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2 HAMBURG 52 · NOTKESTIEG 1

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Abstract

We investigate the meaning of the (infinitely many) conserved non-trivial currents of the classical Sine-Gordon equation for conventional quantum perturbation theory of a scalar field φ with selfinteraction $(\cos(\beta\varphi)-1+\frac{(\beta\varphi)^2}{2})$. Radiation corrections produce for all currents anomalies with contributions on the mass shell.

It is well known that the integrability of the classical Sine-Gordon equation in 1 + 1 dimensions

$$\left(\frac{\partial^2}{\partial t} - \frac{\partial^2}{\partial x} \right) \phi + \frac{1}{\beta} \sin\beta\phi \quad = \quad 0 \tag{1}$$

is neatly connected with the existence of infinitely many conserved local currents[1]. The respective current densities can be found recursively from the so called trace identities[1]. For a derivation of these identities we refer to the work of Zakkarov and Shabat[2]. We only quote the results. To make expressions simpler we use characteristic coordinates.

$$u = \frac{1}{2}(x + t), \qquad v = \frac{1}{2}(x - t)$$

Equ. (1) becomes in these coordinates

$$\frac{\partial^2 \phi}{\partial u \, \partial v} = \frac{1}{\beta} \sin\beta\phi \quad . \tag{2}$$

Define quantities $j_k^{(v)}$ by the recursion relation

$$j_n^{(v)} = \sum_{i+k=n-1} j_i^{(v)} j_k^{(v)} + \phi_{,u} \frac{d}{du} \frac{j_{n-1}^{(v)}}{\phi_{,u}} \tag{3}$$

$$j_1^{(v)} = \frac{\beta^2 \phi^2_{,u}}{4} \quad \left(\phi_{,nu} = \left(\frac{\partial}{\partial u}\right)^n \phi \right)$$

Use the equation of motion to calculate

$$\frac{d}{dv} j_n^{(v)} = \frac{d}{dv} j_n^{(u)} (\phi_{,u}, \cdots \phi_{,nu}) = \sum_{i=1}^{n} \frac{\partial j_n^{(v)}}{\partial \phi_{,iu}} \left(\frac{d}{du}\right)^{i-1} \frac{(\sin \beta\phi)}{\beta} \equiv$$

$$\equiv p_n(\phi, \phi_{,u}, \ldots \phi_{nu})$$

The content of the trace identities is simply that p_n can be expressed identically as a total derivative with respect to u

$$\frac{d}{dv} j_n^{(v)} = p_n \equiv \frac{d}{du} j_n^{(u)} (\phi, \phi_{,u} \cdots \phi_{,(n-1)u}) \tag{4}$$

Interchanging the roles of u and v one obtains another series of conservation laws independent from (4).

$$\bar{j}_n^{(u)} = j_n^{(v)}(\phi_{,v}\cdots\phi_{,nv})$$

$$\frac{d}{du}\,\bar{j}_n^{(u)} = p_n(\phi,\phi_{,v},\dots,\phi_{,nv}) \equiv \frac{d}{dv}\,\bar{j}_n^{(v)} \qquad (4a)$$

$$\bar{j}_n^{(v)} = j_n^{(u)}(\phi,\phi_{,v},\dots\phi_{,(n-1)v})$$

The first two conservation laws $(n=1)$ in (4) and (4a) give energy and momentum conservation. We are interested in the densities j_{2n+1}, $\bar{j}_{2n+1}$ $(1\leq n\leq\infty)$. (The densities with an even index lead to zero charges.)

To start with we look, in tree graph approximation, for consequences of the conservation laws (4) and (4a) for a quantized scalar field ϕ with selfinteraction $\frac{\cos\beta\phi}{\beta^2} - \frac{1}{\beta^2} + \frac{\phi^2}{2} = \mathcal{L}(\phi)$ +. For concreteness we concentrate on the density j_3. The arguments apply equally well to all the other currents j_n, $\bar{j}_n$ $(n\geq 3)$.
We decompose, by an elementary calculation, the integrated Ward identity

$$0 = \int dz_1 \frac{\partial}{\partial v_1} \langle T(:j_3^v:(z_1) \prod_{i=2}^{n} \phi(z_1))\rangle_{tr}$$

With $\langle T(\dots)\rangle_{tr}$ we denote the Gell-Mann-Law expansion of a time ordered vacuum expectation value in tree graph approximation. The double points indicate Wick normal ordering.

$$\begin{aligned} 0 &= \int d^2z_1 \frac{\partial}{\partial v_1} \langle T(:j_3^v:(z_1) \prod_{i=2}^{n}\phi(z_i))\rangle_{tr} \\ &= \int d^2z_1 \langle T\{:(\beta^4\langle -\phi_u^3(\Box + 1)\,\phi + \phi_u^3\,\phi\rangle + \beta^2\,\langle -\phi_{3u}(\Box + 1)\,\phi - \\ &\quad - \phi_u\cdot(\Box + 1)\,\phi_{2u} + \phi_{3u}\,\phi + \phi_u\,\phi_{2u}\rangle):(z_1) \prod_{i=2}^{n} \phi(z_i)\}\rangle_{tr} \\ &= \int dz_1 \frac{\partial}{\partial u_1} \langle T(\beta^2:\phi_{2u}\frac{\sin\beta\phi}{\beta}:(z_1) \prod_{i=2}^{n}\phi(z_i))\rangle_{tr} \\ &\quad + i\sum_{\ell=2}^{n} \langle T(\langle 2\beta^2\,\phi_{3u} + :\beta^4\,\phi_u^3 :\rangle\,(z_\ell) \prod_{\substack{i\neq\ell\\ i=2}}^{n} \phi(z_i))\rangle_{tr}, \quad \Box = \frac{-\partial^2}{\partial u\,\partial v} \end{aligned} \qquad (5)$$

+ We choose units such that the mass of ϕ is equal to 1.

Restricting (5) to the mass shell one obtains after Fourier transformation a relation for the S-matrix

$$0 = \sum_i {}_iP^3_{u,in} + \sum_j {}_jP^3_{u,out} \tag{6}$$

${}_iP_{u,in(out)}$ denote the momenta of the asymptotic particles conjugate to the variables u_i. To arrive at (6) from (5) one has only to note that the first term in (5) $\int d^2z_1 \frac{\partial}{\partial u_1}(\ldots)$ drops out per se and that from the second term the part with a factor β^4 vanishes on the mass shell because of the nonlinearity in the field.

Equ. (6) is the first of an infinite number of relations for the S-matrix emerging from the conservation laws (4), (4a):

$$\left.\begin{aligned} &\sum_i {}_iP^{2n+1}_{u,in} + \sum_j {}_jP^{2n+1}_{u,out} = 0 \\ &\sum_i {}_iP^{2n+1}_{v,in} \quad \sum_j {}_jP^{2n+1}_{v,out} = 0 \end{aligned}\right\} \quad n = 0, 1, 2 \ldots \tag{7}$$

It is an observation due to Polyakov [3] that the relations (7) are equivalent to the statement: the S-matrix is a pure phase.

Turning now to the radiation corrections of equ. (5) we collect several pieces of information.

(I) Whatever normal product prescription " $:\;:$ " we choose for the current density

$$j_3^{(v)} = \frac{\phi_u^4}{4} + \phi_u\,\phi_{3u}$$

(compatible with generalized unitarity and causality) we can write down an identity of the form

$$\begin{aligned} 0 &= \int d^2z_1 \frac{\partial}{\partial v_1} \langle T\left(:\left(\frac{\phi_u^4}{4} + \phi_u\,\phi_{3u}\right):(z_1) \prod_{i=2}^{n} \phi(z_i)\right)\rangle \\ &= 2i \sum_{\ell=2}^{n} \langle T(\phi_{3u}(z_\ell) \prod_{\substack{i=2 \\ i\neq\ell}}^{n} \phi(z_i))\rangle + R \end{aligned}$$

where the (unambigously given) term $2i \sum_{\ell=2}^{n}(\ldots)$ comes from the contractions of the bilinear part of j_3^v with an external propagator leg. The relation (6) for the S-matrix will be true after the inclusion of radiation corrections if and only if the rest R does not contribute on the mass shell. This makes us, in a sense, independent of the peculiarities of any chosen subtraction scheme.

(II) The requirements of a minimal number of subtractions and Lorentz covariance determine uniquely the Green's functions of an arbitrary set of Wick ordered monomials in ϕ with any number of derivatives (in $\frac{\partial}{\partial u}$ or $\frac{\partial}{\partial v}$ exclusively) distributed over one monomial.

Proof: It follows from general principles [4,5] that it is always possible to define minimally subtracted Green's functions, of any collection of fields, which carry as distributions only those representations of the Lorentz group which occur in the tensor product of the same collection of fields. Let $P(\phi)$ be a monomial in the field ϕ with ℓ derivatives $\frac{\partial}{\partial u}$, e.g.

$$P(\phi) = :\phi_u^{i_1} \ldots \phi_{nu}^{i_n}: \left(\sum_{j=1}^{n} i_j \cdot j = \ell \right) \tag{8}$$

Consider a minimally subtracted covariantly defined Green's function

$$<_o T(:P(\phi):(x) \prod_{i=1}^{k} :\phi^{k_i}(y_i):)_o> \tag{9}$$

From the point of a minimal subtraction scheme the allowed addition to (9) is a distribution concentrated at the coincidence point $x = y_1 = \ldots = y_k$, that is a δ-function with $(\ell-2k)$ derivatives at most. The latter number can be derived from power counting for scalar fields in 2 dimensions.
Lorentz covariance on the other hand requires that the δ-function addition is of the form

$$\prod_{i=1}^{k} \left(\frac{\partial}{\partial u_i}\right)^{m_i} \delta(x-y_1) \ldots \delta(x-y_k),$$

$$\sum m_i = \ell, \qquad u_i = (x_u - y_{i,u}).$$

We see that the conditions of a minimal number of subtractions and Lorentz covariance clash. This completes the proof.

(III) Let $P(y)$ be a monomial of the form given in (8). Consider the free field time ordered vacuum expectation values

$$<T(P(\phi)\ (x_1), :\phi^{\Sigma i_k}:\ (x_2))>,\quad \sum_k i_k <T(:\phi^{i_1}_{,u}\cdots\phi^{i_k - 1}_{ku}\cdots\phi^{i_n}_{nu}\ \phi_{(k-1)u} :(x_1)$$

$$, :\phi^{\Sigma i_k} :(x_2))> \equiv \tilde{T}(x_1 - x_2)$$

obeying the conditions of (II). There is a number $C(i_1 \ldots i_n) \neq 0$ such that

$$\frac{\partial}{\partial v_1} <T(P(\phi)(x)_1 :\phi^{\Sigma i_k}: (x)_2> - \tilde{T}(x_1 - x_2) = C(i_1 \ldots i_n)\left(\frac{d}{du_1}\right)^{\ell-1} \delta(x_1 - x_2) \tag{10}$$

Note that the absorptive part of (10) vanishes. For reasons exploited already in (II) the right hand side of (10) can be only of the given form. By inspection of the behaviour at large momenta of the two terms on the left hand side of (10) one easily verifies that $C(i_1 \ldots i_n)$ must be non-zero. This fact is the source of the anomalies.

Putting results (I), (II) and (III) together we inspect now the general Gell-Mann-Low expansion of the integrated Ward identity (5)

$$\begin{aligned} 0 &= \int d^2 z_1 \frac{\partial}{\partial v_1} <T(j_3^{(v)}\ (z_1) \prod_{i=2}^{n+1} \phi(z_i))> \underset{\text{G-M-L}}{\equiv} \\ &\equiv \int d^2 z_1 \frac{\partial}{\partial v_1} <T(j_3^{(v)}\ (z_1) \prod_{i=2}^{n+1} (\ldots)\ \ell^{i\int \mathcal{L} d^2 x})> \end{aligned} \tag{11}$$

Every single term in this formal series is according to the remarks in (II) uniquely determined under the given conditons. (Without mentioning it further we suppose all Green's functions we speak about to be renormalized by the unique "minimal" and covariant prescriptions). The strategy we follow to evaluate (11) consists (as in the tree graph calculations above) in an application of the equation of motion for ϕ including anomalies (c.f. (III)), which we organize as counter terms.

$$\begin{aligned} 0 &= \int d^2 z_1 \frac{\partial}{\partial v_1} <T(j_3^{(v)}\ (z_1) \prod_{i=2}^{n+1} \phi(z_i))>_{\text{G-M-L}} = \\ &= \int d^2 z_1 \, (<T\{:(-\beta^4\ \phi_u^3(\Box + 1)\ \phi - \beta^2(\phi_{3u}(\Box + 1)\ \phi + \\ &\quad + \phi_u(\Box + 1)\ \phi_{2u})):(x_1) \prod_{i=2}^{n+1} (\ldots)\}>_{\text{G-M-L}} + \end{aligned}$$

$$+ \langle T\{:(\beta^4\ \phi_u^3\ \phi + \beta^2\ (\phi\ \phi_{3u} + \phi_u\ \phi_{2u})):(z_1)\prod(\ldots)\}\rangle_{G-M-L} \tag{12}$$

$$= \int d^2z_1 \Big[\frac{\partial}{\partial u_1} \langle T\{:\beta^2\ \phi_{2u}\ \frac{\sin\beta\phi}{\beta}\ :\ (z_1),\prod(\ldots)\}\rangle$$

$$+ i \sum_{\ell=2}^{n+1} \delta(z_1 - z_\ell)\langle T\{:(\beta^4\ \phi_u^3 + 2\beta^2\ \phi_{3u}):(z_\ell)\ \prod_{i\neq\ell}\ \ldots\}\rangle$$

$$+ \beta^4 \langle \beta^2\ \frac{C(4)}{4!} - \frac{C(1,0,3)}{\beta^2 2!}\rangle \left(\frac{\partial}{\partial u_1}\right)^3 \langle T\{:\cos\beta\phi:(z_1)\prod\ldots\ \}\rangle$$

$$+ \frac{\beta^5}{3!}\ C(3) \langle T\{:\phi_u \left(\frac{\partial}{\partial u_1}\right)^2 \sin\beta\phi:(z_1)\prod\ldots\ \}\rangle \tag{12a}$$

$$- \frac{3\beta^4}{4}\ C(2)\langle T\{:\phi_u^2\ \frac{\partial}{\partial u_1}\ \cos\beta\phi:(z_1)\prod\ldots\}\rangle\Big]_{G-M-L}$$

$$= i \sum_{\ell=2}^{n+1} \langle T\{:(\beta^4\ \phi_u^3 + 2\beta^2\ \phi_{3u}):(z_1)\prod_{i\neq\ell}(\ldots)\}\rangle_{G-M-L} \tag{12b}$$

$$+ \int d^2z_1\ \ \beta^5 (C(2)\frac{3}{4} - \frac{\beta^2}{2\cdot 3!}\ C(3))\ \ \langle T\{:(\phi_u^3\ \sin\beta\phi):(z_1)\ \prod_{i=2}^{n+1}(\ldots)\}\rangle_{G-M-L}$$

To justify the equality (12) = (12a) we remark that the listed anomalies with coefficients C(...), which spoil the naiveté of the tree graph calculation, emerge generically from graphs with two vertices. This may be alternatively expressed: An anomalie which apprears for the first time in a graph with more than two vertices would give there a δ-function contribution with the same Lorentz covariance as the normal term. However, such a δ-function is excluded by power counting arguments (c.f. (II)).
Restricting (12b) to the mass shell we see that in general (if $\beta^2 \neq \frac{C(2)}{C(3)}\frac{9}{2} = 4\pi$) the anomalous terms with coefficients C(...) are not cancelled.
It follows from the considerations under (I) that the S-matrix is not a pure phase and that no over subtraction scheme for the renormalization of the current will cure this on mass shell anomaly. We have so far not been able to verify the interesting conjecture that the anomalies of all currents j_n, $\bar{j}_n$ $(n \geq 3)$ happen to drop out for the same value of the coupling constant.

One should note that the exceptional value $\beta^2 = 4\pi$ is the one for which Coleman [6] asserts an equivalence of the Sine-Gordon equation and the massive free Dirac field.

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