DEUTSCHES ELEKTRONEN-SYNCHROTRON

DESY 76/29
June 1976

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THE BOSE FIELD STRUCTURE ASSOCIATED WITH A FREE MASSIVE DIRAC FIELD IN ONE SPACE DIMENSION

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Abstract:

We show that the exponential of the pseudopotential of a free massive Dirac field in one space dimension can be represented in closed form as a Wick-ordered operator. This leads, for this special case, to a modified form of Coleman's equivalence relations which is not based on a mass perturbation expansion.

I. We derive in this note a formula related to the pseudopotential φ of the vector current $:\bar{\psi}\gamma^\mu\psi:$ of a free massive Dirac field in two-dimensional space-time. More precisely, we show how the exponential $E_\lambda = :exp(2\sqrt{\pi}\, i\lambda\varphi):$ defined by Wightman[1] by subtraction of all vacuum parts, can be reduced to its Wick-ordered form $E_\lambda = :exp(\varphi_\lambda):$ and we obtain an explicit expression for the operator φ_λ.

Our interest in this elementary problem is due to Coleman's remarkable paper[2] in which he finds an equivalence between the quantized sine-Gordon and massive Thirring models and, as a special case, between the free massive Dirac field and the sine-Gordon field for a specific value ($\beta^2 = 4\pi$ in Coleman's notation) of its coupling constant. It appeared to us desirable to study this special case by a direct method, rather than treat the massive free field as a perturbation expansion of a massless field. Our formula for the operator φ_λ lends itself to such a direct approach, and we obtain the result that the pseudopotential φ satisfies the equation

$$\Box\varphi = 2m^2 :\varphi[\cos(2\sqrt{\pi}\varphi) - 1]:$$

This shows that, for the case of a free field, modifications occur if Coleman's relations are formulated in a non-perturbative manner. These modifications are connected with the fact that the massive free field has a more singular short-distance behavior than a massless field. For example, the quantity $\bar{\psi}(x)\psi(x)$ (which occurs in the equivalence relations) has an infinite vacuum expectation value in the massive case.

Our results do not settle the equivalence problem between a free Dirac field and the sine-Gordon field. This will depend on the precise definition of the operators $\cos\beta\varphi$ or $\sin\beta\varphi$ in the latter theory which we do not investigate here. In our view, the present state of the equivalence question is that for $\beta^2 < 4\pi$ the results of Fröhlich and Seiler[3] and of Schroer and Truong[4] support Coleman's orginal derivation. For $\beta^2 = 4\pi$ (and for $\beta^2 > 4\pi$ according to Schroer and Truong) a different technique is needed to analyze the equivalence in a convincing manner.

II. Wightman[1] has shown that the current

$$j^\mu = :\bar{\psi}\gamma^\mu\psi: \qquad (1)$$

of a free massive quantized spinor field in one space dimension can be derived from a pseudopotential φ which satisfies

$$j_0 = \frac{1}{\sqrt{\pi}}\,\partial_1\varphi\ , \quad j_1 = \frac{1}{\sqrt{\pi}}\,\partial_0\varphi \quad . \tag{2}$$

Eq. (2) determines φ up to a constant operator which we fix by defining φ as

$$\varphi(x^0, x^1) = \sqrt{\pi}\int_{-\infty}^{x^1} d\xi\ :\psi^+\psi:(x^0,\xi) \quad . \tag{3}$$

($\varphi = \sqrt{\pi}\,(\sigma + \frac{1}{2}Q)$ where σ is Wightman's pseudopotential and Q the charge operator.)

The equation

$$\Box\,\varphi = -2im\sqrt{\pi}:\bar\psi\gamma^5\psi: \tag{4}$$

follows from $\quad \partial_\mu:\bar\psi\gamma^5\gamma^\mu\psi: = -2im:\bar\psi\gamma^5\psi: \quad .$

It is known[1] that $\varphi(x)$ is a local scalar field (under continuous Lorentz transformations). It is non-local relative to ψ with the equal time commutator

$$[\,\varphi(x),\psi(y)]_{x^0=y^0} = -\sqrt{\pi}\,\theta(x^1-y^1)\,\psi(y)\,. \tag{5}$$

$\varphi(x)$ can be defined for fixed x^0 as a distribution in x^1 , while $\Box\varphi$ and Eq. (4) cannot be restricted to a fixed time.

Inserting the standard expression for ψ as a Fourier integral over destruction operators $a(p^1),\ b(p^1)$ and their adjoints, we obtain[1]:

$$\varphi(x) = \int d^2p\, d^2q\, e^{i(p-q)x}\,\delta(p^2-m^2)\,\delta(q^2-m^2):A^+(p)A(q):K(p,q) \tag{6}$$

with

$$A(q) = \theta(q^0)\,a(q^1) + \theta(-q^0)\,b^+(-q^1) \tag{7}$$

$$[\,A(q),A^+(p)]_+\,\delta(p^2-m^2)\,\delta(q^2-m^2) = \delta^{(2)}(p-q)\,\delta(p^2-m^2)$$

and

$$K(p,q) = \frac{im}{\sqrt{\pi}}\left\{\frac{\theta[(p-q)^2]\,\varepsilon(p^0-q^0)}{\sqrt{(p-q)^2}} + \frac{\theta[-(p-q)^2]\,|p^1-q^1|}{(p^1-q^1+i0)\sqrt{-(p-q)^2}}\right\}. \tag{8}$$

Following again Wightman[1], we define powers of φ by

$$\vdots\,\varphi^n(x)\,\vdots = \lim_{x_1,\dots,x_n\to x}\ \vdots\,\varphi(x_1)\dots\varphi(x_n)\,\vdots \tag{9}$$

where the triple dots indicate complete truncation, i.e. subtraction of all vacuum parts:

$$\varphi(x_1)\dots\varphi(x_n) = \vdots\,\varphi(x_1)\dots\varphi(x_n)\,\vdots + \sum_{r=1}^{n}\sum_{P}\langle\varphi(x_{i_1})\dots\varphi(x_{i_r})\rangle_0\ \vdots\,\varphi(x_{j_1})\dots\varphi(x_{j_{n-r}})\,\vdots\ .$$

The sum is over the partitions of 1,...,n into disjoint subsets with

$$i_1 < i_2 < \dots < i_r \quad ; \quad j_1 < j_2 < \dots < j_{n-r}$$

Our main interest lies in the exponential of φ , defined as

$$E_\lambda(x) = \vdots\,\exp\,(2\sqrt{\pi}\,i\lambda\,\varphi(x))\,\vdots \quad . \tag{10}$$

Challifour[5] has shown that, for λ in a complex neighborhood of the origin, $E_\lambda(x)$ is defined for fixed time as a distribution in x^1 and satisfies

$$[E_\lambda(x),\psi(y)]_{x^0=y^0} = \theta(x^1-y^1)\,(e^{-2\pi i\lambda}-1)\,\psi(y)\,E_\lambda(x) \tag{11}$$

as follows formally from (5).

To compare our triple-dot product with other formulations, we remark that on a formal level (or with an appropriate cut-off)

$$E_\lambda(x) = \frac{e^{2\sqrt{\pi}\,i\lambda\,\varphi(x)}}{\langle e^{2\sqrt{\pi}\,i\lambda\,\varphi(x)}\rangle_0} \quad .$$

A number of authors (e.g. Coleman[2]) define products $N(\varphi^n)$ by subtracting only two-point correlations, with $\langle\varphi(x)\varphi(0)\rangle_0$ or $i\Delta^{(+)}(x,\mu^2)$ (for some mass μ) as a contraction function. For exponentials the connection with $E_\lambda(x)$ is given by

$$E_\lambda(x) = \frac{N(e^{2\sqrt{\pi}i\lambda\varphi(x)})}{\langle N(e^{2\sqrt{\pi}i\lambda\varphi(x)})\rangle_0} \quad . \tag{12}$$

Hence, if both definitions are legitimate, they differ only by a c-number. It has been shown by Schroer and Truong[4]) that $N(e^{2\sqrt{\pi}i\lambda\varphi})$ ceases to be meaningful for $\lambda\to\pm 1$.

III. Coleman's equivalence suggests that the exponential E_λ should, for $\lambda=\pm 1$, be related to $:\bar{\psi}(1\pm\gamma^5)\psi:$. To investigate this, it is appropriate to re-express E_λ , defined by triple-point ordering, as an ordinary Wick-product of the free field creation and annihilation operators.

We show the following:

$$E_\lambda(x) = :\exp\,\varphi_\lambda(x):$$

$$\varphi_\lambda(x) = \int d^2p\,d^2q\,e^{i(p-q)x}\,\delta(p^2-m^2)\,\delta(q^2-m^2):A^+(p)A(q):\,K_\lambda(p,q) \tag{13}$$

$$K_\lambda(p,q) = \frac{2i}{\sqrt{\pi}}\sin\pi\lambda\,\left|\frac{pq-\varepsilon^{\mu\nu}p_\mu q_\nu}{m^2}\right|^\lambda e^{\frac{i\pi}{2}\lambda[\varepsilon(p^0)+\varepsilon(q^0)]}K(p,q) \tag{14}$$

with $K(p,q)$ defined in (8). ($\varepsilon^{\mu\nu}$ is the antisymmetric tensor with $\varepsilon^{01}=1$).

For an elementary proof of this relation we note first that a well known theorem on Wick-ordering of an exponential, whose exponent is a bilinear functional of a free field, states:

$$E_\lambda(x) = :\exp\{(e^{2\sqrt{\pi}i\lambda\varphi})_2\}: \quad . \tag{15}$$

The symbol $(e^{2\sqrt{\pi}i\lambda\varphi})_2$ denotes the connected two-operator part in the Wick expansion of $e^{2\sqrt{\pi}i\lambda\varphi}$ obtained as the sum of all pairings which

a) leave two operators unpaired

b) do not contain loops .

Since $(e^{2\sqrt{\pi}i\lambda\varphi})_2$ is bilinear in the Dirac operators, we can represent it as

$$(e^{2\sqrt{\pi}i\lambda\varphi(x)})_2 = \int d^2p\,d^2q\,e^{i(p-q)x}\,\delta(p^2-m^2)\,\delta(q^2-m^2):A^+(p)A(q):\,K_\lambda(p,q)\,. \tag{16}$$

To determine the kernel $K_\lambda(p,q)$ we expand (16) as a power series in λ :

$$(\varphi^{n+1}(x))_2 = \int d^2p\,d^2q\,e^{i(p-q)x}\,\delta(p^2-m^2)\,\delta(q^2-m^2):A^+(p)A(q):K_{(n)}(p,q) \tag{17}$$

$$K_\lambda(p,q) = \sum_{n=0}^{\infty}\frac{(2\sqrt{\pi}i\lambda)^{n+1}}{(n+1)!}K_{(n)}(p,q)\;;\qquad K_{(0)}(p,q) = K(p,q)$$

We indicate contractions by

$$\overline{A(q)A^+(p)}\,\delta(p^2-m^2)\delta(q^2-m^2) = \langle A(q)A^+(p)\rangle_0\,\delta(p^2-m^2)\delta(q^2-m^2) = \theta(p^0)\,\delta^{(2)}(p-q)\,\delta(p^2-m^2) \tag{18}$$

$$\underline{A^+(p)A(q)}\,\delta(p^2-m^2)\delta(q^2-m^2) = \langle A^+(p)A(q)\rangle_0\,\delta(p^2-m^2)\delta(q^2-m^2) = \theta(-p^0)\,\delta^{(2)}(p-q)\,\delta(p^2-m^2)$$

and we denote by

$$\overline{\varphi(x)\varphi(y)} \qquad\text{and}\qquad \underline{\varphi(x)\varphi(y)}$$

a contraction of the corresponding operators from $\varphi(x)$ and $\varphi(y)$.

In this notation

$$(\varphi(x)\varphi(y))_2 = :\overline{\varphi(x)\varphi}(y): + :\underline{\varphi(x)\varphi}(y): \quad .$$

The relation

$$(\varphi^n(x)\varphi(y))_2 = :\overline{(\varphi^n(x))_2\varphi}(y): + :\underline{(\varphi^n(x))_2\varphi}(y): + \sum_{r=1}^{n-1}\binom{n}{r}:\overline{(\varphi^r(x))_2\underline{(\varphi^{n-r}(x))_2\varphi}}(y): \tag{19}$$

can be verified by inspection.

Identifying x and y we have

$$(\varphi^{n+1})_2 = :(\varphi^n)_2\varphi: + :(\varphi^n)_2\varphi: + \sum_{r=1}^{n-1}\binom{n}{r}:(\varphi^r)_2(\varphi^{n-r})_2\varphi: \qquad (20)$$

which leads, when inserted into (17) to a quadratic recursion formula for the kernels $K_{(n)}(p,q)$:

$$K_{(n)}(p,q) = \int d^2k\,\delta(k^2-m^2)\left[\theta(k^0)K_{(n-1)}(p,k)K(k,q) + \theta(-k^0)K(p,k)K_{(n-1)}(k,q)\right] \qquad (21)$$

$$-\sum_{r=1}^{n-1}\binom{n}{r}\int d^2k\,d^2\ell\,\theta(k^0)\theta(-\ell^0)\delta(k^2-m^2)\delta(\ell^2-m^2)K_{(n-1-r)}(p,k)K(k,\ell)K_{(r-1)}(\ell,q)\,.$$

Although Eq.(21) could be handled as it stands, it is easier to first sum over n as in (17) in which case the following functional equation results:

$$\frac{\partial}{\partial\lambda}K_\lambda(p,q) = 2\sqrt{\pi}\,i\int d^2k\left\{K(p,k) + \int d^2\ell\,\theta(\ell^0)\delta(\ell^2-m^2)K_\lambda(p,\ell)K(\ell,k)\right\}\times$$

$$\times\left\{\delta^{(2)}(k-q) - \theta(-k^0)\delta(k^2-m^2)K_\lambda(k,q)\right\}\,. \qquad (22)$$

We can now verify that $K_\lambda(p,q)$ as given in (14) satisfies this equation and is therefore the unique solution of the recursion formula (21). To show this we note the following relations, obtained by standard contour integration:

$$K(p,q) + \int d^2k\,\theta(k^0)\delta(k^2-m^2)K_\lambda(p,k)K(k,q) = \frac{\sqrt{\pi}}{2i}\frac{e^{i\pi\lambda}}{\sin\pi\lambda}K_\lambda(p,q) \qquad (23)$$

$$\int d^2k\,\theta(-k^0)\delta(k^2-m^2)K_\lambda(p,k)K_\lambda(k,q) = \sin\pi\lambda\, e^{-i\pi\lambda}\left\{iK_\lambda(p,q) - \frac{\sin\pi\lambda}{\pi}\frac{\partial}{\partial\lambda}\left[\frac{K_\lambda(p,q)}{\sin\pi\lambda}\right]\right\}.$$

(The expression (14) for $K_\lambda(p,q)$ can be guessed by working out the first few terms in (21)).

IV. It is a remarkable feature of the Wick-ordered expression $E_\lambda = :\exp\varphi_\lambda:$ given in (13) and (14) that φ_λ is a product of $\sin\pi\lambda$ multiplied by an operator which is non-singular for integer values of λ .

An elementary evaluation gives

$$\varphi_\lambda \xrightarrow[\lambda\to 1]{} -\frac{\sin\pi\lambda}{\pi}\left\{2\sqrt{\pi}\,i\varphi + \frac{4\pi}{m}:\bar\psi\left(\frac{1+\gamma^5}{2}\right)\psi:\right\}. \qquad (24)$$

To establish (24) we have made use of

$$:\bar\psi\left(\frac{1+\gamma^5}{2}\right)\psi: = :\psi_1^+\psi_2: = \int d^2p\,d^2q\,e^{i(p-q)x}\delta(p^2-m^2)\delta(q^2-m^2):A^+(p)A(q):\frac{|pq-\varepsilon^{\mu\nu}p_\mu q_\nu|^{1/2}\varepsilon(p^0)}{2\pi}\,.$$

In terms of bilinear forms, i.e. if matrix elements are taken between suitable states, we derive from (24) the following relations:

$$E_\lambda\Big|_{\lambda=\pm 1} = 1 \quad . \qquad (25)$$

Differentiating (10) with respect to λ, we obtain

$$\frac{\partial}{\partial\lambda}E_\lambda = 2\sqrt{\pi}\,i\,\vdots\,\varphi\exp(2\sqrt{\pi}\,i\lambda\varphi)\,\vdots = :\left(\frac{\partial}{\partial\lambda}\varphi_\lambda\right)\exp(\varphi_\lambda): \quad .$$

For $\lambda = \pm 1$:

$$2\sqrt{\pi}\,i\,\vdots\,\varphi\exp(\pm 2\sqrt{\pi}\,i\varphi)\,\vdots = \frac{\partial\varphi}{\partial\lambda}\Big|_{\lambda=\pm 1}$$

and from (24)

$$\frac{\partial\varphi}{\partial\lambda}\Big|_{\lambda=\pm 1} = 2\sqrt{\pi}\,i\varphi \pm \frac{4\pi}{m}:\bar\psi\left(\frac{1\pm\gamma^5}{2}\right)\psi: \quad .$$

Therefore,

$$\vdots\,\varphi\left[\exp(\pm 2\sqrt{\pi}\,i\varphi) - 1\right]\vdots = \pm\frac{2\sqrt{\pi}}{im}:\bar\psi\left(\frac{1\pm\gamma^5}{2}\right)\psi: \qquad (26)$$

or

$$-\frac{m}{\sqrt{\pi}}\,\vdots\,\varphi\sin(2\sqrt{\pi}\varphi)\,\vdots = :\bar\psi\psi: \qquad (27)$$

$$\frac{im}{\sqrt{\pi}}\,\vdots\,\varphi\left[\cos(2\sqrt{\pi}\varphi) - 1\right]\vdots = :\bar\psi\gamma^5\psi: \quad .$$

Inserting the last line into (4) we obtain an "equation of motion" for φ :

$$\Box\varphi = 2m^2 :\varphi[\cos(2\sqrt{\pi}\varphi)-1]: \qquad (28)$$

which is invariant under $\varphi \to \varphi + n\sqrt{\pi}$ if

$$:\cos(2\sqrt{\pi}\varphi): = 1 \ .$$

We have to show now that the relations (25) to (28) are indeed operator equations. To indicate this, we consider the set of all states Ψ (of finite energy) which are obtained by applying polynomials $P(\psi(f), \psi^+(f))$ to the vacuum state. Here

$$\psi(f) = \int d^2x\, \psi(x)\, f(x)$$

with test functions of compact support in momentum space. $\tilde{f}(k) \in \mathcal{D}(R^2)$.

Further, we define

$$E_\lambda(g) = \int d^2x\, E_\lambda(x)\, g(x) \qquad (29)$$

with $\tilde{g}(k) \in \mathcal{D}(R^2)$. $E_\lambda(g)$ when applied to a state can change its energy by only a finite amount and the power series expansion

$$E_\lambda(g)\Psi = \sum_{n=0}^{N} \frac{1}{n!} :\varphi_\lambda^n:(g)\,\Psi \qquad (30)$$

contains only finite powers of φ_λ , i.e. finite numbers of annihilation and creation operators, since Ψ and g have been restricted accordingly.

Therefore, the norm of $E_\lambda(g)\Psi$ is finite since it can be expressed, using (13), as a finite sum of integrals over compact domains.

This shows that all relations obtained in this section are correct operator equations under the given conditons. For $\lambda = \pm 1$ these conditions can be relaxed. In particular

$$\lim_{g\to\delta^{(2)}(x)} \lim_{\lambda\to\pm 1} E_\lambda(g) = 1 \qquad (31)$$

and Eqs.(26) to (28) are correct with tempered $f, g \in S(R^2)$, since the right hand side of (27) is defined as a tempered operator-valued distribution.

We conclude that the Eqs.(26) to (28) replace, for a free Dirac field, the corresponding expression given by Coleman. How Eq.(28) is related to an equation of motion of the quantized sine-Gordon field (for $\beta^2 = 4\pi$) can only be decided after the operators in the sine-Gordon model have been properly defined. This has not yet been done.[6)]

The authors have profited from discussions with D. Buchholz and K. Fredenhagen.

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