

DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 76/30
June 1976



Convergence of Operator Product Expansions on the Vacuum in Conformal Invariant Quantum Field Theory

by

G. Mack

II. Institut für Theoretische Physik der Universität Hamburg

Abstract:

In a conformal invariant quantum field theory (in 4 space time dimensions) Wilson operator product expansions converge on the vacuum, because they are closely related to conformal partial wave expansions.

2 HAMBURG 52 · NOTKESTIEG 1

To be sure that your preprints are promptly included in the
HIGH ENERGY PHYSICS INDEX,
send them to the following address (if possible by air mail) :



Convergence of operator product expansions on the vacuum in
conformal invariant quantum field theory

G. Mack
II. Institut für theoretische Physik der Universität Hamburg

Abstract: In a conformal invariant quantum field theory (in 4 space
time dimensions) Wilson operator product expansions converge on
the vacuum, because they are closely related to conformal partial
wave expansions.

1. Introduction

Let $\phi^i(x)$, $\phi^j(y)$ two local quantum fields.
According to Wilson [1], their product should admit an
asymptotic expansion at short distances of the form

$$\phi^i(\frac{1}{x}x)\phi^j(-\frac{1}{y}y)\Omega = \sum_k C^{ijk}(x)\phi^k(y)\Omega. \quad (1.1a)$$

Herein ϕ^k are local fields, and $C^{ijk}(x)$ are singular c-number
functions. In a scale invariant theory they are homogeneous
functions of x . The expansion is presumably valid for all
states Ω in the field theoretic domain \mathcal{D} which is created
out of the vacuum by polynomials in smeared field operators.
We shall however only consider the special case

$$\Omega = \text{vacuum} \quad (1.1b)$$

Studies in perturbation theory [1] indicate that expansion
(1.1) is then valid as an asymptotic expansion to arbitrary
accuracy for matrix elements $(\Psi, \phi^i(x)\phi^j(y)\Omega), \Psi \in \mathcal{D}$.
This means that the error in a truncated expansion can be
made smaller than any given power of $|x|$ at sufficiently
small distances $|x|$ by taking into account sufficiently many
terms. (For more precise formulation cp. e.g. Appendix A
of [1].)

Asymptotic expansions need not converge. For instance
the asymptotic expansion near $y = 0$ of the function
 $f(y) = \exp(-1/y)$ of one positive real variable y in powers
of y vanishes identically and does therefore not converge to
the function f .

Among the fields ϕ^k there are derivatives of other
local fields. In general there appears $\partial^\mu \phi$ etc. together

with any nonderivative field ϕ . In a conformal invariant theory, non-derivative fields ϕ can be recognized by their conformal transformation law [4], viz. $[\phi(0), \omega^\mu] = 0$. K^μ = generators of special conformal transformations.

From the work of Ferrara, Gatto and Grillo one knows [4] that conformal symmetry imposes strong restrictions on the coefficients C_{ijk} in (1.1): The terms involving non-derivative fields determine all the others. Using this, the terms involving derivatives of one and the same nonderivative local field can be formally summed. Here we will prove more:

Theorem 1: Consider conformal invariant quantum field theory (in four space time dimensions) and suppose that vacuum expansions (1.1) are valid as asymptotic expansions in homogeneous functions of x to arbitrary accuracy for $(\Psi, \phi^i(\frac{1}{2}x) \phi^{j(-\frac{1}{2}x)} \Omega)$, Ψ in \mathcal{D} . Then $\phi^i \phi^j \Omega$ admits a convergent expansion,

$$\phi^i(x) \phi^j(y) \Omega = \sum' \int dz \phi^k(z) \mathcal{B}^{kij}(z, x, y) \quad (1.3)$$

\mathcal{B}^{kij} are generalized c-number functions. Summation is over nonderivative fields ϕ^k only and integration is over Minkowski space. Convergence is strong convergence in Hilbert space after smearing with test functions $f(x, y)$.

The result is valid for nonderivative fields ϕ^i, ϕ^j of any dimensions d_i, d_j transforming according to arbitrary finite dimensional irreducible representations ℓ_i, ℓ_j of the Lorentz group $M \cong SL(2C)$. Multispinor indices have been suppressed.

The functions \mathcal{B}^{kij} are to a large extent determined by conformal symmetry. Let $U \cong SU(2)$ the rotation subgroup

of M and denote by \tilde{M}, \tilde{U} the sets of all finite dimensional irreducible representations of M resp. U . Write $X_i = [\ell_i, d_i]$ etc. We shall show that functions \mathcal{B}^{kij} are linear combinations of a finite number of kinematically determined kernels $\mathcal{B}^{kij}(x_i, x_k, y_j)$. Given X_i, X_j and X_k they are labelled by

$$\ell \in \tilde{M}, s \in \tilde{U} \text{ such that } s \subset \ell \text{ and } \ell \subset \ell_i \otimes \ell_j, s \subset \ell_k \quad (1.3)$$

\otimes stands for the Kronecker product, and \subset means "is contained in". If no pair (s, ℓ) satisfying (1.3) exists, then ϕ^k cannot appear in the operator product expansion of $\phi^i \phi^j$.

Example: ϕ^i, ϕ^j scalar. Then $\ell_i = \ell_j = \text{id.}$, the trivial 1-dimensional representation. So $\ell = \text{id.}$, $s = \text{id.}$ and ℓ_k must be a completely symmetric tensor representation; \mathcal{B}^{kij} is then unique up to normalization.

The proof of the theorem has two ingredients

1. The Hilbert space of physical states carries a unitary representation \mathcal{U} of the conformal group G^* = universal covering of $SO(4, 2)$. It was shown by Lüscher and the author that this is true even if one only assumes weak conformal invariance, i.e. invariance of Euclidean Green functions under $SO_e(5, 1)$ or its 2-fold spin covering [5].
2. All unitary irreducible representations of G^* with positive energy are finite component field representations in the terminology of [6]. This result was proven by the author in [7].

Using these facts one can derive partial wave expansions on G^* , i.e. decompose $\int dx dy f(x, y) \phi^i(x) \phi^j(y) \Omega$ into states which transform irreducibly. Because of the Plancherel theorem, partial wave expansions are strongly convergent.

* For massless free fields ϕ^k there are further restrictions beyond this, cp. end of Sec.7.

They are here at the same time asymptotic expansions. Comparing with (1.1a) one finds that they can be rewritten in the form (1.2).

An independent proof of the theorem for theories in 2 space time dimensions was given by Lüscher [8]. He uses different methods employing a semigroup. Interesting further results on 2-dimensional models were obtained by Rühl and Yunn [9].

We conjectured in [10] that the assertion of theorem 1 would also hold true in realistic theories with mass and without conformal symmetry.

Let us mention that one can also give a dynamical derivation of the vacuum expansions (1.1) themselves in conformal invariant quantum field theory (QFT). This is discussed elsewhere [11]. It is not, however, a derivation from QFT axioms and conformal symmetry alone: One also needs Lagrangean integral equations to identify composite fields, and meromorphy of Euclidean conformal partial waves in dimension must be assumed to get a discrete expansion in the first place.

Finally, the following corollaries of theorem 1 may be of interest.

Let P^μ , K^μ the generators of translations and special conformal transformations, respectively, and

$$H = \frac{1}{2} (P^0 + K^0) \quad \text{the "conformal Hamiltonian"}$$

Assume that the hypothesis of theorem 1 hold for arbitrary products of fields ϕ^i, ϕ_j . Let f test functions and smeared fields.

Then we have

Corollary 2. The Hilbert space \mathcal{H} of physical states

is spanned by states of the form $\phi^k(f)_\Omega$, $\phi^k(f)$ smeared fields, $\Omega = \text{vacuum}$.

Corollary 3. The conformal Hamiltonian H has a purely discrete spectrum with eigenvalues $\omega = 0$ (vacuum) and

$$\omega = d_k + m, \quad m = 0, 1, 2, \dots$$

d_k dimensions of nonderivative fields in the theory.

Corollary 2 is obtained by recalling that finite products $\phi^{i_1}(f_1) \dots \phi^{i_n}(f_n)$ of fields generate a dense set of states out of the vacuum according to the principles of QFT. Then one applies theorem 1 repeatedly.

Corollary 3 follows from corollary 2 because states $\phi^k(f)_\Omega$ for given k span an irreducible representation space of G^* , with spectrum of H determined in ref. [5] to be of the form $\omega = d_k + m$, $d_k = \dim \phi^k$. Because only a discrete number of fields appears in the operator product expansions by hypothesis, the corollary follows.

2. Harmonic analysis

We wish to decompose $\phi^i(x) \phi^j(y) \Omega$ into states which transform irreducibly under G^* . It will suffice to consider scalar products $(\Psi, \phi^i(x) \phi^j(y) \Omega)$ with states Ψ in the dense domain \mathcal{D} .

For simplicity of writing consider first a theory of one hermitean scalar field $\phi(x)$, and $\phi^i \cdot \phi^j = \phi$. The Wightman functions are

$$W(x_1 \dots x_n) = (\Omega, \phi(x_1) \dots \phi(x_n) \Omega) \quad (2.1)$$

Let $\mathcal{F} = \sum_n \mathcal{J}_n$ the space of finite sequences of Schwartz test functions $f_0, f_1(x_1) \dots f_n(x_1 \dots x_n)$. The subspace \mathcal{J}_2 consists of a sequence with only one nonvanishing term $f_2(x_1, x_2)$. The field theoretic domain \mathcal{D} consists of vectors

$$\Psi(f) = \sum_n \int dx_1 \dots dx_n f_n(x_1 \dots x_n) \phi(x_1) \dots \phi(x_n) \Omega ; \quad f \in \mathcal{F}. \quad (2.2)$$

According to the reconstruction theorem, the dense domain \mathcal{D} in the Hilbert space \mathcal{H} of physical states may be identified with a space of continuous linear functionals $F: \mathcal{F} \rightarrow \langle F, f \rangle$ on \mathcal{F} , i.e. sequences $F = (F_n)_{n=0,1,\dots}$ of generalized functions $F_n \in \mathcal{J}'_n$. We shall write $F(x_1 \dots x_n)$ in place of $F_n(x_1 \dots x_n)$ and use functional notation,

$$\langle F, f \rangle = \sum_n \int dx_1 \dots dx_n F(x_1 \dots x_n) f_n(x_1 \dots x_n)$$

The identification is such that

$$f \in \mathcal{D} \text{ if and only if } F = Wf \text{ for an } f \text{ in } \mathcal{F}, \text{ with} \\ W(x_1 \dots x_n) = \sum_n \int dy_1 \dots dy_n f(y_1 \dots y_n) \bar{W}(y_1 \dots y_n \dots x_n). \quad (2.3a)$$

The scalar product on \mathcal{D} becomes

$$\langle \Psi(f_1), \Psi(f_2) \rangle = \langle Wf_1, f_2 \rangle \quad (2.3b)$$

Since the Hilbert space \mathcal{H} carries a unitary representation U of G^* it can be decomposed

$$\mathcal{H} = \int d\mu(\chi) \mathcal{H}^\chi \quad (2.4a)$$

μ a measure on the set $\hat{G}^* = \{\chi\}$ of all unitary irreducible representations ($U|\chi\rangle$) of G^* . \mathcal{H}^χ consists of a direct sum of irreducible representation spaces which carry equivalent UIR's X . In particular, states Wf in \mathcal{D} may be so decomposed,

$$Wf = \int d\mu(\chi) f^\chi \quad , \quad f^\chi \text{ in } \mathcal{H}^\chi \quad (2.4b)$$

Since an irreducible representation space of G^* must be contained in \mathcal{H} as a whole, the spectrum condition allows only UIR's with positive energy. All such have been classified in [7]. First there is of course the trivial 1-dimensional representation. The others can be labelled by $\chi = [\ell, \delta]$, $\ell \in \mathcal{M}$ a finite dimensional irreducible representation of $M = SL(2\mathbb{C})$ ("Lorentz spin") and $\delta > \delta_{\min}(\ell)$ real ("dimension"), cp. proposition 6 below. The UIR X may be realized in a space \mathcal{F}_X of (generalized) functions on Minkowski space with values in the finite dimensional representation space $V\ell$ of the Lorentz group M . Functions φ in \mathcal{F}_X satisfy a spectrum condition, i.e. their Fourier transform is supported in $\text{sptr.}(\chi) \subseteq \bar{V}_+$, the closed forward cone. The action of $\Psi(g)$ of $g \in G^*$ on functions φ in \mathcal{F}_X will be reviewed later on.

Consider isometric intertwining maps "

$$\mathcal{B}^X : \mathcal{F}_X \rightarrow \mathcal{F}_X \quad \text{such that } T_X^*(q) \mathcal{B}^X = \mathcal{B}^X U(q) \quad \text{for } g \text{ in } G^*. \quad (2.5)$$

i.e. \mathcal{B}^X preserves the norm and commutes with the action of the group. Every vector f_X in \mathcal{H}^X may be written in the form $f_X = \mathcal{B}^X \varphi_X$ where $\varphi_X \in \mathcal{F}_X$ is an intertwining map as were just introduced, and $\varphi_X \in \mathcal{F}_X$. Both \mathcal{B}^X and φ_X are uniquely determined by f_X . Both

We may assume that $(\psi(\mathbf{r}), \Omega) = 0$. The trivial 1-dimensional representation of G^* will then not appear in the decomposition, because the vacuum Ω is the only Lorentz invariant state.

The decomposition (2.4b) becomes

$$Wf = \int d\mu(x) \mathcal{B}^X \varphi_X \quad \text{with} \quad \varphi_X \in \mathcal{F}_X$$

We restrict Wf to a continuous linear functional $Wf(x, x_1)$ on \mathcal{F}_X . Since \mathcal{F}_X is a function space there are kernels $\mathcal{B}(x, x_1, x_2)$ associated with maps \mathcal{B}^X such that

$$(\Psi(f), \phi(x_1) \phi(x_2) \Omega) = Wf(x, x_1) - \int dx \varphi(x) \mathcal{B}(x, x_1, x_2) \quad (2.6)$$

The kernels are singular functions with values in V^k ; we write $v_1^* v_2$ for the scalar product of two vectors in V^k . Often, physicists write indices $\overline{a}^k b_a$ (sum over a).

Since functions φ in \mathcal{F}_X satisfy a spectrum condition, kernels $\mathcal{B}(x, x; \cdot)$ are nonunique as functions of x . In particular, the Fourier transform

$$\tilde{\mathcal{B}}(p_X; x, x_1) = \int dx e^{ip_X x} \mathcal{B}(x, x_1, x_2) \quad (2.7)$$

is only relevant for $p \in \text{sptr.}(\mathcal{F}_X) \subseteq \widehat{V}$.

We shall count kernels $\mathcal{B}(x, x; \cdot)$ only as distinct if $\tilde{\varphi}(p) \tilde{\mathcal{B}}(p_X; x, x_1)$ differ for some φ in \mathcal{F}_X .

The intertwining property (2.5) imposes strong covariance conditions on kernels B . Further restrictions come from the spectrum condition for states $\phi(x_i)\Omega$. We write $y_i \gamma_i$ if $y_i, \gamma_i \in V_+$. Spectrum condition and covariance imply

Proposition 4. Let $\tilde{\varphi}(p)$ the Fourier transform of an arbitrary element of \mathcal{F}_X . Then $\tilde{\varphi}(p)^* \tilde{\mathcal{B}}(p_X; x_1, x_2)$ is boundary value of a holomorphic function of $z_j = x_j + iy_j$ ($j=1, 2$) in the tube $y_1 > y_2 > 0$. Kernels $\tilde{\mathcal{B}}(p_X; x_1, x_2)$ are linear combinations of a finite number of kinematically determined functions $\tilde{\mathcal{B}}^k(p_X; x_1, x_2)$ (at most one for scalar $\phi = \phi^i = \phi^j$) which can be labelled as in (1.3). Moreover, they can be analytically continued in p to entire analytic functions of p .

Proof of proposition 4 will be given in the following sections by explicit expressions for $\tilde{\mathcal{B}}^k$ will be given in Sect. 8.

Let us return to expansion (2.6). It remains to be shown that

- i) the measure $\mu(x)$ is discrete so that
- ($\Psi(f), \phi(x_1) \phi(x_2) \Omega) = \sum_k \int dx \varphi^{x_k}(x) * \mathcal{B}(x, x_1, x_2)$
- and ii),

$$\varphi^{x_k}(x)^* = (\Psi(f), \phi^k(x) \Omega)$$

where ϕ^k is a nondervative field appearing in the Wilson expansion (1.1) with dimension d_k and Lorentz spin ℓ_k if $x_k = [\ell_k, d_k]$. Later on we write \mathcal{B}^k for $\mathcal{B}(\cdot, x_k, \cdot)$

We shall use the hypothesis that Wilson expansion (1.1) is valid as an asymptotic expansion at $x = 0$. We will derive from (2.6) an asymptotic expansion at $x = 0$ in homogeneous functions of x . As asymptotic expansions in homogeneous functions are unique, (2.8) can then be deduced by comparison.

By proposition 4, kernels $\tilde{\mathcal{B}}(p_X; x, x_1)$ are entire functions of p . They may therefore be expanded in an everywhere convergent power series

$$\tilde{\mathcal{B}}(p_X; x, x_1) = \sum_{r=0}^{\infty} C^{Xr}_{\beta} (x) p_{\beta_1} \dots p_{\beta_r}; \quad \beta = (\beta_1, \dots, \beta_r) \quad (2.9)$$

For reasons of dilatational invariance one has for real $\lambda > 0$,

$$\begin{aligned} C_{\beta}^{X_r}(\lambda x) &= \lambda^{-2d+\delta+r} C_{\beta}^{X_r}(x) & \text{for } X = [\ell, \delta], \quad d = \dim \phi \\ (\text{or, more generally}) \quad 2d &\equiv d_i + d_j \quad , \quad d_{i,j} = \dim \phi^{i,j}. \end{aligned} \quad (2.10)$$

We will insert power series expansion (2.9) in (2.6).

$$(Y(f), \phi(\frac{1}{t}x)\phi(-\frac{1}{t}x)\Omega) = \int d\mu(x) \int dp \sum_{\beta} C_{\beta}^{X_r}(x) p_{\beta_1} \dots p_{\beta_r} \tilde{\varphi}^X(p)^* \quad (2.11)$$

Suppose that in (2.6) f is a sequence of test functions whose Fourier transforms have compact support. Vectors $Y(f)$ with such f are still dense in the Hilbert space \mathcal{K} of physical states. Because of momentum conservation, the Fourier transforms $\tilde{\varphi}^X(p)$ of $\varphi^X(x)$ will then also have compact support, and so $\varphi^X(x)$ are infinitely differentiable at $x=0$ (even entire in x). Because of homogeneity (2.10), expansion (2.11) implies the following asymptotic expansion

$$\begin{aligned} (Y(f), \phi(\frac{1}{t}x)\phi(-\frac{1}{t}x)\Omega) &= \sum_{\beta} \int d\mu(x) \int dp C_{\beta}^{X_r}(x) p_{\beta_1} \dots p_{\beta_r} \tilde{\varphi}^X(p)^* \\ &= \sum_{\beta} \int d\mu(x) i^r C_{\beta}^{X_r}(x) \nabla_{\beta_1} \dots \nabla_{\beta_r} \varphi^X(0)^* \end{aligned} \quad (2.11')$$

where it is understood that summation and integrations $\sum_{\beta} \int d\mu(x)$ are rearranged in order of increasing $\delta+r$. ($X=[\ell, \delta]$)

Expansion (2.11') can reproduce Wilson expansion (1.1) only if (2.8) holds true. It follows from (2.6) and (2.8) that expansion (1.2) is true on a dense set of vectors $Y(f)$ as described before (2.11'). Being a partial wave expansion it is then generally true and strongly convergent as stated in our theorem.

At the same time we see from (2.9), (2.11) how the coefficients in the Wilson expansion (1.1) are obtained by power series expansion from the kernels $\tilde{\mathcal{B}}(p_X; x_1, x_2)$.

We have written our formulae for a scalar field $\phi = \phi^i \cdot \phi^j$. They will however remain true generally if interpreted correctly, i.e. with appropriate indices supplied. The analysis of the kernels $\tilde{\mathcal{B}}$ will be done in full generality in the following sections.

It only remains to prove proposition 4. The sequel of this paper will be devoted to this problem. At the same time, we will obtain explicit expressions for the kernels $\tilde{\mathcal{B}}(p_X; x_1, x_2)$, cp. Sec. 8.

One could try to determine the kernels $\tilde{\mathcal{B}}$ by imposing infinitesimal conformal invariance. In fact this program was already carried out by Ferrara et al [1] for the scalar case even before global conformal invariance was understood, and proposition 4 is implicit in their work for this case. For general spin the infinitesimal method becomes too complicated. We shall therefore resort to global methods which are more powerful. [In applications one wants to apply theorem 1 repeatedly (as e.g. in corollary 2) and fields of arbitrary Lorentz spin may then appear.]

3. The conformal group G^*

The group G^* is an infinite sheeted covering of $SO_e(4,2)$. Its geometry was examined in [7]. The following picture emerges.

G^* contains the quantum mechanical (q.m.) Lorentz group $M \cong SL(2\mathbb{C})$ and therefore also its two-element-center Γ_1 , whose representation distinguishes between bosons and fermions. Γ_1 is also contained in the center of G^* but does not exhaust it. The group G^*/Γ_1 may be pictured as a group of transformations of superworld \tilde{M} . That is, G^* can act on \tilde{M} , but the action of Γ_1 is trivial. Points η of \tilde{M} may be parametrized

$$\eta = (\tau, \epsilon), \quad -\infty < \tau < \infty; \quad \epsilon = (\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4) \text{ a unit 4-vector} \quad (3.1)$$

viz. $(\epsilon^1)^2 + (\epsilon^2)^2 + (\epsilon^3)^2 + (\epsilon^4)^2 = 1$.

The action of G^* on \tilde{M} is specified by the action of various subgroups.

A subgroup \tilde{K} of G^* acts on \tilde{M} by rotations of ϵ and translations b_σ of τ to $\sigma + \tau$. $\tilde{K} \cong \mathbb{R} \times SU(2) \times SU(2)$, also

$\tilde{m}_\alpha \gamma = \tilde{m}_\alpha$, γ being in the center of G^* . It extends by continuity to all of $\tilde{M} = \overline{\cup \gamma M^4}$ (union over Γ_1).

The action of G^* on \tilde{M} is completely specified by the action of its subgroups K, \bar{N}, M and Λ , for every g in G^* may be written in the form

$$g = k m a \bar{n}, \quad k \in \tilde{K} \quad \text{etc.} \quad (3.2)$$

This decomposition is nonunique. Let $u = \tilde{K} \cap M \cong SU(2)$ the rotation subgroup of M . Then $k m a \bar{n} = k' m' a' \bar{n}$, if and

compact subgroup of G^*/Γ_1 .

A fundamental domain F in \tilde{M} with respect to the discrete subgroup Γ_1 is a submanifold such that

$$F \cap \gamma F = \emptyset \quad \text{for } \epsilon + y \in \Gamma_1, \quad \forall y \in \tilde{M}$$

A fundamental domain $F = M_c^4$ may be chosen as

$$M_c^4 = \{(\tau, \epsilon) \in \tilde{M}; -\pi < \tau < \pi, \epsilon^5 \geq -\cos \tau\} \quad (3.3a)$$

Its interior may be identified with Minkowski space $M^4 = \{x^\mu\}$ through the reparametrization

$$x^\mu = \frac{\sin \tau}{\cos \tau + \epsilon^5} \quad i \quad x^i = \frac{\epsilon^i}{\cos \tau + \epsilon^5} \quad (i=1,2,3) \quad (3.3b)$$

Translations \bar{n} in \bar{N} , Lorentz transformations m in M and dilatations a in A act in the customary way on points of $M^4 \subset M_c^4$ parametrized by x^μ (see below). Their action on translates γM_c^4 of M^4 is then also determined because $\tilde{m}_\alpha \gamma = \tilde{m}_\alpha$, γ being in the center of G^* . It extends by continuity to all of $\tilde{M} = \overline{\cup \gamma M^4}$ (union over Γ_1).

$$g = k m a \bar{n} \quad \text{etc.} \quad (3.4)$$

only if $\mathbf{u}' = \mathbf{u}\mathbf{x}$, $m' = u^{-1}m$ with \mathbf{u} in \mathbf{U} , and $\alpha = \alpha'$, $\bar{n} = \bar{n}'$.

Let $N = \mathbf{R}\bar{N}\mathbf{R}^{-1}$, it is called subgroup of special conformal transformations. The point $\gamma_0 = (0, \varepsilon)$, $\varepsilon = (0, 1)$ is left invariant by MAN , and \tilde{M} is a homogeneous space $\tilde{M} \approx G^*/MAN$. MAN is isomorphic to a Poincaré group since

$$\mathbf{R}m\mathbf{R}^{-1} = \tilde{m} \in M, \quad \mathbf{R}\alpha\mathbf{R}^{-1} = \alpha^{-1} \quad \text{for } m\alpha \in MA. \quad (3.4)$$

The fundamental domain M_c^4 may also be made into a homogeneous space $M_c^4 \approx \bar{M}/\Gamma_2 \approx G^*/P$ with $P = \Gamma_2 MAN$. The action of subgroups Γ_1 , N , M , A , N on cosets $\mathbf{x} = \bar{n}_x P \in M^4$ is the usual one: The center $r = \Gamma_2 \cap \Gamma_1$ acts trivially, and *

M : Lorentz transformation: $\mathbf{x}^\mu \mapsto \Lambda(m)\mathbf{x}^\nu$, $x^\nu \equiv (mx)^\nu$

A : dilatations: $\mathbf{x}^\mu \mapsto |\alpha| \mathbf{x}^\mu$, $|\alpha| > 0$.

\bar{N} : translations \bar{n}_y : $\mathbf{x}^\mu \mapsto \mathbf{x}^\mu + \mathbf{y}^\mu$, \mathbf{y}^μ real.

N : spec. conf. transf. n_y : $\mathbf{x}^\mu \mapsto \sigma(x, y)^{-1} (x^\mu - y^\mu x^\nu)$
 y^μ real, $\sigma(x, y) = 1 - 2xy + x^\lambda y^\lambda$.

R : $\mathbf{x}^\mu \mapsto \frac{\theta \mathbf{x}^\mu}{x^2}$, θ -time reflection

If $\bar{n}_y \in \bar{N}$ then $\mathbf{R}\bar{n}_y\mathbf{R}^{-1} = n_y \in N$.

Elements $m \in M \approx SL(2\mathbb{C})$ may be identified with unimodular two by

* Our metric is $g_{\mu\nu} = \text{diag. (+ ---)}$; $x \cdot y = g_{\mu\nu} x^\mu y^\nu$, $x^2 = x \cdot x$ etc.

two matrices; $\Lambda(m)$ is then given by the fundamental formula of spinor calculus. Let $\mathbf{x} = x^0 \mathbf{A} + \mathbb{L}^3 \mathbf{k} \sigma^k$ then $\Lambda(m)$ is determined by m through

$$m \mathbf{x}^{m*} = \mathbf{x}' \quad \text{with } \mathbf{x}'^\mu = \Lambda(m)^\mu_\nu \mathbf{x}^\nu \quad (3.6)$$

Translations act transitively on M^4 and N^4 is almost all of $M_c^4 = G^*/P$. Therefore the set $\bar{N}P$ fills up all but a lower dimensional submanifold of G^* . Elements in $\bar{N}P$ will be called regular. Every regular element g of G^* may be written in a unique way as

$$g = \bar{n}g_{MAN} \quad \text{with } \bar{n} \in \bar{N}, \quad g \in \Gamma_2, \quad m \in M, \quad \alpha \in A \quad \text{and } n \in N. \quad (3.7)$$

Haar measure of G^* factorizes as $dg = d\bar{n}dmdadn$ in this parametrization. In the following it is understood that restriction to regular elements of G^* is made whenever this is necessary in order that the formulae make sense.

Let \mathbf{x}' and $\mathbf{p}(x, g) \in P = \Gamma_2 \cap MAN$ determined by \mathbf{x} , g through the unique decomposition

$$g^{-1} \bar{n}_x = \bar{n}_x \mathbf{p}(x, g)^{-1} \quad \text{Then } \mathbf{x}' = g^{-1} \mathbf{x} \quad (3.8)$$

\mathbf{x}' is determined by the action (3.5) of G^* on cosets $x \in G^*/P$. From (3.8) one deduces the cocycle condition

$$\mathbf{p}(x, g_1 g_2) = \mathbf{p}(x, g_1) \mathbf{p}(g_1^{-1} x, g_2) \quad (3.9)$$

Special cases: For $\bar{n}_y m \in \bar{N} \cap MA$ one has

$$\mathbf{p}(x, \bar{n}_y m) = \bar{n}_y m \quad \text{independent of } x \quad (3.10)$$

The next lemma gives an explicit expression for $p(x, g)$.

Lemma 5. Let $p(x, g)$ as defined in (3.8), and $\gamma \in G^*$ the reciprocal radius transformation defined after (3.2). Then $p(x, \gamma)$ is MA-covariant in the sense that

$$\tilde{b}p(x, \gamma) = p(bx, \gamma)\tilde{b} \quad \text{for } b \in M_A, \quad \tilde{b} = \gamma b\gamma^{-1} \in M_A$$

It is explicitly given by

$$p(x, \gamma) = \gamma m_x a_x n_x \quad \text{with } y = \tilde{y}^N, \quad N = \text{Sign } x, \quad (3.14)$$

$$|a_x| = |x^1|, \quad m_x = i^{N-1} x^{-1} \tilde{x}, \quad z^{\mu} = -x^{\mu}/x^1.$$

Herein $\tilde{x} = x^0 \tilde{x} + \sum x^k \sigma^k$, and $\text{Sign } x = \pm 1$ for $x \in V_{\pm}$ and 0 otherwise.

The quantity $p(x, g)$ for general regular $g \in G^*$ is expressible in terms of $p(x, \gamma)$. Write $g = \bar{n} \gamma m$, $\bar{n} \in \bar{N}$, $\gamma \in \Gamma_2$ etc. Then

$$p(x, g) = p(x, \gamma)(\gamma m)^{-1} \quad \text{and} \quad p(x, n') = p(x, \gamma)p(\bar{n}n, \gamma).$$

The quantity $p(x, \gamma)$ was computed in [7]. The other assertions of Lemma 5 follow from the cocycle condition (3.9) and (3.10), noting that $\bar{n}n_x \gamma = \bar{n}_x$, $b\bar{n}_x = \bar{n}_x b$ for $b \in M_A$. In particular, it follows from the last relation that $\tilde{b}\bar{n}_x \times \gamma \bar{n}_bx = b$, and so by definition (3.8),

$$\tilde{b}\bar{n}_x p(x, \gamma)' = \bar{n}_x b\bar{n}_x p(x, \gamma)^{-1} = \bar{n}_x b\bar{n}_x p(bx, \gamma)^{-1} b. \quad \text{This shows MA-covariance of } p(x, \gamma). \quad \square$$

Having completed the outline of the group G^* 's geometry, we now turn to its unitary irreducible representations with positive energy.

Let $\lambda = [\ell, \delta]$, δ real and $\ell \in \tilde{M}$ a finite dimensional irreducible representation of M by matrices $\mathcal{D}^{\ell}(m)$ in a vector space V^{ℓ} . We equip V^{ℓ} with a scalar product, written u^*v , of vectors u, v in V^{ℓ} which is such that

$$\mathcal{D}^{\ell}(\tilde{m})^{-1} = \mathcal{D}^{\ell}(m)^* \quad \text{for } \tilde{m} = \tilde{R}m\tilde{R}' = \theta m\theta^{-1}, \quad m \in M_A.$$

We define a finite dimensional representation of $P = \Gamma_2 \text{MAN}$ in V^{ℓ} by

$$\mathcal{D}^{\lambda}(\gamma m) = |\alpha|^c e^{i\text{tr} \tilde{c} \cdot \mathcal{D}^{\ell}(m)} \quad \text{with } c = \delta - 2, \quad \text{for } \gamma \in \tilde{Y}^N \quad (3.12)$$

As usual, $\alpha \in A$ is dilatation by $|\alpha|$, cp.(3.5), etc.

Let \mathcal{E}_X the space of infinitely differentiable functions on G^* with values in V^{ℓ} and having covariance property

$$f(gP) = |\alpha|^2 \mathcal{D}^{\lambda}(P)^* f(g) \quad \text{for } P = \gamma m \in \Gamma_2 \text{MAN}. \quad (3.13)$$

\mathcal{E}_X becomes a representation space for G^* by imposing the transformation law

$$(\tau_X(g)f)(g') = f(g'g) \quad (3.14)$$

Because of covariance property (3.13) and decomposition (3.7) of group elements, functions f in \mathcal{E}_X are uniquely specified by their restriction $f(x) = f(\bar{n}_x)$ to \bar{N} . Transformation law (3.14) becomes in this language

$$(\tau_X(g)f)(x) = |\alpha|^2 \mathcal{D}^{\lambda}(P(x, g)) f(g^{-1}x) \quad (3.15)$$

with $|a|$ from $p(x, g) = \gamma m$. We are dealing with an induced representation on G^*/P . (P is called a parabolic subgroup, it is not the minimal one).

A scalar product on \mathcal{E}_X is constructed with the help of an intertwining map (or operator)

$$\Delta_+^{\lambda} : \mathcal{E}_X \rightarrow \mathcal{F}_X \quad (3.16)$$

where \mathcal{F}_X is a space of generalized functions on G^* with values in V^{ℓ} having covariance property

The Fourier transform of the intertwining kernel (=conformal invariant 2-point function) (3.20) is

$$\varphi(gp) = |a|^2 \mathcal{D}^\chi(p)^* \varphi(g) \quad \text{for } g \in G^*, \quad p \in \text{mane } \Gamma_2^* M \cap$$
(3.17)

\mathcal{T}_χ is made into a representation space for G^* by the transformation law

$$(\mathcal{T}_\chi(g) \varphi)(g') = \varphi(g^{-1}g') \quad \text{for } g \in G^* \quad (3.18)$$

Generalized functions φ in \mathcal{F}_χ are determined by their restriction $\varphi(x) = \varphi(\bar{n}_x)$ to \bar{N} . The transformation law becomes

$$(\mathcal{T}_\chi(g) \varphi)(x) = |a|^2 \mathcal{D}^\chi(p(x, g)^{-1}) \varphi(g^{-1}x) \quad (3.19)$$

with notation as in (3.15). The intertwining map Δ_+^χ is required to commute with the action of the group

$$\Delta_+^\chi \mathcal{T}_\chi(g) f = \mathcal{T}_\chi(g) \Delta_+^\chi f \quad \text{for } f \in \mathcal{E}_\chi$$

It is given explicitly by

$$\begin{aligned} \varphi(x) &= (\Delta_+^\chi f)(x) = n_+(\chi) \int_{\bar{N}} d\bar{n}' f(\bar{n}_x R \bar{n}') \\ &= \int dx' \Delta_+^\chi(x - x') f(x') \quad \text{with} \end{aligned}$$

$$\Delta_+^\chi(x) = n_+(\chi) (-x^2 + i\epsilon x^0)^{-\delta} \delta_{j_1, -l_1} \mathcal{D}^\ell(ix) \quad (3.20)$$

Here (j_1, j_2) is the highest weight of the representation $\ell \in \mathcal{M}$ of M , and \mathcal{D}^ℓ is the extension to $GL(2C)$ of ℓ through $\mathcal{D}^\ell(pm) = p^{2j_1+2l_1} \mathcal{D}^\ell(m)$, $p \in \mathbb{C}$. Eqs. (3.20) were derived in [7].

$\tilde{\Delta}_+^\chi(p) = n'_+(\chi) \mathcal{D}^\ell(-\frac{2}{\partial p}) \theta(p) (p^2)^{-2+\delta_{j_1, l_1}} \quad (3.20')$

with a new normalization factor $n'_+(\chi)$. $\tilde{\Delta}_+^\chi(p)$ vanishes for momenta p outside the closed forward cone. The massless scalar 2-point function is obtained as a limit, $j_1=j_2=0$, $\delta \rightarrow 1$, viz.

$$r(c+1)^{-1} \theta(p) (p^2)^c \rightarrow \theta(p) \delta(p^2) \text{ as } c \rightarrow -1.$$

In ref. [7] a complete classification of all UIR's of G^* with positive energy was given. The result will be quoted as our

Proposition 6. • The UIR's of G^* with positive energy can be labelled by $\chi = [\ell, \delta]$, ℓ a finite dimensional irreducible representation of $M \simeq SL(2C)$ and $\delta \geq \delta_{\min}(\ell)$ real. If (j_1, j_2) is the highest weight of ℓ (viz. $2j_1, 2j_2$ nonnegative integers) then $\delta_{\min}(\ell) = j_1 + j_2 + 2$ if $j_1 \neq 0, j_2 \neq 0$, and $\delta_{\min}(\ell) = j_1 + j_2 + 1$ otherwise, except for the trivial 1-dimensional representation which has $\delta = j_1 + j_2 = 0$. The nontrivial UIR's χ can be realized in the representation spaces \mathcal{E}_χ equipped with scalar product

$$(f_1, f_2) = \int dx_1 dx_2 \bar{f}_1(x_1) \cdot \Delta_+^\chi(x_1, x_2) f_2(x_2)$$

with intertwining kernel (3.20).

Representations with $j_1=0$ or $j_2=0$ and $\delta = \delta_{\min}$ are zero mass representations, the others have continuous mass spectrum, $spt_r(\chi) = \bar{V}_+$.

Remark: An equivalent UIR, χ' is realized in the space $\mathcal{F}_{\chi'} = \Delta_+^\chi \mathcal{E}_\chi$. If $\varphi_1 = \Delta_+^\chi f_1$, $f_1 \in \mathcal{E}_\chi$ the scalar product $(\varphi_1, \varphi_2) = \int dx f_1(x)^* \varphi_2(x)$. Generalized functions φ in $\mathcal{F}_{\chi'}$ satisfy a spectrum condition since the intertwining kernel $\tilde{\Delta}_+^\chi(p)$ does, cp. (3.20').

In the following, we shall often not distinguish in notation between the test function space \mathcal{E}_χ and the Hilbert space constructed from it. If we use functional notation for the elements of this Hilbert space, it is always understood that an arbitrary representative out of the equivalence class of functions modulo zero norm vectors is to be chosen.

4. Implications of the spectrum condition.

Let us use the intertwining map Δ_+^X to introduce

$$V(x\chi; x_1, x_2) = \int dx' \Delta_+^X(x-x')^* \mathcal{B}(x'\chi; x_1, x_2) \quad (4.1)$$

Because UIR's of G^* acting in \mathcal{T}_X and \mathcal{E}_X are equivalent and intertwined by Δ_+^X , the conformal partial wave expansion (2.6) may be rewritten in the equivalent form

$$\langle \Psi(f), \phi(x_1) \phi(x_2) \rangle = \int d\mu(x) \int dx \xi^X(x)^* V(x\chi; x_1, x_2) \\ \text{with } \xi^X \in \mathcal{E}_X \quad (4.2)$$

The kernel B is determined by V to within the arbitrariness discussed in Sec. 2. We shall first determine V and then recover B from it by solving (4.1).

Let us first state implications of the spectrum condition for V .

Lemma 7. The kernels $V(x\chi; x_1, x_2)$ are limits of generalized functions $V(x\chi; z_1, z_2)$ of x which are holomorphic in the complex parameters $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ in the tube $y_2 > y_1 > 0$. The limit is taken by letting $y_1, y_2 \rightarrow 0$ through the tube.

Proof: Consider expansion (4.2). It is well known that $\phi(x_1) \phi(x_2)$ is boundary value of states $\Psi(z_1, z_2) \in \mathcal{H}$ which are holomorphic in $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ in the tube. (This result is reviewed in [5]). Therefore, by decomposing $\Psi(f)$ we obtain an expansion

$$\langle \Psi(f), \Psi(z, z_1) \rangle = \int d\mu(x) \int dx \xi^X(x)^* V(x\chi; z, z_1)$$

in which $\int dx \xi^X(x)^* V(x\chi; z, z_1)$ is a holomorphic function of z_1 , z_2 in the tube and has $\int dx \xi^X(x)^* V(x\chi; x_1, x_2)$ as a limit. If $\xi_0 \in \mathcal{E}_X$ is an arbitrary vector, then, because \mathcal{E}_X carries an irreducible unitary representation, vectors of the form $\int dg f(g) T(g) \xi_0$ with f an infinitely differentiable function with compact support on G^* form a dense set of vectors in the UIR-Hilbert space \mathcal{E}_X . Therefore $\xi^X(x)$ may be considered as an arbitrary element of \mathcal{E}_X . But \mathcal{E}_X contains all Schwartz test functions with values in V_+ ; therefore $V(x\chi; z_1, z_2)$ is a generalized function of x and has the indicated holomorphy property. \square

It follows from lemma 7 that it will suffice to determine $V(x\chi; x_1, x_2)$ for relatively spacelike points x_1, x_2 on Minkowski space.

Lemma 7 cannot be carried over without further ado to $B(x\chi; x_1, x_2)$ because this kernel is nonunique as a function of x .

However, it does imply the first assertion of proposition 4 because every element of \mathcal{T}_X is of the form $\varphi = \Delta_+^X \xi$ with $\xi \in \mathcal{E}_X$.

5. Relatively spacelike pairs of points

Our further analysis is based on the fact that the conformal group G^* acts transitively on pairs of relatively spacelike points on superworld \tilde{M} . This will now be explained.

The manifold \tilde{M} admits a G^* -invariant causal ordering [5].

Two points $\eta_i = (\tau_i, \xi_i)$ and $\eta_s = (\tau_s, \xi_s)$

are relatively spacelike if and only if

$$|\tau_i - \tau_s| < \arccos \xi_i \cdot \xi_s \quad (5.1)$$

$\arccos x$ is the principal value of $\arccos x$ which lies between $0 \dots \pi$.

Lemma 8. a) G^* acts transitively on relatively spacelike pairs of points on \tilde{M} .

b) η_i, η_s in \tilde{M} are relatively spacelike if and only if there exists $k \in G^*$ such that $k\eta_i, k\eta_s$ are relatively spacelike points on Minkowski space

$$M^4 \subset \tilde{M} \quad (\text{cp. Eqs. (3.3)}).$$

c) The little group* in G^* of a pair of relatively spacelike points on \tilde{M} is isomorphic to MA. The manifold of relatively spacelike pairs of points on \tilde{M} may therefore be identified with the homogeneous space G^*/MA .

Proof: Let η_0 the origin of $M^4 \subset \tilde{M}$ and $\eta_\infty = \infty \eta_0$. We call η_∞ the unique point at spatial infinity of Minkowski space. Explicitly $\eta_\infty = (0, \xi), \xi = (\underline{\omega}, \omega^{-1})$

* little group = subgroup of stability

The little group of η_∞ is MAN and the little group of η_∞ therefore $MAN = RMANR^{-1}$.

a) Let (η_i, η_s) relatively spacelike. Since G^* acts transitively on \tilde{M} there is g such that $\eta_i = g\eta_\infty$. By G^* -invariance of causal ordering, $\eta'_i = g^{-1}\eta_i$ is then relatively spacelike to η_∞ . By (5.1) and (3.3a) this means that η'_i must belong to Minkowski space M^4 . The little group MAN of η_∞ acts transitively on M^4 . There is therefore P in MAN such that $\eta'_i = P\eta_0$. Since

P leaves η_∞ invariant we have then $(\eta_i, \eta_s) = (P\eta_0, gP\eta_0)$.

Since every pair of relatively spacelike points may be written in this way, with $gP \in G^*$, we have proven transitivity.

b) The if part follows from G^* -invariance of the causal ordering. Conversely, choose η_3 in M^4 and relatively spacelike to η_0 . By transitivity a) there is g in G^* such that $(\eta_0, \eta_3) = (g\eta_0, g\eta_3)$. Decompose

$g = kMAN$ as in (3.4). Then $(\eta_4, \eta_5) = (MAN\eta_0, MAN\eta_3)$ are relatively spacelike points in M^4 since the Poincaré group carries M^4 into itself, and $(\eta_4, \eta_5) = (k\eta_4, k\eta_5)$

c) The little group of the pair (η_0, η_∞) is $MAN \cap M\bar{A} = MA$.

The assertion of c) follows from this and a). \square

6. Global transformation law.

According to the discussion in Sec. 2, physical states in the dense domain Ω may be thought of as continuous linear functionals $F = Wf$ on the test function space \mathcal{J} . They can be restricted to the subspace \mathcal{J}_2 which consists of Schwartz test functions $h(x_1, x_2)$ with support containing only relatively spacelike pairs of points on Minkowski space. These pairs may at the same time be thought of as relatively spacelike pairs of points in the fundamental domain M_c^4 in superworld \tilde{M} .

The space \mathcal{J}_2 is not globally G^* -invariant. We shall imbed it in a space \mathcal{J}^\sim of test functions on $\tilde{M} \times \tilde{M}$ with compact support containing only relatively spacelike pairs of points. The space \mathcal{J}^\sim is G^* -invariant, i.e. it admits an action

$$\Gamma(g) : \mathcal{J}^\sim \rightarrow \mathcal{J}^\sim \quad (\text{g in } G^*)$$

of the group G^* . Afterwards we will extend functionals $F = Wf$ from \mathcal{J}_2 to \mathcal{J}^\sim by a process of analytic continuation (cp. Sec. 8 of [5]). In doing so a physical state $\Psi(\xi)$ is associated to every h in \mathcal{J}^\sim . The global G^* -transformation law of these special states can be stated explicitly, so that we may thereafter deal with an explicitly known action of G^* in a concrete function space in place of an abstract unitary representation of G^* in an abstract Hilbert space of physical states.

Let us deal with general spin right away. Let $X_i = [\ell_i, d_i]$ and $X_j = [\ell_j, d_j]$ specified by Lorentz transformation law and dimension of the fields ϕ^i and ϕ^j whose operator

product (1.1a) we want to expand. We denote by v^i, v^j the finite dimensional vector spaces in which act the representations ℓ_i and ℓ_j of M . It is understood that they are equipped with a scalar product which is such that $D^I(\theta_m \theta)^{-1} = D^I(m)^*$.

The space \mathcal{J}_2^\sim consists of test functions $h(x_1, x_2)$ with values in the tensor product $v^i \otimes v^j$ and

$$\Psi(h) = \int dx_1 dx_2 h_{\alpha\beta} \phi_\alpha^i(x_1) \phi_\beta^j(x_2) Q \quad \text{for } h \in \mathcal{J}_2^\sim. \quad (6.1)$$

Indices α, β label an orthonormal basis in V^i resp. V^j ; summation over repeated indices α, β is understood.

Let $P^o = \text{MAN}$ so that superworld $\tilde{M} = G^* / P^o$.

Let us restrict the representations D_X (3.12) of $P = P^o \cap P_2$ to P^o . Consider the finite dimensional representation π of $P^o \times P^o$ in $v^i \otimes v^j$ by matrices

$$\pi(p_1, p_2) = [D^{X_i}(p_1)]^* \otimes [D^{X_j}(p_2)]^* \delta_p(p_1)^{\frac{1}{2}} \delta_p(p_2)^{\frac{1}{2}} \quad (6.2)$$

where $\delta_p(\text{man}) = |\alpha|^4$,

$P_1, P_2 \in P^o$. The space \mathcal{J}^\sim consists of all infinitely differentiable cross sections on the homogeneous vector bundle $E = (\tilde{M} \times \tilde{M}) \times_{\pi} (V^i \otimes V^j)$ with compact support containing in its interior only relatively spacelike pairs of points on \tilde{M} (notation of [3]). In other words, \mathcal{J}^\sim consists of infinitely differentiable functions on $G^* \times G^*$ with values in $v^i \otimes v^j$ having covariance property

$$h(g_1, p_1, g_2, p_2) = \pi(p_1; p_2)^{-1} h(g_1, g_2) \quad \text{for } p_i \in P^o, g_i \in G^* \quad (i=1,2) \quad (6.3)$$

The action $\pi(g)$ of G^* is

$$(\pi(g) h)(j_1, j_2) = h(g^{-1} j_1, g^{-1} j_2) \quad (6.4)$$

Evidently such functions h are completely specified if they are known for one representative (g_1, g_2) out of every coset $(\eta_1, \eta_2) \in \tilde{M} \times \tilde{M} = (G^* \times G^*) / (P^0 \times P^0)$. Therefore, if a representative of every coset is fixed in some way, cross sections h may also be considered as vector-valued functions on $\tilde{M} \times \tilde{M}$. The support of h is the closure of the (open) set of all pairs $(\eta_1, \eta_2) \in \tilde{M} \times \tilde{M}$ such that $h(g_1, g_2) \neq 0$ for $(g_1, g_2) \in (\eta_1, \eta_2)$. \tilde{J} is made up of cross sections h with support properties as stated above.

Consider the subspace of \tilde{J} which consists of cross sections which vanish outside Minkowski space $M^4 \times M^4 \subset \tilde{M} \times \tilde{M}$. It may be identified with the space \tilde{J}_2' as follows: Every $g_i \in G^*$ with $g_i P^0 \in M^4 \subset \tilde{M}$ may be written as $g_i = \bar{n}_{x_i} p_i$ with $p_i \in P^0$, \bar{n}_{x_i} = translation by x_i . Therefore by (6.3)

$$\begin{aligned} h(g_1, g_2) &= \pi(p_1, p_2)^{-1} h(\bar{n}_{x_1}, \bar{n}_{x_2}) \\ &\equiv \pi(p_1, p_2)^{-1} \hat{h}(x_1, x_2) \end{aligned} \quad (6.5)$$

for $j_1 = \bar{n}_{x_1} p_1 \in \bar{n}_{x_1} P^0 \in M^4$.

and $\hat{h}(x_1, x_2)$ is in \tilde{J}_2' and determines $h(g_1, g_2)$ everywhere on $G^* \times G^*$ so long as h is in the subspace.

Let $h \in \tilde{J}_2'$ and $\Psi(h)$ defined by (6.1).

It follows from the results of [5] that the Hilbert space of

physical states carries a unitary representation U of G^* whose action on states $\Psi(h)$ is such that

$$U(g)\Psi(h) = \Psi(\pi(g)h) \quad (6.6)$$

provided $g \in G^*$ and $h \in \tilde{J}_2'$ are such that also $\pi(g)h \in \tilde{J}_2'$, i.e. g does not carry any point in the support of h outside Minkowski space. It follows from lemma 8b and compact support of h that every h in \tilde{J} may be written as a finite sum of the form

$$h = \sum_i \pi(g_i) h_i \quad \text{with } h_i \in \tilde{J}_2' \quad \text{and } g_i \in G^* \quad (6.7)$$

We may then define

$$\Psi(h) = \sum_i U(g_i) \Psi(h_i) \quad (6.8)$$

with $\Psi(h_i)$ defined by (6.1). We must show that this is consistent, i.e. independent of the choice of g_i and h_i in (6.7). Suppose that $h = \sum \pi(g'_i) h_i$ is another decomposition of h with h_i in \tilde{J}_2' and $g'_i \in G^*$. By making finer splittings and reordering we may achieve that both sums have equally many terms, and

$$\pi(g'_i) h'_i = \pi(g_i) h_i \quad \text{for all } i. \quad (6.9)$$

The intertwining property (G^* -invariance) of v^χ reads because of (6.10)

$$v^\chi [T_\chi(g)\xi, T(g)\eta] = v^\chi [\xi, \eta] \quad (6.12)$$

But $\Psi(\tau(g_i)\eta'_i) = U(g'_i)\Psi(\eta'_i) = U(g'_i)\Psi(\tau(g'_i; g_i)\eta_i) = U(g'_i; g_i)\Psi(\eta'_i) = U(g'_i)\Psi(\eta'_i) - \Psi(\tau(g'_i)\eta_i)$. We used in turn definition (6.8), hypothesis (6.9), Eq. (6.6), the group law, and definition (6.8) again. By summing over i , we have $\Psi(h) = \sum \Psi(\tau(g_i)\eta_i) = \sum \Psi(\tau(g'_i)\eta'_i)$ which proves consistency.

We have shown that definition (6.8) is meaningful. It is then automatically consistent with (6.1) and, moreover, transformation law (6.6) holds generally true for arbitrary h in J^\sim and g in G^* . Eq. (6.6) is the promised explicit form of the global transformation law.

It follows that the functionals w_f on J_1^\sim extend to continuous linear functionals on J^\sim by virtue of the definition

$$\langle w_f, h \rangle = (\Psi(f), \Psi(h)) \quad \text{for } f \in J_1^\sim,$$

and

$$(U(g)\Psi(f), \Psi(h)) = \langle U(g)w_f, h \rangle = \langle w_f, T(g)h \rangle \quad (6.10)$$

The conformal partial wave expansion of these expressions is obtained by decomposing the states $\Psi(h)$ as described in Sec. 2. We write it in terms of elements of \mathcal{E}_χ as in Sec. 4.

$$\langle w_f, h \rangle = \int d\mu(x) \int dx \xi(x)^* V^\chi(x; h) = \int d\mu(x) V^\chi[\xi(x), h]$$

with $\xi \in \mathcal{E}_\chi$, for $h \in J^\sim$.

If h is in J_1^\sim it is determined by a function $\hat{h}(x, x_2)$ of relatively spacelike pairs (x_1, x_2) of points on Minkowski space through (6.5), and so

$$V^\chi[\xi, h] = \int dx \int dx_2 \int dx \xi(x)^* V(x, x_2) \hat{h}(x, x_2) \quad \text{for } h \in J^\sim \quad (6.11b)$$

Here and everywhere we use vector notation: $h(x_1, x_2)$ takes values in $V^i \otimes V_j$, the kernel $V(x, x_1, x_2)$ is a linear map from $V^i \otimes V_j$ to V^ℓ , $\xi(x)$ takes values in V^ℓ , and we write $v_1^* v_2$ for the scalar product of two vectors v_1, v_2 in V^ℓ .

7. G^* - invariant sesquilinear forms on $\mathcal{E}_X \times \mathcal{J}^{\sim}$.

We wish to determine the most general sesquilinear form $v_X[\cdot, \cdot]$ on $\mathcal{E}_X \times \mathcal{J}^{\sim}$ which is G^* -invariant in the sense of (6.12) and such that the kernel $V(x_1, x_2, x_3)$ determined by it admits analytic continuation as required by lemma 7. This problem can be solved by a standard method of the theory of induced representations, viz. Bruhat theory of intertwining maps [14].

First we give an alternative description of the space \mathcal{J}^{\sim} . It will exhibit the representation acting in \mathcal{J}^{\sim} as an induced representation on G^*/MA . Let us define a representation L of MA by operators $L(ma)$ acting in the vector space $V \otimes V$:

$$L(ma)^{-1} = D^{X_1}(ma)^* \otimes D^{X_2}(ma^{-1})^*, \quad m \in MA. \quad (7.1)$$

Lemma 9. There is a bijective intertwining map Q from \mathcal{J}^{\sim} to the space of all infinitely differentiable cross sections on the homogeneous vector bundle $(G^*/MA)_X(V^1 \otimes V^2)$ with base G^*/MA and fibre $V^1 \otimes V^2$.

Explicitly, $Q\mathcal{J}^{\sim}$ consists of infinitely differentiable functions h_1 on G^* with values in $V^1 \otimes V^2$ having covariance property

$$h_1(gma) = L(ma)^{-1} h_1(g) \quad \text{for } ma \in MA. \quad (7.2)$$

It is made into a representation space for G^* by imposing the transformation law

$$(\tau(g) h_1)(g') = h_1(g^{-1}g') \quad (h_1 \in Q\mathcal{J}^{\sim}) \quad (7.3)$$

The map Q is explicitly given by

$$(Qh_1)(g) = h_1(g, gR) \quad \text{for } h_1 \in \mathcal{J}^{\sim} \quad (7.4)$$

Evidently it commutes with the action of the group, $T(g)Q = QT(g)$ by (6.4). Covariance property (7.2) of Qh_1 follows from (6.3) since Remark 5' by (3.4'). R was defined after (3.2). To prove the lemma it only remains to be shown that Qh_1 determines h_1 . This follows from the fact (lemma 8) that G^* acts transitively on relatively spacelike pairs of points on superworld M . As g ranges over

G^* , the pair $(g\gamma_0, gR\gamma_0) = (g\gamma_0, g\gamma_{\infty})$ ranges over all relatively spacelike pairs of points on superworld M , c.p. the proof of lemma 8. Therefore the set of pairs (g, gR) contains a representative out of every coset $(\gamma_0, \gamma_1) \in M \times \tilde{M} = (G^* \times G^*) / (P^0 \times P^0)$. This suffices to determine h_1 by the discussion following (6.4). \square

Because of lemma 9 we may consider $v_X[\cdot, \cdot]$ as a G^* -invariant sesquilinear form on $\mathcal{E}_X \times Q\mathcal{J}^{\sim}$. This will be helpful.

Elements $h \in Q\mathcal{J}^{\sim}$ admit an integral representation

$$h(g) = \int_{MA} d\mu_{MA} L(ma) h'(gma) \quad \text{for } h \in Q\mathcal{J}^{\sim} \quad (7.5)$$

with h' an infinitely differentiable vector valued function with compact support on G^* ; $d\mu_{MA}$ is (right- and left) invariant Haar measure on MA . This integral representation makes covariance property (7.2) manifest.

According to (3.13), $f \in \mathcal{E}_X$ may also be considered as functions on G^* with values in V^f and admitting an integral representation

$$f(g) = \int_P d\mu_P \delta_P(p)^{\dagger} \mathcal{D}^X(p^{-1})^* f'(gp) \quad (7.6)$$

Here f' is an infinitely differentiable function on G^* with values in V^f and compact support, and $d\mu_P$ is left-invariant Haar measure on $P = \Gamma_1 MAN$. Integration over P includes a summation over Γ_1 . The measure $d\mu_P$ is not right-invariant; instead $d(Pp) = \delta_P(p) dp$ with modulus function [15]

$$\delta_P(p) = |\alpha|^4 \quad \text{for } p = \gamma_{MAN} \in \Gamma_1 MAN \quad (7.7)$$

Integral representation (7.6) fulfills the covariance condition (3.13) for arbitrary f' .

For the sesquilinear form $V^X[\cdot, \cdot]$ we may then make the general Ansatz

$$V^X[f, Q^{-1}h] = \int_{G^* \times G^*} dg dq' f'(q)^* t(g, q') h(q) \quad \text{for } h \in QJ^\sim, f \in \mathcal{E}_X$$

and h', f' related to h, f by (7.5), (7.6), (7.8)

with a kernel $t(g, q')$ which maps $V^X \otimes V^X \rightarrow V^X$. The kernel $t(g, q')$ is a generalized function on $G^* \times G^*$, but we will use functional notation as physicists always do.

Expression (7.8) must depend on h' only through h .

If $h''(q) = L(b)h'(qb)$ with $b \in MA$ then h'' and h' determine the same h . Therefore we must require

$$t(g, q)b)L(b) = t(g, q) \quad \text{for } b \in MA \quad (7.9a)$$

Similarly, $f''(q) = \delta_p(p)^k D^X(p')^* f'(p)$ and $f'(q)$ determine the same f . Since $V^X[f, h]$ should depend on f' only through f we get the consistency condition

$$\delta_p(p)^k D^X(p) t(gp, q) = t(g, q) \quad \text{for } p \in P \quad (7.9b)$$

From transformation law (7.3) and integral representation (7.5) we have

$$(\tau(g)h)(q') = h(q'q) = \int_{MA} db L(b) h(q'q'b) \quad \text{for } h \in QJ^\sim$$

Similarly from (3.14) and (7.6)

$$(\tau_X(g)f)(q') = f(q'q') = \int_P dp \delta_p(p)^k D^X(p^{-1})^* f'(q'q)p \quad \text{for } f \in \mathcal{E}_X$$

Therefore, G^* - invariance (6.12) reads

$$V^X[\tau_X(g)f, \tau(g)h] = \int dg dq' f'(q, q')^* t(g, q') h(q, q) = V^X[f, h].$$

This requires

$$t(g, q, g', q') = t(g', q') \quad \text{for all } g, q \in G^* \quad (7.9c)$$

It remains to determine the general solution of Eqs. (7.9a, b, c).

The general solution of (7.9c) is

$$t(g, q') = t^*(g', q) \quad (7.10)$$

with a (generalized) function t^* on G^* whose values are maps: $v^i \otimes v_j \mapsto v^i t^* v_j$. Covariance conditions (7.9a, b) read then

$$t^*(b', qP) = \delta_p(p)^k D^X(p)^* t^*(q)L(b) \quad \text{for } b \in MA, p \in P \quad (7.11)$$

Let us abbreviate $MA = H$ and let $P = \Gamma_{MAN}$ as before. We define a left action of $H \times P$ on G^* by

$$(b, p)q = bqP^{-1} \quad \text{for } p \in P, b \in H \cap MA$$

Evidently this satisfies the group law $(p_1 b_1)(p_2, b_2)q = (p_1 p_2, b_1 b_2)q$. The manifold G^* decomposes therefore into orbits under $H \times P$ and $H \times P$ acts transitively on each orbit. Let us determine the orbits.

Consider the action of $H \times P$ on cosets in $M_c^4 = G^*/P$ and their elements. P acts transitively within each coset; therefore the problem reduces to determining the orbits in M_c^4 under H . Let us parametrize the finite points of M_c^4 by Minkowskian coordinates $x^\mu(x^\nu)$ as in (3.3b). There are then three open orbits consisting respectively of positive timelike x , negative timelike x , and spacelike x . In addition there are several lower dimensional orbits (They consist of the point $x = 0$, pos. lightlike x , negative lightlike x , the unique point at spatial infinity of M^4 , and the remaining points at infinity, respectively).

Correspondingly, the open orbits on G^* consist of

$$\begin{aligned} G_+^* &= \{ g = \bar{n}_x p \text{ with } p \in P, x \text{ pos. timelike} \} \\ G_-^* &= \{ g = \bar{n}_x p \text{ with } p \in P, x \text{ neg. timelike} \} \\ G_\sim^* &= \{ g = \bar{n}_x p \text{ with } p \in P, x \text{ spacelike} \} \end{aligned}$$

and in addition there are several lower dimensional orbits.

Suppose that $t^*(q)$ is known on one of the open orbits, say G_+^* .

It is clear that $V(x\chi; x_i, x_j)$ will then be determined on an open set of arguments. Analyticity properties (lemma 7) can then be used to determine it everywhere.

Let us choose a standard $\hat{x}^\mu = (1, 0, 0, 0)$. Correspondingly we select \bar{n}_χ as a standard point in G_+^* . Let us determine the little group of \bar{n}_χ in $H \times P$. The rotation group $U(M)$ consists

of $u \in M$ such that $u\hat{x} = \hat{x}$. Suppose $b\bar{n}_\chi p^{-1} = \bar{n}_\chi$.

Consider this equation mod(P). It follows that $b\hat{x} = \hat{x}$. This requires $b \in U$. On the other hand $b\bar{n}_\chi p^{-1} = \bar{n}_\chi b p^{-1}$. Therefore we must have $p = b$. In conclusion

$$b\bar{n}_\chi p^{-1} = \bar{n}_\chi \quad \text{for } b \in H, p \in P \text{ if and only if } (b, p) = (u, u), u \in U \quad (7.12)$$

Thus the little group of \bar{n}_χ in $H \times P$ is isomorphic to the rotation group U .

Let $g \in G_+^*$. Then it can be written in the form

$$g = b\bar{n}_\chi p^{-1} = \bar{n}_x b p^{-1} \quad \text{with } b \in H, p \in P, x = b\hat{x} \quad \text{for } g \in G_+^* \quad (7.13)$$

Covariance condition (7.11) says that

$$t^*(b\bar{n}_\chi p^{-1}) = \delta_p(p)^{-1} D^X(p) \hat{t}_L(b)^{-1} \quad \text{with } \hat{t} = t^*(\bar{n}_\chi) \quad (7.14)$$

For consistency, \hat{t} must be U -invariant

$$\hat{t} = D^X(u) \hat{t}_L(u)^{-1} = D^L(u) \hat{t} [D^L(u) \otimes D^{\ell_i}(u)]^{-1} \quad \text{for } u \in U \quad (7.15)$$

In other words, \hat{t} is a U -invariant map $V^i \otimes V^j \mapsto V^L$.

Next we will classify all such maps.

Finite dimensional irreducible representations of $M = SL(2C)$ \cong Spin(3,1) are constructed by analytic continuation (Weyl's unitary trick) from UL's of Spin(4), the twofold covering of SO(4). The Clebsch-Gordanology of both groups is therefore the same, and they contain U as a common subgroup.

Let us decompose $V^i \otimes V^j$ into irreducibles under M ,

$$V^i \otimes V^j = \sum V^L \quad \text{with Clebsch Gordon maps} \quad * \quad (7.16)$$

$$C(\ell_i, \ell_j; \ell') : V^i \otimes V^j \mapsto V^L$$

Let us decompose representations ℓ and ℓ' of M into irreducible representations $s \in \tilde{U}$ of U ,

$$V^\ell = \sum_{s \in \tilde{U}} v_s \ell_s \quad \text{etc.} \quad (7.17)$$

We identify $V^L = V^{\ell_s} \otimes W^s$. Consider the projection operators $\pi(\ell_s)$ and their adjoints, viz. U -invariant imbeddings $\pi^*(\ell_s)$,

$$\pi(\ell_s) : V^\ell \mapsto W^s, \quad \pi^*(\ell_s) : W^s \mapsto V^L \quad (7.18)$$

The most general U -invariant map from $V^i \otimes V^j \mapsto V^L$ is a linear combination

$$\hat{t} = \sum_{\ell'_s} c_{\ell_s} \hat{t}_{\ell'_s}, \quad \hat{t}_{\ell'_s} = \pi^*(\ell'_s) \pi(\ell_s) C(\ell_s, \ell'_s; \ell) \quad (7.19)$$

sum over $\ell'_s \in \tilde{U}$, $s \in \tilde{U}$ such that $\ell'_s \ell_s \otimes \ell'_s, s \in \tilde{U}$ with complex coefficients c_{ℓ_s} . \tilde{U} is the set of all finite dimensional irreducible representations of M .

With this we have found the most general form of $\hat{t}(\hat{q}, \hat{q}')$ for $\hat{q}, \hat{q}' \in G_+^*$. The result is given by Eqs. (7.10), (7.14) with (7.13), and (7.19). The sesquilinear form $V^X[f, h]$ on $\mathcal{E}_X \times J^*$ is then determined by (7.8) for f, h having suitable support properties. It remains to recover the corresponding kernels $V(x\chi; x_i, x_j)$ and continue them analytically.

* Remember that we write v_i for V^i , the vector space which carries the irreducible representation ℓ_i of M .

Let $h \in \mathcal{F}_2 \subset \mathcal{I}^*$. Then on the one hand h is determined by a function $h(x_1, x_2)$ of Minkowski space arguments x_1, x_2 by (6.5) and, on the other hand, it is also determined by $Qh(q)$ according to Lemma 9. Let us find the connection.

First we observe that

$$\bar{n}_{x_1} p_1 = \bar{n}_{x_2} p_2 \quad \text{for } p_1 = n_{xy}, p_2 = p(y, x), \gamma = x_2 - x_1. \quad (7.20)$$

in the notation of (5.8), and p_1, p_2 are in \mathcal{P}° for spacelike y by lemma 5. It follows from covariance (6.3)

$$\begin{aligned} h(x_1, x_2) &= h(\bar{n}_{x_1}, \bar{n}_{x_2}) = \pi(p_1, p_2) h(\bar{n}_{x_1} p_1, \bar{n}_{x_2} p_2) \\ &= \pi(p_1, p_2) h(\bar{n}_{x_1} p_1, \bar{n}_{x_2} p_2) = \pi(p_1, p_2) Qh(\bar{n}_{x_1} n_{xy}). \end{aligned}$$

and $Qh(q)$ vanishes unless $q = \bar{n}_{x_1} n_{xy} \pmod{\mathcal{M}}$ for some x_1 and spacelike y . In particular it vanishes if $q = \bar{n} n$ may with $y \neq e, v \in \Gamma_1$ etc.

If we write $g = \bar{n} n$ then Haar measure $dg = d\bar{n} dn dm da$ and $d\bar{n}_{x_1} = dx, d\bar{n}_{x_2} = dy$. If $Qh(q) = \int_M db L(b) h'(gb)$ then

$$\begin{aligned} \int_M dg' t(g', h'(q')) &= \int_M \int_{\mathcal{D}} d\bar{n} dn db t(g, \bar{n} n g b) L(b) h'(n g b) \\ &= \int_{\bar{N} \times N} d\bar{n} dn t(g, \bar{n} n) Qh(\bar{n} n), \end{aligned}$$

because of the above mentioned support property. Thus finally

$$\int_M dg' t(g', h'(q')) = \iint (-y^*)^4 dy dx t(g, \bar{n}_x n_{xy}) \pi(p_1, p_2) h(x_1, x_2)$$

with $x_1 = x, x_2 = x + y, p_1 = n_{xy}, p_2 = p(y, x)$

Similarly, let $f \in \mathcal{E}_X$ and write $f(\bar{n}_x) \in \mathcal{F}(x)$. Splitting $g = \bar{n} p$ with $p \in \mathcal{P}$ the measure factorizes $dg = d\bar{n} dp$ as we have just said. Some integrations in (7.8) can therefore be carried out with the help of (7.6) and covariance condition (7.9b). As a result

$$\begin{aligned} V^X[\{f, h\}] &= \iint \iint (-y^*)^4 dy dx dz f(z)^* t(\bar{n}_x, \bar{n}_z n_{xy}) \pi(p_1, p_2)^{-1} h(x, x + y) \\ &= \iint \iint (\bar{x}_2 - x_1)^{-4} dx_1 dx_2 dz f(z)^* t^*(n_{xy} \bar{n}_z n_{xy}) \pi(p_1, p_2)^{-1} h(x, x + y) \end{aligned} \quad (7.21)$$

for $f \in \mathcal{E}_X, h \in \mathcal{F}_2$. Thus by comparison with (6.11b)

$$\begin{aligned} V(x_3 X, x_1 x_2) &= (-x_2^{-1})^{*4} t^*(n_{xy}^{-1} \bar{n}_z n_{xy}) \pi(p_1, p_2)^{-1} \\ \text{with } p_1 &= n_{xy} x_3, p_2 = p(x_1, x_2), x_{ij} = x_i - x_j. \end{aligned} \quad (7.22)$$

It only remains to insert the previously derived expression for t^* .

Evidently, $V(\dots)$ is translationally invariant, i.e. depends only on coordinate differences. We may therefore put $x_i = 0$.

According to definition (3.8)

$$\begin{aligned} n_{xy} \bar{n}_z &= \bar{n}_z p(x, n_{xy})^{-1} = b \bar{n}_z p^{-1} \\ \text{with } z' &= n_{xy}^{-1} z = r_2(xz - xy) \quad \text{and } p = p(x, n_{xy}), b \\ \text{provided } b \bar{x}' &= z', b \in \mathcal{M} \end{aligned} \quad (7.22)$$

A suitable b in \mathcal{M} exists if y, z are such that z' is positive timelike. We have $z'^2 = (rz - xy)^2 = (z - y)^2 / r^2 y^2$. Since y is spacelike by hypothesis, we may put $y^0 = 0$ without loss of generality. We see that z' will be positive timelike if

$$z \text{ pos.timelike, } z' \text{ spacelike or vice versa; } y \text{ spacelike} \quad (7.23)$$

We restrict our attention to this case. It corresponds with the previous assumption that the argument of t^* is in the orbit G_+^* . According to definitions (7.1), (6.2)

$$\begin{aligned} L(b) &= \pi(b, \tilde{b}), \quad \text{with} \\ \tilde{b} &= r_2 b r^{-1} = \tilde{m} a^{-1} \quad \text{for } b = m a \in \mathcal{M}, \tilde{m} = \theta m \theta^{-1} \end{aligned} \quad (7.24)$$

Expression (7.14) for $t^*(\cdot)$ yields then

$$V(zx, oy) = (y^r)^{-k} \delta_p(p)^{-k} D^X(p) \hat{t}^{\pi}(p, b, p, b)^{-1} \quad (7.25)$$

with same b, P, P_1, P_2 as before in (7.21), (7.22). This is valid for z, y as described in (7.23).

Expression (7.22) for P can be simplified. We use lemma 5 repeatedly. $P = p(z, ny) b = p(z, R) p(yz, R) b = p(z, R) \tilde{b} p(yz, R)$

$\circ p(z, R) \tilde{b} p(-x, R)$. Writing

$p(z, R) = y^{-r} \text{mod}(N)$; y -generator of Γ_z .

$p(yz, R) = y m_z q_z n_z$ as in lemma 5 we obtain

$$p = y^{r-1} m_z q_z \tilde{b} \text{ (mod } N) = y^{r-1} m_z \tilde{m} q_z a^{-1} \text{ (mod } N)$$

$$\text{with } r = \text{sign } z; |a|^2 = (z-y)^2 / z^2 y^2; |a_z| = |z|.$$

Similarly

$$p_1 = ny; p_2 = my \alpha \text{ (mod } N) \text{ with } |\alpha_y| = -y^2 \quad (7.26)$$

We will now introduce an M -covariant version of \hat{t} in order to switch $D^X(p)$ through \hat{t} in (7.25). According to spinor calculus, irreducible representations ℓ of M may be labelled by their highest weight $(j_1, j_2); j_1, j_2$ half-integer. Completely symmetric tensor representations of rank j are labelled (j_1, j_1) in this way.

Lemma 10. Given three finite dimensional irreducible representations ℓ, ℓ_i, ℓ_j of M , define

$\Sigma = \max$ rank of any completely symmetric tensor representation of M contained in the tensor product $\ell \otimes \ell_i \otimes \ell_j$

Consider linear maps $t(x) : V^{\ell_i} \otimes V^{\ell_j} \rightarrow V^\ell$ such that

- 1) $t(x)$ is a homogeneous polynomial of x of degree Σ .
- 2) $t(x)$ are M -covariant in the sense that

$$D^{\ell_i}(m) t(x) [D^{\ell_i}(m) \otimes D^{\ell_j}(m)]^{-1} = t(mx) \text{ for } m \in M$$

All such are obtained from U -invariant maps \hat{t} as were

classified in (7.19) by setting

$$\hat{t}(x) = |x^2|^{\frac{1}{2}} \tau D^{\ell(m)} \hat{t} [D^{\ell_i}(m) \otimes D^{\ell_j}(m)]^{-1} \quad (7.27)$$

for positive timelike $x = |x^2|^{\frac{1}{2}} m \hat{x}$.

Conversely let $t(x)$ defined by (7.27) for positive timelike x . Then it can be analytically continued to all x and satisfies 1) and 2).

We shall relegate the proof of this lemma to Appendix A.

[Since representations D^X of MAN are trivial on N we have

$$D^X(p) = D^X(m_z q_z \tilde{b}) \text{ for } z \in V_+, \text{ with } \Lambda(m_z)^\mu_\nu |a_z| = 2z^\mu z^\nu \cdot \tilde{g}^{\mu\nu} z^2$$

by lemma 5. Moreover $(m_z q_z \tilde{b} \hat{x})^\mu = (z^2)^{-1} \Lambda(m_z)^\mu_\nu |a_z| (\theta z)^\nu$, whence

$$m_z q_z \tilde{b} \hat{x} = \frac{(z-y)^2 z^2}{y^2} \left\{ \frac{z}{z-y} - \frac{z-y}{(z-y)^2} \right\} \quad (7.27)$$

Now we are ready to use lemma 10 to switch $D^X(p)$ through \hat{t} in (7.25). At the same time we insert the definitions of $\delta_p(p)$ and $\pi(\cdot, \cdot)$. They give $\delta_p(p) = |a_z a^{-1}|^4$,

$$\pi(p_1 b, p_2 \tilde{b})^{-1} = D^{\ell_i}(ma)^* \otimes D^{\ell_j}(my) \tilde{m} a^{-1} * |a_j|^2 \quad \text{Altogether}$$

$$V(zx, oy) = |y|^{-2+\epsilon - c_1 - c_2} (iz||z-y|)^{-2-\epsilon + \Sigma} \left(\frac{|z|}{|z-y|} \right)^{c_1 - c_2} \cdot \hat{t} \left(\frac{z}{z-y} - \frac{z-y}{(z-y)^2} \right) [D^{\ell_i}(m_z) \otimes D^{\ell_j}(m_z \tilde{m} a^{-1} \tilde{m} y)] \quad (7.27)$$

with $|y| = |y^2|^{\frac{1}{2}}$ etc. We use lemma 5 again to evaluate the argument of D^{ℓ_j} . One has $m_z \tilde{m} a^{-1}$ and so $m_z \tilde{m} m \tilde{m} = m_z \tilde{m} \tilde{m} = \tilde{m}_{Rz-Ry}$. Thus $m_z \tilde{m} m^{-1} \tilde{m}_y^{-1} = m_z \tilde{m}^{-1} \tilde{m}_{Rz-Ry} \tilde{m}_y^{-1} = m_{-y}$.

Irreducible representations ℓ of $M \cong SL(2)$ are extended to $GL(2C)$ in a standard way. Suppose ℓ has highest weight (j_1, j_1) , then one defines

$$|\ell| = j_1 + j_2; D^{\ell}(pm) = p^{2|\ell|} D^{\ell}(m) \quad \text{for } m \in SL(2C), p \in C. \quad (7.28)$$

With this notation, Eq. (7.27) becomes

$$V(zx, oy) = |y|^{-2+\epsilon - c_1 - c_2 - \epsilon_j - \epsilon_{j-1}} (iz||z-y|)^{-2-\epsilon + \Sigma - 2|\ell_{j-1}-2|} (f_{j-1} \left(\frac{|z|}{|z-y|} \right)^{c_1 - c_2})$$

$$\cdot \hat{t} \left(\frac{z}{z-y} - \frac{z-y}{(z-y)^2} \right) [D^{\ell_i}(z) \otimes D^{\ell_j}(z-y)]. \quad (7.29)$$

This is valid for z positive timelike, y and z -y spacelike. An expression for arbitrary arguments is obtained by using the spectrum condition, viz. Lemma 7. We note that expression (7.29) is real analytic in its domain of validity. This guarantees uniqueness of analytic continuation to the whole domain of holomorphy given in lemma 7. As a result we have the following proposition 11. Of course kernels $V(x_1, x_2, x_3)$ depend also on spin and dimension $\chi_i = [\ell_i, d_i]$, $\chi_{ij} = [\ell_{ij}, d_{ij}]$ of the fields $\phi^i(x_i), \phi^{ij}(x_i)$ whose product we want to expand. We shall therefore indicate this dependence by writing $V(x_1, x_2, x_3, x_4) = V(x_1, x_2, x_3)$.

Proposition 11. Let $V(x_1, x_2, x_3, x_4, x_5)$ a 3-point function which satisfies the spectrum condition (lemma 7) and which is conformal invariant in the sense explained earlier, with transformation law specified by $x_i \rightarrow [\ell_i, 2c_i], x_i - [\ell_i, 2c_i]x_i + [c_i, 2c_i]$. In this, c_i, c_i, c are real, ℓ_i, ℓ_i, ℓ finitedimensional irreducible representations of $M \cong SL(2\mathbb{C})$ acting in vector spaces $V\ell_i, V\ell_i, V\ell$. Then

$$V(x_1, x_2, x_3, x_4, x_5) = (-x_{12}^3)^{-\delta_{12}} (-x_{34}^2)^{-\delta_{34}} (-x_{25}^2)^{-\delta_{25}}. \quad (7.30)$$

+ $\left(\frac{x_{35}}{(-x_{12}^3)} - \frac{x_{12}}{(-x_{35}^3)}\right) \left[\mathcal{D}^{\ell_1}(x_{35}) \otimes \mathcal{D}^{\ell_2}(x_{12}) \right]$
 with $\delta_{12} = \frac{1}{2}(z - c_1 + c_2 + \Sigma)$, $\delta_{34} + \delta_{25} = 2 + c - \Sigma + 2(\ell_1 + 2\ell_2 + 1)$, $\delta_{35} - \delta_{12} - c_1 - c_2$, and $t(x)$ are linear maps: $V\ell_i \otimes V\ell_i \rightarrow V\ell$ which satisfy the hypothesis of lemma 10. If ℓ etc. and Σ are defined in (7.28) and lemma 10; if $\ell_i \otimes \ell_i \otimes \ell$ does not contain a completely symmetric tensor-representation of M , then a conformal invariant 3-point function does not exist. An if -prescription is understood,

$$(-x_{ij}^2)^{\alpha} = [- (x_i - x_j)^2 + i\varepsilon(x_i - x_j)^2]^{\alpha} \quad (7.31)$$

$x = x^\mu \mathbf{1} + \sum_i x^i \sigma_i$, σ_i Pauli matrices.

Expression (7.30) is a well defined distribution for arbitrary c, c_1, c_2 .

Corollary 12. Let $V(x_1, x_2, x_3, x_4, x_5)$ a conformal invariant 3-point function which satisfies the spectrum conditions for a 3-point Wightman function. Then it can be analytically continued to the permuted extended tube and satisfies all the Wightman axioms for a 3-point Wightman function $(\Omega, \phi^k(x)^* \phi^j(x_i) \Omega)$ of three possibly distinct local fields (with Lorentz spin and dimension $\chi_k = [\ell_k, d_k]$ etc.). When two of the fields are identical, the Wightman 3-point function has further symmetry properties. These are not automatically ensured by (7.30).

Remark. The kernels $V(x_1, x_2, x_3, x_4, x_5)$ are not Clebsch-Gordan kernels for the tensor product $\chi_i \otimes \chi_2$ of UIR's of G^* . Indeed, states $\phi^i(x_i) \phi^j(x_i) \Omega$ transform in general according to a unitary representation of G^* which is not a Kronecker product, sp. epilogue of ref. [5]. In particular it restricts to a nontrivial representation of the center of G^* , while for a Kronecker product of irreducible representations every element of the center would have to be represented by a multiple of the identity. |

We add some remarks on zero mass representations. Most of the UIR's χ of G^* with positive energy have continuous mass spectrum, but there are also zero mass representations (cp. [7] and proposition 6). A priori they could appear in the conformal partial wave expansion (4.2) and then also in the light cone expansion (1.2). We shall now argue that this only happens in exceptional cases.*

Let us first discuss the meaning of this. Suppose $\phi(x)$ is a local field and $\square \phi(x) \Omega = 0$. Then also $\square \phi(x) = 0$ because a local field can never annihilate the vacuum. Therefore $\phi(x)$ is a free zero mass field. Appearance of zero mass representations in the conformal partial wave expansion would therefore mean that there appear massless free fields in the operator product expansion. This can happen. (Example : The expansion of the product

* This observation originates in a remark made by L. Castell some years ago.

of a massless free field $\phi(x)$ with its stress energy tensor must contain $\phi(x)$ again). But it happens only in special cases.

The reason lies in the nonexistence of a suitable 3-point function.

Considered as functions of x , 3-point functions $V(x_1 x_2 x_3 \chi_1 \chi_2)$ must be in the representation space \mathcal{F}_X . As such they must satisfy a spectrum condition. For continuous mass representations it says that the Fourier transform $\tilde{V}(px; x_1 x_2 x_3 \chi_1 \chi_2)$ has support concentrated in the closed forward light cone, $p \in \tilde{V}_+$.

Because of the $i\varepsilon$ -prescription, expression (7.30) satisfies this condition.

If χ is a zero mass representation, however, elements of \mathcal{F}_X satisfy certain differential equations, in particular their Fourier transform is concentrated at $p^1 = 0$. Expression (7.30) does not meet this condition in general. Consider for instance the scalar case $\ell_1 = \ell_2 = \text{id}$, $c = -1$.

The Fourier transform \tilde{V} is given by Eq. (8.4) below for this case (Caution: the limit $c \rightarrow -1$ must be taken with care in order not to lose contributions concentrated at $p^1 = 0$, cp. after (3.20')). We see that \tilde{V}^* cannot vanish identically for p in the interior of the forward lightcone unless the argument of one of the r -functions in front is a nonpositive integer, i.e. $c_1 - c_2$ is an odd integer. More careful inspection reveals that $\tilde{V}^*(p_{-1}; x_1 c_1 x_2 c_2)$ is concentrated at $p^1 = 0$ if and only if $c_1 - c_2 = \pm 1$.

8. Recovery of kernels $\tilde{\mathcal{B}}(px; x_1 x_2)$.

We introduce the Fourier transform of 3-point functions with respect to the first argument

$$\tilde{V}(px; x_1 x_2 x_3 \chi_1 \chi_2) = \int dx e^{ipx} V(x_1 x_2 x_3 \chi_1 \chi_2) \quad (8.1)$$

The kernels $\tilde{\mathcal{B}}$ are obtained from them by Eq. (4.1), viz.

$$\tilde{V}(px; x_1 x_2 x_3 \chi_1 \chi_2) = \Delta_+^\chi(p) \tilde{\mathcal{B}}(px; x_1 x_2 x_3) \quad (8.2)$$

where $\Delta_+^\chi(p)$ is the Fourier transform of the 2-point function (intertwining kernel) (3.20). As we discussed earlier (in Sec. 2), kernels \mathcal{B} are nonunique and determined only to the extent that (8.2) determines them.

We consider the scalar case first. We introduce a special notation for this case

$$\begin{aligned} V^o(x_1 c_1 x_2 c_2) &= V(x_1 x_2 x_3 \chi_1 \chi_2) \\ \mathcal{B}^o(x_1 c_1 x_2 c_2) &= \mathcal{B}(x_1 x_2 x_3 \chi_1 \chi_2) \quad \text{etc.} \end{aligned} \quad (8.3)$$

With $\chi_i = [\text{id}, 2+c_i]$, $\chi_{\alpha} = [\text{id}, 2+c_\alpha]$, $\chi = [\text{id}, 2+c]$, where id stands for the trivial 1-dimensional representation of \mathbb{N} .

From proposition II we obtain (same notation 7.31)

$$\tilde{V}^o(px; x_1 c_1 x_2 c_2) = \int dx_3 e^{ipx_3} (-x_{13})^{\frac{1}{2}(c-c_1-c_2-2)} (-x_{23})^{\frac{1}{2}(c-c_1-c_2-2)} (-x_{31})^{\frac{1}{2}(c-c_1-c_2-2)}$$

$$= \Gamma(c+2) \Gamma\left(\frac{c+c_1-c_2+2}{2}\right)^{-1} \Gamma\left(\frac{c-c_1+c_2+2}{2}\right)^{-1} \int_u^1 u^{\frac{1}{2}(c+c_1-c_2-2)} \int_u^1 u^{\frac{1}{2}(c-c_1+c_2)} (1-u)^{\frac{1}{2}(c-c_1+c_2)} (1-u)^{\frac{1}{2}(c-c_1+c_2)} \cdot \int dx e^{ipx} [r(z(u)-x)^2 - u(1-u)x_{12}^2 - i\epsilon(z(u)^0 - x^0)]^{-c-2}$$

with $z(u) = ux_1 + (1-u)x_2$. The second equation was obtained by inserting the standard integral representation

$$A^{-v} B^{-\mu} = \Gamma(v+\mu) \Gamma(v)^{-1} \Gamma(\mu)^{-1} \int du u^{v-1} (1-u)^{\mu-1} \int du u^{v-1} (1-u)^{\mu-1} [uA + (1-u)B]^{-v-\mu}.$$

The Fourier transform of the generalized function $[-x^2 + \alpha^2 + ix\omega]^{\lambda}$ is well known for $\alpha^2 > 0$, and so we obtain, for $x_{12}^2 < 0$,

$$\tilde{V}_c(p; x_1, c, x_2, c_2) = 2x^3 r\left(\frac{c+c_2-c_1+2}{2}\right)^{-1} r\left(\frac{c-c_2+c_1+2}{2}\right)^{-1} (-x_{12}^2)^{\frac{1}{2}(c_1+c_2+2)} \cdot$$

$$\cdot \theta(p) \int_0^1 du \left(\frac{u}{1-u}\right)^{\frac{1}{2}(c_1-c_2)} e^{-ip[ux_1 + (1-u)x_2]} \left(\frac{1}{4}p^2\right)^{c/2} J_c\left([-u(1-u)x_{12}^2 p^2]\right). \quad (8.4)$$

J_c is the Bessel function, $\theta(p) = 1$ for $p \in \bar{V}_+$ and 0 otherwise.

The u -integral is regularized by analytic continuation in c [16].

Validity of (8.4) for arbitrary x_1, x_2 follows by uniqueness of analytic continuation. Dividing by $\Delta X(p)$ we obtain finally

$$\tilde{B}_c^0(p; x_1, c, x_2, c_2) = n'_+(c)^{-1} 2\pi^3 r\left(\frac{c+c_2-c_1+2}{2}\right)^{-1} r\left(\frac{c-c_2+c_1+2}{2}\right)^{-1} (-x_{12}^2)^{\frac{1}{2}(c_1+c_2+2)} \cdot$$

$$\cdot \int_0^1 du \left(\frac{u}{1-u}\right)^{\frac{1}{2}(c_1-c_2)} e^{-ip[ux_1 + (1-u)x_2]} \left(4p^2\right)^{-c/2} J_c\left([-u(1-u)x_{12}^2 p^2]\right)^{1/2}, \quad (8.5)$$

for $p \in \bar{V}_+$, with the prescription (7.31).

with a constant $n'_+(c)$ which is determined by the normalization of the scalar 2-point function, cp. Eq. (3.20').

We see by inspection that \tilde{B}_c^0 has the holomorphy properties in p which were stated in proposition 4. It is equal to

$(-x_{12}^2 + ix\omega)^{\lambda}$ times an entire holomorphic function in x_{12} , x_1 and p , and so it is a generalized function of x_{12} and x_2 which is holomorphic in the parameter p .

Let us now turn to the general case. The first two assertions of proposition 4 are clear from Eq. (8.2), and proposition 11, viz. the classification of 3-point functions V . It remains to demonstrate holomorphy in p . This can be simplified very much by remembering once more the arguments of Sec. 7.

Let $\tilde{h}(x, x_1)$ an arbitrary Schwartz test function with values in the dual of $V^L \otimes V^L$, and

$$\mathcal{B}_h^X(x) = \int dx_1 dx_2 \mathcal{B}(x_X; x_1, x_2) \tilde{h}(x, x_1) \quad (8.6)$$

The kernels \mathcal{B} have the following properties which define them (Eq. (8.2) is a consequence, cp. Sec. 4).

1. As functions of x_1 and x_2 kernels $\mathcal{B}(x_X; x_1, x_2)$ transform in the same way as $V(x_X; x_1, x_2)$. I.e. they are both restrictions of cross sections on $\tilde{M} \times \tilde{M}$, at least for $x_{12}^2 < 0$, cp. Sec. 6 [1].
2. As functions of x , kernels $\mathcal{B}(x_X; x_1, x_2)$ transform like elements of \mathcal{E}_X . The smeared kernels $\mathcal{B}_h^X(x)$ are in the Hilbert space \mathcal{E}_X , viz.

$$(\mathcal{B}_h^X, \mathcal{B}_h^X) = \int dp \tilde{\mathcal{B}}_h^X(p)^* \Delta_+^X(p) \tilde{\mathcal{B}}_h^X(p) < \infty \quad (8.7)$$

3. Kernels $\mathcal{B}(x_X; x_1, x_2)$ are conformal invariant.

The statement of the transformation laws 1. and 2. gives meaning to 3.

Let f a function in the representation space $\mathcal{F}_{\tilde{X}}$, $\tilde{x} = [\tilde{t}; \tilde{r}; \tilde{c}]$ real, and define $f'(x) = f(-x)$. Then f' transforms like an element of \mathcal{E}_X , $x = [t; r; c]$. This is seen by comparing Eqs. (3.15) and (3.19) and noting that the phase factor $e^{i\pi N_C}$ in definition (3.12) can be reverted by a space time reflection $\Pi\Theta$ (It takes $t \mapsto -t$, $\tilde{t} \mapsto -\tilde{t}$ in the notation of Sec. 3), while

$$\mathcal{D}^L(m)^* = \mathcal{D}^L(m)^{-1} = (-)^{2|\ell|} \mathcal{D}^L(\pi \Theta m \Theta \Pi)^{-1}.$$

It follows that $\mathcal{B}(x_X; x_1, x_2)$ transforms in the same way as a function of x as $V(-x_{\tilde{X}}; -x_1, -x_2)$. They are both conformal invariant and they also have the same transformation law as functions of x_1, x_2 . This is so because $V(-x_{\tilde{X}}; -x_1, -x_2)$ transforms in the same way as a function of x_1, x_2 as $V(x_X; x_1, x_2)$ [for $(x_1, x_2) \in \mathcal{O}$], since only the restriction of representation Δ_X to $\mathcal{P}^0 = \text{MAN}$ enters now (cp. Eqs. (6.2), (6.3)) for which the phase factor $e^{i\pi N_C}$ in (3.12) is absent.

In conclusion, $\mathcal{B}(x_X; x_1, x_2)$ has the same conformal covariance properties as $V(-x_{\tilde{X}}; -x_1, -x_2)$.

Moreover, we see from proposition 11 that

$$\tilde{t} : V^{\ell_1} \otimes V^{\ell_2} \rightarrow V^{\ell} \quad \text{is at the same time a } U\text{-invariant map from } V^{\ell_1} \otimes V^{\ell_2} \text{ to } V^{\tilde{\ell}} \quad \text{and vice versa. Therefore there are a fortiori}$$

equally many linearly independent ones.

with $t(x)$ a matrix valued polynomial which satisfies the hypothesis of lemma 10. This motivates the Ansatz

$$\tilde{B}(px; x, \chi, x_1, \chi_1) = t(x_{31}^1 x_{32} - x_{31}^2 x_{32}) B^\circ(xc'; x_{41} c_1' x_{42} c_2') \quad (8.8a)$$

$$-c' = -c + \Sigma + 2|\ell_1| + 2|\ell_2| ; \quad c'_k = c_k + \Sigma + |\ell_1| + |\ell_2| \quad (\ell_{1,2} \geq 0) ; \quad \chi_{x, [\ell_1, \ell_2]} .$$

with $t(x)$ a matrix valued polynomial which satisfies the hypothesis of lemma 10 with $\tilde{\ell}_1, \ell_1, \ell_2$ substituted for ℓ, ℓ_1, ℓ_2 . Correspondingly

$$\tilde{B}(px; x, \chi, x_1, \chi_1) = t(x_{31}^1 x_{32} - x_{31}^2 x_{32}) \tilde{B}^\circ(px'; x_{41} c_1' x_{42} c_2') \quad (8.8b)$$

$$\text{where now } x_{32} = -i \frac{\partial}{\partial p} - x_1 ; \quad x_{31} = i \frac{\partial}{\partial p} - x_2 .$$

It is clear that this defines an entire function of p because the same is true of \tilde{B}° , and application of a differential operator of finite order cannot destroy holomorphy.

Therefore, proposition 4 will be proven if we can show that Ansatz (8.8a) is general and satisfies the conformal covariance requirements 1-3 supra and (8.7). It suffices to do so for relatively spacelike x, x_1, x_2 because $\tilde{B}(px; x, \chi, x_1, \chi_1)$ shares the analyticity properties (lemma 7) in x and x_1 for $p \in \text{sptr}\chi$.

Concerning generality, we only have to count. Given x, χ, x_1, χ_1 there are according to Eq. (8.2) as many linearly independent kernels $\tilde{B}(px; x, \chi, x_1, \chi_1)$ as 3-point functions $V(x\chi; x, \chi, x_1, \chi_1)$. In view of proposition 11 it only remains to verify that the number of linearly independent polynomials $t(x)$ satisfying the hypothesis of lemma 10 remains unchanged when $\tilde{\ell}$ is substituted for ℓ . These polynomials $t(x)$ are in one to one correspondence with U -invariant maps \tilde{t} . The vector spaces $V^{\tilde{\ell}}$ and V^ℓ are the same, and representations \tilde{t} and t agree on U . Therefore every U -invariant map

$$\Theta_\pm(x; x_1, x_2) = \begin{cases} 1 & \text{if } x \cdot x_1 \in \tilde{V}_1, x \cdot x_2 \text{ spacelike, or vice versa} \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{t} : V^{\ell_1} \otimes V^{\ell_2} \rightarrow V^{\ell} \quad \text{is at the same time a } U\text{-invariant map from } V^{\ell_1} \otimes V^{\ell_2} \text{ to } V^{\tilde{\ell}} \quad \text{and vice versa. Therefore there are a fortiori}$$

equally many linearly independent ones.

Next we discuss finiteness condition (8.7). It follows from (7.30) that $\tilde{V}(px; x, \chi, x_1, x_2)$ is a tempered distribution, and therefore, by (8.2), $\tilde{B}(px; x, \chi, x_1, x_2)$ is polynomially bounded in p for $p \in \tilde{V}_+$. The Fourier transform $\tilde{h}(p; p_1, p_2, x, \chi, x_1)$ of any Schwartz test function $h(x, \chi, x_1)$ falls off faster than any power of total momentum $p_1 + p_2$. Because of momentum conservation (translation invariance), also $\tilde{B}_k^\chi(p)$ falls then off faster than any power of p for $p \in \tilde{V}_+$. Since it is also ∞ -differentiable (even holomorphic) in p , it agrees with a test function on the support of $\Delta_+^\chi(p)$, and therefore

$$\int dp \cdot \tilde{B}_h(p)^\ast \Delta_+^\chi(p) \tilde{B}_k^\chi(p) < \infty .$$

This proves (8.7). We turn to conformal covariance of Ansatz (8.8a). We need only consider the case of relatively spacelike x_1 and x_2 .

According to the discussion of Sec. 7, conformal invariant 3-point functions are determined by matrix-valued functions $t^\#(q)$ on the group G^* which satisfy a covariance condition, viz. (7.11). G^* decomposes into three open orbits G_+^* , G_-^* and G_\sim^* plus some lower dimensional submanifolds. On each of the orbits $t^\#(q)$ is fixed once it is known at one point. Conformal invariance alone does not relate the values of $t^\#(q)$ on different orbits however. We showed that $t^\#(q)$ for $g \in G_+$ determines the 3-point function for arguments x_1, x_2, x such that $x \cdot x_1$ is positive timelike and $x \cdot x_2$ spacelike, or vice versa ($x \cdot x_1$ is spacelike by hypothesis). G_-^* is obtained from G_+^* by spacetime reflection, and G_\sim^* is the open interior of what is left. Let us introduce step functions to match

$$\Theta_\pm(x; x_1, x_2) = \begin{cases} 1 & \text{if } x \cdot x_1 \in \tilde{V}_1, x \cdot x_2 \text{ spacelike, or vice versa} \\ 0 & \text{otherwise} \end{cases}$$

The vector spaces $V^{\tilde{\ell}}$ and V^ℓ are the same, and representations \tilde{t} and t agree on U . Therefore every U -invariant map

$\text{sign } x_i$ is defined to be ± 1 if $x \in V_{\pm}$, and 0 if x is spacelike. Note that $x - x_i \in V_+$, $x - x_i \in V_-$ is impossible if $x_i - x_j$ is spacelike, therefore $\theta_+ + \theta_- + \theta_\sim = 1$ for all x . It follows from the orbit structure that

$$[a, \theta_+(x, x_i) + a_2 \theta_-(x, x_i) + a_3 \theta_\sim(x, x_i)] V(-x \tilde{x}; -x_i x_i) \quad (8.9)$$

has the same conformal covariance properties as $V(-x \tilde{x}; -x_i x_i)$ for arbitrary constants a_1, a_2, a_3 (for $x_i - x_j$ spacelike). Moreover, in the scalar case t, t_1, t_2, t -id, expression (8.9) is the most general conformal invariant 3-point function because then $t^*(ij)$ at any point gG^* is simply a number. It follows that the kernel $B(x, x_i x_i)$ is of the form (8.9) in the scalar case. But then the Ansatz (8.8a) ensures that the same is true in general, because of identity (8.8) for V , and so the Ansatz (8.8a) is indeed conformal invariant.

There is one technical subtlety involved here. Our discussion so far has been for singular functions of x_i and x , that is functions which are defined everywhere except on some lower dimensional submanifolds. What we need is distributions, though. So the question arises whether there exists a conformal invariant regularization. The regularization is unique (within the limits discussed in Sec. 2) if it exists, because $\tilde{B}(px; x, x_i x_i)$ is boundary value of an analytic function of x , and x_i for $p \in \text{sp}^r X$. For some range of c , it is an integrable function of x , and x_i . Elsewhere it can be defined by analytic continuation in c . Explicit expressions (8.8b) and (8.5) show that this is possible*, at least after a change of normalization has been effected through multiplication by $n'_+(c)$.

In conclusion, we have found the kernels $\tilde{B}(px; x, x_i x_i x_j)$ which enter into the conformal partial wave expansion of Sec. 2. They are given explicitly by Eqs. (8.8b), (8.5) and Lemma 10, and they have the properties listed in proposition 4.

* This is consistent with the remark at the end of Sec. 7 since $\tilde{B}(px; x_i x_i x_j)$ may vanish at $p=0$.

Acknowledgement. The author is indebted to M. Lüscher, B. Schroer and I. Todorov for discussions.

Appendix A. Proof of lemma 10

The first part is easy. Given $t(x)$, define \hat{t} by $\hat{t} = t(\hat{x})$. Then \hat{t} is U -invariant by covariance condition ii), and formula (7.27) follows from (ii) and homogeneity. As for the converse, we note first that definition (7.27) of $t(x)$ for positive timelike x makes sense, i.e. $t(x)$ depends on m only through $mx = x/|x|^{1/2}$ because \hat{t} is invariant under the little group U of \hat{x} . It remains to show that $t(x)$ is a polynomial.

Let $E = L(v^i, v^i \otimes v^j)$ the vector space of all linear maps from V^i to $V^i \otimes V^j$. It carries a representation of M given by $D(m)^\sigma = D(m)^\nu [D(m) \otimes D(m)]^\nu$. This representation is isomorphic to the tensor product $L \otimes L \otimes L$. Because of Fermi-superselection rule, it is a 1-valued representation of $M/U \cong SO(3,1)$. It may therefore be decomposed into irreducibles which are all tensor representations of M . Thus $E = \bigoplus E^k$, sum over irreducible representations of M contained in $L \otimes L \otimes L$, with multiplicities. \hat{t} is a U -invariant vector in E , it decomposes as $\hat{t} = \sum c_k v^k$ with complex coefficients c_k , and v^k a normalized U -invariant vector in E^k . Such a vector exists only if E^k carries a completely symmetric tensor representation; let its rank also be denoted by k . The components of the vectors $\hat{y}_k(mx) - D(m)v^k$ are called spherical functions for M . It is well known that $\hat{y}_k(x) = (x^2)^{k/2} y_k(x/\sqrt{x^2})$ are polynomials. So $\hat{t} = \sum c_k |x|^{1/2} \hat{y}_k(x)$. According to Weyl's unitary trick,

representations of $M/U \cong SO(3,1)$ are obtained from representations of $SO(4)$ by analytic continuation. $SO(4)$ has nontrivial center, $SO(4)/Z_2 \cong SO(3) \times SO(3)$. Now $L \otimes L \otimes L$ either comes from a one-valued or from a two-valued representation of $SO(3) \times SO(3)$. In the first (second) case it contains only completely symmetric tensor representations of even (odd) rank k . In any case Z is always even and > 0 , because \sum is the maximal value of k by definition. This shows that $t(x)$ is a polynomial.

References:

1. K. Wilson, Phys. Rev. 179 (1969) 1499.
2. W. Zimmermann, Commun. Math. Phys. 15 (1969) 1969 and in "Lectures on Elementary Particles and Quantum Field Theory", Brandeis University, vol. I, MIT press (1970); J.M. Lowenstein, Phys. Rev. D₄, 2281 (1971).
3. G. Mack, Nucl. Phys. B5 (1971) 592.
4. S. Ferrara, R. Gatto, A.F. Grillo, Springer Tracts in Modern Physics 67, Springer Verlag Heidelberg 1973, and references therein; Ann. Phys. (N.Y.) 26, (1973) 116; see also S. Ferrara, A.F. Grillo, G. Parisi and R. Gatto, Nucl. Phys. B₄₉ (1972) 77; Lett. Nuovo Cim. 4, 115 (1972).
5. M. Lüscher, G. Mack, Commun. Math. Phys. 41, 203 (1975).
6. Abdus Salam, G. Mack, Ann. Phys. (N.Y.) 53, 174 (1969).
7. G. Mack, All unitary ray representations of the Conformal Group $SU(2,2)$ with positive energy. DESY - report DESY 75/50, Dec. 1975, to be published in Commun. Math. Phys.
8. M. Lüscher, Commun. Math. Phys. (to appear).
9. W. Rühl, B.C. Yunn, Commun. Math. Phys. (in press). J. Kupsch, W. Rühl, B.C. Yunn, Ann. Phys. (N.Y.) 89, 115 (1975).
10. G. Mack, Duality in Quantum Field Theory, DESY - report DESY 75/44, (Nov. 1975).
11. G. Mack, J. de physique (Paris) 24 colloque Cl (1973) 99; and in E.R. Caianiello (Ed.), Renormalization and Invariance in Quantum Field Theory, Plenum Press, New York 1974; V. Dobrov, V. Petkova, S. Petrova, I. Todorov, Dynamical derivation of vacuum operator product expansions in conformal quantum field theory, IAS preprint, Princeton 1976 (to be publ. in Phys. Rev.).

- V. Dobrev, G. Mack, V. Petkova, S. Petrova, I. Todorov, Harmonic analysis on the n -dimensional Lorentz group and its application to conformal quantum field theory (to be publ. in Lecture notes in physics).
12. W. Rühl, Commun. Math. Phys. **30**, 267; **34**, 149 (1973).
 13. N. Wallach, Harmonic analysis on homogeneous spaces, Marcel Dekker Inc., New York 1973.
 14. G. Warner, Harmonic Analysis on semisimple Lie groups, vol. I, Springer Verlag Berlin-Heidelberg-New York 1972, Ch. 5 .
J. Strathdee, J.F. Boyce, R. Delbourgo, Abdus Salam, Partial wave analysis (Part I), Trieste IC/67/9 , Feb. 1967 .
 15. J. Dieudonné, Treatise on analysis, vol. 2 , Academic press, New York 1970.
 16. I.M. Gelfand, G.E. Shilov, Generalized functions, vol. I , Academic press, New York 1964.

