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Mass Formulas for Broken SU(4)

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Abstract: We derive mass formulas for broken SU(4) using Okubo's tensor formalism. The mass splitting term is a superposition of an SU(3) scalar operator and the I = 0 component of the SU(3) octet. To accomodate the new resonances, mixing between the SU(4) singlet and fifteenplet is discussed. In addition we consider also the electromagnetic mass shift.

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## 1. Introduction

Recently SU(4) symmetry already considered earlier [1] had a revival in connection with the discovery of the new mesons $\Psi(3.1)$, $\Psi(3.7)$ and $\Psi(4.1)$ [2]. One way to understand these new resonances is their interpretation as vector mesons composed predominantly of charmed quarks. The basic symmetry of strong interaction then becomes SU(4) instead of SU(3). One considers the $\Psi$ as part of an SU(4) fifteenplet together with the usual vector mesons $\rho$, $K^*$, $\omega$ and $\phi$. Because of the large mass differences inside such a multiplet SU(4) must be badly broken. In principle the structure of the symmetry breaking term is unknown. To incorporate the usual well satisfied SU(3) mass splitting, we assume that the breaking operator $H'$ has a term $H_8$ which transforms as the $I = 0$ component of the SU(3) octet. The mass splitting of the different SU(3) submultiplets inside an SU(4) multiplet is described by $H_{15}$ which transforms as the SU(3) scalar component of the SU(4) fifteenplet, so that $H' = H_8 + H_{15}$. For the tensors of SU(4) we use the Okubo notation. Then it is easy to derive the Wigner-Eckart theorem for matrix elements of SU(4) tensor operators within a given SU(4) multiplet. We find that the matrix elements of $H'$ and also of the electromagnetic current $J_\mu$ can be expressed by the eigenvalues of various operators which appear in the reduction $SU(4) \supset SU(3) \otimes U(1)$ and $SU(4) \supset SU(2) \otimes SU(2) \otimes U(1)$.[3]

In section 2 we derive the mass formulas for broken SU(4) neglecting electromagnetism and apply them to the 15-plet representation of the vector mesons. In section 3 we consider mixing between singlet and fifteenplet which leads to a five parameter mass formula.[+)]

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[+)] This problem has been considered also in ref. 4.

This reduces to a three parameter formula if ideal mixing is imposed. In section 4 we take up the problem of electromagnetic mass splitting for both, the unmixed and the mixed case. In section 5 we draw some conclusions.

## 2. Mass Formulas for Broken SU(4)

In this section we shall derive mass formulas for SU(4) multiplets which are the generalization to SU(4) of the well-known Okubo mass formula in SU(3). We shall consider the mass splitting inside the full SU(4) multiplet and not just the mass splitting between SU(3) submultiplets [5]. In SU(3) the mass operator H is

$$H = H_o + H_1 \quad , \tag{2.1}$$

where $H_o$ is the SU(3) symmetric part and $H_1$ transforms like the isoscalar generator of SU(3), the hypercharge operator $Y_8 \sim F_8$.

In analogy in SU(4) the mass operator is

$$H = H_o + H_1 + H_2 \quad , \tag{2.2}$$

where $H_o$ is now a SU(4) scalar and $H_1$ and $H_2$ transform as $F_8$ and $F_{15}$ . $F_8$ is the isoscalar generator of the SU(3) octet and $F_{15}$ is the SU(3) - scalar generator of SU(4).

For the derivation we need the properties of the SU(4) tensor operators. We shall use the Okubo notation for these. Then it is a simple matter to derive the Wigner-Eckart theorem for matrix elements of a SU(4) tensor operator within a given SU(4) multiplet.

Usually one denotes the SU(4) generators by $F_i$ (i = 1,2,....., 15) which obey

$$[F_i, F_j] = i f_{ijk} F_k \tag{2.3}$$

and which can be represented in terms of a set of standard hermitean matrices $\lambda_i$ by $F_i = \frac{1}{2}\lambda_i$ when acting on the quadruplet representation of SU(4). In the Okubo notation the SU(4) generators are given by a set of sixteen operators $F^i_{\ j}$ (i,j = 1,2,3,4)

$$F^i_{\ j} = \sum_{n=1}^{15} F_n (\lambda_n)_{ij} \tag{2.4}$$

which satisfy

$$[F^i_{\ j}, F^k_{\ l}] = \delta^i_l F^k_{\ j} - \delta^k_j F^i_{\ l} \tag{2.5}$$

$$\sum_{i=1}^{4} F^i_{\ i} = 0 \tag{2.6}$$

$$(F^i_{\ j}) = (F^j_{\ i})^+ \tag{2.7}$$

The $F^i_{\ j}$ transform against charge conjugation $\mathcal{C}$ as:

$$\mathcal{C} F^i_{\ j} \mathcal{C}^\dagger = - F^j_{\ i} . \tag{2.8}$$

Now, for any regular tensor operator $T^i_{\ j}$ we have within a given irreducible representation R of SU(4) [6)]

$$T^i_{\ j} = a\, F^i_{\ j} + b \sum_{k=1}^{4} F^i_{\ k}\, F^k_{\ j} + c \sum_{k,l=1}^{4} F^i_{\ k}\, F^k_{\ l}\, F^l_{\ j} \ , \qquad (2.9)$$

where a,b and c are reduced matrix elements. In the special case that $R \otimes 15$ contains R at most twice, which covers all the cases of physical interest, we drop the last term (c = o), so that we have only two reduced matrix elements.

In accordance with (2.2) the mass operator H is:

$$H = m_o T_o + m_{33}\, T^3_{\ 3} + m_{44}\, T^4_{\ 4} \ , \qquad (2.10)$$

where $T_o$ is the SU(4) symmetric part, while $T^3_{\ 3}$ transforms like a linear combination of $F_8$ and $F_{15}$, and $T^4_{\ 4}$ is proportional to $F_{15}$. The exact relation will be given below.

As we show later $T^4_{\ 4}$ can be expressed in terms of those operators whose eigenvalues label the states of the SU(3) submultiplets. This is no longer true for the second term $T^3_{\ 3}$ which describes the SU(3) splitting. The SU(3) states are characterized by the eigenvalues of $I^2$, $I_3$ and $Y_8$ . But $T^3_{\ 3}$ cannot be expressed by them . On the contrary $T^3_{\ 3}$ connects states of the same $I^2$,$I_3$ and $Y_8$ within different SU(3) submultiplets of an SU(4) multiplet. This is similar to the situation occurring in SU(3) in connection with the electromagnetic mass splitting where different I-spin multiplets are mixed. It is wellknown that the electromagnetic mass splitting in SU(3) is simplified considerably if another SU(2) subgroup of SU(3) , namely U-spin, is used to label the states. The situation is quite similar in connection with $T^3{}_3$ where part of it can be expressed by operators of another SU(2) subgroup of SU(4), called Z-spin.

Therefore we express $T^3{}_3$ and $T^4{}_4$ in (2.10) through the generators of the different SU(2) - subgroups of SU(4)[3]. Whereas in SU(3) we had only three SU(2) - subgroups defining the I-, U- and V-spin, respectively, in SU(4) we have six different SU(2) -subgroups which define six spins: I,U,V,W,X and Z:

$$
\begin{aligned}
&I_- = F^1{}_2 = F_1 - iF_2 , \qquad I_+ = F^2{}_1 = F_1 + iF_2 ,\\
&\quad I_3 = \tfrac{1}{2}(F^1{}_1 - F^2{}_2) = F_3 ;\\[2ex]
&U_- = F^2{}_3 = F_6 - iF_7 , \qquad U_+ = F^3{}_2 = F_6 + iF_7 ,\\
&\quad U_3 = \tfrac{1}{2}(F^2{}_2 - F^3{}_3) = -\tfrac{1}{2} F_3 + \tfrac{\sqrt{3}}{2} F_8 ;\\[2ex]
&V_- = F^1{}_3 = F_4 - iF_5 , \qquad V_+ = F^3{}_1 = F_4 + iF_5 ,\\
&\quad V_3 = \tfrac{1}{2}(F^1{}_1 - F^3{}_3) = \tfrac{1}{2} F_3 + \tfrac{\sqrt{3}}{2} F_8 ;\\[2ex]
&W_- = F^1{}_4 = F_9 - iF_{10} , \qquad W_+ = F^4{}_1 = F_9 + iF_{10} ,\\
&\quad W_3 = \tfrac{1}{2}(F^1{}_1 - F^4{}_4) = \tfrac{1}{2} F_3 + \tfrac{1}{2\sqrt{3}} F_8 + \sqrt{\tfrac{2}{3}} F_{15} ;\\[2ex]
&X_- = F^2{}_4 = F_{11} - iF_{12} , \qquad X_+ = F^4{}_2 = F_{11} + iF_{12} ,\\
&\quad X_3 = \tfrac{1}{2}(F^2{}_2 - F^4{}_4) = -\tfrac{1}{2} F_3 + \tfrac{1}{2\sqrt{3}} F_8 + \sqrt{\tfrac{2}{3}} F_{15} ;\\[2ex]
&Z_- = F^3{}_4 = F_{13} - iF_{14} , \qquad Z_+ = F^4{}_3 = F_{13} + iF_{14} ,\\
&\quad Z_3 = \tfrac{1}{2}(F^3{}_3 - F^4{}_4) = -\tfrac{1}{\sqrt{3}} F_8 + \sqrt{\tfrac{2}{3}} F_{15} ;\\[2ex]
&F_8 = -\tfrac{1}{2\sqrt{3}}(3F^3{}_3 + F^4{}_4), \qquad F_{15} = -\sqrt{\tfrac{2}{3}} F^4{}_4 .
\end{aligned}
\tag{2.11}
$$

It is now easy to obtain from (2.9) for c = o:

$$T^3{}_3 = (a-2b)\left(-Y_8 + \tfrac{1}{3} Y_{15}\right) + \frac{b}{3}\left[3\left(\vec{Z}^2 - \vec{I}^2\right) + \left(\mathscr{C}_3 - Y_{15}^2\right)\right], \qquad (2.12)$$

$$T^4{}_4 = b\,\mathscr{C}_4 - (a-2b)\,Y_{15} - \frac{b}{3}\left(\mathscr{C}_3 - Y_{15}^2\right). \qquad (2.13)$$

In (2.12) and (2.13) $\mathscr{C}_4$ is one of the Casimir operators of SU(4)

$$\mathscr{C}_4 = \sum_{i=1}^{15} \left(F_i\right)^2, \qquad (2.14)$$

and $\mathscr{C}_3$ is one of SU(3),

$$\mathscr{C}_3 = 3\sum_{i=1}^{8} \left(F_i\right)^2, \qquad (2.15)$$

whereas $Y_8$ and $Y_{15}$ stand for

$$Y_8 = \frac{2}{\sqrt{3}} F_8 \quad , \qquad Y_{15} = \sqrt{\frac{3}{2}}\, F_{15} \, . \qquad (2.16)$$

According to (2.10) the mass operator is a linear combination of $T^3{}_3$ and $T^4{}_4$ together with an invariant term.

Finally we get the following four-parameter formula for the mass splitting:

$$H = m + (m_1 - m_2)(\mathcal{C}_3 - Y_{15}^2) + 3m_1(\vec{Z}^2 - \vec{I}^2) \qquad (2.17)$$
$$+ 3f m_1 Y_8 - f(m_1 - 3m_2) Y_{15} .$$

Here the contribution of $\mathcal{C}_4$ has been lumped together with that from $T_o$, and we have introduced $m_1 = m_{33}b/3$, $m_2 = m_{44}b/3$, $f = (2b-a)/b$. For mesons (2.17) stands for the squares of the masses. Since mesons are assigned to self conjugate multiplets the C odd terms in (2.17) must vanish, i.e. $f = o$. For baryons we take (2.17) as a formula for the masses themselves where C even and odd terms contribute.

To apply the mass formula to actual cases we need the classification of the SU(4) states in terms of the eigenvalues for the operators appearing in (2.17). According to the usual reduction scheme $SU(4) \supset SU(3) \otimes U(1)$ we can choose for the classification of states the irreducible SU(3) representations (p,q) (in the "highest weight" notation) contained in a given SU(4) multiplet. The states within the SU(3) representations are classified as usual by I, $I_3$ and $Y_8$ . For the additional U(1) we have then $Y_{15}$, which commutes with the SU(3) generators $F_1$, $F_2$ ... $F_8$. The Casimir operator $\mathcal{C}_3$ when acting on states of an SU(3) multiplet (p,q) has the eigenvalues [3]

$$\mathcal{C}_3(p,q) = p^2 + q^2 + pq + 3p + 3q . \qquad (2.18)$$

For the evaluation of the third term in (2.17) we need the classification of SU(4) states according to the reduction $SU(4) \supset SU_I(2) \otimes SU_Z(2) \otimes U_P(1)$, where $P = Y_8 + \frac{2}{3} Y_{15}$ commutes with all components of $\vec{I}$ and $\vec{Z}$. Since $Y_8$ and $Y_{15}$ do not commute with the $\vec{Z}$ operators the formula (2.17) automatically leads to mixing between states belonging to different SU(3) multiplets.

In the following we shall apply (2.17) to problems of current interest. There our SU(4) operators $Y_8$ and $Y_{15}$ are connected in a specific form to the physical operators for hypercharge $Y$, charm C and baryon number B:

$$
\begin{aligned}
B &= \frac{2\sqrt{2}}{3} F_0 \qquad \left( F_0 = \frac{1}{2}\lambda_0 = \frac{1}{2\sqrt{2}} \mathbb{1} \right) \\
Y &= \frac{1}{4} B + Y_8 - \frac{1}{3} Y_{15} , \qquad\qquad (2.19) \\
C &= \frac{3}{4} B - Y_{15} .
\end{aligned}
$$

The charge Q is then

$$
Q = I_3 + \frac{1}{2}(Y + C) = I_3 + \frac{1}{2}(B + Y_8) - \frac{2}{3} Y_{15} . \qquad (2.20)
$$

The definition (2.19) for $Y$ and C corresponds to the usual assignment of the fundamental quark representation (u,d,s,c) with $3Y$ = diag (1,1,-2,1) , C = diag (0,0,0,1) and 3Q = diag (2,-1,-1,2).

As an example let us work out the mass matrix for the 15-plet of the vector mesons $\rho$, $K^*$, $\omega$, $\phi$, $D^*$ and $F^*$.

The classification of these states as SU(3) multiplets and their SU(2) quantum numbers for I, U, W and Z spin together with $Y$, C and $P \sim Y - C$ are given in table 1 . We listed also the U and W spins. They will be used later in connection with the electromagnetic mass splitting.

As is well known the SU(4) fifteenplet consists of an SU(3) singlet, octet, triplet and antitriplet. The triplet has C = 1 and consists of $D^{*+}$, $D^{*0}$ and $F^{*+}$. The I = 0 member of the octet is denoted by $\omega_8$, whereas $\omega_{15}$ is the SU(3) singlet component . The physical $\omega$ and $\phi$ will be the mixings of $\omega_8$ and $\omega_{15}$ .

In terms of these states the I-, Z-, U- and, W- multiplets are:

<u>I-spin:</u> singlets $\omega_8$; $\omega_{15}$; $F^{*+}$; $F^{*-}$;

doublets $(K^{*+}, K^{*0})$; $(\overline{K^{*0}}, -K^{*-})$;

$(\overline{D^{*0}}, D^{*-})$; $(D^{*+}, -D^{*0})$;

triplet $(-\rho^+, \rho^0, \rho^-)$. (2.21)

<u>Z-spin:</u> singlets $\rho^+$; $\rho^0$; $\rho^-$; $\sqrt{\frac{2}{3}}\omega_8 + \frac{1}{\sqrt{3}}\omega_{15}$;

doublets $(K^{*-}, D^{*0})$; $(\overline{D^{*0}}, -K^{*+})$;

$(\overline{K^{*0}}, D^{*+})$; $(D^{*-}, -K^{*0})$;

triplet $\left(-F^{*-}, -\frac{1}{\sqrt{3}}\omega_8 + \sqrt{\frac{2}{3}}\omega_{15}, F^{*+}\right)$. (2.22)

<u>U-spin:</u> singlets $\omega_{15}$; $D^{*0}$; $\overline{D}^{*0}$; $\frac{1}{2}(\omega_8 + \sqrt{3}\rho^0)$;

doublets $(\rho^-, K^{*-})$; $(K^{*+}, -\rho^+)$;

$(D^{*-}, F^{*-})$; $(F^{*+}, -D^{*+})$;

triplet $\left(-K^{*0}, \frac{1}{2}(\sqrt{3}\omega_8 - \rho^0), \overline{K}^{*0}\right)$. (2.23)

<u>W-spin:</u> singlets $K^{*o}$ ; $\bar{K}^{*o}$ ; $\frac{1}{2}(\sqrt{3}\,\omega_8 - \varrho^o)$ ; $\frac{1}{\sqrt{6}}(\sqrt{3}\,\varrho^o + \omega_8 - \sqrt{2}\,\omega_{15})$ ;

doublets $(\varrho^+, D^{*+})$ ; $(D^{*-}, -\varrho^-)$ ;
$(K^{*+}, F^{*+})$ ; $(F^{*-}, -K^{*-})$ ;

triplet $\left(-\bar{D}^{*o}, \frac{1}{\sqrt{12}}(\sqrt{3}\,\varrho^o + \omega_8 + 2\sqrt{2}\,\omega_{15}), D^{*o}\right)$ (2.24)

From (2.21) we see that the particles belonging to the same I-spin multiplet have the same quantum numbers for $Y$ and C, according to the reduction $SU(4) \supset SU_I(2) \otimes U_Y(1) \otimes U_C(1)$.
Similarly the other SU(2) reductions are:
$SU(4) \supset SU_Z(2) \otimes U_{Q-C}(1) \otimes U_{Q-Y}(1)$, $SU(4) \supset SU_U(2) \otimes U_Q(1) \otimes U_C(1)$ and $SU(4) \supset SU_W(2) \otimes U_Q(1) \otimes U_Y(1)$.
Of course $Y - C$ is equivalent to $P = Y_8 + \frac{2}{3} Y_{15}$ .
Then for U- and W-spin the reduction is $SU(4) \supset SU_U(2) \otimes SU_W(2) \otimes U_Q(1)$.

All these relations are intimately connected with our quantum number assignment and classification of the fundamental quark representation $4 = (1,0,0)$ of SU(4). The I-, Z-, U- and W-spin content of the quark representation is exhibited in table 2. Then the application of the operators $I_+$, $Z_+$, $U_+$ and $W_+$ on the quark representation has the effect: $I_+ d = u$, $Z_+ c = s$, $U_+ s = d$, $W_+ c = u$ for the quarks.
Similarly for the antiquarks: $I_+ \bar{u} = -\bar{d}$, $Z_+ \bar{s} = -\bar{c}$, $U_+ \bar{d} = -\bar{s}$, $W_+ \bar{u} = -\bar{c}$.
We apply now (2.17) to the vector meson 15-plet. From (2.22) we see that all states except $\omega_8$ and $\omega_{15}$ are also diagonal in Z-spin.
Therefore we have the following equations for the squared masses of $\varrho$ , $K^*$ , $D^*$ and $F^*$ (denoted by the particle symbols):

$$
\begin{aligned}
\rho &= m + 3m_1 - 9m_2 \quad , \\
K^* &= m + 9m_1 - 9m_2 \quad , \\
D^* &= m + 3m_1 - 3m_2 \quad , \\
F^* &= m + 9m_1 - 3m_2 \quad ,
\end{aligned}
\tag{2.25}
$$

whereas the matrix M for the singlet - octet mixing has the form:

$$
M = \begin{pmatrix} m + 11 m_1 - 9 m_2 & -2\sqrt{2}\, m_1 \\ -2\sqrt{2}\, m_1 & m + 4 m_1 \end{pmatrix} . \tag{2.26}
$$

The eigenvalues of this matrix are identified with the squared masses of the physical $\omega$ and $\phi$ meson. (2.25) yields one mass relation through elimination of $m_o$, $m_1$ and $m_2$

$$
\rho + F^* = K^* + D^* . \tag{2.27}
$$

From (2.25) and (2.26) two further relations follow if one uses the trace which is equal to $\omega+\phi$ and the determinant of (2.26) which is equal to $\omega\phi$ . These are

$$
3F^* + \rho = 2(\omega+\phi) , \tag{2.28}
$$

$$
(\omega-\rho)(\phi-\rho) = \frac{2}{3}(K^*-\rho)(2\omega+2\phi-\rho-3K^*). \tag{2.29}
$$

The eigenfunctions of $\omega$ and $\Phi$ are then respectively (up to normalization):

$$|\omega\rangle \sim (m+4m_1-m_\omega)|\omega_8\rangle + 2\sqrt{2}\, m_1 |\omega_{15}\rangle,$$
$$|\phi\rangle \sim (m+4m_1-m_\phi)|\omega_8\rangle + 2\sqrt{2}\, m_1 |\omega_{15}\rangle. \qquad (2.30)$$

The formulas (2.27) - (2.29) have been given before [7]. If the SU(3) splitting term proportional to $T^3_3$ is neglected we have [5]

$$3M_3 = M_8 + 2M_1 \qquad (2.31)$$

where $M_1$, $M_3$ and $M_8$ are the average mass squared of the SU(3) singlet, triplet and octet, respectively.

Already (2.31) shows that the average mass squared of the charmed triplet ($D^{*+}$, $D^{*o}$, $F^{*+}$) is rather low of the order of 1 $\mathrm{GeV}^2$.

For completeness we give solutions of (2.27) to (2.29) using the $\omega$, $\Phi$ and $K^*$ masses as input. The two solutions are (masses in GeV):

(i) $m_\rho = 0.64$ , $m_{D^*} = 0.76$, $m_{F^*} = 0.98$;

(ii) $m_\rho = 1.07$ , $m_{D^*} = 1.06$, $m_{F^*} = 0.85$.

The SU(4) symmetry breaking has the nice feature to lead to definite SU(3) singlet - octet mixing for $\omega$ and $\Phi$ (see 2.30).

On the other hand the results (i) and (ii) are not tenable since the $\rho$-mass cannot be fitted. The solution of the SU(4) mass breaking for the vector mesons lies in additional SU(4) singlet - fifteenplet mixing which will be discussed in the next section.

Here we still note that the mass formula (2.17) , when applied to the quark representation gives, according to table 2:

$$\frac{1}{2}(m_u + m_d + m_s + m_c) = m, \quad m_s - m_u = -3fm_1,$$
$$m_u = m_d, \qquad m_c - m_u = -3fm_2, \qquad (2.32)$$

so that $R \equiv (m_c - m_u) / (m_s - m_u) = m_2/m_1$. In deducing (2.32) we have taken into account the fact that for the fundamental representation $\underline{4}$ only one reduced matrix element appears in (2.9), i.e. $b = c = 0$, since $\underline{15}$ is contained in $\underline{4} \times \underline{\bar{4}}$ only onces.

## 3. Mixing Analysis.

In the last section we have seen that a mass formula based on SU(4) is contradicted by the empirical masses of the vector mesons. With the discovery of the new vector meson $\Psi(3095)$ a new situation arose. There are now three $I = 0$ vector mesons $\omega$, $\Phi$ and $\Psi$ which in principle could be mixed states. Apparently the $\Psi$ has no SU(4) partners. It is appropriate then to consider the mixing of the fifteenplet with a singlet. In this case the mass matrix (2.26) is replaced by

$$M = \begin{pmatrix} m + 11 m_1 - 9 m_2 & -2\sqrt{2}\, m_1 & A \\ -2\sqrt{2}\, m_1 & m + 4 m_1 & B \\ A & B & m_0 \end{pmatrix} . \qquad (3.1)$$

$m_0$ stands for the SU(4) singlet mass squared and A and B are the nondiagonal matrix elements which mix the singlet $\omega_0$ with $\omega_8$ and $\omega_{15}$, respectively. (3.1) has six free parameters whereas only five masses are known, $\rho$, $K^*$, $\omega$, $\phi$ and $\Psi$, to be used as input. Therefore (3.1) has no predictive power without further assumptions.

(2.17) shows that $F_8$ and $F_{15}$ appear in a definite ratio. We assume that the same determines B/A, the ratio of the $\omega_{15}$ and $\omega_8$ coupling to the singlet. This gives the ratio

$$\alpha \equiv \frac{B}{A} = -\frac{m_1 - 3m_2}{2\sqrt{2}\, m_1} \quad . \tag{3.2}$$

The five-parameter mass matrix thus obtained agrees with the matrix used recently in ref. [4]. Other authors based their mass mixing on the nonet symmetry for vector mesons [8]. In our notation this means to fix A by

$$A = -2\sqrt{6} \quad m_1 \quad , \tag{3.3}$$

but we shall not use this additional constraint.
In (3.1) two parameters are fixed by the $\rho$ and $K^*$ masses from (2.25)

$$m_1 = \frac{1}{6}\,( K^* - \rho \quad ), \tag{3.4}$$

$$m - \; 9m_2 = \frac{1}{2}\,( 3\rho - K^* ) . \tag{3.5}$$

As the three unknown in (3.1) we use $\alpha$ , A and $m_0$. $m_2$ is determined via (3.2)

$$m_2 = \frac{m_1}{3}\,( 1 + 2\sqrt{2}\,\alpha ) . \tag{3.6}$$

Expressing the three tensor invariants of the matrix (3.1) by the eigenvalues (squared masses of $\omega$, $\phi$ and $\Psi$) gives us three equations for $\alpha$, A and m. Elimination of A and m leads to a third order equation in $\alpha$. This equation has three solutions. Only for two of them A is real. In one solution the masses of charmed particles are around 1 GeV. We must take the solution where $m_{D^*}$ and $m_{F^*}$ are around 2 GeV, otherwise the $\Psi$ can copiously decay into these mesons. Then with the input

$$
\begin{array}{ll}
m_{\rho} = 0.76737 \text{ GeV}, & m_{\omega} = 0.78266 \text{ GeV}, \\
m_{K^*} = 0.89433 \text{ GeV}, & m_{\phi} = 1.01969 \text{ GeV}, \\
 & m_{\psi} = 3.095 \text{ GeV},
\end{array}
\tag{3.7}
$$

we obtain

$$
\begin{array}{ll}
\alpha = 21.4184, & m_1 = 0.0352 \text{ GeV}^2, \\
m = 6.9786 \text{ GeV}^2, & m_2 = 0.7218 \text{ GeV}^2, \\
A = -0.1843, & m_o = 3.2414 \text{ GeV}^2.
\end{array}
\tag{3.8}
$$

The corresponding masses of the charmed particles are

$$
\begin{aligned}
m_{D^*} &= 2.218 \text{ GeV}, \\
m_{F^*} &= 2.265 \text{ GeV}.
\end{aligned}
\tag{3.9}
$$

The large value of $\alpha$ means that the SU(4) breaking is roughly twenty times larger than the SU(3) breaking which is determined by $m_1$.

The wave functions for the mixed states $\omega$, $\Phi$ and $\Psi$ in terms of the 8, 15 and 0 components are:

$$|\omega\rangle = 0.6595\ |\omega_0\rangle + 0.6301\ |\omega_8\rangle + 0.4098\ |\omega_{15}\rangle ,$$

$$|\phi\rangle = 0.5342\ |\omega_0\rangle - 0.7765\ |\omega_8\rangle - 0.3342\ |\omega_{15}\rangle , \quad (3.10)$$

$$|\Psi\rangle = 0.5288\ |\omega_0\rangle - 0.0015\ |\omega_8\rangle - 0.8488\ |\omega_{15}\rangle .$$

It is easy to express $|\omega\rangle$, $|\phi\rangle$ and $|\Psi\rangle$ in terms of the ideally mixed states $|\omega_\sigma\rangle$, $|\omega_s\rangle$ and $|\omega_c\rangle$ defined by

$$|\omega_\sigma\rangle = \frac{1}{\sqrt{3}}|\omega_8\rangle + \frac{1}{\sqrt{6}}|\omega_{15}\rangle + \frac{1}{\sqrt{2}}|\omega_0\rangle ,$$

$$|\omega_s\rangle = -\sqrt{\frac{2}{3}}|\omega_8\rangle + \frac{1}{\sqrt{12}}|\omega_{15}\rangle + \frac{1}{2}|\omega_0\rangle , \quad (3.11)$$

$$|\omega_c\rangle = -\frac{\sqrt{3}}{2}|\omega_{15}\rangle + \frac{1}{2}|\omega_0\rangle .$$

The quark content of these states is $(u\bar{u} + d\bar{d})/\sqrt{2}$ , $s\bar{s}$ and $c\bar{c}$, respectively. The physical states then are:

$$|\omega\rangle = 0.9975\ |\omega_\sigma\rangle - 0.0664\ |\omega_s\rangle - 0.0251\ |\omega_c\rangle ,$$

$$|\phi\rangle = 0.0659\ |\omega_\sigma\rangle + 0.9976\ |\omega_s\rangle - 0.0223\ |\omega_c\rangle , \quad (3.12)$$

$$|\Psi\rangle = 0.0265\ |\omega_\sigma\rangle + 0.0206\ |\omega_s\rangle + 0.9994\ |\omega_c\rangle .$$

We see that the physical $\omega$, $\phi$ and $\Psi$ states deviate very little from the ideally mixed states (3.11). Whether the amount of deviation from the ideally mixed situation is small enough to understand the small total and partial decay widths of the $\Psi$ will be taken up in a forthcoming paper. From (3.10) we note that the $\omega$-$\Phi$ mixing comes out reasonable, the mixing angle being 39°.

It is of interest to see what are the mass constraints resulting from a pure ideal mixing. Imposing that the eigenfunctions of (3.1) are exactly given by (3.11) we get the following constraints on the parameters in (3.1)

$$
\begin{aligned}
m_o &= m + 6\,(m_1 - m_2) = m + 4m_1\,(1 - \sqrt{2}\,\alpha)\ , \\
A &= -\,2\,\sqrt{6}\,m_1\ .
\end{aligned}
\qquad (3.13)
$$

As can be read off from (2.17) the quantity $m + 6\,(m_1 - m_2)$ is just the mean mass $\overline{m}_{15}$ of the 15-plet. The constraint (3.13) results in the mass formulas for ideal mixing, i.e.

$$
\begin{aligned}
\rho &= \omega\ , & 2D^* &= \omega + \Psi\ , \\
2K^* &= \omega + \phi\ , & 2F^* &= \phi + \Psi\ .
\end{aligned}
\qquad (3.14)
$$

The constraints for $\rho$ and $K^*$ are the well known nonet symmetry constraints which are well satisfied. As is well known for the pseudoscalar mesons the corresponding relations for $\eta$, $\eta'$, $\pi$ and $K$ are not satisfied, whereas for the tensor mesons the agreement is similar to that of the vector mesons.

## 4. Electromagnetic Current in SU(4).

In our scheme where the charges of the four fundamental quarks u, d, s and c are (2/3, -1/3, - 1/3, 2/3) the electric charge operator Q has the form (2,20)

$$Q = \frac{\sqrt{2}}{3} F_0 + F_3 + \frac{1}{\sqrt{3}} F_8 - \sqrt{\frac{2}{3}} F_{15} . \qquad (4.1)$$

Therefore the electromagnetic current J and any electromagnetic operator related linearly to it is the same superposition of $F_0$, $F_3$, $F_8$ and $F_{15}$ as in (4.1). In Okubo's notation this means that

$$J = e\, T_0 + e'( T^1{}_1 + T^4{}_4) \quad , \qquad (4.2)$$

where $T_o$ is the SU(4) invariant part and $T^1{}_1$ and $T^4{}_4$ are irreducible tensor operators whose matrix elements for an irreducible SU(4) representation are obtained again from (2.9). We observe that J commutes with all U- and W- spin components. Therefore J is a U -spin scalar as in SU(3) and in addition a W-spin scalar. $T^4{}_4$ has been expressed already in (2.13) by $Y_{15}$ and the SU(3)-Casimir operator $\mathcal{C}_3$.

Similarly to $T^3{}_3$ we can transform $T^1{}_1$ into

$$T^1{}_1 = (a-2b)(I_3 + \frac{1}{2} Y_8 + \frac{1}{3} Y_{15}) + \frac{b}{3} \left[ 3(\vec{W}^2 - \vec{U}^2) + \mathcal{C}_3 - Y^2{}_{15} \right]. \quad (4.3)$$

We see that the breaking caused by $T^1{}_1$ is controlled by U- and W-spin in analogy to I- and Z-spin which essentially determine $T^3{}_3$.
Thus we have

$$T^1{}_1 + T^4{}_4 = b\,\mathcal{C}_4 + (a-2b)\,(Q - \frac{1}{2}B) + b(\vec{W}^2 - \vec{U}^2) \qquad (4.4)$$

The SU(4) invariant contributions in (4.4) can be combined with $T_0$ and introducing $f = 3(2b-a)/b$ as in (2.17), J turns out to be of the form:

$$J = e_0 + 3e_1(\vec{W}^2 - \vec{U}^2) - fe_1 Q \qquad (4.5)$$

J consists of a charge-conjugation even part and a charge-conjugation odd part (proportional to Q). The $\mathcal{C}$-even part for example is of interest for radiative transitions $V \to P + \gamma$, where $V$ are members of the vector and P of the pseudoscalar meson fifteenplet. Both terms contribute for example to the electromagnetic form factors of the baryons. The application of (4.5) to the radiative decays of the meson fifteenplet is easily done with the help of the decomposition into U- and W-spin submultiplets. This will be done in a forthcoming paper.

Now we shall apply (4.5) to the problem of electromagnetic mass splitting. As an example we consider the vector meson 15 plet as in section 2. Since electromagnetic mass shifts are of second order in the electromagnetic interaction, that is, quadratic in the current, some caution is necessary in connection with the application of (4.5). If we assume that the electromagnetic mass breaking term has nevertheless the transformation properties of $J \sim (T^1{}_1 + T^4{}_4)$ we deduce from (4.5) and the classification (2.23) and (2.24) of the particles into U- and W-spin multiplets the following relations for the non-SU(4)-invariant electromagnetic mass shifts $\delta V = \langle V|(J - e_0)|V\rangle$ :

$$\begin{aligned} &\delta\rho^+ = \delta\rho^0 = \delta K^{*+} = \delta D^{*+} = \delta F^{*+} = 0 \\ &\delta K^{*0} = -\,\delta D^{*0} = \tfrac{3}{2}\,\delta\omega_8 = -\tfrac{3}{2}\,\delta\omega_{15} \\ &\qquad = -\sqrt{3}\,\delta\omega_{3,8} = -\sqrt{6}\,\delta\omega_{3,15} = -3\sqrt{2}\,\delta\omega_{8,15} \\ &\qquad = -\,6\,e_1 \end{aligned} \qquad (4.6)$$

Because of our simplifying assumption the electromagnetic mass difference between $\rho^+$ and $\rho^0$ vanishes in the case of no mixing between $\rho^0$, $\omega_8$ and $\omega_{15}$ components. This agrees with well-known results for SU(3)[9]. To overcome this difficulty and to obtain relations for electromagnetic mass shifts valid to all orders, we go back to the statement that J and any power $J^n$ are U- and W-spin scalars. Then the matrix elements of $J^n$ are related in the following way:

$$\langle \rho^+ | J^n | \rho^+ \rangle = \langle K^{*+} | J^n | K^{*+} \rangle = \langle D^{*+} | J^n | D^{*+} \rangle = \langle F^{*+} | J^n | F^{*+} \rangle \; ,$$

$$2 \langle K^{*o} | J^n | K^{*o} \rangle = 3 \langle \omega_8 | J^n | \omega_8 \rangle - \langle \rho^o | J^n | \rho^o \rangle \; ,$$

$$2 \langle D^{*o} | J^n | D^{*o} \rangle = \langle \omega_8 | J^n | \omega_8 \rangle + 4 \langle \omega_{15} | J^n | \omega_{15} \rangle - 3 \langle \rho^o | J^n | \rho^o \rangle \; ,$$

$$2 \langle \rho^o | J^n | \omega_8 \rangle = \sqrt{3} \langle \rho^o | J^n | \rho^o \rangle - \sqrt{3} \langle \omega_8 | J^n | \omega_8 \rangle \; ,$$

$$2\sqrt{2} \langle \omega_{15} | J^n | \omega_8 \rangle = \langle \omega_8 | J^n | \omega_8 \rangle + 2 \langle \omega_{15} | J^n | \omega_{15} \rangle - 3 \langle \rho^o | J^n | \rho^o \rangle \; ,$$

$$\langle \rho^o | J^n | \omega_{15} \rangle = \sqrt{3} \langle \omega_{15} | J^n | \omega_8 \rangle \; , \qquad (4.7)$$

for the states of the fifteenplet, and

$$\langle \rho^o | J^n | \omega_o \rangle = \sqrt{3} \langle \omega_8 | J^n | \omega_o \rangle = - \sqrt{\tfrac{3}{2}} \langle \omega_{15} | J^n | \omega_o \rangle \qquad (4.8)$$

for the matrix elements between 15-plet and singlet.

It follows from (4.7) that the electromagnetic mass splitting of the 15- plet is determined by four parameters: $\langle \rho^+ | J^n | \rho^+ \rangle$ , $\langle \rho^o | J^n | \rho^o \rangle$ , $\langle \omega_8 | J^n | \omega_8 \rangle$ and $\langle \omega_{15} | J^n | \omega_{15} \rangle$. One further constraint is obtained from (4.5) by evaluating it between I = 1/2 and I = 0 states. In this case only that part of the operator $J^n$ contributes which has the same structure as $J$ . This constraint is

$$2 \langle \rho^o | J^n | \rho^o \rangle = \langle \omega_{15} | J^n | \omega_{15} \rangle + \langle \omega_8 | J^n | \omega_8 \rangle \; . \qquad (4.9)$$

So we are left with three parameters for the electromagnetic mass shifts to all orders in the electromagnetic coupling which we denote by $\Delta_\varrho = \langle \varrho^+ | J^n | \varrho^+ \rangle$ ,

$\Delta_8 = \langle \omega_8 | J^n | \omega_8 \rangle$ , $\Delta_{15} = \langle \omega_{15} | J^n | \omega_{15} \rangle$.

Then we have the following mass formulas instead of (2.25):

$$
\begin{aligned}
\varrho^+ &= m + 3m_1 - 9m_2 + \Delta_\varrho \; , \\
K^{*+} &= m + 9m_1 - 9m_2 + \Delta_\varrho \; , \\
K^{*o} &= m + 9m_1 - 9m_2 + \tfrac{5}{4}\Delta_8 - \tfrac{1}{4}\Delta_{15} \; , \\
D^{*+} &= m + 3m_1 - 3m_2 + \Delta_\varrho \; , \\
D^{*o} &= m + 3m_1 - 3m_2 + \tfrac{5}{4}\Delta_{15} - \tfrac{1}{4}\Delta_8 \; , \\
F^{*+} &= m + 9m_1 - 3m_2 + \Delta_\varrho \; .
\end{aligned}
\tag{4.10}
$$

We see that we have still the mass relation (2.27), but only for the charged components

$$\varrho^+ + F^{*+} = K^{*+} + D^{*+} \, . \tag{4.11}$$

This is a consequence of the first line in (4.7) that the electromagnetic mass shifts of all charged members of the 15 -plet are equal. The masses of the physical $\varrho^o$, $\omega$ and $\phi$ are obtained by diagonalizing the symmetric mass matrix M

$$
M = \begin{pmatrix}
m+3m_1-9m_2+\frac{1}{2}(\Delta_8+\Delta_{15}) & \frac{\sqrt{3}}{4}(-\Delta_8+\Delta_{15}) & \frac{\sqrt{6}}{8}(-\Delta_8+\Delta_{15}) \\
\frac{\sqrt{3}}{4}(-\Delta_8+\Delta_{15}) & m+11m_1-9m_2+\Delta_8 & -2\sqrt{2}m_1+\frac{\sqrt{2}}{8}(-\Delta_8+\Delta_{15}) \\
\frac{\sqrt{6}}{8}(-\Delta_8+\Delta_{15}) & -2\sqrt{2}m_1+\frac{\sqrt{2}}{8}(-\Delta_8+\Delta_{15}) & m+4m_1+\Delta_{15}
\end{pmatrix}
\tag{4.12}
$$

which replaces (2.26). The electromagnetic mass shifts in (4.12) reduce to the relations (4.6), obtained in first order, if $\Delta_8 = -\Delta_{15}$ . In (4.12) the parameters $m_1$ and $m$ can be eliminated by:

$$m_1 = \frac{1}{6}(K^{*+} - \rho^+) ,$$

$$m - 9m_2 = \frac{3}{2}(\rho^+ - K^{*+}) + K^{*o} - \frac{5}{4}\Delta_8 + \frac{1}{4}\Delta_{15} , \tag{4.13}$$

so that only three parameters are left over to be determined by the input masses of $\rho^o$, $\omega$ and $\Phi$.

Then we have only two unknowns

$$\Delta = \frac{1}{4}(\Delta_{15} - \Delta_8) ,$$

$$\overline{m} = m + \Delta_{15} , \tag{4.14}$$

and M has the following form:

$$M = \begin{pmatrix} 3m_1 + k + 3\Delta & \sqrt{3}\,\Delta & \sqrt{\frac{3}{2}}\,\Delta \\ \sqrt{3}\,\Delta & 11m_1 + k + \Delta & -2\sqrt{2}\,m_1 + \frac{1}{\sqrt{2}}\Delta \\ \sqrt{\frac{3}{2}}\,\Delta & -2\sqrt{2}\,m_1 + \frac{1}{\sqrt{2}}\Delta & 4m_1 + \overline{m} \end{pmatrix} , \tag{4.15}$$

with $\quad k = \frac{3}{2}(\rho^+ - K^{*+}) + K^{*o}$

The tensor invariants of the matrix M give us three equations for the two unknowns in terms of $m_1$, k and the masses of $\rho^o$, $\omega$ and $\Phi$. By eliminating $\Delta$ and $\overline{m}$ we obtain a very complicated mass formula. We do not expect useful solutions from this mass formula similar to the case without electromagnetic mass splitting. The masses of the charmed mesons are determined from (4.11) and

$$\rho^0 + \omega + \Phi = \frac{3}{2} D^{*0} + \frac{3}{2} K^{*0} ,$$

$$D^{*+} - K^{*+} = \frac{2}{3}(\rho^0 + \omega + \Phi) - 18 m_1 - 2k - 6\Delta \qquad (4.16)$$

In order to incorporate the $\Psi$ meson we consider the case of singlet and 15-plet mixing together with electromagnetic mass shifts. Then, compared to (4.15), we have three more matrix elements $\Delta_{03}$, $\Delta_{08}$ and $\Delta_{015}$, which describe the contribution of the electromagnetic mass shift in the nondiagonal elements between the singlet and the 3, 8 and 15 component of the 15-plet. In analogy to the procedure in section 3 we fix the ratios of these nondiagonal elements by the ratios of the corresponding components in the electromagnetic current (see (4.5) :

$$\Delta_{015} : \Delta_{08} : \Delta_{03} = -\sqrt{\frac{2}{3}} : \frac{1}{\sqrt{3}} : 1 \qquad (4.17)$$

Then the mass matrix in the 3, 8, 15, 0 representation has the following form:

$$M = \begin{pmatrix} 3m_1 + k + 3\Delta & \sqrt{3}\,\Delta & \sqrt{\frac{3}{2}}\,\Delta & \Delta_{03} \\ \sqrt{3}\,\Delta & 11 m_1 + k + \Delta & -2\sqrt{2} m_1 + \frac{1}{\sqrt{2}}\Delta & A + \frac{1}{\sqrt{3}}\Delta_{03} \\ \sqrt{\frac{3}{2}}\,\Delta & -2\sqrt{2} m_1 + \frac{1}{\sqrt{2}}\Delta & 4m_1 + \overline{m} & \alpha A - \sqrt{\frac{2}{3}}\Delta_{03} \\ \Delta_{03} & A + \frac{1}{\sqrt{3}}\Delta_{03} & \alpha A - \sqrt{\frac{2}{3}}\Delta_{03} & m_0 \end{pmatrix} \qquad (4.18)$$

Since $\overline{m} = 3m_1(1 + 2\sqrt{2}\alpha) + k + 5\Delta$ the matrix (4.18) has still five unknowns $\alpha$, A, $m_o$, $\Delta$ and $\Delta_{03}$ , so that the mass spectrum cannot be predicted from the masses of $\rho^o$, $\omega$, $\Phi$ and $\Psi$. We eliminate $\Delta_{03}$ by the constraint

$$\Delta_{03} = - \frac{\sqrt{3}\, A}{4 m_1} \Delta \ . \tag{4.19}$$

(4.19) follows from the assumption that

$$m_{30} : m_{80} = m_{3,15} : m_{8,15} \ . \tag{4.20}$$

(4.19) agrees for $A = -2\sqrt{6}\ m_1$ (see (3.3) ) with the relation which follows from the assumption that the electromagnetic mass breaking term has the transformation properties of J, so that it obeys (4.5).

With the constraint (4.19) we have four unknowns which can be determined from the masses of $\rho^o$, $\omega$, $\Phi$ and $\Psi$ . The explicit solution of this mixing problem is rather complicated and will be considered elsewhere.

## 5. Conclusions

In this paper we derived formulas for mass breaking in SU(4) multiplets using Okubo's tensor formalism. These formulas are applied to mass breaking of the vector meson 15-plet with and without singlet - 15 -plet mixing. Without mixing no sensible solutions are obtained. For the case with mixing

we obtain two solutions. In only one of them the masses of the charmed vector mesons are around 2 GeV. For this solutions the $\Psi$(3.1) meson is dominantly a $c\bar{c}$ system. We emphazise that a parameter-free solution for the masses of the charmed mesons comes out only if the nondiagonal matrix elements between singlet $\omega_0$ and $\omega_8$ and $\omega_{15}$ respectively are constrained by (3.2).
The electromagnetic mass splitting is considered also in some detail. The mass matrix for the 15-plet is derived including singlet - 15-plet mixing. To obtain parameter free solutions for the charmed mesons one needs three further constraints for the nondiagonal elements of the electromagnetic mass breaking matrix.

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Table 1: Quantum numbers of the 15-plet.

| | Q | Y=S | C | Y-C | $(p,q)_{SU_3}$ | $(I, I_3)$ *) | $(Z,Z_3)$ *) | $(U,U_3)$ *) | $(W,W_3)$ *) |
|---|---|---|---|---|---|---|---|---|---|
| $\rho^+$ | +1 | 0 | 0 | 0 | 8= (1,1) | (1,+1) | (0,0) | $(\frac{1}{2},-\frac{1}{2})_a$ | $(\frac{1}{2},+\frac{1}{2})_a$ |
| $\rho^0$ | 0 | 0 | 0 | 0 | 8= (1,1) | (1,0) | (0,0) | (0,0)+ (1,0) | (0,0)+(0,0)'+(1,0) |
| $\rho^-$ | -1 | 0 | 0 | 0 | 8= (1,1) | (1,-1) | (0,0) | $(\frac{1}{2},+\frac{1}{2})_b$ | $(\frac{1}{2},-\frac{1}{2})_b$ |
| $K^{*+}$ | +1 | +1 | 0 | +1 | 8= (1,1) | $(\frac{1}{2},+\frac{1}{2})_a$ | $(\frac{1}{2},-\frac{1}{2})_a$ | $(\frac{1}{2},+\frac{1}{2})_a$ | $(\frac{1}{2},+\frac{1}{2})_c$ |
| $K^{*0}$ | 0 | +1 | 0 | +1 | 8= (1,1) | $(\frac{1}{2},-\frac{1}{2})_a$ | $(\frac{1}{2},-\frac{1}{2})_b$ | (1, +1) | (0,0) |
| $\bar{K}^{*0}$ | 0 | -1 | 0 | -1 | 8= (1,1) | $(\frac{1}{2},+\frac{1}{2})_b$ | $(\frac{1}{2},+\frac{1}{2})_c$ | (1, -1) | (0,0) |
| $K^{*-}$ | -1 | -1 | 0 | -1 | 8= (1,1) | $(\frac{1}{2},-\frac{1}{2})_b$ | $(\frac{1}{2},+\frac{1}{2})_d$ | $(\frac{1}{2},-\frac{1}{2})_b$ | $(\frac{1}{2},-\frac{1}{2})_d$ |
| $D^{*+}$ | +1 | 0 | +1 | -1 | 3= (1,0) | $(\frac{1}{2},+\frac{1}{2})_c$ | $(\frac{1}{2},-\frac{1}{2})_c$ | $(\frac{1}{2},-\frac{1}{2})_c$ | $(\frac{1}{2},-\frac{1}{2})_a$ |
| $D^{*0}$ | 0 | 0 | +1 | -1 | 3= (1,0) | $(\frac{1}{2},-\frac{1}{2})_c$ | $(\frac{1}{2},-\frac{1}{2})_d$ | (0,0) | (1, -1) |
| $\bar{D}^{*0}$ | 0 | 0 | -1 | +1 | $\bar{3}$= (0,1) | $(\frac{1}{2},+\frac{1}{2})_d$ | $(\frac{1}{2},+\frac{1}{2})_a$ | (0,0) | (1, +1) |
| $D^{*-}$ | -1 | 0 | -1 | +1 | $\bar{3}$= (0,1) | $(\frac{1}{2},-\frac{1}{2})_d$ | $(\frac{1}{2},+\frac{1}{2})_b$ | $(\frac{1}{2},+\frac{1}{2})_d$ | $(\frac{1}{2},+\frac{1}{2})_b$ |
| $F^{*+}$ | +1 | +1 | +1 | 0 | 3= (1,0) | (0,0) | (1, - 1) | $(\frac{1}{2},+\frac{1}{2})_c$ | $(\frac{1}{2},-\frac{1}{2})_c$ |
| $F^{*-}$ | -1 | -1 | -1 | 0 | $\bar{3}$= (0,1) | (0,0) | (1, + 1) | $(\frac{1}{2},-\frac{1}{2})_d$ | $(\frac{1}{2},+\frac{1}{2})_d$ |
| $\omega_8$ | 0 | 0 | 0 | 0 | 8= (1,1) | (0,0) | (0,0)+(1,0) | (0,0)+(1,0) | (0,0)+(0,0)'+(1,0) |
| $\omega_{15}$ | 0 | 0 | 0 | 0 | 1= (0,0) | (0,0) | (0,0)+(1,0) | (0,0) | (0,0)'+(1,0) |

*) Particles belonging to the same doublet $(\frac{1}{2},\pm\frac{1}{2})_i$ are indicated by the same index i = a,b,c and d.

Table 2: Quantum numbers of the fundamental 4-plet.

| | Q | Y | B | S | C | $(p,q)_{SU_3}$ | $(I,I_3)$ | $(Z,Z_3)$ | $(U,U_3)$ | $(W,W_3)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| u | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 | $3=(1,0)$ | $(\frac{1}{2}, +\frac{1}{2})$ | $(0,0)$ | $(0,0)$ | $(\frac{1}{2}, +\frac{1}{2})$ |
| d | $-\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 | $3=(1,0)$ | $(\frac{1}{2}, -\frac{1}{2})$ | $(0,0)$ | $(\frac{1}{2}, +\frac{1}{2})$ | $(0,0)$ |
| s | $-\frac{1}{3}$ | $-\frac{2}{3}$ | $\frac{1}{3}$ | -1 | 0 | $3=(1,0)$ | $(0,0)$ | $(\frac{1}{2}, +\frac{1}{2})$ | $(\frac{1}{2}, -\frac{1}{2})$ | $(0,0)$ |
| c | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 1 | $1=(0,0)$ | $(0,0)$ | $(\frac{1}{2}, -\frac{1}{2})$ | $(0,0)$ | $(\frac{1}{2}, -\frac{1}{2})$ |

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