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Large Momentum Behaviour of the Feynman Amplitudes in the ϕ_4^4-Theory

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Abstract: The complete asymptotic expansion of the Feynman amplitudes for large values of the scale parameter is derived in the ϕ_4^4-theory for Euclidean and Minkowski metrics.

I. Introduction

The large momentum behaviour of Feynman amplitudes has attracted attention since the early days of renormalization theory [1,2]. Weinberg's power counting theorem, proved for convergent graphs and Euclidean metrics, found innumerable applications in Lagrangian field theory. In 1968, Fink [3] obtained more detailed information concerning the logarithms which accompany the leading power of the scale parameter Λ . In 1973, Slavnov [4] showed that every (inverse) power of the scale parameter Λ in the asymptotic expansion of the Feynman amplitude is accompanied by a polynomial in $\ln\Lambda$ and nothing else. Quite recently, Bergère and Lam determined all the coefficients of the logarithms going along with the leading power of Λ [5].
In 1970, the (leading) asymptotic form of the full vertex functions of renormalizable theories was derived from the Callan Symanzik equations [6]. The validity of this asymptotic form is not restricted to perturbation theory. Never-the-less, in this context numerous questions remained open onto which the behaviour of individual Feynman amplitudes may shed some light e.g. the way in which the perturbation series sums up to produce this asymptotic form, the details of the asymptotic form, the asymptotic form for exceptional momenta, several differently scaled subsets of the momenta etc. This alone might already motivate our interest in the large momentum behaviour of Feynman amplitudes. Still, there is yet another (though intimately related) aspect to these investigations. In view of the popularity of theories involving massless particles the transition from massive to massless fields deserves attention. Although in general summation over the various perturbation theoretic contributions changes qualitatively the approach to the zero mass limit, detailed knowledge about this approach for individual graphs is desirable.
The present work gives the complete large momentum and small mass behaviour of Feynman amplitudes for individual vertex graphs and thereby for arbitrary individual graphs in the ϕ^4_4-theory to which we restrict ourselves for the sake of transparency. It should be pointed out here that no restrictions are imposed on the momenta carried by the external lines of the graph in question neither

linear ones (exceptionality) nor quadratic ones (mass shell confinements). Restrictions of this or a similar kind require a special consideration because we view the asymptotic expansion in the context of distributions and not for every configuration of the external momenta separately. This distribution theoretic formulation of the problem turns out to be both adequate and helpful for Minkowski metrics.
The appropriate frame for the derivation of the asymptotic expansion of the Feynman amplitudes appears to be the analytic renormalization scheme [7,8,9] (and possibly the version of it which uses complex space-time dimension [10]). In this scheme integrations over contours in the complex plane achieving analytic continuation take the place of the cumbersome Taylor operator in the Bogoliubov-Parasiuk-Hepp-Zimmermann scheme (cf. e.g. ref [5]).
Also, the concept of labeled (singularity-) s-families $(\mathcal{L},\sigma)$ in the analytic renormalization scheme [9] corresponding to the resolution of the ultra violet singularities of the Feynman integrand [11] lends itself in a natural way to a generalization: the concept of labeled s_∞-families $(\mathcal{L}_\infty,\sigma_\infty)$ (explained below in section II) corresponding to the resolution of the combined ultra violet and infra red singularities. In order to derive the complete asymptotic expansion for the scale parameter Λ tending to plus infinity the degeneracies of the quadratic form of the external momenta entering the Feynman integrand need to be extracted. In section III this is achieved by diagonalization. Section IV recalls the analytic renormalization procedure. In section V the asymptotic expansion is derived and stated in a form which allows to read off the error committed when truncating after a finite number of terms.
At the end of section V we indicate a way to handle mass insertions.

II. Notations and Definitions

A <u>graph</u> G is a collection $\mathcal{V}(G)$ of $V(G)$ vertices v and a collection $\mathcal{L}(G)$ of $L(G)$ (internal) lines l such that for every line $l \in \mathcal{L}(G)$ there is assigned an initial vertex $i(l) \in \mathcal{V}(G)$ and a final vertex $f(l) \in \mathcal{V}(G)$: $i(l) \neq f(l)$ (no tad-poles!). The vertices $i(l)$ and $f(l)$ are called the endpoints of the line l .

$$G = (\mathcal{V}(G), \mathcal{L}(G) ; i, f)$$

The <u>union</u> G of two graphs G_1 and G_2 : $G = G_1 \cup G_2$ defined by $\mathcal{V}(G) = \mathcal{V}(G_1) \cup \mathcal{V}(G_2)$, $\mathcal{L}(G) = \mathcal{L}(G_1) \cup \mathcal{L}(G_2)$, $i(l) = i_j(l)$ for $l \in \mathcal{L}(G_j)$, $f(l) = f_j(l)$ for $l \in \mathcal{L}(G_j)$ $j = 1,2$ is again a graph.

A <u>subgraph</u> G' of a graph G is a graph $(\mathcal{V}(G'), \mathcal{L}(G'); i', f')$ such that $\mathcal{V}(G') \subset \mathcal{V}(G)$, $\mathcal{L}(G') \subset \mathcal{L}(G)$, $i'(l) = i(l)$, $f'(l) = f(l)$ for every line $l \in \mathcal{L}(G')$.

A subgraph G'' of a subgraph G' of a graph G is a subgraph of G .

A graph G is said to be <u>connected</u> if for every pair $v_1, v_2 \in \mathcal{V}(G)$ there exists a sequence of different lines $\{ l_1, \ldots, l_k \,/\, l_i \neq l_j$ for any $i \neq j$ $1 \leq i,j \leq k$, $l_i \in \mathcal{L}(G)$ for all i , $1 \leq i \leq k \}$ such that v_1 is one of the endpoints of l_1 the second endpoint of l_1 being one of the endpoints of l_2 , the second endpoint of l_2 being one of the endpoints of l_3 , ..., the second endpoint of l_k being v_2 .Otherwise, G is said to be disconnected.

A graph G can uniquely be decomposed into a union of connected subgraphs G_i' of G $i = 1,2,\ldots,c(G)$:

$$G = \bigcup_{i=1}^{c(G)} G_i'$$

The G_i' s $i = 1,2,\ldots,c(G)$ are called the <u>connectivity components</u> (c-components) of G .

A graph G is called <u>one-particle-irreducible</u> (IPI) if it is connected and if any subgraph obtained from G by the removal of one line is connected. Otherwise, G is called one-particle-reducible (IPR).

Any c-component G_i' of a graph G can uniquely be decomposed into $K(G_i')$ IPI-components joined by $(K(G_i') - 1)$ lines, the <u>connectivity-reducing lines</u> (CR-lines).

A graph G is called <u>one-vertex-irreducible</u> (IVI) if G is connected and if any subgraph obtained from G by the removal of one vertex and the lines having this vertex as one of their endpoints is connected. Otherwise, G is called one-vertex-reducible (IVR).

Any c-component G_i' of a graph G can uniquely be decomposed into $M(G_i')$ IVI-components joined by $(M(G_i') - 1)$ vertices, the <u>connectivity-reducing vertices</u> (CR-vertices).

A graph G is said to be <u>irreducible</u> if G is both IPI and IVI. Otherwise, G is said to be reducible.

Let G be a graph. The <u>number of independent loops</u> of G will be denoted by $N(G)$.

The numbers $V(G)$, $L(G)$, $c(G)$ and $N(G)$ are related by the equation

$$N(G) = L(G) + c(G) - V(G) .$$

A subgraph T of G is called a 1-tree or a <u>tree</u> of G if $V(T) = V(G)$, $L(T) = V(G) - 1$, $c(T) = 1$ i.e. if T connects all vertices of G to each other and if $\mathcal{L}(T)$ does not form loops ($N(T) = 0$).

A subgraph T_r of G is called an <u>r-tree</u> of G if $V(T_r) = V(G)$, $L(T_r) = V(G) - r$, $c(T_r) = r$ i.e. if T_r effects a partition of the vertices of G into r mutually disjoint sets any two vertices of the same set being connected in T_r and if $\mathcal{L}(T_r)$ does not form loops ($N(T_r) = 0$).

A subgraph T_r' of G being related to an r-tree of G as follows:

$$\mathcal{V}(T_r') = \{ v \,/\, v = i(l) \text{ or } v = f(l) \text{ for any } l \in \mathcal{L}(G) \smallsetminus \mathcal{L}(T_r) \} ,$$

$$\mathcal{L}(T_r') = \mathcal{L}(G) \smallsetminus \mathcal{L}(T_r)$$

is called a <u>co-r-tree</u> of G .

Now, consider an irreducible graph G^o and set $\mathcal{V}(G^o) = \mathcal{V}$, $V(G^o) = V$, $\mathcal{L}(G^o) = \mathcal{L}$, $L(G^o) = L$, $N(G^o) = N$.

Among the various subsets of $\mathfrak{V}$ we distinguish the set $\mathfrak{U}$ of all U <u>external vertices</u> u. For every subgraph G of G^{o} we define

$$\mathfrak{U}(G) = \mathfrak{V}(G) \cap \mathfrak{U}$$

We add to G^{o} one more vertex v_{∞} and connect this "infinite" vertex v_{∞} to the external vertices u by U lines l_u, $u \in \mathfrak{U}$. The graph

$$\mathfrak{V} \cup \{v_{\infty}\}, \mathcal{L} \cup \{l_u \,/\, u \in \mathfrak{U}\}; \quad i_{\infty}(l) = i(l), \quad f_{\infty}(l) = f(l) \quad \text{for}$$

$$l \in \mathcal{L} \quad \text{and} \quad i_{\infty}(l_u) = v_{\infty}, \quad f_{\infty}(l_u) = u \quad \text{for} \quad u \in \mathfrak{U}$$

will be denoted by G^{o}_{∞}.

Similarly for a subgraph G of G^{o}, the pair

$$\mathfrak{V}(G) \cup \{v_{\infty}\}, \ \mathcal{L}(G) \cup \{l_u \,/\, u \in \mathfrak{U}(G)\}$$

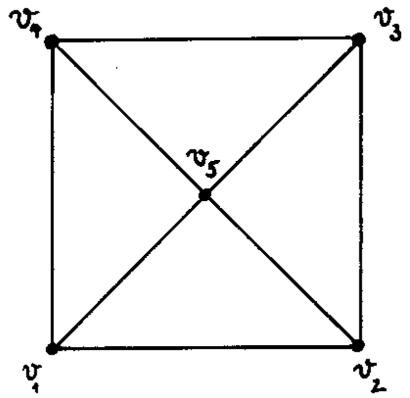
defines a subgraph G_{∞} of G^{o}_{∞}.

The graph G^{o} as well as any subgraph G of G^{o} are frequently thought of as subgraphs of G^{o}_{∞}.

A subgraph G of the graph G^{o} is said to be <u>irreducible in view of infinity</u> (I_{∞}) if either G is irreducible or if G_{∞} is IPI and if with the possible exception of v_{∞} none of the vertices of G_{∞} is a CR-vertex. Otherwise, G is called reducible in spite of infinity (R_{∞}).

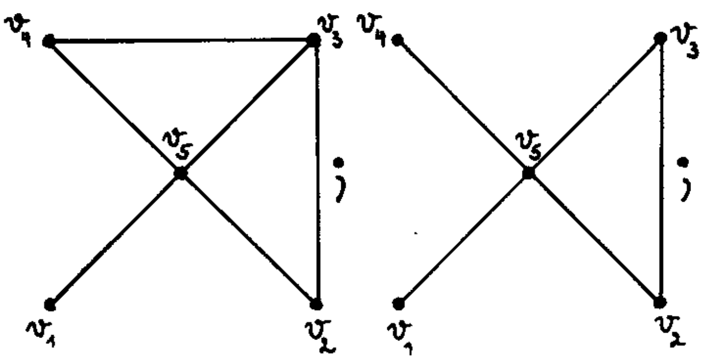
<u>Example</u>

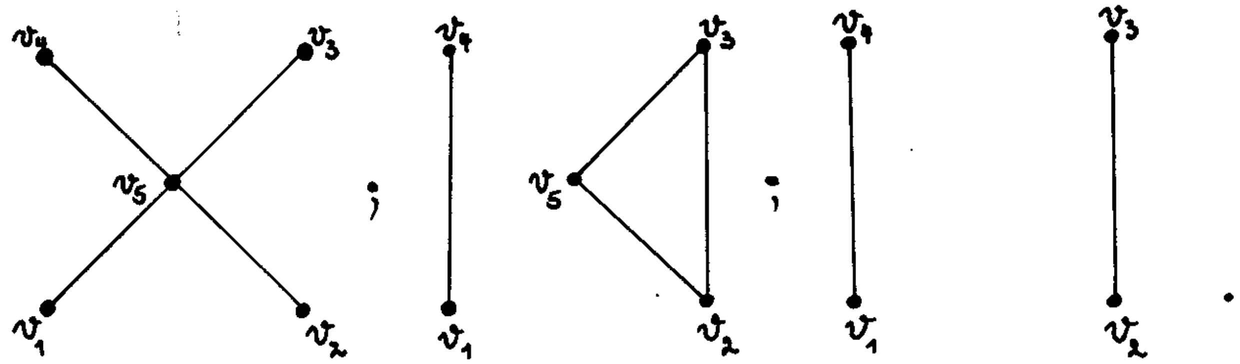
G^{o} :



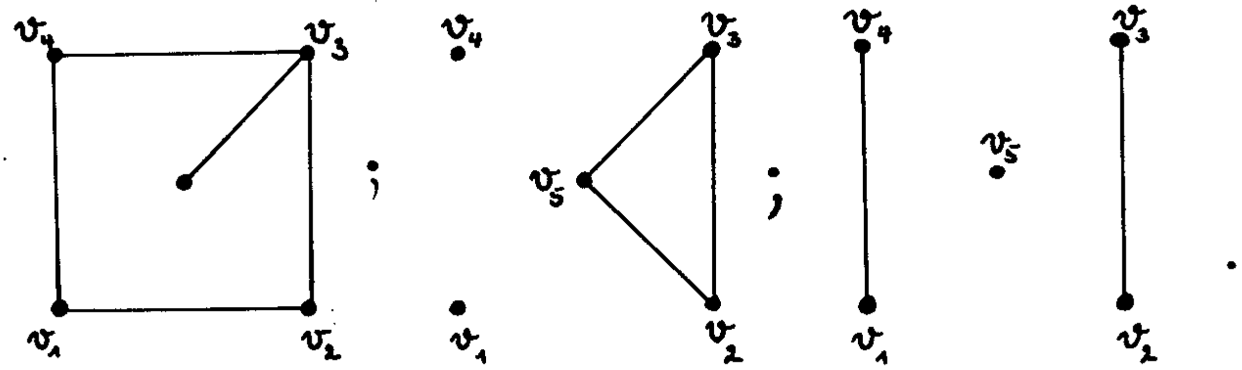
$$\mathfrak{U} = \{v_1, v_2, v_3, v_4\}$$

Some of the I_{∞}-subgraphs of G^{o}:





Some of the R_∞-subgraphs of G^o :



<u>Remarks</u>: (a) G^o is I_∞.

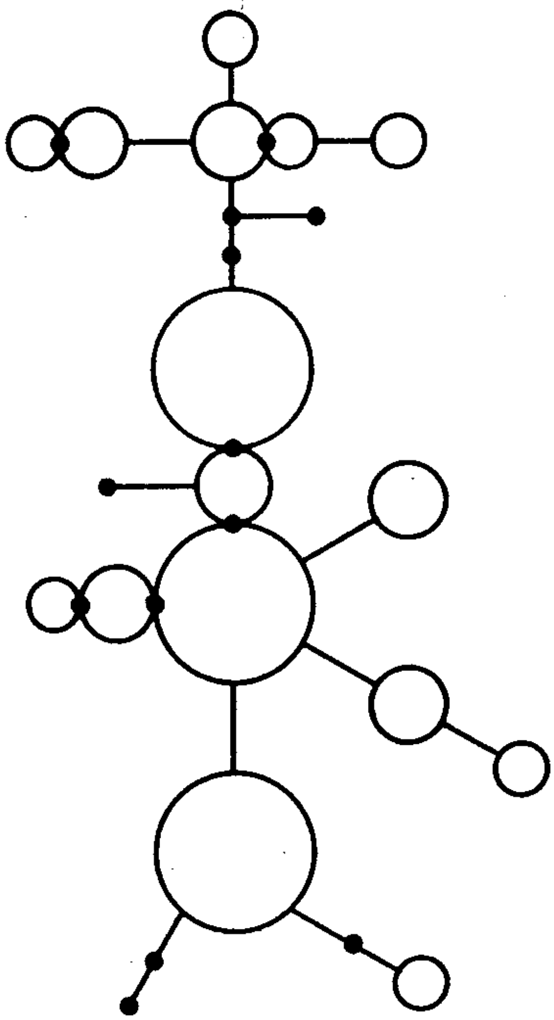
(b) Every irreducible subgraph G of G^o is I_∞.

(c) Let G be a reducible subgraph of G^o . Then G is I_∞ iff G is a union of one or more disconnected tree arrangements of "stalks" and "cactuses" - a stalk consisting of external and/or connectivity-reducing vertices joined to each other and to the rest of the tree arrangement by CR-lines, a cactus consisting of irreducible components of G^o joined by CR-vertices- with the following properties

i) each tree arrangement has at least two different external vertices

ii) every "free end" of each of these tree arrangements has at least one external vertex which is not connectivity-reducing.

<u>Example</u> of a tree arrangement of stalks and cactuses:



Next, we define an s_∞-family $\mathcal{E}_\infty$ for G^o as a maximal collection of I_∞-subgraphs G of G^o with the following properties (cf. ref[8])

(S_∞-0) $$\mathcal{L}(G) \neq \emptyset$$

(S_∞-1) If G, $G' \in \mathcal{E}_\infty$, then either $G \subset G'$, $G' \subset G$ or $\mathcal{L}(G) \cap \mathcal{L}(G') = \emptyset$.

(S_∞-2) If $G_1, \ldots, G_k \in \mathcal{E}_\infty$ and $\mathcal{L}(G_i) \cap \mathcal{L}(G_j) = \emptyset$ for any $i \neq j$, then $\bigcup_{i=1}^{k} G_i$ is R_∞.

Moreover, we define a labeled s_∞-family for G^o to be a pair $(\mathcal{E}_\infty, \sigma_\infty)$ where $\mathcal{E}_\infty$ is an s_∞-family for G^o and σ_∞ a mapping $\sigma_\infty : \mathcal{E}_\infty \to \mathcal{L}$ satisfying

(S_∞-3) $$\sigma_\infty(G) \in \mathcal{L}(G)$$

(S_∞-4) If $G' \in \mathcal{E}_\infty$ is a proper subset of $G \in \mathcal{E}_\infty$, then $\sigma_\infty(G) \notin \mathcal{L}(G')$.

Finally, if $(\mathcal{E}_\infty, \sigma_\infty)$ is a labeled s_∞-family for G^0 we define $\mathcal{D}_\infty = \mathcal{D}(\mathcal{E}_\infty, \sigma_\infty)$ to be the subset of α-space given by

$$\mathcal{D}_\infty = \{\underline{\alpha} = (\alpha_l)_{l\in\mathcal{L}} \,/\, \alpha_l \geqslant 0 \text{ for all } l \in \mathcal{L} \,,\ \alpha_l \leqslant \alpha_{\sigma_\infty(G)} \text{ for all } l \in \mathcal{L}(G) \,,\ G \in \mathcal{E}_\infty\}.$$

Let $\mathcal{E}_\infty$ be an s_∞-family for G^0 , G an element of $\mathcal{E}_\infty$. The collection of all elements of $\mathcal{E}_\infty$ which are (proper) subgraphs of G will be denoted by $\mathcal{E}^+_\infty(G)$ $(\mathcal{E}^-_\infty(G))$, the collection of all elements of $\mathcal{E}_\infty$ which contain G as a (proper) subgraph will be denoted by $\mathcal{E}^\infty_+(G)$ $(\mathcal{E}^\infty_-(G))$.

The following statements can be proved along the lines of ref[8] (cf. Lemmas 3.2 and 3.3)

i) For every $G \in \mathcal{E}_\infty$ there exists a line $l \in \mathcal{L}(G)$ not contained in $\mathcal{L}(G')$ for some $G' \in \mathcal{E}^-_\infty(G)$.

ii) $\#[\mathcal{E}^-_\infty(G)] = N(G_\infty) - 1$, $\#[\mathcal{E}_\infty] = N(G^0_\infty)$.

iii) Every s_∞-family $\mathcal{E}_\infty$ for G^0 may be labeled, i.e. there exists a mapping $\sigma_\infty : \mathcal{E}_\infty \longrightarrow \mathcal{L}$ such that $(\mathcal{E}_\infty, \sigma_\infty)$ is a labeled s_∞-family for G^0 .

iv) If $(\mathcal{E}_\infty, \sigma_\infty)$ is a labeled s_∞-family for G^0 , then $T_U = T_U(\mathcal{E}_\infty, \sigma_\infty)$: $(\mathcal{V}, \mathcal{L} \setminus \sigma_\infty(\mathcal{E}_\infty))$ is a U-tree of G^0 each of the U c-components of tree structure containing exactly one external vertex $u \in \mathcal{U}$ (One or more c-components of T_U may consist of just one external vertex).

v) $\cup\, \mathcal{D}_\infty = \cup\, \mathcal{D}(\mathcal{E}_\infty, \sigma_\infty) = \{\underline{\alpha} \,/\, \alpha_l \geqslant 0 \text{ for all } l \in \mathcal{L}\}$ where the union extends over all labeled s_∞-families for G^0 .

vi) If $(\mathcal{E}_\infty, \sigma_\infty)$ and $(\mathcal{E}'_\infty, \sigma'_\infty)$ are two different labeled s_∞-families for G^0 , then $\mathcal{D}_\infty \cap \mathcal{D}'_\infty = \mathcal{D}(\mathcal{E}_\infty, \sigma_\infty)\ \mathcal{D}(\mathcal{E}'_\infty, \sigma'_\infty)$ has Lebesgue measure zero.

Comparison of the number of elements in an s-family $\mathcal{E}$ for G^0 (cf. ref [8]) and in an s_∞-family $\mathcal{E}_\infty$ for G^0 yields

$$\#[\mathcal{E}_\infty] = \#[\mathcal{E}] + U - 1$$

since $\#[\mathcal{E}] = N = L + 1 - V$ and $\#[\mathcal{E}_\infty] = N(G^0_\infty) = L + U + 1 - (V + 1)$.

Consider the subset $\mathcal{F}_\infty$ of $\mathcal{E}_\infty$ consisting of all elements F of $\mathcal{E}_\infty$ with $N(F) = N(F_1) + 1$ where F_1 is the following subgraph of G^o : $\mathcal{V}(F_1) = \{v \,/\, v$ is an endpoint of at least one line of $\mathcal{L}(F)\setminus\sigma_\infty(F)\}$, $\mathcal{L}(F_1) = \mathcal{L}(F)\setminus\sigma_\infty(F)$. For any $G \in \mathcal{E}_\infty$ define

$$\mathcal{F}^+_\infty(G) = \{F \,/\, F \in \mathcal{F}_\infty ,\ F \subseteq G\}$$
$$\mathcal{F}^-_\infty(G) = \{F \,/\, F \in \mathcal{F}_\infty ,\ F \subsetneq G\} .$$

We claim that $T = T(\mathcal{E}_\infty, \sigma_\infty) : (\mathcal{V}, \mathcal{L}\setminus\sigma_\infty(\mathcal{F}_\infty))$ is a tree of G^o. For the proof we note that $V(T) = V$, $\# [\sigma_\infty(\mathcal{F}_\infty)] = N$, $L(T) = L - N = V - 1$, $N(T) = 0$. Hence $c(T) = 1$.

Next, consider the complementary subset $\mathcal{H}_\infty = \mathcal{E}_\infty \setminus \mathcal{F}_\infty$ of $\mathcal{E}_\infty$. For any $G \in \mathcal{E}_\infty$ we denote the subset of $\mathcal{H}_\infty$ consisting of all elements H of $\mathcal{H}_\infty$ which are (proper) subgraphs of G by $\mathcal{H}^+_\infty(G)$ $(\mathcal{H}^-_\infty(G))$ and the subset of $\mathcal{H}_\infty$ consisting of all elements of $\mathcal{H}_\infty$ which contain G as a (proper) subgraph by $\mathcal{H}^\infty_+(G)$ $(\mathcal{H}^\infty_-(G))$. Furthermore, we denote the number of elements in $\mathcal{H}^\infty_+(G)$ by $h(G)$.

We claim that $T_{h(G)+1} = T_{h(G)+1}(\mathcal{E}_\infty, \sigma_\infty) : (\mathcal{V}, \mathcal{L}\setminus\sigma_\infty(\mathcal{F}_\infty)\setminus\sigma_\infty(\mathcal{H}^\infty_+(G)))$ is an $(h(G) + 1)$ - tree of G^o with at least one external vertex in every c-component.

In order to prove this claim we note that $V(T_{h(G)+1}) = V$, $L(T_{h(G)+1}) = L - N - h(G) = V - (1 + h(G))$, $N(T_{h(G)+1}) = 0$. Hence $c(T_{h(G)+1}) = 1 + h(G)$.
The rest of the claim follows from the observation that the removal of lines from $\mathcal{H}_\infty(G)$ reduces line by line the connectivity of the external vertices.

For any line $l \in \mathcal{L}$ we define $G(l)$ to be the minimal element of $\mathcal{E}_\infty$ containing the line l .
With this notation the subset $\mathcal{D}_\infty$ of the α-space can be parametrized as follows:

$$\alpha_l = \begin{cases} \prod\limits_{G' : \mathcal{E}_\infty \ni G' \supset G} t_{G'} & \text{if } l = \sigma_\infty(G) \text{ for some } G \in \mathcal{E}_\infty \\ \beta_l \cdot \prod\limits_{G' : \mathcal{E}_\infty \ni G' \supset G(l)} t_{G'} & \text{if } l \neq \sigma_\infty(G) \text{ for any } G \in \mathcal{E}_\infty \end{cases}$$

where $0 \leqslant t_{G^o} < \infty$, $0 \leqslant t_G \leqslant 1$ for any $G \in \mathcal{E}^-_\infty(G^o)$, $0 \leqslant \beta_l \leqslant 1$ $l \neq \sigma_\infty(G)$ for any $G \in \mathcal{E}_\infty$ or writing the symbol $\underline{t}$ for $(t_G)_{G \in \mathcal{E}^-_\infty(G)}$ and the symbol $\underline{\beta}$ for $(\beta_l)_{l \neq \sigma_\infty(G)}$ for any $G \in \mathcal{E}_\infty$:

$$0 \leqslant t_{G^o} < \infty , \quad (\underline{t}, \underline{\beta}) \in I^{L-1} \quad \text{with} \quad I = [0 , 1] .$$

III. Diagonalization of the Relevant Quadratic Forms

To every external vertex $u \in \mathcal{U}$ we associate a real four vector variable, the external momentum $p_u \in \mathbb{R}^4$ and set $\underline{p} = (p_u)_{u \in \mathcal{U}}$. We assume that a real scalar product is given on $\mathbb{R}^4$. In the applications we have in mind the quadratic form associated with this scalar product is not positive semidefinite. Hence we suppose that there exists a vector $p_o \in \mathbb{R}^4$ such that $p_o^2 = p_o \cdot p_o = -1$.

The removal of any line l from $\mathcal{L}(T(\mathcal{E}_\infty, \sigma_\infty))$ results in a 2-tree of G^o with the two c-components $\Theta_l = \Theta_l(\mathcal{E}_\infty, \sigma_\infty)$ and $\Theta^l = \Theta^l(\mathcal{E}_\infty, \sigma_\infty)$ where $i(l) \in \mathcal{V}(\Theta_l)$ and $f(l) \in \mathcal{V}(\Theta^l)$. By k_l we denote the partial sum of external momenta "that flow into the line l in the tree $T(\mathcal{E}_\infty, \sigma_\infty)$":

$$k_l = \sum_{u \in \mathcal{U}(\Theta_l)} p_u = \sum_{u \in \mathcal{U}} p_u - k^l$$

$$k^l = \sum_{u \in \mathcal{U}(\Theta^l)} p_u .$$

By k_H and k^H, $H \in \mathcal{H}_\infty$ we denote the partial sums of external momenta

$$k_H = \sum_{u \in \mathcal{U}(\Theta_{\sigma_\infty(H)})} p_u$$

and

$$k^H = \sum_{u \in \mathcal{U}(\Theta^{\sigma_\infty(H)})} p_u$$

respectively. We set

$$\underline{k} = (k_H)_{H \in \mathcal{H}_\infty}, \quad k_\emptyset = \sum_{u \in \mathcal{U}} p_u .$$

$k_\emptyset$ and k_H, $H \in \mathcal{H}_\infty$ are U linearly independent partial sums of the external momenta. Thus every linear combination of the external momenta can uniquely be written as a linear combination of $k_\emptyset$ and k_H, $H \in \mathcal{H}_\infty$. The change of basis in the U-dimensional real space $\mathbb{R}^U$ from $(p_u)_{u \in \mathcal{U}}$ to $(k_\emptyset, (k_H)_{H \in \mathcal{H}_\infty})$ is achieved by a non-singular (α-independent) transformation $R = R(\mathcal{E}_\infty, \sigma_\infty)$

$$\begin{pmatrix} k_\emptyset \\ (k_H)_{H \in \mathcal{H}_\infty} \end{pmatrix} = R \left((p_u)_{u \in \mathcal{U}} \right) .$$

Let χ be an arbitrary subset of $\mathfrak{V}$. Denote the complementary subset $\mathfrak{V}\setminus\chi$ by χ'.

How is $(\sum_{u\in\chi} p_u)$ linearly expressed in terms of $k_\emptyset$ and $(k_H)_{H\in\mathcal{H}_\infty}$?

In order to answer this question, we define the subgraph $\Theta_\chi = \Theta_\chi(\mathcal{E}_\infty,\sigma_\infty)$ of $T(\mathcal{E}_\infty,\sigma_\infty)$ as follows:

$\mathcal{V}(\Theta_\chi) = \chi \cup \{v \,/\, v\in\mathcal{V},\ v$ is an endpoint of a line $l\in\mathcal{L}(\Theta_\chi)\}$

$\mathcal{L}(\Theta_\chi) = \{l \,/\, l$ belongs to a sequence of different lines from $\mathcal{L}(T(\mathcal{E}_\infty,\sigma_\infty))$ that connects two external vertices u_1 and u_2 from χ to each other without at the same time connecting the vertices u_1 and u_2 to an external vertex u' from $\chi'\}$

$\cup\,\{l \,/\, l$ belongs to a sequence of different lines from $\mathcal{L}\setminus\sigma_\infty(\mathcal{E}_\infty)$ that connects an external vertex u from χ to an internal vertex $v\in\mathcal{V}\setminus\mathfrak{V}\}$.

We denote the c-components of Θ_χ by $\Theta_\chi^j = \Theta_\chi^j(\mathcal{E}_\infty,\sigma_\infty)$ $j=1,\ldots,J=J_\chi(\mathcal{E}_\infty,\sigma_\infty)$. We set $\chi_j = \mathfrak{V}(\Theta_\chi^j)$. Let $\mathcal{H}^j_{\chi;i}$ and $\mathcal{H}^j_{\chi;f}$ be the sets of all elements $H\in\mathcal{H}_\infty$ with $\sigma_\infty(H)\notin\mathcal{L}(\Theta_\chi)$, $i(\sigma_\infty(H))\in\mathcal{V}(\Theta_\chi^j)$ and $f(\sigma_\infty(H))\in\mathcal{V}(\Theta_\chi^j)$, respectively. We set

$$\mathcal{H}_{\chi;i} = \bigcup_{j=1}^{J}\mathcal{H}^j_{\chi;i},\qquad \mathcal{H}_{\chi;f} = \bigcup_{j=1}^{J}\mathcal{H}^j_{\chi;f}$$

Then, for $H\in\mathcal{H}^j_{\chi,i}$:

$$k_H = \Big(\sum_{u\in\chi_j} p_u\Big) + \Big(\sum_{H'\in\mathcal{H}^j_{\chi;f}} k_{H'}\Big) + \Big(\sum_{H'\in\mathcal{H}^j_{\chi;i},\,H'\neq H}(k_\emptyset - k_{H'})\Big)$$

or

$$\Big(\sum_{u\in\chi_j} p_u\Big) = k_\emptyset\Big(1 - \#[\mathcal{H}^j_{\chi;i}]\Big) + \sum_{H'\in\mathcal{H}^j_{\chi;i}} k_{H'} - \sum_{H'\in\mathcal{H}^j_{\chi;f}} k_{H'}.$$

Summation over j leads to

$$\Big(\sum_{u\in\chi} p_u\Big) = \Big(J - \#[\mathcal{H}_{\chi;i}]\Big)k_\emptyset + \sum_{H'\in\mathcal{H}_{\chi;i}} k_{H'} - \sum_{H'\in\mathcal{H}_{\chi;f}} k_{H'}.$$

This is the explicit linear expression of $(\sum_{u\in\chi} p_u)$ in terms of $k_\emptyset$ and k_H, $H\in\mathcal{H}_\infty$ we were asking for.

Now, consider $d(\underline{\alpha})$

$$d(\underline{\alpha}) = \sum_T \prod_{l\in\mathcal{L}\setminus\mathcal{L}(T)} \alpha_l = \sum_{T'} \prod_{l\in\mathcal{L}(T')} \alpha_l$$

where the sum $\sum_T$ extends over all trees of G^0 and the sum $\sum_{T'}$ over all co-trees of G^0.

Take an arbitrary co-tree T' from the sum $\sum_{T'} \prod_{l\in\mathcal{L}(T')} \alpha_l$.

We argue with Speer and Westwater [12] that the intersection of $\mathcal{L}(T')$ with $\mathcal{L}(F)$ for every $F\in\mathcal{F}_\infty$ contains at least $N(F)$ lines $l_{F'}\in\mathcal{L}(F')$, $F'\in\mathcal{F}_\infty$, $F'\subset F$. Hence for $\underline{\alpha}\in\mathcal{D}_\infty$

$$d(\underline{\alpha}) = \prod_{G\in\mathcal{E}_\infty} t_G^{N(G)}\cdot \hat{d}(\underline{t},\underline{\beta})$$

where the function $\hat{d}(\underline{t},\underline{\beta})$ is a polynomial in $\underline{t}$ and $\underline{\beta}$. On the other hand, for $\underline{\alpha}\in\mathcal{D}_\infty$

$$d(\underline{\alpha}) \geqslant \prod_{\ell\in\mathcal{L}\setminus(\mathcal{L}\setminus\sigma_\infty(\mathcal{F}_\infty))}\alpha_\ell = \prod_{\ell\in\sigma_\infty(\mathcal{F}_\infty)}\alpha_\ell = \prod_{G\in\mathcal{E}_\infty} t_G^{N(G)} .$$

Hence the polynomial $\hat{d}(\underline{t},\underline{\beta})$ is larger or equal to one for $(\underline{t},\underline{\beta})\in I^{L-1}$.

Analogously, examine $d_2(\chi|\chi')(\underline{\alpha})$

$$d_2(\chi|\chi')(\underline{\alpha}) = \sum_{T_2}\prod_{\ell\in\mathcal{L}\setminus\mathcal{L}(T_2)}\alpha_\ell = \sum_{T_2'}\prod_{\ell\in\mathcal{L}(T_2')}\alpha_\ell$$

where the sum $\sum_{T_2}$ extends over all 2-trees of G^0 such that for u_1 , $u_2\in\mathcal{V}$ there exist a sequence of different lines in $\mathcal{L}(T_2)$ connecting u_1 and u_2 iff u_1 and u_2 are contained either both in χ or both in $\chi' = \mathcal{V}\setminus\chi$, and where the sum $\sum_{T_2'}$ extends over the co-2-trees T_2' of G^0 constructed from all such 2-trees T_2 of G^0.

Take an arbitrary co-2-tree T_2' from the sum $\sum_{T_2'} \prod_{l\in\mathcal{L}(T_2')}\alpha_l$.

Again we argue with Speer and Westwater [12] that the intersection of $\mathcal{L}(T_2')$ with $\mathcal{L}(F)$ for every $F\in\mathcal{F}_\infty$ contains at least $N(F)$ lines $l_{F'}\in\mathcal{L}(F')$, $F'\in\mathcal{F}_\infty$, $F'\subset F$ and that in

$$\Big(\mathcal{V}(H_\chi)\,,\ \mathcal{L}(H_\chi)\setminus\bigcup_{F\in\mathcal{F}_\infty^-(H_\chi)}\{\ell_F\}\Big)\ ,\quad H_\chi = \bigcap_{H\in\mathcal{H}_{\chi;i}\cup\mathcal{H}_{\chi;f}} H$$

a vertex u from χ is connected to a vertex u' from χ' by a sequence of different lines. Hence

$$\mathcal{L}(T_2')\cap\mathcal{L}(H_\chi)\supset\bigcup_{F\in\mathcal{F}_\infty^-(H_\chi)}\{\ell_F\}\cup\{\ell_{H_\chi}\}$$

where $l_{H_\chi}\in\mathcal{L}(H_\chi)$. Thus, for $\underline{\alpha}\in\mathcal{D}_\infty$ $d_2(\chi|\chi')(\underline{\alpha})$ can be written as

$$d_2(\chi|\chi')(\underline{\alpha}) = \prod_{G\in\mathcal{E}_+^\infty(H_\chi)} t_G\cdot\prod_{G\in\mathcal{E}_\infty} t_G^{N(G)}\cdot\hat{d}_2(\chi|\chi')(\underline{t},\underline{\beta})$$

where the function $d_2(\chi|\chi')(\underline{t},\underline{\beta})$ is a polynomial in $\underline{t}$ and $\underline{\beta}$.

Next, we study the quadratic form $A_{\underline{\alpha}}(\underline{p},\underline{p})$

$$A_{\underline{\alpha}}(\underline{p},\underline{p}) = \frac{-\frac{1}{2}\sum_{\chi\subset\mathcal{V}} s(\chi)\, d_2(\chi|\chi')(\underline{\alpha})}{d(\underline{\alpha})} \Big/ \sum_{u\in\mathcal{V}} p_u = 0$$

$$= -\frac{1}{2}\sum_{\chi\subset\mathcal{V}} s(\chi) \prod_{G\in\mathcal{E}^{\infty}_{+}(H_\chi)} t_G \cdot \left(\frac{\hat{d}_2(\chi|\chi')(\underline{t},\underline{\beta})}{\hat{d}(\underline{t},\underline{\beta})}\right) \Big/\!\!\Big/ \sum_{u\in\mathcal{V}} p_u = 0$$

for $\underline{\alpha}\in\mathcal{D}_\infty$ with $s(\chi) = (\sum_{u\in\chi} p_u)^2$.

When expressing $(p_u)_{u\in\mathcal{V}}$ by $k_\emptyset$, $(k_H)_{H\in\mathcal{H}_\infty}$, $A_{\underline{\alpha}}(\underline{p},\underline{p})$ goes over into $t_{G_0}\cdot D_{\underline{t},\underline{\beta}}(\underline{k},\underline{k}) = t_{G_0}\cdot D^{\mathcal{H}_\infty}_{\underline{t},\underline{\beta}}(\underline{k},\underline{k})$

$$A_{\underline{\alpha}}(\underline{p},\underline{p}) = -t_{G_0}\cdot \sum_{H',H''\in\mathcal{H}_\infty} d_{H',H''}(\underline{t},\underline{\beta})\, k_{H'}\cdot k_{H''} = t_{G_0}\cdot D_{\underline{t},\underline{\beta}}(\underline{k},\underline{k})$$

$$= -\sum_{H',H''\in\mathcal{H}_\infty} \Big(\prod_{G\in\mathcal{E}^{\infty}_{+}(H'\cap H'')} t_G\Big)\, \hat{d}_{H',H''}(\underline{t},\underline{\beta})\, k_{H'}\cdot k_{H''}$$

where the coefficients $d_{H',H''}(\underline{t},\underline{\beta})$ and $\hat{d}_{H',H''}(\underline{t},\underline{\beta})$ are quotients of two polynomials in $\underline{t}$ and $\underline{\beta}$ the divisor polynomials being larger or equal to one for $(\underline{t},\underline{\beta})\in I^{L-1}$ in both cases.

The quadratic form $D''_{\underline{t},\underline{\beta}}(\underline{k},\underline{k})$

$$D''_{\underline{t},\underline{\beta}}(\underline{k},\underline{k}) = t_{G_0}\cdot D_{\underline{t},\underline{\beta}}\left(\Big(k_H\Big/\prod_{G\in\mathcal{E}^{\infty}_{+}(H)} t_G^{1/2}\Big)_{H\in\mathcal{H}_\infty}, \Big(k_H\Big/\prod_{G\in\mathcal{E}^{\infty}_{+}(H)} t_G^{1/2}\Big)_{H\in\mathcal{H}_\infty}\right)$$

$$= -\sum_{H',H''\in\mathcal{H}_\infty} \Big(\prod_{G\in\mathcal{E}^{\infty}_{+}(H'\cap H'')\cap\mathcal{E}^{-}_{\infty}(H'\cup H'')} t_G^{1/2}\Big)\, \hat{d}_{H',H''}(\underline{t},\underline{\beta})\, k_{H'}\cdot k_{H''}$$

is a finite-valued, non-degenerate quadratic form for $(\underline{t},\underline{\beta})\in I^{L-1}$.

The non-degeneracy is seen as follows. Set $k_H = x_H\cdot p_0$, $x_H\in\mathbb{R}^1$ for all $H\in\mathcal{H}_\infty$, $p_0^2 = -1$. Then

$$D''_{\underline{t},\underline{\beta}}\big((x_H p_0)_{H\in\mathcal{H}_\infty},(x_H p_0)_{H\in\mathcal{H}_\infty}\big)$$

$$= \sum_{H',H''\in\mathcal{H}_\infty} \Big(\prod_{G\in\mathcal{E}^{\infty}_{+}(H'\cap H'')\cap\mathcal{E}^{-}_{\infty}(H'\cup H'')} t_G^{1/2}\Big)\, \hat{d}_{H',H''}(\underline{t},\underline{\beta})\, x_{H'}\, x_{H''}$$

is a sum of non-negative terms $\frac{1}{2}\,\hat{s}_{\underline{t}}(\chi)\,\dfrac{\hat{d}_2(\chi\,\chi')(\underline{t},\underline{\beta})}{\hat{d}(\underline{t},\underline{\beta})}$ where

$$\hat{s}_{\underline{t}}(\chi) = \Big[\sum_{H\in\mathcal{H}_{\chi;i}}\Big(\prod_G t_G^{1/2}\Big)x_H - \sum_{H\in\mathcal{H}_{\chi;s}}\Big(\prod_G t_G^{1/2}\Big)x_H\Big]^2$$

the products extending over $\mathcal{E}_+^{\infty}(H_\chi)\cap\mathcal{E}_\infty^-(H)$.

In particular, if $\chi = \mathcal{V}(\Theta_{\sigma_\infty(H)})$ for some $H\in\mathcal{H}_\infty$ then $\hat{s}_{\underline{t}}(\chi)$ is equal to x_H^2. In this case, for $(\underline{t},\underline{\beta})\in I^{L-1}$

$$d_2(\chi|\chi')(\underline{\alpha}) \geq \prod_{\ell\in\mathcal{L}\setminus\{\mathcal{L}(T(\mathcal{E}_\infty,\sigma_\infty))\setminus\sigma_\infty(H)\}}\alpha_\ell$$

$$= \prod_{\ell\in\sigma_\infty(\mathcal{F}_\infty')\cup\{\sigma_\infty(H)\}}\alpha_\ell = \prod_{G\in\mathcal{E}_+^{\infty}(H)} t_G\cdot\prod_{G\in\mathcal{E}_\infty} t_G^{N(G)}.$$

Hence $\hat{d}_2(\chi\,\chi')(\underline{t},\underline{\beta})$ is larger or equal to one and, since $\hat{d}(\underline{t},\underline{\beta})$ is positive and bounded from above in $(\underline{t},\underline{\beta})\in I^{L-1}$, the quadratic form $D''_{\underline{t},\underline{\beta}}((x_Hp_o)_{H\in\mathcal{H}_\infty};(x_Hp_o)_{H\in\mathcal{H}_\infty})$ is larger or equal to zero, the latter if and only if all x_H, $H\in\mathcal{H}_\infty$ vanish.

The positive definiteness of the quadratic form $D''_{\underline{t},\underline{\beta}}((x_Hp_o)_{H\in\mathcal{H}_\infty};(x_Hp_o)_{H\in\mathcal{H}_\infty})$ implies the positivity of the determinants

$$\Big\|\big(d'_{H',H''}(\underline{t},\underline{\beta})\big)_{H',H''\in\mathcal{H}_\infty^+(H)}\Big\| = \Big\|\big(d''_{H',H''}(\underline{t},\underline{\beta})\big)_{H',H''\in\mathcal{H}_\infty^+(H)}\Big\|$$

for any $H\in\mathcal{H}_\infty$ and any $(\underline{t},\underline{\beta})\in I^{L-1}$. Here we have set

$$d'_{H',H''}(\underline{t},\underline{\beta}) = \Big(\prod_G t_G\Big)\,\hat{d}_{H',H''}(\underline{t},\underline{\beta})$$

$$d''_{H',H''}(\underline{t},\underline{\beta}) = \Big(\prod_G t_G^{1/2}\Big)\,\hat{d}_{H',H''}(\underline{t},\underline{\beta})$$

the products extending over $\mathcal{E}_+^{\infty}(H'\cap H'')\cap\mathcal{E}_\infty^-(H')$ and $\mathcal{E}_+^{\infty}(H'\cap H'')\cap\mathcal{E}_\infty^-(H'\cup H'')$ respectively. By removing appropriate factors from the rows of

$$\Big\|\big(d_{H',H''}(\underline{t},\underline{\beta})\big)_{\substack{H'\in\mathcal{H}_\infty^-(H)\cup\{H\}\\ H''\in\mathcal{H}_\infty^-(H)\cup\{\overline{H}\}}}\Big\|$$

we obtain the following result:

$$\left\| \left(d_{H',H''}(\underline{t},\underline{\beta})\right)_{\substack{H'\in\mathcal{H}_\infty^-(H)\cup\{H\}\\ H''\in\mathcal{H}_\infty^-(H)\cup\{\bar{H}\}}} \right\| = \left(\prod_{H'\in\mathcal{H}_\infty^+(H)} \alpha_{\sigma_\infty(H')}\right)\cdot \left\| \left(d'_{H',H''}(\underline{t},\underline{\beta})\right)_{\substack{H'\in\mathcal{H}_\infty^-(H)\cup\{H\}\\ H''\in\mathcal{H}_\infty^-(H)\cup\{\bar{H}\}}} \right\|$$

and in particular

$$\left\| \left(d_{H',H''}(\underline{t},\underline{\beta})\right)_{H',H''\in\mathcal{H}_\infty^+(H)} \right\| = \left(\prod_{H'\in\mathcal{H}_\infty^+(H)} \alpha_{\sigma_\infty(H')}\right)\cdot \left\| \left(d'_{H',H''}(\underline{t},\underline{\beta})\right)_{H',H''\in\mathcal{H}_\infty^+(H)} \right\|$$

where the second factors on the right hand sides of the last two equations are independent of t_{Go} and depend on $\underline{t}$ and $\underline{\beta}$ in the form of a quotient of two polynomials the divisor polynomial having no zeros in I^{L-1}.

Starting from the quadratic form $t_{Go}D_{\underline{t},\underline{\beta}}(\underline{k},\underline{k})$, Jacobi's diagonalization procedure [13] enables us to determine a linear transformation $S_{\underline{t},\underline{\beta}} = S_{\underline{t},\underline{\beta}}(\xi_\infty,\sigma_\infty)$ and a quadratic form $E_{\underline{t},\underline{\beta}}(\underline{q},\underline{q}) = E^{\mathcal{D}_\infty}_{\underline{t},\underline{\beta}}(\underline{q},\underline{q})$

$$E_{\underline{t},\underline{\beta}}(q,q) = -\sum_{H\in\mathcal{H}_\infty} e_H(\underline{t},\underline{\beta})\, q_H^2$$

with the following properties:

i) $S_{\underline{t},\underline{\beta}}$ is non-singular for $(\underline{t},\underline{\beta})\in I^{L-1}$.

ii) $(S_{\underline{t},\underline{\beta}})_{H,\bar{H}}$ is independent of t_{Go} and depends on $\underline{t}$ and $\underline{\beta}$ in the form of a quotient of two polynomials the divisor polynomial having no zeros in I^{L-1}.

iii) When substituting $((q_H)_{H\in\mathcal{H}_\infty}) = ((q_H(\underline{t},\underline{\beta}))_{H\in\mathcal{H}_\infty}) = S_{\underline{t},\underline{\beta}}\,((k_H)_{H\in\mathcal{H}_\infty})$ the form $t_{Go}\cdot E_{\underline{t},\underline{\beta}}(\underline{q},\underline{q})$ goes over into the form $t_{Go}\cdot D_{\underline{t},\underline{\beta}}(\underline{k},\underline{k})$.

The transformation

$$\left(S_{\underline{t},\underline{\beta}}\right)_{H,\bar{H}} = \begin{cases} 0 & \text{if } H \supsetneq \bar{H} \\ 1 & \text{if } H = \bar{H} \\ \dfrac{\left\| \left(d'_{H',H''}(\underline{t},\underline{\beta})\right)_{\substack{H'\in\mathcal{H}_\infty^-(H)\cup\{H\}\\ H''\in\mathcal{H}_\infty^-(H)\cup\{\bar{H}\}}} \right\|}{\left\| \left(d'_{H',H''}(\underline{t},\underline{\beta})\right)_{H',H''\in\mathcal{H}_\infty^+(H)} \right\|} & \text{if } H\subset\bar{H} \end{cases}$$

and the quadratic form

$$E_{\underline{t},\underline{\beta}}(q,q) = -t_{G_0}^{-1} \sum_{H \in \mathcal{H}_\infty} \Big(\prod_{G \in \mathcal{E}_+^\infty(H)} t_G \Big) e'_H(\underline{t},\underline{\beta})\, q_H^2$$

with

$$e'_H(\underline{t},\underline{\beta}) = \frac{\left\| \left(d'_{H',H''}(\underline{t},\underline{\beta}) \right)_{H',H'' \in \mathcal{H}_\infty^+(H)} \right\|}{\left\| \left(d'_{H',H''}(\underline{t},\underline{\beta}) \right)_{H',H'' \in \mathcal{H}_\infty^-(H)} \right\|}$$

match the requirements listed above. Thus, $t_{GO} \cdot E_{\underline{t},\underline{\beta}}(\underline{q(\underline{t},\underline{\beta})},\underline{q(\underline{t},\underline{\beta})})$ is the diagonal form of $t_{GO} \cdot D_{\underline{t},\underline{\beta}}(\underline{k},\underline{k})$.
Note that $e'_H(\underline{t},\underline{\beta})$ also is independent of t_{GO} and depends on $\underline{t}$ and $\underline{\beta}$ in the form of a quotient of two polynomials both positive in I^{L-1} [14].

IV. Analytically Renormalized Feynman Amplitudes

For the sake of simplicity and definiteness, we shall restrict our discussion to Feynman amplitudes occurring in the perturbation expansion of a $P(\emptyset)_4$ Lagrangian field theory describing a polynomial self-interaction of one sort of neutral scalar massive (m) particles in one time and three space dimensions. The generalization to theories involving massive particles with spin and derivative coupling in one time and arbitrarily many space dimensions is straight-forward.

With Speer [7] we associate with every line l of the vertex graph G^0 a complex variable λ_l, $\underline{\lambda} = (\lambda_l)_{l\in\mathcal{L}}$, and modify the propagators according to

$$\frac{i}{(2\pi)^2}\,\frac{1}{k^2-m^2+i0} \longrightarrow \frac{e^{i\pi\lambda}\,\Gamma(\lambda)}{i\,(2\pi)^2}\left[k^2-m^2+i0\right]^{-\lambda},$$

$$\Delta_F\left(x_{i(l)}-x_{f(l)};m\right) \longrightarrow \Delta_F^{\lambda_l}\left(x_{i(l)}-x_{f(l)};m\right)$$

$$= \mathcal{F}_k^{-1}\left\{\frac{e^{i\pi\lambda_l}\,\Gamma(\lambda_l)}{i\,(2\pi)^2}\left[k^2-m^2+i0\right]^{-\lambda_l}\right\}\left(x_{i(l)}-x_{f(l)}\right)$$

This modification of the propagators results in the replacement of the amplitude

$$i^{V-1}\left[\frac{(4\pi^2)^{V+1}}{(4\pi^2)^{U+L}\,4^N}\right]^{-1}\int\cdots\int\prod_{v\in\mathcal{V}\setminus\mathcal{U}} d^4x_v \prod_{l\in\mathcal{L}}\Delta_F\left(x_{i(l)}-x_{f(l)};m\right)$$

which in general is ill-defined by the analytically regularized amplitude

$$\mathcal{T}_{\underline{\lambda}}\left((x_u)_{u\in\mathcal{U}};m\right) = i^{V-1}\,\frac{(4\pi^2)^{U+L}\,4^N}{(4\pi^2)^{V+1}}\int\cdots\int\prod_{v\in\mathcal{V}\setminus\mathcal{U}} d^4x_v\prod_{l\in\mathcal{L}}\Delta_F^{\lambda_l}\left(x_{i(l)}-x_{f(l)};m\right)$$

which is well-defined for $\underline{\lambda}\in\Omega_2 = \{\underline{\lambda}\ /\ \mathrm{Re}\,\lambda_l > 2 \text{ for all } l\in\mathcal{L}\}$.

In Ω_2 the Fourier transform of $\mathcal{T}_{\underline{\lambda}}$ can be expressed with the help of the parameters $\underline{t}$ and $\underline{\beta}$ as follows (cf. ref.[8])

$$\tilde{\mathcal{T}}_{\underline{\lambda}}(\underline{p};m) = \sum \tilde{\mathcal{T}}_{\underline{\lambda}}^{\mathcal{D}_0}(\underline{p};m)$$

where

$$\tilde{\mathcal{C}}_{\underline{\lambda}}^{\mathcal{D}_\infty}(\underline{p};m) = \delta\left(\sum_{u\in\mathcal{U}} p_u\right) \prod_{\ell\in\mathcal{L}\setminus\sigma_\infty(\mathcal{E}_\infty)} \left[\int_0^1 d\beta_\ell\, \beta_\ell^{\lambda_\ell-1}\right] \cdot$$

$$\cdot \prod_{G\in\mathcal{E}_\infty(G^o)} \left[\int_0^1 dt_G\, t_G^{\nu(G)-1}\right] \frac{\Gamma(\nu)}{[\hat{d}(\underline{t},\underline{\beta})]^2} \Big[E_{\underline{t},\underline{\beta}}\left(\underline{q(\underline{t},\underline{\beta})}, \underline{q(\underline{t},\underline{\beta})}\right)$$

$$+ m^2 \cdot \sum_{\ell\in\mathcal{L}} \left(\alpha_\ell / \alpha_{\sigma_\infty(G^o)}\right) - i0 \Big]^{-\nu},$$

$\nu(G) = \sum_{l\in\mathcal{L}(G)} (\lambda_l - 1) + n(G)$, $n(G) = L(G) - 2\cdot N(G)$, $\nu = \nu(G^o)$, $n = n(G^o)$ and where the sum extends over all labeled s_∞-families $(\mathcal{E}_\infty, \sigma_\infty)$ for G^o .

Using the fact that $\left[E_{\underline{t},\underline{\beta}}(\underline{q(\underline{t},\underline{\beta})}, \underline{q(\underline{t},\underline{\beta})}) + m^2 \cdot \sum_{\ell\in\mathcal{L}} (\alpha_l / \alpha_{\sigma_\infty}(G^o)) - i0\right]^{-\nu}$ is an infinitely differentiable distribution-valued function of $\underline{t}$ and $\underline{\beta}$ as long as m is larger than zero, we may convince ourselves that

$$\prod_{G\in\mathcal{E}_\infty} \Gamma(\nu(G))^{-1} \cdot \tilde{\mathcal{C}}_{\underline{\lambda}}^{\mathcal{D}_\infty}(\underline{p};m)$$

is an entire distribution-valued function of $\underline{\lambda}$ for every labeled s_∞-family. Hence the distribution-valued function of $\underline{\lambda}$

$$\prod_{G} \Gamma\Big(\sum_{\ell\in\mathcal{L}(G)} (\lambda_\ell - 1) + n(G)\Big)^{-1} \cdot \tilde{\mathcal{C}}_{\underline{\lambda}}(\underline{p};m)$$

is entire. Here, the product $\prod_G$ extends over all I_∞-subgraphs of G^o .

Speer's generalized evaluator $\mathcal{W} = \{\mathcal{W}_L \ / \ L = 1,2,\ldots\}$ is applicable to the amplitudes $\tilde{\mathcal{C}}_{\underline{\lambda}}(\underline{p};m)$. The result of the application $\mathcal{W}_L \tilde{\mathcal{C}}_{\underline{\lambda}}(\underline{p};m)$ is the analytically renormalized Feynman amplitude of the vertex graph G^o contributing in ν^{th} order perturbation theory to the vertex function of the momenta carried by the external lines of G^o .

V. Asymptotic Expansion of Analytically Renormalized Feynman Amplitudes

It is our long range goal to understand the way in which the perturbation series sums up to produce the asymptotic form of the vertex functions as obtained from the Callan Symanzik equations.
As a first step in this direction, we shall determine the complete asymptotic expansion of the (Λ-) parameter dependent distribution

$$\tilde{\mathcal{J}}_\Lambda(\underline{p};m) = \mathcal{W}_L\, \tilde{\mathcal{C}}_{\underline{\lambda}}(\underline{\Lambda p};m)$$

for Λ tending to plus infinity. By contrast to other authors having contributed to this subject, we do not discuss the asymptotic behaviour in Λ of $\tilde{\mathcal{J}}_\Lambda(\underline{p};m)$ pointwise, i.e. for a fixed configuration of the external momenta $(p_u)_{u\in\mathfrak{V}}$. Instead, we rather establish the asymptotic behaviour of the complex-valued function of Λ

$$\langle \tilde{\mathcal{J}}_\Lambda, \tilde{\varphi}\rangle = \int d^{4U}\underline{p}\;\; \tilde{\varphi}(\underline{p})\tilde{\mathcal{J}}_\Lambda(\underline{p};m)$$

for any $\tilde{\varphi}\in \mathcal{S}(\mathbb{R}^{4U})$. At the first sight, this seems to complicate matters unnecessarily. For Minkowski metrics, however, the latter formulation of the problem turns out to be both adequate and helpful.

In order to establish the asymptotic expansion of the parameter dependent distribution $\mathcal{J}_\Lambda(\underline{p};m)$ for Λ tending to plus infinity, it suffices to determine the asymptotic behaviour of

$$\begin{aligned}
[\Lambda^2]^{n+2}\, \tilde{\mathcal{J}}^{\mathcal{D}_\infty}_\Lambda(\underline{p};m) &= [\Lambda^2]^{n+2}\, \mathcal{W}_L\, \tilde{\mathcal{C}}^{\mathcal{D}_\infty}_{\underline{\lambda}}(\underline{\Lambda p};m) \\
= \mathcal{W}_{\Lambda,L}\, \tilde{\mathcal{C}}^{\mathcal{D}_\infty}_{\underline{\lambda}}\left(\underline{p};\frac{m}{\Lambda}\right) &= \delta\Big(\sum_{u\in\mathfrak{V}} p_u\Big)\, \mathcal{W}_L \Big\{ [\Lambda^2]^{-\sum_{\ell\in\mathcal{L}}(\lambda_\ell - 1)}\, \Gamma(\nu) \\
\prod_{\ell\in\mathcal{L}\setminus \mathcal{G}_\infty(\mathcal{E}_\infty)} \Big[\int_0^1 d\beta_\ell\, \beta_\ell^{\lambda_\ell - 1}\Big] &\cdot \prod_{G\in\mathcal{E}_{\mathcal{D}_\infty}(G^\circ)} \Big[\int_0^1 dt_G\, t_G^{\nu(G)-1}\Big] \frac{1}{[\hat{d}(\underline{t},\underline{\beta})]^2} \\
\Big[E_{\underline{t},\underline{\beta}}\big(\underline{q(\underline{t},\underline{\beta})}, \underline{q(\underline{t},\underline{\beta})}\big) &+ \frac{m^2}{\Lambda^2} e(\underline{t},\underline{\beta}) - i0 \Big]^{-\nu} \Big\}
\end{aligned}$$

for every labeled s_∞-family where we have set

$$e(\underline{t},\underline{\beta}) = e^{(\mathcal{E}_\infty,\sigma_\infty)}(\underline{t},\underline{\beta}) - \sum_{l\in\mathcal{L}} \left(\alpha_l / \alpha_{\sigma_\infty(G^0)} \right) \geq 1$$

and

$$\mathcal{W}_{\Lambda,L} = \mathcal{W}_L \left[\Lambda^2\right]^{-\sum_{l\in\mathcal{L}}(\lambda_l-1)}$$

Without loss of generality we may assume that U is larger or equal to two.
The limit $\Lambda \longrightarrow +\infty$ corresponds formally to the transition from the Feynman amplitude with massive lines to the Feynman amplitude (for the same vertex graph G^0) with massless lines. In the zero mass case, however, we are dealing with a complex power of a homogeneous quadratic form: $\left[E_{\underline{t},\underline{\beta}}(\underline{q}(\underline{t},\underline{\beta}),\underline{q}(\underline{t},\underline{\beta})) - i0\right]^{-\nu}$ which fails to be an infinitely differentiable function of $\underline{t}$ whenever and wherever the quadratic form $E_{\underline{t},\underline{\beta}}(\underline{q},\underline{q})$ degenerates. It is this lack of infinite differentiability which prevents us from finding the answer to our problem right-away and, moreover, forces us to introduce the subsets of α-space $\mathcal{D}_\infty = \mathcal{D}(\mathcal{E}_\infty,\sigma_\infty)$ instead of $\mathcal{D}(\mathcal{E},\sigma)$ (cf. ref[8]).

In order to contrôl the formation of the singularity under consideration, we convert the additive occurrence of $-q_H^2$ and $\frac{m^2}{\Lambda^2}$ in

$$\left[\sum_{H\in\mathcal{H}_\infty}\left(\prod_{G\in\mathcal{E}_+^\infty(H)\cap\mathcal{E}_\infty^-(G_0)} t_G\right)e'_H(\underline{t},\underline{\beta})(-q_H^2)+\frac{m^2}{\Lambda^2}\,e(\underline{t},\underline{\beta})-i0\right]^{-\nu}$$

into a multiplicative occurrence - with the help of Mellin transforms. The result is

$$\Gamma(\nu)^{-1}\left[\frac{\Lambda^2}{m^2\cdot e(\underline{t},\underline{\beta})}\right]^{\nu}\cdot\prod_{H\in\mathcal{H}_\infty}\left[\frac{1}{2\pi i}\int_{-\gamma_H-i\infty}^{-\gamma_H+i\infty} ds_H\,\Gamma(-s_H)\left(\frac{e'(\underline{t},\underline{\beta})\Lambda^2}{e(\underline{t},\underline{\beta})m^2}\prod_{G\in\mathcal{E}_+^\infty(H)\cap\mathcal{E}_\infty^-(G^0)} t_G\right)^{s_H}\right.$$

$$\left.\cdot\left(-q_H^2-i0\right)^{s_H}\right]\Gamma\left(\nu+\sum_{H\in\mathcal{H}_\infty}s_H\right)$$

where the γ_H's , $H\in\mathcal{H}_\infty$ are real numbers between zero and two.

For $\underline{\lambda}$ contained in a compact subset of $\Omega_{2(U-1)} = \{\underline{\lambda}\,/\,\mathcal{R}e\,\lambda_l > 2(U-1)$ for all $l\in\mathcal{L}\}$ and for $(\underline{t},\underline{\beta})$ contained in I^{L-1}, the integrations over s_H converge uniformly.

In order to prove the uniform convergence, we note the identity

$$\left(-q^2-i0\right)^s = \frac{\left(-q^2-i0\right)^{s+j}}{[(s+1)\cdots(s+j+1)]\,[(s+2)\cdots(s+j)]}\left(-\frac{\Box_q}{4}\right)^j$$

for any $j=0,1,\ldots$ In view of this identity and the above Mellin representation we obtain

$$\left(\prod_{G\in\mathcal{E}^-_\infty(G_0)} t_G^{\nu(G)-1}\right)\left[E_{\underline{t},\underline{\beta}}(\underline{q},\underline{q})+\frac{m^2}{\Lambda^2}e(\underline{t},\underline{\beta})-i0\right]^{-\nu}$$

$$= \prod_{H\in\mathcal{H}_\infty}\left[\frac{1}{2\pi i}\int_{-\gamma_H-i\infty}^{-\gamma_H+i\infty} ds_H \frac{\Gamma(-s_H-j_H-1)}{(s_H+2)\cdots(s_H+j_H)}\left(\frac{\Lambda^2\cdot e'_H(\underline{t},\underline{\beta})}{m^2\cdot e(\underline{t},\underline{\beta})}\prod_{G\in\mathcal{E}^\infty_+(H)\cap\mathcal{E}^-_\infty(G^0)} t_G\right)^{s_H}\left(-q_H^2-i0\right)^{s_H}\right]$$

$$\Gamma\left(\nu+\sum_{H\in\mathcal{H}_\infty}s_H\right)\left(\prod_{G\in\mathcal{E}^-_\infty(G^0)} t_G^{\nu(G)-1}\right)\frac{(-1)^{U-1}}{\Gamma(\nu)}\left[\frac{\Lambda^2}{m^2\cdot e(\underline{t},\underline{\beta})}\right]^{\nu}\cdot\prod_{H\in\mathcal{H}_\infty}\left(\frac{\Box_{q_H}}{4}\right)^{j_H}$$

where the following estimate for $\gamma_H \neq -1$, $j_H \geqslant 2$. $H\in\mathcal{H}_\infty$ can be used

$$\left|\prod_{H\in\mathcal{H}_\infty}\left[\frac{\Gamma(-s_H-j_H-1)}{(s_H+2)\cdots(s_H+j_H)}\left(\frac{\Lambda^2\cdot e'_H(\underline{t},\underline{\beta})}{m^2\cdot e(\underline{t},\underline{\beta})}\prod_{G\in\mathcal{E}^\infty_+(H)\cap\mathcal{E}^-_\infty(G^0)} t_G\right)^{s_H}\left(-q_H^2-i0\right)^{s_H+j_H}\right]\cdot\right.$$

$$\left.\cdot\,\Gamma\left(\nu+\sum_{H\in\mathcal{H}_\infty}s_H\right)\right|\cdot\prod_{G\in\mathcal{E}^-_\infty(G^0)} t_G^{\mathcal{R}e\,\nu(G)-1} < \text{const}\left[1+\left|\sum_{H\in\mathcal{H}_\infty}\eta_H\right|\right]^{\mathcal{R}e\,\nu-\sum\gamma_H-\frac{1}{2}}$$

$$\prod_{H\in\mathcal{H}_\infty}\frac{\left[1+\|q_H\|^2\right]^{j_H}}{\left[1+|\eta_H|\right]^{\frac{1}{2}-\gamma_H+2j_H}}$$

for $s_H = -\gamma_H + i\eta_H$.

In the pointwise discussion for Minkowski metrics, on the other hand, even in the case that all $q_H^2 > 0$, $H\in\mathcal{H}_\infty$, the corresponding s-integrations would not converge uniformly in $\underline{\lambda}$, $\underline{t}$ and $\underline{\beta}$ provided that $\mathcal{R}e\,\nu$ is larger or equal to zero.

For $\underline{\lambda}\in\Omega_{2(U-1)}$ we have shown the following representation to be valid

$$\tilde{\mathcal{C}}^{\mathcal{D}_\infty}_{\underline{\lambda}}(\underline{\Delta p};m) = \delta\Big(\sum_{u\in\mathcal{U}} p_u\Big)\prod_{H\in\mathcal{H}_\infty}\Big[\frac{1}{2\pi i}\int_{-\gamma_H-i\infty}^{-\gamma_H+i\infty} ds_H \Gamma(-s_H)\Gamma(s_H+2)\Big][\Lambda^2]^{\sum_{H\in\mathcal{H}_\infty} s_H-2} g_{\underline{\lambda};\underline{s}}(\underline{p};m)$$

where

$$g_{\underline{\lambda};\underline{s}}(\underline{p};m) = g^{(\mathcal{E}_\infty,\sigma_\infty)}_{\underline{\lambda};\underline{s}}(\underline{p};m) = \prod_{G\in\mathcal{E}_\infty}\Big[\Gamma\big(\nu(G)+\sum_{H\in\mathcal{H}^-_\infty(G)} s_H\big)\Big] h_{\underline{\lambda};\underline{s}}(\underline{p};m)$$

with the entire function of $\underline{\lambda}$ and $\underline{s}$

$$h_{\underline{\lambda};\underline{s}}(\underline{p};m) = h^{(\mathcal{E}_\infty,\sigma_\infty)}_{\underline{\lambda};\underline{s}}(\underline{p};m) = \prod_{\ell\in\mathcal{L}\setminus\sigma_\infty(\mathcal{E}_\infty)}\Big[\int_0^1 d\beta_\ell\, \beta_\ell^{\lambda_\ell-1}\Big]\cdot$$

$$\cdot\prod_{G\in\mathcal{E}^-_\infty(G^\circ)}\Big[\Gamma\big(\nu(G)+\sum_{H\in\mathcal{H}^-_\infty(G)} s_H\big)^{-1}\int_0^1 dt_G\, t_G^{\{\nu(G)+\sum_{H\in\mathcal{H}^-_\infty(G)} s_H-1\}}\Big]\big[\hat{d}(\underline{t},\underline{\beta})\big]^{-2}$$

$$\big[m^2\cdot e(\underline{t},\underline{\beta})\big]^{-\nu}\prod_{H\in\mathcal{H}_\infty}\Big(\frac{e'_H(\underline{t},\underline{\beta})}{m^2\cdot e(\underline{t},\underline{\beta})}\Big)^{s_H}\Gamma(2+s_H)^{-1}\big(-q_H(\underline{t},\underline{\beta})^2-i0\big)^{s_H}\Big]$$

and where the s-integrations converge uniformly for $\underline{\lambda}$ contained in any compact subset of $\Omega_{2(U-1)}$.

We define $K = K(\mathcal{E}_\infty,\sigma_\infty)$ to be the minimal element of $\mathcal{H}_\infty$ with the property $n(G)>0$ for every $G\in\mathcal{E}^\infty_+(K)$. Specializing to the quartic self-interaction and to vertex graphs with more than two external lines we notice that $n(G)$ is larger or equal to zero for all $G\in\mathcal{E}^\infty_+(H)$, $H\in\mathcal{H}_\infty$. We shift some of the s-contours to the right and obtain

$$\prod_{G\in\mathcal{E}_\infty}\Big[\Gamma\Big(\sum_{\ell\in\mathcal{L}(G)}(\lambda_\ell-1)+n(G)\Big)^{-1}\Big]\cdot\tilde{\mathcal{C}}^{\mathcal{D}_\infty}_{\underline{\lambda}}(\underline{\Delta p};m)$$

$$=\sum_{\mathbb{H}\in\mathcal{H}^+_\infty(K)}\delta\Big(\sum_{u\in\mathcal{U}} p_u\Big)\prod_{H\in\mathcal{H}^\infty_-(\mathbb{H})}\Big[\frac{1}{2\pi i}\int_{-\gamma_H-i\infty}^{-\gamma_H+i\infty} ds_H\Gamma(-s_H)\Gamma(2+s_H)\Big]\frac{1}{2\pi i}\int_{S_{\mathbb{H}}-i\infty}^{S_{\mathbb{H}}+i\infty} ds_{\mathbb{H}}\Gamma(-s_{\mathbb{H}})\Gamma(2+s_{\mathbb{H}})$$

$$[\Lambda^2]^{\sum_{H\in\mathcal{H}^\infty_+(\mathbb{H})} s_H-2}\Big\{\prod_{G\in\mathcal{E}_\infty}\Big[\Gamma\Big(\sum_{\ell\in\mathcal{L}(G)}(\lambda_\ell-1)+n(G)\Big)^{-1}\Big]g_{\underline{\lambda};\underline{s}}(\underline{p};m)\Big\}_{\substack{s_H=0 \text{ for}\\ H\in\mathcal{H}^-_\infty(\mathbb{H})}}$$

where $0<\gamma_H<1$, $0<\sum_{H\in\mathcal{H}^\infty_-(\mathbb{H})}\gamma_H < S_{\mathbb{H}} < 1$ for $\mathbb{H}\in\mathcal{H}^-_\infty(K)$ and $0<\sum_{H\in\mathcal{H}^\infty_-(K)}\gamma_H - S_K < -S_K < 1$. The s-integrations converge uniformly not only when $\underline{\lambda}$ varies over any compact subset of $\Omega_{2(U-1)}$ but also after analytic continuation of the integrand when $\underline{\lambda}$ varies over any compact subset of $\Omega_{1-\epsilon}=\{\underline{\lambda} \,/\, \mathcal{R}e\,\lambda_l > 1-\epsilon \text{ for all } l\in\mathcal{L}\}$. Thus, for the quartic self-interaction and vertex graphs with more than two external lines - we shall restrict the subsequent discussion to this case - we may continue $\tilde{\mathcal{C}}^{\mathcal{D}_\infty}_{\underline{\lambda}}(\Delta\underline{p};m)$ analytically from $\Omega_{2(U-1)}$ to a neighbourhood of $\lambda_l = 1^-$, $l\in\mathcal{L}$. Again in view of the uniform convergence of the above s-integrations, the generalized evaluator $\mathcal{W}_L$ operates directly on the integrand:

$$\mathcal{W}_L\,\tilde{\mathcal{C}}^{\mathcal{D}_\infty}_{\underline{\lambda}}(\Delta\underline{p};m) = \sum_{\mathcal{H}\in\mathcal{H}^+_\infty(K)} \tilde{\mathcal{C}}^{\mathcal{D}_\infty}_{\mathcal{H};\mathcal{W}}(\Delta\underline{p};m)$$

The terms $\tilde{\mathcal{C}}^{\mathcal{D}_\infty}_{\mathcal{H};\mathcal{W}}(\Delta\underline{p};m)$ stand for

$$\delta\Big(\sum_{u\in\mathfrak{V}} p_u\Big)\frac{1}{2\pi i}\int_{\varrho_{h(\mathcal{H})}-i\infty}^{\varrho_{h(\mathcal{H})}+i\infty} dz_{h(\mathcal{H})}\cdots\frac{1}{2\pi i}\int_{\varrho_2-i\infty}^{\varrho_2+i\infty}dz_2\,\frac{1}{2\pi i}\int_{\varrho_1-i\infty}^{\varrho_1+i\infty}dz_1\prod_{j=1}^{h(\mathcal{H})-1}\Big[\Gamma(z_{j+1}-z_j)\Gamma(\lambda+z_j-z_{j+1})\Big]$$

$$\Gamma(-z_{h(\mathcal{H})})\Gamma(\lambda+z_{h(\mathcal{H})})\,\big[\Lambda^2\big]^{z_1-2} f^{(\mathcal{W})}_{z_1,\dots,z_{h(\mathcal{H})}}(\underline{p};m)$$

with $1>\varrho_{h(\mathcal{H})}>\dots\dots>\varrho_2>\varrho_1>0$ for $\mathcal{H}\in\mathcal{H}^-_\infty(K)$
and $0>\varrho_{h(K)}>\dots\dots>\varrho_2>\varrho_1>-1$ for $\mathcal{H}=K$.

We replaced the integration variables s_H and $s_{\mathcal{H}}$ by the new variables z_j , $1\leq j\leq h(\mathcal{H})$, defined by

$$z_j = \sum_{i=j}^{h(\mathcal{H})} s_{H_i}$$

after having enumerated the elements of $\mathcal{H}_\infty$ according to

H_1 = maximal element of $\mathcal{H}_\infty$

H_j = maximal element of $\mathcal{H}^-_\infty(H_{j-1})$ $j = 2,\dots,h(\mathcal{H})$.

Finally, $f^{(\mathcal{W})}_{z_1,\dots,z_{h(\mathcal{H})}}(\underline{p};m)$ stands for

$$\mathcal{W}_L\{g_{\underline{\lambda};\underline{s}}(\underline{p};m)/_{s_H=0 \text{ for } H\in\mathcal{H}^-_\infty(\mathcal{H})}\} ,$$

a distribution-valued meromorphic function of z_j , $1\leq j\leq h(\mathcal{H})$, with poles at $z_j = -n_j, -n_j-1, -n_j-2, \dots$, $n_j = n_j(\mathcal{E}_\infty,\sigma_\infty) = \min\{n(G) \,/\, G\in\mathcal{E}^\infty_j\}$ $j = 1,2,\dots,h(\mathcal{H})$ and

$$\mathcal{E}^\infty_j = \begin{cases}\mathcal{E}^\infty_+(H_j)\cap\mathcal{E}^-_\infty(H_{j-1}) & \text{for } j = 2,\dots,h(\mathcal{H})\\ \mathcal{E}^\infty_+(H_1) & \text{for } j = 1\end{cases}$$

The order of the pole at $z_j=-\mu_j$, $\mu_j = n_j, n_j+1, n_j+2, \dots$, for every j separately, is at most equal to

$$\#\left[\mathcal{E}^{\infty}_{j,\mu_j}\right] + \#\left\{ G' \,/\, G' \in \mathcal{E}^{-}_{\infty}(G) \setminus \mathcal{E}^{\infty}_{+}(\mathcal{H}) \text{ for some } G \in \mathcal{E}^{\infty}_{j,\mu_j},\ n(G') \leq 0 \right\}$$

where

$$\mathcal{E}^{\infty}_{j,\mu_j} = \left\{ G / G \in \mathcal{E}^{\infty}_{j},\ n(G) \leq \mu_j \right\} .$$

We move the z_1-contour to the left and obtain

$$\tilde{\mathcal{C}}^{\mathcal{D}_\infty}_{\mathcal{H};\mathcal{W}}(\underline{\Delta p}\,; m) = \sum_{\mu=\mu_0(\mathcal{H})}^{M} \frac{1}{2\pi i} \oint_{|z_1+\mu|=\epsilon} dz_1 \left[\Lambda^2\right]^{z_1-2} \tilde{\mathcal{C}}^{\mathcal{D}_\infty; z_1}_{\mathcal{H};\mathcal{W}}(\underline{p}\,; m)$$
$$+ \frac{1}{2\pi i} \int_{\varrho - i\infty}^{\varrho + i\infty} dz_1 \left[\Lambda^2\right]^{z_1-2} \tilde{\mathcal{C}}^{\mathcal{D}_\infty; z_1}_{\mathcal{H};\mathcal{W}}(\underline{p}\,; m)$$

for $\mathcal{H} \in \mathcal{H}^{+}_{\infty}(K)$, any integer $M \geq \mu_0(\mathcal{H})$, $-(M+1) < \varrho < -M$, $\epsilon > 0$ sufficiently small.
Here, for every $\mathcal{H} \in \mathcal{H}^{+}_{\infty}(K)$, $\tilde{\mathcal{C}}^{\mathcal{D}_\infty; z_1}_{\mathcal{H};\mathcal{W}}(\underline{p};m)$ is a distribution-valued meromorphic function of z_1 with poles at $z_1 = -\mu$, $\mu = -\mu_0(\mathcal{H})$, $-\mu_0(\mathcal{H})-1, -\mu_0(\mathcal{H})-2, \ldots$

$$\mu_0(\mathcal{H}) = \mu_0(\mathcal{E}_\infty, \sigma_\infty; \mathcal{H}) = \min_{1 \leq j \leq h(\mathcal{H})} \left\{ n_j + 2(j-1) \right\}$$

of order $m_\mu(\mathcal{H})$

$$m_\mu(\mathcal{H}) = m_\mu(\mathcal{E}_\infty, \sigma_\infty; \mathcal{H}) = \#\left[\bigcup_{j=1}^{h(\mathcal{H})} \mathcal{E}^{\infty}_{j,\mu_j - 2(j-1)} \right] + \#\left\{ G' / G' \in \mathcal{E}^{-}_{\infty}(G) \setminus \mathcal{E}^{\infty}_{+}(\mathcal{H}) \text{ for some } G \in \bigcup_{j=1}^{h(\mathcal{H})} \mathcal{E}_{j,\mu_j - 2(j-1)},\ n(G') \leq 0 \right\}$$
$$+ \begin{cases} 1 & \text{if } \mu \geq 2\cdot(h(\mathcal{H}) - 1) \text{ for } \mathcal{H} \in \mathcal{H}^{-}_{\infty}(K) \\ 1 & \text{if } \mu \geq 2\cdot h(K) \text{ for } \mathcal{H} = K \\ 0 & \text{otherwise} \end{cases}$$

For $-\mu > \mathcal{R}e\, z_1 > -(\mu+1)$, $\tilde{\mathcal{C}}^{\mathcal{D}_\infty; z_1}_{\mathcal{H};\mathcal{W}}(\underline{p};m)$ is given by $\delta\left(\sum_{u \in \mathcal{V}} p_u\right) \cdot$

$$\cdot \sum_{r=1}^{h(\mathcal{H})} \sum_{\mu_2=2}^{\mu-\Theta(2)} (-1)^{\mu_2}(\mu_2 - 1) \cdots \sum_{\mu_r=2}^{\mu-\Theta(r)-[\mu_2+\cdots+\mu_{r-1}]} (-1)^{\mu_r}(\mu_r - 1) \frac{1}{2\pi i} \int_{\varrho_{r+1} - i\infty}^{\varrho_{r+1} + i\infty} dz_{r+1} \cdots$$

$$\cdots \frac{1}{2\pi i} \int_{\varrho_{h(\mathcal{H})}-i\infty}^{\varrho_{h(\mathcal{H})}+i\infty} dz_{h(\mathcal{H})} \; \Gamma\left(\left[2+\mu_2+\cdots+\mu_r+z_1\right]-z_{r+1}\right)\Gamma\left(z_{r+1}-\left[\mu_2+\cdots+\mu_r+z_1\right]\right)$$

$$\prod_{j=r+1}^{h(\mathcal{H})-1}\left[\Gamma(2+z_j-z_{j+1})\Gamma(z_{j+1}-z_j)\right]\Gamma(2+z_{h(\mathcal{H})})\Gamma(-z_{h(\mathcal{H})}) \, f^{(\mathcal{W})}_{z_1,\ldots,z_{h(\mathcal{H})}}(\underline{p};m)\Big/_{\substack{z_2=\mu_2+z_1 \\ \vdots \\ z_r=\mu_2+\cdots+\mu_r+z_1}}$$

with

$$\theta(j) = \begin{cases} 1 & \text{if } \; j \leq h(K) \\ 0 & \text{otherwise} \end{cases}$$

and

$$0 < \theta(r+1) + \varrho_{r+1} < 1 + \mu + \mathrm{Re}\, z_1$$

$$0 < \theta(r+1) + \varrho_{r+1} < \cdots < \theta(h(\mathcal{H})) + \varrho_{h(\mathcal{H})} < +1 .$$

This formula may be proved by induction on μ .

As to the possible values of $\mu_0(\mathcal{H})$ we note the inequality

$$c(G) \leq j \quad \text{for } \; G \in \mathcal{E}^\infty_j$$

and take into account the following relation valid for the ϕ^4_4 - theory

$$n(G) = \frac{1}{2}\sum_{i=1}^{c(G)}\left\{\#\left[\text{external lines of } \; G^!_i\right] - 4\right\}$$

where the $G^!_i$ s denote the c-components of G .
From this we infer for the ϕ^4_4 - theory

$$n_1 = n \,, \quad n_j \geq n+2-j \quad j = 2,3,\ldots,h(\mathcal{H})$$

Actually, these relations are true for all monomial interactions apart from the cubic one.
Hence $\mu_0(\mathcal{H}) = \mu_0(\mathcal{E}_\infty,\sigma_\infty;\mathcal{H})$ is equal to n for all $\mathcal{H} \in \mathcal{H}^+_\infty(K)$ and all labeled s_∞-families $(\mathcal{E}_\infty,\sigma_\infty)$ for G^0 .
Moreover, the order of the poles at $z_1 = -\mu = -n,-n-1,-n-2,\ldots$ of

i) $\sum_{\mathcal{H}\in\mathcal{H}^+_\infty(K)} \tilde{\mathcal{T}}^{\mathcal{D}_\infty;z_1}_{\mathcal{H};\mathcal{W}}(\underline{p};m)$ is equal to $m_\mu(\mathcal{E}_\infty,\sigma_\infty) = \underset{\mathcal{H}\in\mathcal{H}^+_\infty(K)}{\mathrm{Max}} m_\mu(\mathcal{E}_\infty,\sigma_\infty;\mathcal{H})$

ii) $\sum_{(\mathcal{E}_\infty,\sigma_\infty)}\sum_{\mathcal{H}\in\mathcal{H}^+_\infty(K)} \tilde{\mathcal{T}}^{\mathcal{D}_\infty;z_1}_{\mathcal{H};\mathcal{W}}(\underline{p};m) = \tilde{\mathcal{T}}^{z_1}_{\mathcal{W}}(\underline{p};m)$ is equal to $m_\mu = \underset{(\mathcal{E}_\infty,\sigma_\infty)}{\mathrm{Max}} m_\mu(\mathcal{E}_\infty,\sigma_\infty)$.

Now, we have all the necessary information at hand to write down the asymptotic expansion for the analytically renormalized Feynman amplitude $\mathcal{W}_L\tilde{\mathcal{T}}_{\underline{\lambda}}(\Lambda\underline{p};m)$ of the vertex graph G^0:

$$\sum_{\mu=n}^{M}\sum_{\varkappa=0}^{m_\mu-1}[\Lambda^2]^{\mu-2}[\ln\Lambda^2]^{\varkappa}\,\tilde{\mathcal{T}}^{(\mathcal{W})}_{\mu,\varkappa}(\underline{p};m)+\mathcal{R}_M$$

with

$$\tilde{\mathcal{T}}^{(\mathcal{W})}_{\mu,\varkappa}(\underline{p};m)=\frac{1}{\varkappa!}\,\frac{1}{2\pi i}\oint_{|z_1+\mu|=\epsilon}dz_1\,(z_1+\mu)^{\varkappa}\,\tilde{\mathcal{T}}^{z_1}_{\mathcal{W}}(\underline{p};m)$$

and

$$\mathcal{R}_M=\frac{1}{2\pi i}\int_{\varrho-i\infty}^{\varrho+i\infty}dz_1\,[\Lambda^2]^{z_1-2}\,\tilde{\mathcal{T}}^{z_1}_{\mathcal{W}}(\underline{p};m)$$

where $-(M+1)<\varrho<-M$.

If the number of external lines is equal to two (and $U=2$) we adopt the same procedure as before with the only difference that in the beginning we push the s-contour further to the right. In this way we obtain

$$\begin{aligned}\mathcal{W}\mathcal{T}_{\underline{\lambda}}(\Lambda\underline{p};m)&=[\Lambda^2]^{-1}\,\delta(p_1+p_2)\,\frac{1}{2\pi i}\oint_{|s-1|=\epsilon}ds\,\frac{\pi\,(s+1)}{\sin\pi(s-1)}\\&\quad\{[\Lambda^2]^{s-1}\mathcal{W}g_{\underline{\lambda};s}(\underline{p},m)-\mathcal{W}g_{\underline{\lambda};1}(\underline{p};m)\}-[\Lambda^2]^{-2}\delta(p_1+p_2)\cdot\\&\quad\frac{1}{2\pi i}\oint_{|s|=\epsilon}ds\,\frac{\pi(s+1)}{\sin\pi s}\{[\Lambda^2]^{s}\mathcal{W}g_{\underline{\lambda};s}(\underline{p};m)-\mathcal{W}g_{\underline{\lambda};0}(\underline{p};m)\}\\&\quad+\sum_{\mu=1}^{M}\delta(p_1+p_2)\,\frac{1}{2\pi i}\oint_{|s+\mu|=\epsilon}ds\,\Gamma(2+s)\Gamma(-s)[\Lambda^2]^{s-2}\mathcal{W}g_{\underline{\lambda};s}(\underline{p};m)\\&\quad+\delta(p_1+p_2)\,\frac{1}{2\pi i}\int_{\varrho-i\infty}^{\varrho+i\infty}ds\,\Gamma(2+s)\Gamma(-s)[\Lambda^2]^{s-2}\mathcal{W}g_{\underline{\lambda};s}(\underline{p};m)\end{aligned}$$

for any positive integer M , $-(M+1)<\varrho<-M$ and $\epsilon>0$ sufficiently small.

The order of the pole of the integrand at $s=-\mu$, $\mu=-1,0,1,2,..$ is equal to

$$1 - \delta_{\mu,1} + \#\{G / G \in \mathcal{E}_{\infty},\ \mathfrak{v} \subset \mathcal{V}(G),\ n(G) \leq \mu\}$$
$$+ \#\{G / G \in \mathcal{E}_{\infty},\ \mathfrak{v} \not\subset \mathcal{V}(G),\ n(G) \leq 0\}.$$

Now, the asymptotic expansion in powers of Λ^{-2} and $\ln \Lambda^2$ of the Feynman amplitude corresponding to G^0 which contributes in V^{th} order perturbation theory to the two point vertex function can be read off easily.

An arbitrary number j of mass insertions can be incorporated into the above scheme by partitioning j in all possible ways into a sum of L non-negative integers $j_1, \ldots, j_L$, replacing the propagator of the line l in the amplitude $\tilde{\mathcal{C}}_{\underline{\lambda}}(\underline{\Delta p};m)$

$$\frac{e^{i\pi\lambda_l}\,\Gamma(\lambda_l)}{i\,(2\pi)^2}\left[k_l^2 - m^2 + i0\right]^{-\lambda_l}$$

by

$$(m^2)^{j_l}\,\frac{e^{i\pi\sum_{\nu=1}^{j_l+1}\lambda_{l,\nu}}\prod_{\nu=1}^{j_l+1}\Gamma(\lambda_{l,\nu})}{\left[i\,(2\pi)^2\right]^{j_l+1}}\left[k_l^2 - m^2 + i0\right]^{-\sum_{\nu=1}^{j_l+1}\lambda_{l,\nu}}$$

multiplying subsequently by the combinatorial factor $\frac{j!}{j_1!\ldots j_L!}$, summing over all different partitions and applying finally an appropriate generalized evaluator $\mathcal{W}'_{L+j}$.

The author profited from various discussions with Dr. H. Trute.

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