(\phi - z)$$

$$(48)\qquad G_b(z,\phi,n) = \mathrm{dn}\,\frac{K_1}{\pi}(\phi_+ - z) + \mathrm{dn}\,\frac{K_1}{\pi}(\phi_- - z)$$

Consider a scattering process involving solitons only. The conservation laws (45) are then equivalent to the statement that

$$(49)\qquad G_s(z,\phi_1) + \ldots + G_s(z,\phi_m) = G_s(z,\phi_1') + \ldots + G_s(z,\phi_{m'}')$$

for all z.

The Jacobian elliptic function dnu is a doubly periodic meromorphic function of u. It has elementary periods $2K_1$ and $i4K_1'$ and poles at $u = iK_1'$ and $u = i3K_1'$ (modulo $2K_1$ resp. $i4K_1'$) Hence $G_s(z,\phi)$ has got simple poles for

$$(50)\qquad z = \phi + 2\pi\nu + i\pi\frac{K_1'}{K_1}(2\varkappa+1) \qquad (\nu,\varkappa \in \mathbb{Z})$$

Recalling $0 \leq \phi < 2\pi$ we see that the poles in eq.(49) cancel if and only if $m = m'$ and $\{\phi_1,\ldots,\phi_m\} = \{\phi_1',\ldots,\phi_m'\}$.

Similar arguments apply if we consider an arbitrary scattering process. The outcome is that

a) the total number of fermions (solitons, antisolitons) and the number of breather modes with internal quantum number n are conserved separately,

b) the sets of momenta p_i (p_i') of incoming and outgoing fermions are equal,

c) the same as b) holds for each type of breather mode separately.

Of course, these conservation laws remain true in the continuum limit $a \searrow 0$. We have thus proven that there is no production of fermions nor breathers in the (continuum) renormalized massive Thirring model.

Let us finally remark that there is no soliton-antisoliton reflection in the classical sine-Gordon theory. Such a process is however not forbidden by the conservation laws established here, because the soliton and the antisoliton carry the same charges[10]. This fits perfectly with the perturbation theoretic result (e.g. [13]) that the reflection amplitude does not vanish in the quantized massive Thirring model. This statement is furthermore confirmed by a semi-classical calculation done by Korepin [14].

VI. Conclusions

In this paper we have considered the quantized massive Thirring model with a (noncovariant) cutoff provided by a space lattice. Infinitely many conserved, dynamical charges were constructed and the corresponding conservation laws for scattering processes were derived. Although the charge densities could in principle be expressed in terms of field operators, the amount of labour necessary to produce such a result becomes very large for the "higher" charges. This obstacle prevented the direct comparision with the works of Berg et al. [3] resp. Flume et al. [4] so far.

The fact that the lattice massive Thirring model is partially exactly soluble makes it interesting also from another point of view. With the aim of explaining quark confinement in nonabelian quark-gluon theories some authors [15], [16] have considered lattice versions of these models. Although quark confinement was then shown to occur, it remained unclear what effects dominate in the continuum limit and wether colour is still confined. These questions were studied [17] in the massive Schwinger model using Padé-extrapolation techniques. A critique of these calculations can now be drawn from the experience with the lattice massive Thirring model: to obtain the correct relativistic dispersion laws (41) we were forced to renormalize the velocity of light by introducing the renormalization constant v(G) (c.f.(2),(5)). Attributing this effect to the use of a nonrelativistic cutoff we are lead to conjecture that the velocity of light must be nontrivially renormalized in (hamiltonian) lattice theories.

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Appendix: Proof of locality of the charges Q_n

We first seek a more explicit expression for the quantities G_n (eq.(23)). To this end we reshuffle the σ-matrices in the definition (16) of $\mathbb{R}(\alpha,\beta)$ to obtain [8]:

$$\text{(A1)}\qquad \mathbb{R}(\alpha,\beta)^{\lambda}{}_{\lambda'} = \sum_{j=1}^{4} p_j\, (\sigma^j)^{\lambda}{}_{\beta}\, (\sigma^j)^{\alpha}{}_{\lambda'}$$

with

$$\text{(A2)}\qquad \begin{aligned} p_1 &= \tfrac{1}{2}(w_1 - w_2 - w_3 + w_4)\;; & p_2 &= \tfrac{1}{2}(-w_1 + w_2 - w_3 + w_4) \\ p_3 &= \tfrac{1}{2}(-w_1 - w_2 + w_3 + w_4)\;; & p_4 &= \tfrac{1}{2}(w_1 + w_2 + w_3 + w_4) \end{aligned}$$

For $V = \Psi$ we have $p_1 = p_2 = p_3 = 0$, $p_4 = 2$ and therefore

$$\text{(A3)}\qquad \mathbb{R}(\alpha,\beta)^{\lambda}{}_{\lambda'}\Big|_{V=\Psi} = 2\,\delta^{\lambda}{}_{\beta}\,\delta^{\alpha}{}_{\lambda'}$$

and by eq.(17)

$$\text{(A4)}\qquad T(\Psi)^{\alpha}{}_{\beta} = 2^{N}\,\delta^{\alpha_{-r+1}}_{\beta_{-r+2}}\cdot\delta^{\alpha_{-r+2}}_{\beta_{-r+3}}\cdot\ldots\cdot\delta^{\alpha_{r}}_{\beta_{-r+1}}$$

$2^{-N}T(\Psi)$ is thus a unitary shift operator for the spin-operators. Obviously

$$\text{(A5)}\qquad T(\Psi)^{-1\,\gamma}{}_{\alpha} = 2^{-N}\,\delta^{\gamma_{-r+2}}_{\alpha_{-r+1}}\cdot\delta^{\gamma_{-r+3}}_{\alpha_{-r+2}}\cdot\ldots\cdot\delta^{\gamma_{-r+1}}_{\alpha_{r}}\,.$$

$$\text{(A6)}\qquad \left[T(\Psi)^{-1}\cdot T(V)\right]^{\alpha}{}_{\beta} = 2^{-N}\,\mathrm{Tr}\left\{\mathbb{R}(\alpha_{-r+1},\beta_{-r+1})\cdot\mathbb{R}(\alpha_{-r+2},\beta_{-r+2})\cdot\ldots\cdot\mathbb{R}(\alpha_{-r+1}\right.$$

To express the derivatives of eq.(A6) with respect to V at $V = \Psi$ in terms of spin-operators σ^j_k it is convenient to introduce operatorvalued n-point functions $G_n(x_1,\ldots,x_n)$ by the following rule: for each n-tupel $(x_1,\ldots,x_n)$ of (not necessarily different) lattice points let m_j denote the number of x_k's being equal to j. Define

$$\text{(A7)}\qquad G_n(x_1,\ldots,x_n) = R^{(m_{-r+1})}_{-r+1}\cdot R^{(m_{-r+2})}_{-r+2}\cdot\ldots\cdot R^{(m_r)}_{r}\,.$$

where (c.f.(20))

$$R^{(m)}_k \doteq U\, \tfrac{1}{2} \sum_{j=1}^{4} P_j^{(m)} \sigma^j_k \sigma^j_{k+1} U^{-1} \quad ; \quad P_j^{(m)} \doteq \frac{\partial^m}{\partial v^m} P_j \Big|_{v=\varphi} \qquad \text{(A8)}$$

Periodic boundary conditions mean that

$$R^{(m)}_r = U\, \tfrac{1}{2} \sum_{j=1}^{4} P_j^{(m)} \sigma^j_r \sigma^j_{-r+1} U^{-1} \qquad \text{(A9)}$$

To account for the trace operation in eq.(A6) it is understood that the σ^j_{-r+1}-matrices appearing in the last factor of (A7) are to be put to the left of $R^{(m-r+1)}_{-r+1}$.

The operators G_n can now be written in the form

$$G_n = \sum_{x_1,\dots,x_n=-r+1}^{r} G_n(x_1,\dots,x_n) \qquad \text{(A10)}$$

This is clearly not a local charge.

We are now going to list some elementary properties of the n-point functions $G_n(x_1,\dots,x_n)$. Note first that

$$R_k^{(o)} = 1 \quad \text{for all } k \qquad \text{(A11)}$$

Hence $G_n(x_1,\dots,x_n)$ is in fact a product of operators $R^{(m)}_{x_i}$. Therefore:

a) $G_o = 1$; $G_1(x) = R_x^{(1)}$

b) $G_n(x_1,\dots,x_n)$ is a totally symmetric function of its arguments.

c) $G_n(x_1,\dots,x_n)$ is translationally covariant:

$$U T(\varphi) U^{-1} G_n(x_1,\dots,x_n) U T(\varphi)^{-1} U^{-1} = G_n(x_1-1, x_2-1, \dots, x_n-1) \qquad \text{(A12)}$$

d) $G_n(x_1,\dots,x_n)$ has "cluster properties". Let $A_1,\dots,A_k$ be a partition of $(x_1,\dots,x_n)$ into k clusters with $n_1,\dots,n_k$ elements respectively. For the corresponding G_n-functions we write simply: $G_{n_1}(A_1)$, $G_{n_2}(A_2)$ etc.. Then, if the distance between all pairs of clusters A_i, A_j $(i \neq j)$ becomes strictly larger than one lattice unit, the operators $G_{n_i}(A_i)$ and $G_{n_j}(A_j)$ commute. Furthermore

$$G_n(x_1,\dots,x_n) = G_{n_1}(A_1)\cdot\ldots\cdot G_{n_k}(A_k) \qquad \text{(A13)}$$

This last property of $G_n(x_1,\dots,x_n)$ suggests that we consider truncated n-point functions $G_n^T(x_1,\dots,x_n)$ well known in field theory and statistical mechanics. Thus, we define recursively

$$G_n(x_1,\dots,x_n) = \sum_{k=1}^{n} \sum_{\substack{\text{Partitions of } (x_1,\dots,x_n) \\ \text{into } k \text{ clusters } A_1,\dots,A_k}} \$\{G^T_{n_1}(A_1)\cdot\ldots\cdot G^T_{n_k}(A_k)\} \qquad \text{(A14)}$$

Here, $\$$ denotes a symmetrization operation, which must be included because the operators $G^T_{n_i}(A_i)$ do not always commute. Explicitly we have

$$\$\{G^T_{n_1}(A_1)\cdot\ldots\cdot G^T_{n_k}(A_k)\} \doteq \frac{1}{k!} \sum_{\text{Permutations } \pi} G^T_{n_{\pi(1)}}(A_{\pi(1)})\cdot\ldots\cdot G^T_{n_{\pi(k)}}(A_{\pi(k)} \qquad \text{(A15)}$$

Clearly $G_1^T(x) = G_1(x)$ and it is not hard to prove by induction that $G_n^T(x_1,\dots,x_n)$ has the same properties b), c) and d) as $G_n(x_1,\dots,x_n)$ with (A13) replaced by: $G_n^T(x_1,\dots,x_n) = ($

As is well known (e.g. [18]) eq.(A14) can be formulated elegantly with the help of generating functionals:

$$\text{(A16)} \quad \begin{aligned} Z(J) &\doteq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1,\dots,x_n=-r+1}^{r} G_n(x_1,\dots,x_n)\, J(x_1)\ldots J(x_n) \\ Z^T(J) &\doteq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1,\dots,x_n=-r+1}^{r} G^T_n(x_1,\dots,x_n)\, J(x_1)\ldots J(x_n) \end{aligned}$$

J is an arbitrary testfunction on the lattice. Eq.(A14) is then equivalent to stating that

(A17) $$Z(J) = \$\, \exp Z^T(J) \qquad \text{for all } J.$$

Setting J = constant = $V - \varphi$ and using (A10),(A23) we see that

(A18) $$Z(V-\varphi) = UT^{-1}(\varphi)T(V)U^{-1} = \$\, \exp Z^T(V-\varphi)$$

The operators G_n commute. By induction we can prove that the quantities

$$\sum_{x_1,\dots,x_n=-r+1}^{r} G_n^T(x_1,\dots,x_n)$$

commute, too. The symmetrization operation $\$$ in eq.(A18) is therefore superfluous. From the definition (24) of the charges C_n we conclude that

(A19) $$C_n = \frac{\partial^{n+1}}{\partial V^{n+1}} Z^T(V-\varphi)\Big|_{V=\varphi} = \sum_{x_1,\dots,x_n=-r+1}^{r} G_n^T(x_1,\dots,x_n)$$

Let us define a charge density

(A20) $$C_n(x) \doteq \sum_{x_1,\dots,x_n=-r+1}^{r} G_n^T(x,x_1,\dots,x_n)$$

This is a translationally covariant (c.f.(A12)) operator and

(A21) $$C_n = \sum_{x=-r+1}^{r} C_n(x)$$

$C_n(x)$ is also a local operator in the sense that it is a sum of products of factors $R_y^{(m)}$ with $|x-y| \leq n$. This is so because $G_{n+1}^T(x,x_1,\dots,x_n)$ satisfies cluster properties as explained above. We have therefore proved that C_n is a local charge for the spin-chain.

The charges $\tilde{C}_n$ can be obtained from C_n just by replacing $R_r^{(m)}$ (eq.(A9)) by

(A22) $$\tilde{R}_r^{(m)} = U \tfrac{1}{2}\{ p_1^{(m)} \sigma_r^1 \sigma_{-r+1}^1 - p_2^{(m)} \sigma_r^2 \sigma_{-r+1}^2 - p_3^{(m)} \sigma_r^3 \sigma_{-r+1}^3 + p_4^{(m)} \} U^{-1}$$

$\tilde{C}_n$ is hence a local operator and

(A23) $$U\tilde{T}(\varphi)U^{-1}\, \tilde{C}_n(x)\, U\tilde{T}(\varphi)^{-1}U^{-1} = \tilde{C}_n(x-1) \quad ; \quad \tilde{C}_n(-r) \doteq \tilde{C}_n(r)$$

We now observe that the projection operators P_+, P_- (eq.(13)) commute with $R_k^{(m)}, \tilde{R}_r^{(m)}$ and hence with $C_n, \tilde{C}_n$. It is then not difficult to prove that up to a constant the charge iQ_n (eq.(25)) can be gotten from C_n by substituting in eqs.(A20),(A21) the fermion expression

(A24) $$[iJ_x \operatorname{sn}(2\varphi;\ell)]^m \Big\{ \tfrac{i}{2}(p_3^{(m)} + p_2^{(m)})(\phi_k^+ \phi_{k+1} - \phi_{k+1}^+ \phi_k) + (-1)^k \tfrac{1}{2}(p_3^{(m)} - p_2^{(m)})(\phi_k^+ \phi_{k+1}^+ + \phi_{k+1}\phi_k) + 2p_1^{(m)}(\phi_k^+\phi_k - \tfrac{1}{2})(\phi_{k+1}^+\phi_{k+1} - \tfrac{1}{2}) + \tfrac{1}{2} p_4^{(m)} \Big\}$$

for $R_k^{(m)}$. Therefore the Q_n's are local with respect to the fermion field theory.

Footnotes

1) This is the form of v(G) expected when the lattice theory is naively "derived" from the continuum field theory taking Wick-ordering into account. The more complicated expression (5) is however needed to recover the correct dispersion law $E = (M^2 + p^2)^{1/2}$ for one particle states in the continuum limit (c.f.sec.IV).

2) For the definition of g we use the same convention as Coleman [2]. This amounts to a normalization of the interacting (electric) current j^μ such that the charge $Q = \int_{t=\text{const.}} dx\, j^0(x)$ takes on integer values only.

3) In order to recover the conventional formulation of the continuum theory in terms of a spinor field Ψ and with

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; ; \quad \gamma_5 = \gamma_0 \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

one has to substitute

$$\Psi_1 = \frac{1}{\sqrt{2}}(\chi_1^+ + \chi_2^+) \quad ; \quad \Psi_2 = \frac{1}{\sqrt{2}}(\chi_1 - \chi_2)$$

4) In the coupling constant range $J_x > J_y > |J_z|$ this quantum number is equal to $(-1)^{\nu'}$ in Baxter's notation.

5) This transformation is needed because we are interested in the coupling constant region $J_x > J_y > |J_z|$ instead of the "fundamental" domain $-J_z > -J_y > |J_x|$.

6) The relation between G_n and C_n is the same as that between disconnected and connected Green's functions at vanishing external momenta in field theory.

7) The parameters τ, λ, μ, k_1 and k_2 are all functions of γ, ℓ or (by (19)) of J_x, J_y, J_z.

8) I apologize to the patient reader for introducing so many symbols that make the basically simple reasoning somewhat cumbersome. I however think that in the long run they will prove useful. Also, the notions have been choosen in such a way as to conform (wherever possible) with the referenced literature.

9) This statement can be checked explicitely for the charges Q_k in the two fermion channel by inspection of the exact scattering state eigenvalue of the transfer matrix T(V) obtained by [9].

10) Of course, they have opposite electric (topological) charge.

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