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Integrable Cylinder Functionals for an Integral of Feynman-Type

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Abstract

Integrability of cylinder functionals is investigated, for the Feynman-type integral as defined by Itô. Several classes of integrable cylinder functionals are specified. An inductive property of the integral is established.

1. Introduction

In this note we establish the integrability of several classes of cylinder functionals, for an integral of the Feynman-type. These functionals are characterized as certain analytic functions, or in terms of bounded measures or distributions.

The subject of Feynman-type integrals (and their physical applications) was reviewed recently in [1]. The results that follow are only a modest contribution. However, we feel that it is worthwhile to place them on record, since still very little rigorous material on such integrals is available.

For convenience of the reader, we repeat here Ito's definition [2] of the integral. The integral is over an arbitrary (separable) Hilbert space $\mathcal{H}$, which may but need not be a space of paths.

Let T be an operator on $\mathcal{H}$ which is of trace class, strictly positive, and symmetric. Let $d\mu_{T,\alpha}$ be the Gaussian measure on $\mathcal{H}$ with the covariance operator T and the mean vector $\alpha \in \mathcal{H}$. Then the integral is defined by

$$I(f) = \lim_{T\to\infty} \frac{1}{c_T} \int_{\mathcal{H}} d\mu_{T,\alpha}(\eta)\, e^{\frac{1}{2} i\langle\eta,\eta\rangle} f(\eta) \tag{1a}$$

where

$$c_T = \int d\mu_{T,0}(\eta)\, e^{\frac{1}{2} i\langle\eta,\eta\rangle} \tag{1b}$$

The limit is to be taken by following the partially ordered set of the T's (where $T' \geq T'' \leftrightarrow T' - T'' \geq 0$). The limit must also be independent of α.

Our present interest is in cylinder functionals f, satisfying $f(\eta) = f(P\eta)$ for some orthogonal, finite-dimensional projection P. For definiteness, we will take

$$\dim P = n. \tag{2}$$

We will refer to $P\mathcal{H}$ as the base-space for f.

The contents of the paper is as follows. In Sec. 2 we establish a property of the integral, which allows the enlargement of the space of integration. Section 3 contains the description of certain classes of integrable functions, while some of the proofs are postponed to Sec. 4. Some explicit formulas bearing on the decomposition of Gaussian measures are included in Appendix A. In Appendix B we consider variants to the definition of Itô, and in Appendix C, some connections with Tauberian theorems.

2. Enlargement of the Space of Integration

We will now examine the following question: Are the functions which are integrable over R^n, or more generally over a Hilbert space $\mathcal{H}_1$, also integrable over a larger space $\mathcal{H} \geq \mathcal{H}_1$. The answer does not follow immediately from Itô's definition.

Our result is the following inductive property.

Proposition 1. Consider two Hilbert spaces, $\mathcal{H}_1 \leq \mathcal{H}$. Let f_1 be a functional which is integrable over $\mathcal{H}_1$. Then f_1 extends to a functional f over $\mathcal{H}$ by

$$f(\eta_1 + \eta_2) = f_1(\eta_1), \qquad \eta_1 \in \mathcal{H}_1, \; \eta_2 \in \mathcal{H}_1^{\perp}, \tag{3a}$$

and f is integrable over $\mathcal{H}$. The respective integrals are equal,

$$I(f) = I_1(f_1) . \tag{3b}$$

Proof. We introduce the following quantities, whose meaning is obvious:

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 = P_1\mathcal{H} + P_2\mathcal{H}, \tag{4a}$$

$$\alpha = \alpha_1 + \alpha_2, \qquad \eta = \eta_1 + \eta_2 \quad (\alpha, \eta \in \mathcal{H}), \tag{4b}$$

$$T : \mathcal{H} \to \mathcal{H} \quad (\mathrm{tr}\, T < \infty, \; T > 0), \tag{4c}$$

$$T_j = P_j \, T \, P_j \restriction \mathcal{H}_j, \qquad j = 1, 2. \tag{4d}$$

We next recall a theorem on decomposition of measures [3,4]. This theorem also applies to complex measures, if the total absolute variation is finite. For the case at hand, it implies the existence of measures $d\lambda_{\eta_1}(\eta_2)$, depending on T and α, such that

$$\frac{1}{c_T}\int_{\mathcal{H}} d\mu_{T,\alpha}(\eta)\; e^{\frac{1}{2}i\langle\eta,\eta\rangle}\; f(\eta) \;=\; \frac{1}{c_{T_1}}\int_{\mathcal{H}_1} d\mu_{T_1,\alpha_1}(\eta_1)\; e^{\frac{1}{2}i\langle\eta_1,\eta_1\rangle}\; f_1(\eta_1)$$

$$\times \int_{\mathcal{H}_2} d\lambda_{\eta_1}(\eta_2). \tag{5}$$

The measure λ_{η_1} here is normalized so that

$$\lambda_{\eta_1}(\mathcal{H}_2) = 1 \quad \text{for} \quad \mu_{T_1,\alpha_1} - \text{almost all} \quad \eta_1, \tag{6}$$

and it includes the effect of the factors $e^{\frac{1}{2}i\langle\eta_2,\eta_2\rangle}$ and c_{T_1}/c_T. Now, as one takes $\lim(T\to\infty)$ on the l.h.s., this entails $\lim(T_1\to\infty)$ on the r.h.s., and by hypothesis, the integral over $\mathcal{H}_1$ then tends to a limit $I(f)$, independently of α_1. The integrability over $\mathcal{H}$ follows.

Note that in case of a distribution-theoretic integral over a finite-dimensional $\mathcal{H}_1$, one can use eq. (5) to define the corresponding integral over $\mathcal{H}_1$.

3. Integrability for Finite Dimensions

We will be concerned with the integrability of the following three classes of complex-valued functions and distributions on R^n.

(1) The first class consists of functions of the form

$$f(x) = (2\pi)^{-\frac{1}{2}n}\int d\mu(p)\; e^{ipx}, \tag{7}$$

where μ is a complex Borel measure on R^n of bounded total absolute variation. (Cf. (2)).

(2) The second class consists of entire functions of order less than two, i. e. satisfying

$$|g(z)| \leq C_\varepsilon \exp(M_\varepsilon |z|^{2-\varepsilon}) \tag{8}$$

for some constants C_ε, M_ε, and $\varepsilon > 0$.

(3) The third class consists of the following subclasses:

(3.a) The set of functions and distributions f of the form

$$f(x) = \partial^{(m)} h(x), \tag{9a}$$

where (m) is an n-index quantity, h is continuous, and

$$|m| \leq n, \quad |h(x)| \leq (1 + |x|)^{-n} \ell(x) \quad \text{for some } \ell \in L_1. \tag{9b}$$

(3.b) The set of functions f whose derivatives satisfy (Δ being the Laplacian in n dimensions)

$$|(1 - \Delta)^N f(x)| \leq (1 + |x|^2)^M \ell(x) \tag{10a}$$

for some $N, M \in \mathbb{N}$ and ℓ such that

$$N - M > \frac{n}{2} \quad \text{and} \quad \ell \in L_1. \tag{10b}$$

We conjecture that our classes (3.a) and (3.b) contain the spaces $\mathcal{O}_M$ and $\mathcal{O}_C$ respectively (cf. e. g. [5]). However, for completeness we shall give an independent argument establishing the integrability of these distributions.

We next presuppose an extension of Ito's definition to distributions over R^n. The elements of $\mathcal{A}_n$ (below) are obviously integrable in this extended sense. It is convenient for us to refer to the space $\mathcal{S}^{1/2}_{1/2}(R^n)$ of [6]. This space is invariant under Fourier transformation, and includes the Gaussian functions $e^{-\frac{1}{2}(x,Bx)}$ if $\mathrm{Re}\, B > 0$. Its dual $\mathcal{S}^{1/2\,\prime}_{1/2}$ is larger than $\mathcal{S}'$ and contains in particular the preceding classes.

<u>Definition 2.</u> Let $\mathcal{A}_n$ be the class of distributions f in $\mathcal{S}^{1/2}_{1/2}(R^n)'$ for which

$$\lim_{B \to -i} \int d^n x \; e^{-\frac{1}{2}\langle x, Bx\rangle} f(x) \tag{11a}$$

exists. The nxn-matrices B are restricted by the conditions

$$B = B^T, \quad BB^* = B^*B, \quad \mathrm{Re}\, B > 0. \tag{11b}$$

We note two immediate properties of such matrices.

Lemma 3. Let A satisfy (11b). Then A can be diagonalized by a real orthogonal matrix. Furthermore, A is invertible, and A^{-1} also satisfies (11b).

Our basic result is the following.

Proposition 4. The classes (1) - (3) (for a given n) are included in $\mathcal{A}_n$.

In order to establish the inclusion for class (1), we can utilize the preceding lemma and adapt the proof of theorem 2 of [2]. (In view of the finite dimensionality, an elementary direct proof can also be given.) The proof for the classes (2) and (3) is outlined in the subsequent section.

We next note:

Proposition 5. The following subset $\mathcal{A}_n^o$ of $\mathcal{A}_n$ is invariant under Fourier transformation:

$$\mathcal{A}_n^o = \{f : f,\ f^* \in \mathcal{A}_n\}. \tag{12}$$

Proof. Without loss of generality we may assume that $f \in \mathcal{A}_n^o$ is real. Now

$$\begin{aligned}(a_B^{-1} \int d^n x\, f(x)\, e^{-\frac{1}{2}<x,Bx>})^* &= ((2\pi)^{-\frac{1}{2}n} \int d^n p\, f^*(p)\, e^{-\frac{1}{2}<p,B^{-1}p>})^* \\ &= (2\pi)^{-\frac{1}{2}n} \int d^n p\, f(p) \exp(-\tfrac{1}{2}<p,B^{-1}_{\text{compl.conj.}}\, p>).\end{aligned} \tag{13a}$$

The constant a_B is defined by the conditions that $f = 1$ must integrate to one. The limits $B \to -i$ and $B^{-1}_{c.c.} \to -i$ are equivalent, in view of the preceding lemma. Furthermore, $\lim_{B\to -i} a_B = (2\pi i)^{n/2}$, and therefore, if $f \in \mathcal{A}_n^o$,

$$\lim_{B\to -i} a_B^{-1} \int d^n p\, f(p)\, e^{-\frac{1}{2}<p,Bp>} \tag{13b}$$

exists*.

---

* Note that this proof depends on allowing, in the definition of $\mathcal{A}_n$, more general matrices than A - i with A real.

The preceding propositions tell us in particular that the Fourier transforms of classes (1 – (3) (cf. [6]) belong to $\mathcal{A}_n$. From the proofs in the following section one can furthermore see that some limiting cases are also included. Consider, e. g., the sequence of entire functions

$$g_m(x) = \int_{-m}^{m} dp\ (1 + p^2)^{-1}\ e^{ip(1-i\kappa)x}, \quad \kappa > 0\ . \tag{14a}$$

It is easy to show that

$$\lim_{m\to\infty} g_m \in \mathcal{A}_1\ . \tag{14b}$$

4. Proofs of Integrability

The proof of proposition 4 for classes (2) and (3) is quite direct, and we only indicate the main points.

First of all, we note that the normalizing factor a_B (as in the proof of proposition 5) is proportional to $(\det B)^{-1/2}$, hence depends continuously on B if B is non-singular, and, therefore, need not be taken explicitly into account.

For class (2), we utilize lemma 3 and transform the coordinates x^k with an orthogonal matrix U, so as to diagonalize B. For the eigenvalues of B (assumed near $-i$), we take

$$\lambda_\kappa - i\mu_\kappa \quad \text{with} \quad \lambda_\kappa > 0 \ \text{and}\ |\mu_\kappa - 1| < 1. \tag{15}$$

Consider now the Cartesian product of sect ors in C^n, where the z^k satisfy (for $\forall k$ and for some $0 < \kappa < 1$)

$$0 \leq \operatorname{Im} z^k \leq \kappa \operatorname{Re} z^k \quad \text{or} \quad 0 \geq \operatorname{Im} z^k \geq \kappa \operatorname{Re} z^k. \tag{16}$$

In these sectors we have the estimate

$$\left|\exp(-\tfrac{1}{2}\langle z,(UBU^{-1})z\rangle)\ g(z)\right| \leq$$

$$\leq C_\varepsilon \exp(-\frac{1}{2}\ \frac{1-\kappa^2}{1+\kappa^2}\ \lambda_{min}\ |z|^2 + M_\varepsilon\ |z|^{2-\varepsilon})\ . \tag{17}$$

(Note that $\langle z,(\cdot)z\rangle$ remains bilinear.)

Hence we can rotate the paths of integration to the lines $\mathrm{Im}\, z^k = \kappa\ \mathrm{Re}\, z^k$, and there is no contribution coming from the boundaries (at infinity).

The resulting integral is seen to be invariant under rotations in the new hyperplane, and therefore we obtain the equality

$$\int d^n x\ e^{-\frac{1}{2}\langle x,Bx\rangle}\ g(x) = \int d^n x\ \exp(-\tfrac{1}{2}(1 + i\kappa)^2 \langle x,Bx\rangle) g((1 + i\kappa)x). \qquad (18)$$

The bounded convergence theorem then allows the passage to the limit $B \to -i$.

We now come to the class (3) and its subclasses. For (3.a), one first shows by induction that

$$|\partial^{(m)} (e^{-\frac{1}{2}\langle x,Ax\rangle} - e^{-\frac{1}{2}\langle x,Bx\rangle})| \le C_{|m|}(1 + |x|^m)\ \ \|A^{-1} - B^{-1}\|, \qquad (19a)$$

where A and B must satisfy (11b).

Here $C_{|m|}$ is independent of x, A, and B, provided

$$\|A^{-1} - i\| < \varepsilon < 1 \quad \text{and} \quad \|B^{-1} - i\| < \varepsilon < 1 . \qquad (19b)$$

The expression (9) for f and integrations by parts lead to the estimate

$$|\int d^n x (e^{-\frac{1}{2}\langle x,Ax\rangle} - e^{-\frac{1}{2}\langle x,Bx\rangle})|f(x)| \le \|A^{-1} - B^{-1}\|\ C_{|m|}\int d^n x \ell(x), \qquad (20)$$

which proves the integrability.

For the class (3.b), one starts with the equation

$$(1 - \Delta)^N\ h_N^B(x) = e^{-\frac{1}{2}\langle x,Bx\rangle} \qquad (21a)$$

whose solution is

$$h_N^B(x) = (\mathrm{const.})\int d^n p\ e^{ipx}\ \frac{e^{-\frac{1}{2}\langle p,B^{-1}p\rangle}}{(1 + p^2)^N} . \qquad (21b)$$

Integration by parts and the estimate (10) yield

$$\left|\int d^n x \left(e^{-\frac{1}{2}\langle x,Ax\rangle} - e^{-\frac{1}{2}\langle x,Bx\rangle}\right) f(x)\right| \leq$$

$$\leq \int d^n x (1 + x^2)^M \left|h_N^A(x) - h_N^B(x)\right| \ell(x). \tag{22}$$

Since the multiplication of $|h_N^{(A)}(x) - h_N^B(x)|$ by $(1 + x^2)^M$ can be reduced to a differentiation in momentum space, one can utilize the estimate (20). The remainder of the proof is straightforward.

For the classes $\mathcal{O}_C$ and $\mathcal{O}'_M$, we note that the integral in the limit $B = -i$ is already defined in the sense of distribution theory, since $e^{\frac{1}{2} i\langle\cdot,\cdot\rangle}$ is in both $\mathcal{O}'_C$ and in $\mathcal{O}_M$. In order to prove convergence as specified in the definition of $\mathcal{A}_n$, it suffices therefore to verify that, for $\forall \phi \in \mathcal{S}$,

$$\lim_{B\to -i} e^{-\frac{1}{2}\langle\cdot,B\cdot\rangle} \phi = e^{\frac{1}{2} i\langle\cdot,\cdot\rangle} \phi, \tag{23}$$

the relevant topology being that of $\mathcal{S}$. The verification of (23) is straightforward, and this yields the convergence of integrals for $f \in \mathcal{O}'_M$. The case $f \in \mathcal{O}_C$ follows by applying Fourier transformation.

Appendix A. Explicit Form of the Decomposition of the Measure μ_T

Let us assume for simplicity that

$$P_j\ D(T^{-1}) \subseteq D(T^{-1}), \tag{A.1}$$

and let us introduce the notation

$$T_{oj} = ((P_j\ T^{-1}\ P_j \restriction \mathcal{H}_j)^{\sim})^{-1}, \tag{A.2}$$

where $\sim$ denotes the Friedrichs extension. This is the extension appropriate for the problem, since T^{-1} and related operators enter primarily in quadratic forms. In general, if $A \geq 0$, then such a form is defined on $D(A^{1/2})$, and only $A^{\sim}$ satisfies [7,8]

$$D(A) \subseteq D(A^{\sim}) \subseteq D((A^{1/2})^{-}). \tag{A.3}$$

These inclusions and the strict positivity of $T^{-\frac{1}{2}}$ imply that the (outer) inverse in (A.2) is well-defined.

If we now start with the relation (where x, y are real and $\neq 0$)

$$(x\ P_1 - y\ P_2)\ T^{-1}\ (x\ P_1 - y\ P_2) \geq 0, \tag{A.4a}$$

or equivalently

$$(x^2 + xy)\ P_1\ T^{-1}\ P_1 + (y^2 + xy)\ P_2\ T^{-1}\ P_2 \geq xy\ T^{-1}, \tag{A.4b}$$

then we can reverse the inequality by taking inverses [9]. We then easily get, for $0 \leq w \leq 1$,

$$0 < (1-w)\ T_{o1} + w\ T_{o2} \leq T. \tag{A.5}$$

It follows in particular that the T_{oj} are of trace class, and that they are the covariance operators of measures on $\mathcal{H}_j$.

For brevity in writing we now take $\alpha = 0$ in eq. (1), we assume an (arbitrary) function F for the integrand, we drop the factors $e^{\frac{1}{2} i\langle\eta_j,\eta_j\rangle}$, and we utilize invariant generalized measures [10]. Then the form of the decomposition is

$$\int \mathcal{D}(\eta)\, e^{-\frac{1}{2}\langle\eta,T^{-1}\eta\rangle} F(\eta) = \tag{A.6}$$

$$= \int \mathcal{D}_1(\eta_1)\, e^{-\frac{1}{2}\langle\eta_1,T_{o1}^{-1}\eta_1\rangle} \int \mathcal{D}_2(\eta_2)\, e^{-\frac{1}{2}\langle\eta_2,T_{o2}^{-1}\eta_2\rangle}\, e^{\langle T^{-1}\eta_1,\eta_2\rangle} F(\eta_1+\eta_2).$$

Let us take for $\mathcal{D}_2$ the generalized measure whose weight is $e^{-\frac{1}{2}\langle\eta_2,T_{o2}^{-1}\eta_2\rangle}$ and for F, a functional not depending on η_2, which can then be put in front of the second integral. This integral then yields

$$\exp \frac{1}{2}\langle\eta_1,T^{-1}\,T_{o2}\,T^{-1}\,\eta_1\rangle =: e(\eta_1). \tag{A.7}$$

In view of (6), we conclude that in the present (simplified) case,

$$d\lambda_{\eta_1}(\eta_2) = (e(\eta_1))^{-1}\,\mathcal{D}_2(\eta_2)\, e^{-\frac{1}{2}\langle\eta_2,T_{o2}^{-1}\eta_2\rangle}\, e^{\langle T^{-1}\eta_1,\eta_2\rangle}. \tag{A.8}$$

If we now integrate

$$F(\eta_1+\eta_2) = \exp i\langle\eta_1,\beta\rangle, \tag{A.9}$$

we see that $\mathcal{D}_1$ must have the weight $e^{-\frac{1}{2}\langle\eta_1,T_1^{-1}\eta_1\rangle}$, to which contribute $e(\eta_1)$ and the exponential involving T_{o1}^{-1}. We conclude, from an examination of the Fourier transforms of the measures (recall that the operators are assumed real and symmetric) that

$$T_1^{-1} = T_{o1}^{-1} - (T^{-1}\,T_{o2}\,T^{-1}) \upharpoonright \mathcal{H}_1. \tag{A.10}$$

In the finite-dimensional case we can get another relation by comparing the normalizing factors of the integrals of (A.6):

$$\det T = \det T_1 \det T_{o2}. \tag{A.11}$$

Appendix B. Integrability for Alternate Definitions of the Integral

Various definitions of Feynman-type integrals were suggested in recent years. One particular kind of definition has apparently not appeared in print, and this is the direct adaptation of the constructions of the canonical integral over a Hilbert space. We have in mind here such constructions as that of Friedrichs and Shapiro [11,12] and that of Segal.

The following definitions come close to the presentation in [12].

Let $\{P_m\}$ be a family of increasing orthogonal projections satisfying

$$\dim P_m = m, \quad P_{m+1} \geq P_m, \quad \lim_{m\to\infty} P_m = 1. \tag{B.1}$$

Let $x \in \mathcal{H}$, and let u denote the values of $P_m x$, so that u ranges over R^m. Let us set

$$I_{\{P\},m,\varepsilon}(f) = \frac{1}{(2\pi(i-\varepsilon))^{m/2}} \int d^m u \; e^{\frac{1}{2}(i-\varepsilon)\langle u,u\rangle} f(P_m x). \tag{B.2}$$

For the present discussion, we define Feynman integrability over R^m as the existence of the limit $\varepsilon \downarrow 0$. This is an (apparently) weaker condition than Itô's. For an infinite dimensional space, we have to take the limits $m \to \infty$, $\varepsilon \downarrow 0$, and there are three natural ways to do this:

(a) $\lim(m\to\infty)$, then $\lim(\varepsilon\downarrow 0)$,

(b) $\lim(\varepsilon\downarrow 0)$, then $\lim(m\to\infty)$,

(c) pick $\sigma > 0$, let $\varepsilon = \sigma/m$, and take $\lim(m\to\infty)$.

One must require, moreover, that the value of the limit(s) be independent of the family $\{P_m\}$ and, in case (c), also of the choice of σ.

Each of the three alternatives (a) - (c) defines an integral. Each definition is new, and all might be mutually inequivalent. The integrability of cylinder functionals can be investigated by adapting the discussion of [12], pp.V-10-11. We forego the details, except to point out (without proof) the following link with our previous conclusions:

Lemma 6. If F, upon restriction to its base-space, is in $\mathcal{A}_n$, then f is integrable in the sense of the definition (c).

Appendix C. A Connection with Tauberian Theorems

In this appendix we state one result which provides another criterion for integrability. We were led to this while trying to exploit the Tauberian theorems, as presented e. g. in [13].

In the Tauberian theorems one generally assumes for the integrand a bound like $|f(u)| < (\text{const.})\, u^{-1}$. In the present context, such a restriction is inconvenient for applications, and is avoided in the following partial converse to the usual theorems. We confine ourselves to the case of one diemsnion, and make a simplifying assumption about f.

Proposition 7. Let the function f be locally in L_1, with a finite number of changes of sign in a bounded interval. Then

$$\lim_{x\to\infty}\int_0^x du\, f(u) = K \Rightarrow \lim_{\varepsilon\downarrow 0}\lim_{x\to\infty}\int_0^x du\, e^{-\varepsilon u^2} f(u) = K. \tag{C.1}$$

I.e., the existence of the first limit implies that of the limits in the r.h.s. and the equality. In particular, if $f(u) = 0$ for $u < 0$, $f\, e^{-\varepsilon(\cdot)^2} \in L_1$ for $\forall\varepsilon > 0$, and $f = f_o\, e^{1/2\, i(\cdot)^2}$, then the hypothesis in (C.1) implies that f_o is Feynman-integrable.

We remark that the implication (C.1) remains valid also when $K = \pm\infty$, and this case was referred to in [1].

Let us consider successive contributions to the integral, corresponding to positive and to negative values of f, as u increases. The proposition will then follow from the following lemma.

Lemma 8. Let $C_{\alpha,j}$ be numbers defined for $j = 0,1,2,\ldots$ and for $0 < \alpha < \delta$ (for some δ), and satisfying

$$0 \le C_{\alpha,j} \le 1, \quad C_{\alpha,j+1} \le C_{\alpha,j}, \quad \lim_{\alpha\downarrow 0} C_{\alpha,j} = 1. \tag{C.2}$$

Assume that one has convergence (in general, conditional) for the series

$$S = a_o - a_1 + a_2 -+\ldots, \tag{C.3}$$

Then one also has the (conditional) convergence and the limit,

$$S^{\alpha} = C_{\alpha,o}\, a_o - C_{\alpha,1}\, a_1 + - \ldots; \qquad \lim_{\alpha \downarrow 0} S^{\alpha} = S. \tag{C.4}$$

Proof. Without loss of generality we may suppose that $a_j > 0$, $\forall j$. Let us set

$$S_n = \sum_{1}^{n} (-)^k a_k, \quad {}_{\alpha}S_n = \sum_{1}^{n} (-)^k C_{\alpha,k}\, a_k. \tag{C.5}$$

The lemma will follow easily from the bounds,

$$\min\{S_o,\ldots,S_n\} \le {}_{\alpha}S_n \le \max\{S_o,\ldots,S_n\}, \tag{C.6a}$$

or equivalently, from

$$\min\{S_1,S_3,\ldots,S_{2n+1}\} \le {}_{\alpha}S_{2n+1} \text{ and } {}_{\alpha}S_{2n} \le \max\{S_o,S_2,\ldots,S_{2n}\}. \tag{C.6b,c}$$

(Such bounds should be applied to interior partial sums.)

We will prove only (C.6c), by induction. The case $n = 0$ is immediate. From the induction hypothesis and the inequality $C_{\alpha,j+1} \le C_{\alpha,j}$ we conclude

$$\begin{aligned} {}_{\alpha}S_{2n+2} &\le C_{\alpha,0}\, a_0 - C_{\alpha,1}\, a_1 + C_{\alpha,1} \max\{S_2 - S_1,\ldots,S_{2n+2} - S_1\} \\ &\le S_o + C_{\alpha,1} \max\{S_2 - S_o,\ldots,S_{2n+2} - S_o\}. \end{aligned} \tag{C.7}$$

Furthermore,

$$C_{\alpha,1} \max\{\ldots\} \le \max\{0,\ldots\}, \tag{C.8}$$

and (C.6c) for $2n+2$ follows from (C.7-8).

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