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ABSTRACT

The ultraviolet asymptotic freedom of a large class of nonabelian gauge theories enable the renormalization constants to be computed exactly in perturbation theory. This exact knowledge is used in investigating renormalized field equations of such theories. They are shown to be invariant under an abelian gauge transformation on the renormalized vector field. Consequences of the abelian invariance are derived in the form of Ward-Takahashi identities for the renormalized proper vertices $\Gamma_R^{\cdots\mu\cdots}(\cdots, q, \cdots)$: $q_\mu \Gamma_R^{\cdots\mu\cdots}(\cdots, q, \cdots) = 0$.

I. INTRODUCTION[1]

The organic union of a local nonabelian group symmetry with ordinary (abelian) gauge invariance has led to the introduction of nonabelian gauge (YM) fields.[2] The very existence of the gauge field is then intimately tied to the nonabelian group symmetry present. The assignment of the gauge field as a carrier of the symmetry has been a rather attractive way of incorporating dynamically a given symmetry group in quantum field theory.

Ordinary gauge invariance originated in classical electrodynamics because only the field strength $F_{\mu\nu}$ and not the potential A_μ is directly measurable. It acquires much greater importance in the quantum theory, where the field A_μ is the quantity more directly associated with the particle (photon) in the theory. In fact, the renormalization of quantum electrodynamics (QED) makes specific use of the consequence of gauge invariance, namely the original Ward identity: $Z_1 = Z_2$.

In the nonabelian version, the renormalization is more complicated,[3-6] and again the consequence of gauge invariance, now in the form of the generalized Ward-Takahashi (WT), or the Slavnov[7] identity, plays a critical role in the execution of the renormalization program. The nonabelian nature of the theory necessitates the introduction of ghost fields, which transparently manifest the lack of positivity[8] peculiar to the theory.

It is this very lack which allows a desirable state of affairs to emerge. People have been interested in the large momentum (ultraviolet) behavior of field theories, and have found that a useful tool for discussing these behaviors has been the renormalization group differential equations[9] satisfied by these theories. The asymptotic behavior was found to be determined in terms of the fixed points of the Callan-Symanzik function $\beta(g)$. It turns out that there is always a fixed point at the origin,[6] and it is then possible to determine the asymptotic behavior from the information furnished by low-order perturbation theory, presumably valid at that point.[10] The catch is that the ultraviolet (UV) behavior is determined for a negative slope of $\beta(g)$ at $g = 0$, and the infrared (IR) behavior in the case of a positive slope. These two cases are referred to as UV and IR free respectively. By positivity, all known field theories have a positive slope, all, that is, except for YM theories, many of which have indeed a negative slope and are therefore UV free.

Thus YM theories occupy the privileged status that their true [10] UV asymptotic behavior is easily determined. In particular, the renormalization constants are known exactly,[11] so that it is now possible to scrutinize the renormalized field equations of the theory for any new features arising, so to speak, from renormalization.

We have performed just that, and in this paper we report on one aspect of the investigation. <u>The renormalized field equation is invariant under an additional symmetry, namely, that of ordinary abelian gauge tranformation</u>. Moreover, the nonabelian gauge theory satisfies a set of abelian WT identities, valid at <u>all</u> energies.

The existence of this symmetry arises from the fact that the renormalization constants can be computed exactly[11] in these UV free theories. When a particular constant or relevant ratio vanishes, new symmetries can arise which are not present in the classical Lagrangian. Such new symmetries are a consequence of renormalization, and furthermore are not present order by order in perturbation theory. In each order, the renormalization constants are infinite (when the cutoff tends to infinity) and it is only the sum which may become zero. To deduce the presence of such a symmetry, the group transformation must be interchanged with the cutoff removal limit. This means that such symmetries have a distinct, and perhaps more speculative, status than those which are present classically and in each order of perturbation theory. Since Green's functions can only be computed asymptotically in the relevant gauge theories the existence of the symmetry can only be directly checked in an asymptotic limit, although the symmetry is predicted to be present at all energies.

The route to the emergence of such renormalization symmetries is indirect. First we use the known asymptotic behavior to calculate renormalization constants. The obtained (suitably vanishing) behaviors of these constants implies the presence of (usually spontaneously broken) new symmetries and the consequences of these symmetries are valid at all energies for the exact theory.

In the course of our analysis we have found it expedient to work with field equations, commutation relations, etc. obtained by canonical manipulation of the YM Lagrangian with ghost fields. The Slavnov identity, as an example, has been derived in this manner from the field equations. Thus we feel confident that the physical content of the YM theory should be embodied in its local field equations, so that the symmetries present there should be true symmetries of the theor y.

In Sec. II we discuss in general terms how statements invalid in the unrenormalized theory can be valid as a consequence of renormalization, and cite previous use of the technique. Sec. III contains a resumé of nonabelian gauge field theory in terms of field equations as well as functionals. Sec. IV introduces renormalization constants, and the Lagrangian and field equations are rewritten in terms of renormalized fields. It is shown how these constants are computed via the renormalization group equations, with gauge-independent results. In Sec. V we show that the renormalized field equations and Lagrangian are invariant under an abelian gauge transformation on the renormalized field. We deduce the WT identities associated with the abelian gauge symmetry both via equal-time commutators and by the use of functional methods. We discuss their consistency with the nonabelian WT (Slavnov) identities usually obtained. Sec. VI concludes the paper.

II. SYMMETRY AS A CONSEQUENCE OF RENORMALIZATION

Before becoming entangled in the complexities of nonabelian gauge theories, we will illustrate our ideas in a simpler context. Consider a typical term

$$\mathcal{J}(A) = Z\, A(x)\, A(x) \tag{2.1}$$

in a formal renormalized Lagrangian or field equation. Here $A(x)$ is a renormalized quantum field and Z is a combination of renormalization constants. In the cutoff theory, both Z and A^2 have expansions in powers of the renormalized coupling g:

$$Z = Z(K) = \sum_{n=0}^{\infty} g^n Z_n(K)\,, \tag{2.2}$$

$$A^2(x) = \sum_{n=0}^{\infty} g^n\, \mathcal{O}_n(x; K)\,, \tag{2.3}$$

where K is the cutoff parameter. In each order of perturbation theory, the renormalization constant $Z_n(K)$ and the ordinary field product are divergent when the cutoff is removed:

$$Z_n(K) \xrightarrow[K\to\infty]{} \infty\,, \tag{2.4}$$

$$\mathcal{O}_n(x; K) \xrightarrow[K\to\infty]{} \infty\,. \tag{2.5}$$

Typically, in a renormalizable theory, the divergences become logarithmically worse in higher orders; e.g.,

$$Z_n(K) \sim a_n (\ln K)^n\,. \tag{2.6}$$

These infinities combine with others to produce finite (for $K \to \infty$) expressions for the renormalized Green's functions in each order of g. That is,

$$\lim_{K\to\infty} \langle 0 | T\, A(x_1) \cdots A(x_N) | 0 \rangle$$

exists.

In each order, the expression (2.1) has no interesting symmetry property. For example, under the "R transformation"

$$R:\; A(x) \longrightarrow A(x) + r\,, \qquad r = \text{const}\,, \tag{2.7}$$

one has

$$\mathcal{F}(A) \longrightarrow \mathcal{F}(A+r) = \mathcal{F}(A) + 2rZ\,A(x) + r^2 Z\,, \tag{2.8}$$

so that

$$\mathcal{F}(A+r) - \mathcal{F}(A) \xrightarrow[K\to\infty]{} \infty\,, \tag{2.9}$$

since r and A(x) are finite. So (2.1) is not invariant under (2.7). Now suppose that the exact[10] Z(K) given by the sum (2.2) <u>vanishes</u> when the cutoff is removed:

$$Z(K) \xrightarrow[K\to\infty]{} 0\,. \tag{2.10}$$

Then

$$\mathcal{F}(A+r) - \mathcal{F}(A) \xrightarrow[K\to\infty]{} 0\,, \tag{2.11}$$

and so (2.1) becomes R symmetric in the exact theory.

In the above circumstance, a new symmetry can arise as a consequence of renormalization. The R symmetry is a typical (spontaneously broken perhaps) symmetry which can arise in this way. Such symmetries imply interesting WT identities and low energy theorems. Consider, for example, a formal field equation of the form

$$\Box A = Z A^2 + \mathcal{G}(A, \cdots), \tag{2.12}$$

with $\mathcal{G}(A, \cdots)$ R invariant. If $Z = 0$ as in (2.10), (2.12) is R invariant in the exact theory even though it is not invariant in any finite order of perturbation theory.

A more precise formulation of such possibilities can be given in terms of finite local field equations.[12,13] Eq. (2.12), for example, can be given a mathematical status in the form

$$\Box A(x) = J(x), \tag{2.13}$$

with

$$J(x) = \lim_{\xi \to 0} \left[z(\xi)\, A(x+\xi) A(x) + \mathcal{G}_\xi(A, \ldots) \right]. \tag{2.14}$$

Here, in each order of perturbation, $z(\xi)$ is a well-defined function with singularities at $\xi = 0$ corresponding to (2.4): $z(0) = Z(\kappa) = \infty$. Thus, under (2.7), $J(x) \longrightarrow J(x) + \infty$, and the field equation (2.13) is not R invariant. If (2.10) obtains for the exact theory, then

$$\mathcal{Z}(0) = 0 \tag{2.15}$$

[this zero cancels an infinity in the field product $A(x+\xi)\ A(x)$ for $\xi \to 0$] and so

$$\begin{aligned} J(x) \longrightarrow \lim_{\xi \to 0} \Big[& \mathcal{Z}(\xi)\, A(x+\xi)\, A(x) + \mathcal{G}_{\xi}(A, \cdots) \\ & + \mathcal{Z}(\xi)\, A(x+\xi)\, r + \mathcal{Z}(\xi)\, r A(x) + \mathcal{Z}(\xi)\, r^2 \Big] \\ = {} & J(x)\,, \end{aligned} \tag{2.16}$$

and the field equation becomes R invariant.

The finite field equation approach to ordinary gauge invariance in QED proceeds in precisely the same way.[12] There the Maxwell equations read

$$\begin{aligned} \partial^{\nu} F_{\mu\nu}(x) &= J_{\mu}(x) = \lim_{\xi \to 0} J_{\mu}(x;\xi) \\ &= \lim_{\xi \to 0} \left[\mathcal{Z}(\xi)\, \bar{\psi}(x+\xi)\, \gamma_{\mu}\, \psi(x) + \cdots \right]. \end{aligned} \tag{2.17}$$

Under a local gauge transformation,

$$J_{\mu}(x;\xi) \longrightarrow J_{\mu}(x;\xi) + R_{\mu}(x;\xi)\,, \tag{2.18}$$

with

$$\lim_{\xi \to 0} R_{\mu}(x;\xi) = 0\,. \tag{2.19}$$

Eq. (2.12) is thus gauge invariant and this, together with the analogous gauge covariance of the Dirac equation, is equivalent to the gauge invariance of QED. Note that here the gauge invariance is true order by order, whereas (2.17) was only symmetric for the exact theory.

In the following sections we will apply similar considerations to nonabelian gauge theories. We will argue on the basis of formal field equations of the form (2.11). We are confident that our conclusion would also follow from using the more meaningful finite local field equations of the form (2.17), although, because of the complexity of such equations in YM theories, we have not shown this in detail. It is our basic assumption that such employment of the field equations to determine the symmetries of the theory is legitimate even for the exact theory in which results of the form (2.10) are valid.

In a nonperturbative context, we have previously used local field equations in this way.[14] There we studied the problem of consistently incorporating scale invariance in operator product expansions implied by canonical commutations. In a $g\varphi^4$ theory for example, it was shown that there must then exist two distinct operators $j(x)$ and $k(x)$ of scale dimension two, which form a two-dimensional reducible representation of the scale group. They appear in the short-distance expansion of $\varphi(x)\,\varphi(0)$:

$$\varphi(x)\,\varphi(0) \underset{x\to 0}{\longrightarrow} (\lambda_1 \ln x^2 + \lambda_2)\, j(0) + \lambda_1\, k(0) \,, \qquad (2.20)$$

and they transform under a scale change as

$$U_\rho \begin{bmatrix} j(x) \\ k(x) \end{bmatrix} U_\rho^{-1} = \rho^2 \begin{bmatrix} j(\rho x) \\ \ln\rho \;\; j(\rho x) + k(\rho x) \end{bmatrix} . \qquad (2.21)$$

The transformation law (2.21) enables the presence of logarithms in (2.20) to be compatible with reducible scale invariance. From (2.20), j and k have explicit representations in terms of φ :

$$j(x) = \lim_{\xi \to 0} \frac{:\varphi(x+\xi)\varphi(x):}{\lambda_1 \ln \xi^2 + \lambda_2} , \tag{2.22}$$

$$k(x) = \lim_{\xi \to 0} \frac{1}{\lambda_1} \left[:\varphi(x+\xi)\varphi(x): - (\lambda_1 \ln x^2 + \lambda_2) j(x) \right] . \tag{2.23}$$

The two expressions highlight the very distinct manners the two operators j and k behave under an R transformation $\varphi \to \varphi + r$: because of the presence of the singular function in the denominator in (2.22), $j(x)$ is R invariant, while $k(x)$ is not. Now from reducible scale invariance only, $j(x)j(0)$ has the expansion

$$j(x)\, j(0) \xrightarrow[x \to 0]{} \frac{1}{x^2} (-b_1 \ln x^2 + a_1)\, j(0) + \frac{1}{x^2}\, b_1\, k(0) , \tag{2.24}$$

which is not a canonical structure because of the presence of logarithms. If, however, we have taken measures to implement R invariance in the system, then R invariance can be applied to the expansion (2.24). The result is of course that the R noninvariant k cannot appear, so that $b_1 = 0$, and the expansion then assumes a purely canonical structure:

$$j(x)\, j(0) \xrightarrow[x \to 0]{} \frac{1}{x^2}\, a_1\, j(0) . \tag{2.25}$$

In this model, R invariance combined with reducible scale invariance assures a canonical structure for the operator product expansion of composite operators. It provides a mechanism to reconcile canonical Bjorken scaling with greater than free-field singularities in field products.

III. NONABELIAN GAUGE THEORIES

The classical theory of nonabelian gauge fields is specified by

$$\mathcal{L}_{cl}(x) = -\frac{1}{4} G_{\mu\nu}^{a}(x)\, G_{a}^{\mu\nu}(x), \tag{3.1}$$

where

$$G_{\mu\nu}^{a}(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + g\, f^{abc} A_\mu^b(x) A_\nu^c(x), \tag{3.2}$$

with f^{abc} the structure constants of the gauge group, and g the bare coupling constant. Sometimes we write

$$f^{abc} L^b M^c \equiv (\vec{L} \times \vec{M})^a. \tag{3.3}$$

We define the covariant derivative

$$\mathcal{D}_\mu^{ab}(x) \equiv \delta^{ab} \partial_\mu^x + g\, f^{acb} A_\mu^c(x), \tag{3.4}$$

and the Lagrangian is then invariant under the infinitesimal gauge transformation

$$A_\mu^a(x) \longrightarrow A_\mu^a(x) + \frac{1}{g} \mathcal{D}_\mu^{ab}(x)\, \omega^b(x), \tag{3.5}$$

where $\vec{\omega}(x)$ is a c-number function of spacetime. We also define

$$\vec{F}_{\mu\nu}(x) = \partial_\mu \vec{A}_\nu(x) - \partial_\nu \vec{A}_\mu(x). \tag{3.6}$$

By antisymmetry,

$$\partial^{\mu} \partial^{\nu} \vec{F}_{\mu\nu}(x) = 0 , \tag{3.7}$$

$$\partial^{\mu} \partial^{\nu} \vec{G}_{\mu\nu}(x) = 0 . \tag{3.8}$$

By virtue of the group-theoretic structure of , we have

$$\mathcal{D}^{\mu}_{ab} \mathcal{D}^{\nu}_{bc} G^{cd}_{\mu\nu}(x) = 0 , \tag{3.9}$$

and (3.9) is valid for arbitrary $\vec{A}_{\mu}(x)$. The Lagrangian (3.1) yields the classical field equation

$$\mathcal{D}^{\nu}_{ab} G^{b}_{\mu\nu}(x) = 0 . \tag{3.10}$$

Note that no use of (3.10) has been made in deriving (3.7)-(3.9).

Naive application of canonical quantization to (3.1) leads to contradiction.[4-6] It is necessary to introduce a gauge-fixing term in the Lagrangian, and the associated term involving fictitious, scalar, anticommuting ghost fields. For calculation of Feynman amplitudes, it is most convenient to choose the gauge function

$$F[A] = \partial_{\mu} A^{\mu} , \tag{3.11}$$

and one gets the modified Lagrangian

$$\mathcal{L}(x) = \mathcal{L}_{cl}(x) - \frac{1}{2\alpha}\left(\partial \cdot \vec{A}(x)\right)^2 + \partial_{\mu} c_1{}^a(x) \mathcal{D}^{\mu}_{ab} c_2{}^b(x) , \tag{3.12}$$

where α is a constant (unrenormalized) gauge parameter, and c_1 and c_2 are anticommuting scalar ghost fields. Now it is possible to write down the canonical equal-time commutation relations

$$\delta(x^0-y^0)\left[G^{0i}_a(x),\, A^j_b(y)\right] = i\,\delta_{ab}\, g^{ij}\, \delta^4(x-y), \quad (3.13)$$

$$\delta(x^0-y^0)\left[\alpha^{-1}\partial\cdot A_a(x),\, A^0_b(y)\right] = i\,\delta_{ab}\, \delta^4(x-y). \quad (3.14)$$

In order to discuss the invariance of the quantized theory under various symmetry operations, we need the equations of motion of the system. Variation of the effective Lagrangian $\mathcal{L}(x)$ (3.12) gives the $\vec{A}_\mu$ field equation

$$\mathcal{D}^\nu_{ab}(x) G^b_{\mu\nu}(x) + \alpha^{-1}\partial_\mu\, \partial\cdot A_a(x) + g f^{abc}\left[\partial_\mu c_1^b(x)\right] c_2^c(x) = 0, \quad (3.15)$$

and the ghost field equations

$$\mathcal{D}^\mu_{ab}\, \partial_\mu\, c_1^b(x) = 0, \quad (3.16)$$

$$\partial_\mu\, \mathcal{D}^\mu_{ab}\, c_2^b(x) = 0. \quad (3.17)$$

The YM field can of course be coupled to other fields in a gauge invariant manner. For example, fermions can be incorporated by adding[15]

$$\mathcal{L}_F = i\bar{\psi}\gamma_\mu D^\mu \psi, \quad (3.18)$$

where

$$D_\mu = \mathbb{1}\,\partial_\mu - i g A^a_\mu T^a, \quad (3.19)$$

and T^a are the fermion representation matrices. The field equation would now be

$$0 = \mathcal{D}^{\nu}_{ab}(x)\, G^{b}_{\mu\nu}(x) + \alpha^{-1} \partial_{\mu} \partial \cdot A_a(x)$$
$$+ g f^{abc} c_1^{b}(x)\, c_2^{c}(x) + g \bar{\psi}(x) \gamma_{\mu} T_a \psi(x) . \qquad (3.20)$$

It is also customary to discuss gauge theories in terms of the vacuum functional $W[J_{\mu}]$ given by the functional integral

$$W[J^{\mu}, K_1, K_2] = \int [dA][dc_1][dc_2]$$
$$\times \exp i \int d^4x \left[\mathcal{L}(x) - \vec{J}_{\mu}(x) \cdot \vec{A}^{\mu}(x) - \vec{K}_1 \cdot \vec{c}_1 - \vec{K}_2 \cdot \vec{c}_2 \right] , \qquad (3.21)$$

with $\mathcal{L}(x)$ given by (3.12). Here $J_{\mu}(x)$, $K_1(x)$, and $K_2(x)$ are classical external sources coupled to the respective fields. The connected parts of the Green's functions are then generated by

$$Z[J] = \ln W[J, \cdots]_{K_1 = K_2 = 0} . \qquad (3.22)$$

In order to discuss proper vertices, the functional $\Gamma[\mathcal{A}]$ is introduced via the Legendre transformation

$$\Gamma[\mathcal{A}] = Z[J] - \int d^4x\, \vec{J}_{\mu}(x) \cdot \vec{\mathcal{A}}^{\mu}(x) , \qquad (3.23)$$

and we have

$$\vec{\mathcal{A}}^{\mu}(x) = \frac{\delta Z[J]}{\delta \vec{J}_{\mu}(x)} , \qquad (3.24)$$

$$\vec{J}^{\mu}(x) = -\frac{\delta\Gamma[\vec{A}]}{\delta\vec{A}_{\mu}(x)} . \tag{3.25}$$

The nonabelian gauge invariance of the Lagrangian leads to the generalized WT identities connecting unrenormalized Green's functions or proper vertices. To derive these identities, we perform the gauge transformation (3.5) on the integration variables A_μ in (3.21). The only change in the integrand comes from the gauge fixing term and the source terms:

$$\begin{aligned} W[J^\mu, K_1, K_2]_{K_1=K_2=0} &\longrightarrow \int [dA][dc_1][dc_2] \\ &\times \exp i\int d^4x \Big\{ \mathcal{L}(x) + \big[-\alpha^{-1}\partial\cdot A_a(x)\,\partial_\mu + \\ &+ g f^{abc}[\partial_\mu c_1{}^b(x)]\,c_2{}^c(x) - J_\mu^a(x)\big]\,\mathcal{D}^\mu_{ab}\, g^{-1}\omega_b(x)\Big\} . \end{aligned} \tag{3.26}$$

The transformation of the integration variable does not affect the value of the integral, and so we may put the coefficient of the arbitrary function $\omega_b(x)$ equal to zero. In this way we obtain

$$\begin{aligned} 0 = \mathcal{L}^\mu\Big[\frac{\delta}{\delta J}\Big]\Big\{ &\alpha^{-1}\partial_\mu\partial_\lambda \frac{\delta}{\delta\vec{J}_\lambda(x)} - \vec{J}_\mu(x) \\ &+ g\Big[\partial_\mu \frac{\delta}{\delta\vec{K}_1(x)}\Big]\times\frac{\delta}{\delta\vec{K}_2(x)}\Big\}\, W[J^\mu, K_1, K_2] , \end{aligned} \tag{3.27}$$

which then yields[16] the Slavnov identity.[7] For the three-point vertex, (3.27) gives

$$\begin{aligned} &\frac{k^\nu}{k^2}\frac{q^\rho}{q^2}\Big(g^{\mu\sigma} - \frac{p^\mu p^\sigma}{p^2}\Big) D_\sigma{}^\lambda(p)\,\Gamma_{\lambda\nu\rho}(p,k,q) \\ &= \frac{k^\nu}{k^2}\,G(q)\Big(g^{\mu\sigma} - \frac{p^\mu p^\sigma}{p^2}\Big)\gamma_{\sigma\nu}(p,k,q) , \end{aligned} \tag{3.28}$$

where Γ is the proper vertex for three YM particles, D is the YM propagator, G is the ghost propagator, and $\gamma_{\sigma\nu}$ is the proper vertex for

$$g f^{agh} \langle 0| T A^g_\sigma(x) c_2^h(x) c_1^b(y) A_\nu^c(z) |0\rangle ,$$

related to the proper vertex Υ_ν for the coupling of a YM particle with two ghosts by

$$p^\sigma \Upsilon_{\sigma\nu}(p,k,q) = \Upsilon_\nu(p,k,q) . \qquad (3.29)$$

The above identity (3.27) can also be derived using the equation of motion approach. Combining (3.10), which is a consequence of the gauge covariance of $\vec{G}_{\mu\nu}$, with the equation of motion (3.15), we have

$$\alpha^{-1} \mathcal{D}\cdot\partial\, \partial\cdot\vec{A}(x) + g\, \mathcal{D}^\mu[\partial_\mu \vec{c}_1(x)] \times \vec{c}_2(x) = 0 . \qquad (3.30)$$

Using (3.30), and the equal-time commutation relation (3.13) and (3.14), it is easy to deduce the WT identity by pulling derivatives through time-ordered products:[17]

$$\begin{aligned} &\alpha^{-1} \langle 0| T \mathcal{D}_x\cdot\partial_x\, T\, \partial\cdot A(x)\, A^\alpha(y)\, A^\beta(z) |0\rangle \\ &= i\Big[\langle 0| T \mathcal{D}_\alpha\, \delta^4(x-y)\, A^\beta(z)|0\rangle + \langle 0| T \mathcal{D}_\beta\, \delta^4(x-z)\, A^\alpha(y)|0\rangle\Big] \\ &\quad - g \langle 0| T \mathcal{D}^\mu [\partial_\mu \vec{c}_1(x)] \times \vec{c}_2(x)\, A^\alpha(y)\, A^\beta(z) |0\rangle , \qquad (3.31) \end{aligned}$$

which is identical with the result of taking functional derivatives twice on (3.27).

Thus the use of functional integrals and of equations of motion are equivalent for deducing the consequences of nonabelian gauge invariance. A symmetry that exists on the level of equations of motion should be exploitable also in the functional integral framework.

IV. RENORMALIZATION AND ASYMPTOTIC FREEDOM

The nonabelian gauge field theories have been shown to be renormalizable. We can express the Lagrangian in terms of renormalized fields and coupling constants through the introduction of renormalization constants:

$$\vec{A}^{\mu} = Z_3^{1/2} \vec{A}^{\mu}_R , \tag{4.1a}$$

$$\psi = Z_2^{1/2} \psi_R , \tag{4.1b}$$

$$\vec{c}_i = \tilde{Z}_3^{1/2} \vec{c}_{iR} , \quad i = 1,2 , \tag{4.1c}$$

$$g = \frac{Z_1}{Z_3^{3/2}} g_R , \tag{4.1d}$$

$$\alpha = Z_3 \alpha_R , \tag{4.1e}$$

and we also have[7,16]

$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} \tag{4.2}$$

as a consequence of nonabelian gauge invariance. The renormalized field equation for gauge fields interacting with fermions would be:[18]

$$0 = \left[\delta^{ab}\partial^{\nu} + \frac{Z_1}{Z_3} g_R f^{acb} A^{\nu}_{cR}(x)\right]\left[\partial_{\mu} A^{b}_{\nu R}(x) - \partial_{\nu} A^{b}_{\mu R}(x)\right.$$
$$\left. + \frac{Z_1}{Z_3} g_R f^{bcd} A^{c}_{\mu R}(x) A^{d}_{\nu R}(x)\right] + Z_3^{-1}\alpha_R^{-1}\partial^{\mu}\partial\cdot A^{a}_{R}(x)$$
$$+ \frac{\tilde{Z}_1}{Z_3} g_R f^{abc}\partial^{\mu} c^{b}_{1R}(x) c^{c}_{2R}(x) + \frac{Z_1 Z_2}{Z_3^{2}} g_R \bar{\psi}_R(x)\gamma^{\mu} T_a \psi_R(x), \quad (4.3)$$

$$\left[\delta^{ab}\partial^{\mu} + \frac{Z_1}{Z_3} g_R f^{acb} A^{\nu}_{cR}(x)\right]\partial_{\mu} c^{b}_{1R}(x) = 0, \quad (4.4)$$

$$\partial_{\mu}\left[\delta^{ab}\partial^{\mu} + \frac{\tilde{Z}_1}{Z_3} g_R f^{acb} A^{\nu}_{cR}(x)\right] c^{b}_{2R}(x) = 0, \quad (4.5)$$

$$\gamma_{\mu}\left[\partial^{\mu} - i\,\frac{Z_1}{Z_3} g_R \vec{A}^{\mu}_{R}(x)\cdot\vec{T}\right]\psi_R(x) = 0. \quad (4.6)$$

In terms of renormalized fields, the Lagrangian is

$$\mathcal{L}(x) = -\frac{1}{4} Z_3\left[\partial_{\mu} A^{a}_{\nu R}(x) - \partial_{\nu} A^{a}_{\mu R}(x) + \frac{Z_1}{Z_3} g_R f^{abc} A^{b}_{\mu R}(x) A^{c}_{\nu R}(x)\right]^2$$
$$- \frac{1}{2\alpha_R}\left(\partial\cdot\vec{A}_R(x)\right)^2 + \tilde{Z}_3\,\partial_{\mu} c^{a}_{1R}(x)\left[\delta^{ab}\partial^{\mu} + \right.$$
$$\left. + \frac{\tilde{Z}_1}{Z_3} g_R f^{acb} A^{\mu}_{cR}(x)\right] c^{b}_{2R}(x)$$
$$+ Z_2\bar{\psi}_R(x)\left[\partial^{\mu} - i\,\frac{Z_1}{Z_3} g_R \vec{A}^{\mu}_{R}(x)\cdot\vec{T}\right]\gamma_{\mu}\psi_R(x). \quad (4.7)$$

Similarly, for the functional integral approach, if we make the scale change

$$\vec{J}^{\mu} = Z_3^{-1/2}\,\vec{J}^{\mu}_{R}, \quad (4.8)$$

$$\vec{\mathcal{A}}^{\mu} = Z_3^{1/2}\,\vec{\mathcal{A}}^{\mu}_{R}, \quad (4.9)$$

in the definition of the generating functionals, then the functional derivatives with respect to these renormalized quantities would give the renormalized Green's functions and proper vertices.

All the Z's occurring can be exactly computed from perturbation theory via the renormalization group. We briefly recapitulate the results here. The renormalization constant Z_i as a function of $\eta \equiv K/M$, where K is the cutoff and M is the subtraction point, satisfies the differential equation

$$0 = \left[-\eta \frac{\partial}{\partial \eta} + \beta(g_R, \alpha_R) \frac{\partial}{\partial g_R} + \gamma_3(g_R, \alpha_R)\, \alpha_R \frac{\partial}{\partial \alpha_R} + \gamma_i(g_R, \alpha_R) \right] Z_i(\eta, g_R, \alpha_R) , \tag{4.10}$$

where

$$\begin{bmatrix} \beta(g_R, \alpha_R) \\ -\gamma_i(g_R, \alpha_R) \end{bmatrix} = M \frac{\partial}{\partial M} \begin{bmatrix} g_R \\ \ln Z_i \end{bmatrix}_{g, \alpha} , \tag{4.11}$$

These parameters can be obtained by low-order perturbative calculations

$$\beta(g_R) = -b\, g_R^3 , \tag{4.12a}$$

$$b = \frac{1}{8\pi^2} \left(\frac{11}{3} c_1 - \frac{4}{3} c_2 \right) ; \tag{4.12b}$$

$$\gamma_1(g_R, \alpha_R) = \frac{g_R^2}{8\pi^2} \left[\left(\frac{17}{6} - \frac{3}{2} \alpha_R \right) c_1 - \frac{8}{3} c_2 \right] , \tag{4.13a}$$

$$\gamma_2(g_R, \alpha_R) = -\frac{g_R^2}{8\pi^2} \alpha_R\, S_2(f) , \tag{4.13b}$$

$$\gamma_3(g_R, \alpha_R) = \frac{g_R^2}{8\pi^2}\left[\left(\frac{13}{3} - \alpha_R\right) c_1 - \frac{8}{3} c_2\right], \qquad (4.13c)$$

$$\tilde{\gamma}_1(g_R, \alpha_R) = \frac{-g_R^2}{8\pi^2} \alpha_R c_1, \qquad (4.13d)$$

$$\tilde{\gamma}_3(g_R, \alpha_R) = \frac{g_R^2}{8\pi^2}\left(\frac{3}{2} - \frac{\alpha_R}{2}\right) c_1, \qquad (4.13e)$$

where

$$f^{acd} f^{bcd} = 2 c_1 \delta^{ab}, \qquad (4.14a)$$

$$\mathrm{tr}(T^a T^b) = 2 c_2 \delta^{ab}, \qquad (4.14b)$$

$$(T^a T^a)_{ij} = S_2(f)\, \delta_{ij}. \qquad (4.14c)$$

For UV freedom, we need $b > 0$, or

$$\frac{c_2}{c_1} < \frac{11}{4}. \qquad (4.15)$$

When (4.15) holds, then the leading behavior of $Z_i(K/M)$ as $K \to \infty$ is obtained from $[Z_i(1,0,\alpha_c)$ is an unknown constant$]$

$$Z_i(\eta, g_R, \alpha_R) \xrightarrow[\eta\to\infty]{} Z_i(1,0,\alpha_c) \exp \int_{g_R}^{g^*} dg' \frac{\gamma_i(g', \alpha_c)}{\beta(g')}. \qquad (4.16)$$

We can distinguish two cases, depending on which fixed point α_c the effective gauge parameter α^* approaches:

$$\alpha_c = \begin{cases} \frac{13}{3} - \frac{8}{3}\frac{c_2}{c_1}, & \text{if } \frac{c_2}{c_1} < \frac{13}{8}, \\ 0, & \text{if } \frac{c_2}{c_1} > \frac{13}{8}. \end{cases} \qquad (4.17)$$

To calculate a quantity like Z_1/Z_3 occurring in (4.3)-(4.7), we notice

$$\frac{Z_1}{Z_3} \xrightarrow[\eta\to\infty]{} (\ln\eta)^{\frac{\bar{\gamma}}{4b}} , \tag{4.18}$$

where

$$\bar{\gamma} = \left(-\frac{3}{2} - \frac{\alpha_c}{2}\right) c_1 . \tag{4.19}$$

Thus, for $\alpha_c = \frac{13}{3} - \frac{8}{3}\frac{c_2}{c_1}$,

$$\bar{\gamma} = -\frac{11}{3} c_1 + \frac{4}{3} c_2 , \tag{4.20}$$

and, for $\alpha_c = 0$,

$$\bar{\gamma} = -\frac{3}{2} c_1 . \tag{4.21}$$

We thus conclude that

$$\frac{Z_1}{Z_3} = 0 , \tag{4.22}$$

and (4.22) holds for all choices of the gauge parameter α_R .

Other ratios of Z's can of course be computed in a similar manner. For example,

$$\frac{Z_1}{Z_3^{\,2}} = 0 , \tag{4.23}$$

$$\frac{\tilde{Z}_1}{Z_3} = 0 , \tag{4.24}$$

all valid for any gauge parameter.

V. ABELIAN GAUGE INVARIANCE

By abelian gauge transformation we mean the transformation on the renormalized fields

$$\vec{A}^{\mu}_{R}(x) \longrightarrow \vec{A}^{\mu}_{R}(x) + \frac{1}{g_R}\,\partial^{\mu}\vec{\Lambda}(x), \qquad (5.1)$$

with $\vec{\Lambda}(x)$ a c-number function, and no compensatory change is made in any matter field interacting with the YM vector field. When we make the transformation (5.1) on the various terms in the renormalized field equation (4.3), the change in the first two terms are

$$\begin{aligned}
&\Delta\Big\{\Big[\delta^{ab}\partial^{\nu} + \frac{Z_1}{Z_3}\,g_R\, f^{acb} A^{\nu}_{cR}(x)\Big] \\
&\times\Big[\partial_{\mu} A^{b}_{\nu R}(x) - \partial_{\nu} A^{b}_{\mu R}(x) + \frac{Z_1}{Z_3} g_R f^{bcd} A^{c}_{\mu R}(x)\, A^{d}_{\nu R}(x)\Big]\Big\} \\
&= \frac{Z_1}{Z_3} f^{acb}\partial^{\nu}\Lambda_c(x)\Big[\partial_{\mu} A^{b}_{\nu R}(x) - \partial_{\nu} A^{b}_{\mu R}(x) + \frac{Z_1}{Z_3} g_R f^{bcd} A^{c}_{\mu R}(x)\, A^{d}_{\nu R}(x)\Big] \\
&+ \Big[\delta^{ab}\partial^{\nu} + \frac{Z_1}{Z_3} g_R f^{acb} A^{\nu}_{cR}(x)\Big]\frac{Z_1}{Z_3} f^{bcd} \\
&\times\Big[\partial_{\mu}\Lambda^{c}(x) A^{d}_{\nu R}(x) + A^{c}_{\mu R}\,\partial_{\nu}\Lambda^{d}(x) + \partial_{\mu}\Lambda^{c}(x)\partial_{\nu}\Lambda^{d}(x)\Big]
\end{aligned} \qquad (5.2)$$

$$\Delta\Big\{\frac{1}{Z_3\alpha_R}\,\partial^{\mu}\,\partial\cdot\vec{A}_R(x)\Big\} = \frac{1}{Z_3 g_R \alpha_R}\,\partial^{\mu}\,\Box\,\vec{\Lambda}(x). \qquad (5.3)$$

The other terms in (4.3) are of course unchanged. Similar considerations apply in the transformation property of the other field equations (4.4)-(4.6). As we already discussed in Sec. II, the product $(Z_1/Z_3)\; A_{\mu}(x)\, A_{\nu}(x)$ is a finite local operator, and so if $Z_1/Z_3 \to 0$, we must have

$$\left(\frac{Z_1}{Z_3}\right)^2 \; A_\mu(x)\, A_\nu(x) \longrightarrow 0 \; . \tag{5.4}$$

Thus the right-hand side of (5.2) vanishes as a consequence of renormalization. The only term in the renormalized field equation (4.3) that changes under the abelian gauge transformation (5.1) is then the gauge-fixing term, as given by (5.3). Similarly, the field equations (4.4)-(4.6) are also invariant. Thus we have precisely the same situation as in an abelian gauge theory like QED.[19]

Similar considerations apply in performing the abelian gauge transformation on the Lagrangian (4.7). The rule to be observed here is that any term that gives zero contribution upon variation to the equation of motion is dropped. For example, the ghost term gives

$$\begin{aligned}&\Delta\left\{\frac{Z_1\tilde{Z}_3}{Z_3}\, g_R \left[\partial_\mu \vec{c}_{1R}(x) \times \vec{c}_{2R}(x)\right]\cdot \vec{A}_R^\mu(x)\right\}\\ &= \frac{Z_1\tilde{Z}_3}{Z_3}\, g_R\, \partial^\mu \vec{\Lambda}(x)\cdot\left[\partial_\mu \vec{c}_{1R}(x)\times \vec{c}_{2R}(x)\right],\end{aligned} \tag{5.5}$$

and this term gives a vanishing contribution to the ghost equations of motion since $Z_1/Z_3 = 0$. The $(\vec{G}_{\mu\nu})^2$ and the fermion terms are likewise abelian invariant. The only change in $\mathcal{L}(x)$ thus comes from the gauge fixing term

$$\Delta\left[\mathcal{L}(x)\right] = -\alpha_R^{-1}\; \partial\cdot\vec{A}\cdot\Box\vec{\Lambda}\, . \tag{5.6}$$

Again, (5.6) is identical to what prevails in QED.[19]

The origin of the abelian invariance is more transparent if we consider the nonabelian transformation (3.5) and express it in terms of renormalized quantities:

$$\vec{A}_{\mu R}(x) \longrightarrow \vec{A}_{\mu R}(x) + 2\,\vec{\omega}(x) \times \vec{A}_{\mu R}(x) + \frac{Z}{Z_1}\,\frac{1}{g_R}\,\partial_\mu \vec{\omega}(x) \tag{5.7}$$

If we now choose $\vec{\omega}(x)$ such that

$$\vec{\Lambda}(x) \equiv \frac{Z_3}{Z_1}\,\vec{\omega}(x) \tag{5.8}$$

is a finite c-number function, and use $Z_1/Z_3 = 0$, (5.7) becomes the abelian gauge transformation (5.1). Given that (5.7) is a symmetry transformation for all $\vec{\omega}(x)$, this shows that (5.1) is a symmetry transformation for all finite and smooth functions $\vec{\Lambda}(x)$. In view of the unrenormalized fermion transformation law

$$\psi(x) \rightarrow \left[1 - 2i\,\vec{T} \times \vec{\omega}(x)\right] \psi(x), \tag{5.9}$$

and its renormalized counterpart

$$\psi_R(x) \longrightarrow \left[1 - 2i\,\vec{T} \times \vec{\omega}(x)\right] \psi_R(x), \tag{5.10}$$

we see that in the presence of fermions the appropriate abelian transformations are (5.1) and

$$\psi_R(x) \longrightarrow \psi_R(x). \tag{5.11}$$

It might be instructive to compare the situation with what prevails in QED. There the gauge transformation expressed in terms of renormalized quantities would be

$$A_{\mu R}(x) \longrightarrow A_{\mu R}(x) + Z_3^{-1/2}\,\partial_\mu \omega(x) \qquad (5.12)$$

$$\psi_R(x) \longrightarrow \psi_R(x) \exp i\, Z_3^{-1/2} e_R\, \omega(x) \qquad (5.13)$$

and the choice

$$\Lambda(x) = Z_3^{-\frac{1}{2}}\, \omega(x) \qquad (5.14)$$

makes the gauge transformation form invariant under renormalization. The point is that in QED the Ward identity mandates the same renormalization constant Z_3 for both coupling constant and photon wave function renormalizations.

There are of course further implications of invariance under (5.7). These correspond to other choices for $\vec{\omega}(x)$. For example, if $\vec{\omega}(x)$ is chosen so that (5.8) is a finite operator such that

$$\vec{A}_{\mu R} \times \vec{\omega} = \frac{Z_1}{Z_3}\, \vec{A}_{\mu R} \times \vec{\Lambda}$$

is a finite operator, the transformation (5.7) remains nonabelian and gives the usual nonabelian WT identities.

There is yet another way of arriving at the existence of the abelian symmetry, and that is via the use of equal-time commutators. We consider the conserved (by (3.8)) current

$$\vec{j}^\mu \equiv \partial_\nu \vec{G}^{\mu\nu}$$

$$= -g\vec{A}_\nu \times \vec{G}^{\mu\nu} - \alpha^{-1}\partial^\mu \partial\cdot\vec{A} - g\partial_\mu \vec{c}_1 \times \vec{c}_2, \quad (5.15)$$

and write down the divergence condition for its time-ordered products with n vector fields:

$$\partial_\mu \langle 0|T\, j_a^\mu(x)\, A_{b_1}^{\alpha_1}(y_1) \cdots A_{b_n}^{\alpha_n}(y_n)|0\rangle$$

$$= -\sum_{i=1}^{n} \langle 0|g^{\mu 0}\delta(x^0-y_i^0)\, T\Big\{g\left[(\vec{A}_\nu \times \vec{G}^{\mu\nu})_a(x),\, A_{b_i}^{\alpha_i}(y_i)\right]$$

$$+ \alpha^{-1}\left[\partial^\mu \partial\cdot A_a(x),\, A_{b_i}^{\alpha_i}(y_i)\right] + g\left[(\partial_\mu \vec{c}_1 \times \vec{c}_2)_a(x),\, A_{b_i}^{\alpha_i}(y_i)\right]\Big\}$$

$$\times\; A_{b_1}^{\alpha_1}(y_1) \cdots \widehat{A_{b_i}^{\alpha_i}(y_i)} \cdots A_{b_n}^{\alpha_n}(y_n)|0\rangle, \quad (5.16)$$

where the hat over $A_{b_i}^{\alpha_i}(y_i)$ indicates that it is omitted. We can evaluate as usual the equal-time commutators involving the unrenormalized fields to give[17]

$$\partial_\mu \langle 0|T\, j_a^\mu(x)\, A_{b_1}^{\alpha_1}(y_1) \cdots A_{b_n}^{\alpha_n}(y_n)|0\rangle$$

$$= -i\sum_{i=1}^{n} \langle 0|T\, \mathcal{D}_{ab_i}^{\alpha_i}\, \delta^4(x-y_i)\, A_{b_1}^{\alpha_1}(y_1)$$

$$\times \cdots \widehat{A_{b_i}^{\alpha_i}(y_i)} \cdots A_{b_n}^{\alpha_n}(y_n)|0\rangle. \quad (5.17)$$

We now write (5.17) in terms of renormalized operators, with

$$\vec{j}_R^\mu = -g_R\frac{Z_1}{Z_3}\vec{A}_\nu^R \times \left[\partial^\mu \vec{A}_R^\nu - \partial^\nu \vec{A}_R^\mu + g_R\frac{Z_1}{Z_3}(\vec{A}_R^\mu \times \vec{A}_R^\nu)\right]$$

$$-\frac{1}{Z_3\alpha_R}\partial^\mu \partial\cdot\vec{A}_R(x) + \frac{\tilde{Z}_1}{Z_3}\, g_R\, \partial^\mu \vec{c}_{1R}(x) \times \vec{c}_{2R}(x), \quad (5.18)$$

and we get the renormalized version of (5.17):

$$\partial_\mu \langle 0| T\, j^\mu_{aR}(x)\, A^{\alpha_1}_{b_1 R}(y_1) \cdots A^{\alpha_n}_{b_n R}(y_n)|0\rangle$$
$$= -\frac{i}{Z_3} \sum_{i=1}^{n} \langle 0| T\Big[\delta^{ab_i}\partial^{\alpha_i} + g_R \frac{Z_1}{Z_3} f^{acb_i} A^{\alpha_i}_{cR}(x)\Big]$$
$$\times \delta^4(x-y_i)\, A^{\alpha_1}_{b_1 R}(y_1)\cdots \widehat{A^{\alpha_i}_{b_i R}(y_i)} \cdots A^{\alpha_n}_{b_n R}(y_n)|0\rangle . \quad (5.18)$$

Because of the vanishing of $Z_1/Z_3^{\,2}$ from (4.23), the coupling constant dependent term disappears, so that (5.18) is in effect an abelian WT identity. In particular,

$$[Q^a,\ A^\alpha_{bR}(y)] = 0 , \qquad (5.19)$$

where

$$Q_a = \int d^3x\ j^0_{aR}(x) . \qquad (5.20)$$

Thus again the theory shows abelian features beyond ordinary perturbation theory.

Finally we shall now derive the WT identities corresponding to the abelian gauge invariance using functional methods. We recall the generating functional

$$W[J] = \int [dA][dc_1][dc_2] \exp i\int d^4x \Big[\mathcal{L}(x) - \vec{J}^R_\mu(x)\cdot \vec{A}^\mu_R(x)\Big] \quad (5.21)$$

with $\mathcal{L}(x)$ given by (4.7). As we saw earlier, the transformation (5.1) on (5.21) gives

$$W[J] \longrightarrow \int [dA][dc_1][dc_2] \exp i\int d^4x \Big[\mathcal{L}(x) - J^R_\mu(x)\cdot A^\mu_R(x)$$
$$- \alpha_R^{-1}\, \partial\cdot\vec{A}_R(x)\cdot \Box\vec{\Lambda}(x) - \vec{J}^R_\mu(x)\cdot\partial^\mu\vec{\Lambda}(x)\Big] . \quad (5.22)$$

The transformation of the integration variables leaves the value of the functional integral invariant, so that we must have

$$\left[-i\alpha_R^{-1}\,\Box\,\partial_\mu \frac{\delta}{\delta J_{\mu R}(x)} + \partial_\mu J_R^\mu(x)\right] W[J] = 0 \,. \quad (5.23)$$

The result is more conveniently stated in terms of proper vertices. The Legendre transformations (3.23)-(3.25) give

$$i\,\alpha_R^{-1}\,\Box\,\partial_\mu \vec{A}_{\mu R}(x) + \partial_\mu \frac{\delta \Gamma}{\delta \vec{A}_{\mu R}(x)} = 0 \,. \quad (5.24)$$

Eqs. (5.23) and (5.24) are just the WT identities for abelian gauge invariance.[19] For example, for the renormalized n point proper vertex, we have

$$q_\mu\, \Gamma_R^{(n)\,\cdots\mu\cdots}(\cdots, q, \cdots) = 0 \,, \quad n > 2, \quad (5.25)$$

etc.

Finally, let us note that the consistency of (5.23) with the nonebelian WT identity (3.27) would require a new relation

$$0 = \left\{ \frac{Z_1}{Z_3^{3/2}}\, g_R \frac{\delta}{\delta \vec{J}_{\mu R}(x)} \times \left[\alpha_R^{-1}\,\partial_\mu \partial_\lambda \frac{\delta}{\delta \vec{J}_{\lambda R}(x)} - \vec{J}_{\mu R}(x)\right] \right.$$
$$+ \frac{\tilde{Z}_1}{Z_3^{1/2}}\, g_R \left[\partial_\mu + \frac{Z_1}{Z_3}\, g_R \frac{\delta}{\delta \vec{J}^\mu_R(x)} \times\right] \partial^\mu$$
$$\left. \times \frac{\bar{\partial}}{\partial K_{1R}(x)} \times \frac{\bar{\partial}}{\delta K_{2R}(x)} \right\} W[J, K_1, K_2] \,. \quad (5.26)$$

The abelian identity (5.25) gives zero for both sides of the Slavnov identity (3.28), so that (3.28) is satisfied trivially.

VI. DISCUSSION

The UV freedom of YM theories is seen to give us precise information on the singularity structures of the theory, and that enables us to draw conclusions valid at all energies. The very feature that allows this to be done, namely the non-abelian nature of the gauge group, itself disappears from the WT identities, usually reliable indicators of the presence of a group structure. This is of course connected with the intrinsic link between the nonabelian group structure and the interaction: they occur as the product $g\ f^{abc}$. The UV freedom means that g can be neglected somehow,[20] and in those cases the theory also behaves as an abelian one.

We should reiterate the warning that our approach is not rigorous. The field equation we used with explicit Z's is a crude instrument indeed; limits are freely exchanged whenever necessary;[21] the use of functional methods is formal at best. We feel that the derivations are plausible.

The abelian nature of nonabelian theories would have other consequences; these are being studied and will be reported elsewhere.

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18. Strictly speaking we need to split points along the lines of Refs. 12 and 13.
19. With the difference that matter fields do not participate in gauge transformation in our case.
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21. The following discussion flows from the remarks of K. Symanzik. A way to circumvent the need for limit interchange in our derivation of the abelian gauge invariance would be to formulate the WT identity for renormalized quantities due to an abelian gauge transformation on the YM theory in $4-\varepsilon$ spacetime dimensions. The WT identity would contain extra terms. The renormalization constants can again be computed via the renormalization group, and the $\varepsilon \rightarrow 0$ limit can be investigated to see if an abelian WT identity like ours is obtained. A complication seems to arise here in that in (sufficiently high orders of) perturbation theory the YM theory is IR divergent even off-shell for all positive rational ε. However, there will be nonanalytic terms of the form $g^{k/\varepsilon + n}$ that compensate these singularities [See K. Symanzik, DESY 73/58, and G. Parisi, lectures at the 1973 Cargèse Summer School, Columbia preprint.] and so for the resulting finite (e.g. β and γ) functions of g and ε the ordinary perturbation expansions should be asymptotic in g for ε sufficiently small. This implies that the nonanalytic terms should not affect the required estimate of the Z factors. Instead of going to $4-\varepsilon$ dimensions, it might also be possible to employ the Slavnov regularization [A.A. Slavnov, Theor. Math. Phys. 13, 174 (1972)] .