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Feynman-type Integrals for Spin and the Functional Approach to Quantum Field Theory

by

Jan Tarski
II. Institut für Theoretische Physik der Universität Hamburg

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Jan Tarski

II. Institut für Theoretische Physik
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Abstract. A construction of Feynman-type integrals for systems with spin is outlined. In particular, in case of fields with half-odd-integer spin, one integrates over a space of pairs $\{\eta, -\eta\}$. While the problems of analysis are only briefly considered, those of geometry are treated in greater detail, and some ideas from the theory of symplectic manifolds are utilized. As a sequel to this construction, a preliminary set of axioms for quantized fields in functional form is suggested.

1. Preliminaries

1A. Introduction.

The functional formalism appears to be useful in an ever-increasing variety of applications in quantum physics. A number of such applications involve both general and specific problems in quantum field theory. Under this circumstance it was natural to ask the question: Could quantum field theory be based on the functional formalism, in a mathematically precise way?

Now, there are two other well-known mathematical approaches to quantum field theory, i.e. the algebraic one and that of Wightman. In each approach, a set of axioms for quantum field theory was suggested. In this paper we present a preliminary set of axioms for the functional framework, after a discussion of some preliminary problems. We hope to provide in this way a better orientation into the possibilities and the limitations of this framework.

One aspect of this framework which has been in a rather unsatisfactory state is that relating to spin. In fact, spin has long been recognized as a particularly awkward entity to handle in this way. A number of attempts to construct Feynman-type (i.e. path or history) integrals for spin and for fermi fields were carried out, to be sure. But the approaches in those works have been developed only to a minimal extent. Moreover, they seen seem to give the feeling that they are not quite what one would really like. (For these reasons we cite only a few of these works in the references.)

We describe here an independent attempt to construct Feynman-type integrals for spin and for fermi fields. We chose to describe the construction in terms of the concepts of symplectic manifolds, including the elaborations which arise in Kostant's theory of group representations.

The fragments of this theory which we need can be reduced to familiar facts about SU_2 and about the Poincaré group, and the general notions of the theory could be bypassed. However, the theory of Kostant points to several useful entities, and it enabled us to see more clearly the analogies between spin and the translational modes.

Our study is not rigorous. We described the simple geometric notions in detail, but we left many problems of analysis open. Typically, we would suggest that an integral could be constructed in a certain way (where the details may be only vaguely sketched), and that it would have certain properties. We feel that our suggestions are reasonable, in view of the prior experience with similar constructions. Of course, it was then pointless to discuss fine points, like the question of differentiability, in detail.

We consider in the text several other aspects of the functional formalism. One of these is the spin-statistics correlation. We develop here further the idea, which was suggested on several occasions in recent years, that this correlation should stem from the homotopy properties of fields. In such a case one should be able to extend it also to nonrelativistic fields. Furthermore, some of the problems that we encounter are not related to spin. We are led to consider, in particular, the use of nonstandard analysis to define fields at a sharp time, and to consider a stochastic interpretation of fields.

In the remainder of Part 1 we review certain facts concerning symplectic manifolds and path integrals. Part 2 deals with nonrelativistic systems with spin. We treat in turn particles, fields, and specifically the commutation rules. In Part 3 we make brief remarks about relativistic systems with spin. Part 4 contains comments on the stochastic interpretation, the axioms, and some concluding remarks. A short appendix is devoted to functional integrals over a quotient space.

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1B. Symplectic manifolds and sections.

The present section summarizes certain facts about symplectic manifolds, and about Kostant's theory of group representations. The sources for this section are [1 - 3].

A symplectic manifold has by definition an everywhere-defined (and sufficiently smooth) bilinear, nonsingular, skew-symmetric form ω on the tangent vectors X_j . Thus one has

$$\omega\left(X_i, X_j\right) = -\omega\left(X_j, X_i\right). \tag{1.1}$$

A symplectic manifold has an even dimension $2n$, and the form ω can always be expressed locally as follows,

$$\omega = \sum_{i=1}^{n} dp^i \wedge dq^i . \tag{1.2}$$

Familiar examples of symplectic manifolds are the phase spaces of classical physics. Such spaces are identifiable with cotangent bundles T^*M, on which there are symplectic structures coming from the Poisson brackets.

Symplectic manifolds are also basic in Kostant's theory. One important notion in this theory is that of polarization. Physically, this corresponds to the selection of a complete set of observables when a system is quantized. In mathematical terms this means selecting n vector fields $Y_1,\dots,Y_n$ which are combinations (in general, complex) of the X_i, which are complex-linearly independent, and which satisfy

$$\omega(Y_i, Y_j) = 0 \quad \text{for } 1 \le i, j \le n. \tag{1.3}$$

We remark that the approach of Kostant, in the particular context of the quantization of classical systems, was discovered independently by Souriau [4].

To a given polarization corresponds a space of admissible wave functions f satisfying $Y_i f = 0$ for $\forall i$. (Strictly speaking, one should use here certain covariant derivatives.) For instance, by selecting $Y_i = \partial/\partial p^i$, with ω as in (1.2), one gets the coordinate space representation of wave mechanics. If $Y_i = \partial/\partial q^i$, one, gets the momentum representation. A still different possibility is to set

$$z^j = 2^{-\frac{1}{2}}(q^j + i p^j), \qquad Y_j = \partial/\partial (z^j)^*, \tag{1.4}$$

so that the conditions $Y_j f = 0$ yield analytic functions. This device is familiar in case of the harmonic oscillators, where one can identify a space of states with a Fock space [5, 6].

The polarizations, to be suitable, must satisfy certain conditions which we do not state in detail. One of these is that of group invariance, if a group action is relevant. This invariance eliminates e.g. the possibility that the coordinate representation might be transformed into the momentum representation by the group.

Another important notion is that of a line bundle L over a manifold M. Such a bundle is constructed by associating to each point $c \in M$ a complex plane, to be denoted by L_c, so that $L = \cup_{c \in M} L_c$. A <u>section</u> of L is a map $\sigma : M \to L$ such that $\sigma(c) \in L_c$ for $\forall c$. If a reference section σ_o is chosen such that $\sigma_o(c) \neq 0$ for $\forall c$, then we can identify σ with a complex-valued function f on M by

$$\sigma(c) = f(c)\, \sigma_o(c). \tag{1.5}$$

An obvious example of a line bundle is $M \times C^1$. Here the choice $\sigma_o(c) = 1$ allows a natural identification of $M \times C^1$ with functions on M. We describe below a slightly more intricate situation, where the use of sections avoids some of the ambiguities associated with the "double-valued functions".

We turn to group representations. Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, let $v \in \mathfrak{g}'$ (the dual to $\mathfrak{g}$), let G_v be the subgroup of G which leaves v invariant, let $\mathfrak{g}_v$ be the Lie algebra of G_v, and let M be the orbit Gv. One can define on M an invariant sympletic structure by the equation

$$(\underline{X}, \underline{Y}) = \langle v, [X, Y] \rangle \quad \text{for } \underline{X}, \underline{Y} \in \mathfrak{g}/\mathfrak{g}_v. \qquad (1.6)$$

Finally, let h be a homomorphism of G_v into $C^1 \setminus \{0\}$ that extends to a representation of G. Such an h exists e.g. if G is semisimple and simply connected, and if v is a weight.

There exists a bijection between the $\mathcal{C}^\infty$ sections Σ on M and a subset $\hat{\Sigma}$ of functions $\mathcal{C}^\infty(G)$. Note first that G_v acts naturally on $G \times C^1$ [or on $\mathcal{C}^\infty(G)$] by

$$g : (\gamma, z) \to (\gamma g, h(g^{-1})z). \qquad (1.7)$$

The quotient space $L = (G \times C^1)/G_v$ is a linear bundle over M, but need not be identifiable with $M \times C^1$.

We define

$$\hat{\Sigma} = \{\hat{\sigma} \in \mathcal{C}^\infty(G) : \hat{\sigma}(\gamma g) = h(g^{-1})\hat{\sigma}(\gamma),\ \forall \gamma \in G, g \in G_v\}, \qquad (1.8)$$

and to each $\hat{\sigma} \in \hat{\Sigma}$ we associate $\sigma \in \Sigma$ by

$$\sigma(\rho_v \gamma) = \tau_v(\gamma, \hat{\sigma}(\gamma)), \qquad (1.9)$$

where ρ_v and τ_v are the natural maps from G onto M, and from $G \times C^1$ onto L , respectively.

In the sequel we use the correspondence $\Sigma \leftrightarrow \hat{\Sigma}$, but without the restriction to $\mathcal{C}^\infty$ functions and sections.

We should like to give next some explicit relations for SU_2 . (A fuller account can be found in [3] .) Here M is a sphere, and there are two group-invariant polarizations, which are complex conjugates. In a suitable parametrization, the respective vectors are proportional to $\partial/\partial z$ or to $\partial/\partial z^*$, cf. eq. (1.4).

Let υ be the weight of the representation with spin s . Then the condition (1.7) , which becomes

$$\hat{\sigma}(\gamma e^{tJ_3}) = e^{ist}\hat{\sigma}(\gamma), \tag{1.10}$$

restricts the function on SU_2 to the linear combinations of the representation matrix elements $D^{(j)s}_{m}$, where $j = s, s+1, \ldots$, and $m = j, j-1, \ldots, -j$. The polarization defined by $\partial/\partial z^*$ restricts the functions further, namely to those where $j = s$, so that one has an irreducible representation. (One encounters here analytic sections, which are defined in terms of covariant derivatives, and which differ from analytic functions on M. The latter can only be constants.)

We note the following. In the case of integral spins, the condition (1.10) yields directly functions on the sphere M , while for the half-odd-integer spins, (1.10) yields functions on M defined only up to a sign. Thus we can introduce the unit spinors for the group SO_3, and then, for $\hat{\sigma}_\pm = D^{(\frac{1}{2})\frac{1}{2}}_{\pm\frac{1}{2}}$, and for the appropriate υ and h, we have the correspondence between the (unordered) pairs, like the following,

$$\{\tau_{\sigma}(\gamma, \pm\hat{\sigma}_{+}(\gamma))\} \leftrightarrow \{\pm(1,0)\}. \qquad (1.11)$$

For some other σ, h, and $\hat{\sigma}$, we can identify the resulting section with the vector (1,0,0), etc. A suitable normalization for the sections is of course presupposed. To spinor-valued functions, such as encountered in wave mechanics, we can assign pairs of section-valued functions,

$$\{\pm(\psi_1, 0)\} \leftrightarrow \{\tau_{\sigma}(\gamma, \pm\psi_1\hat{\sigma}_{+}(\gamma))\}, \qquad (1.12)$$

where $\psi_1 : R^3 \to C^1$. The covering group of the Euclidean group acts on such functions in the natural way. I.e., for a translation T and for $\rho \in SU_2$, to which corresponds the rotation R, one has the rules

$$T\psi_1(\cdot) = \psi_1(T^{-1}\cdot), \quad \rho\psi_1(\cdot) = \psi_1(R^{-1}\cdot), \qquad (1.13a)$$

$$T D^{(j)s}_{m}(\cdot) = D^{(j)s}_{m}(\cdot), \quad \rho D^{(j)s}_{m}(\cdot) = D^{(j)s}_{m}(\rho^{-1}\cdot). \qquad (1.13b)$$

Let us suppose further that the functions ψ_1 form a linear topological space (of dimension $2 < n \leq \infty$), and let us leave out the point $\psi_1 = 0$. Then a standard result [7] asserts that the spaces of pairs in (1.12) have double connectivity.

1C. Path integrals, perturbations, and degeneracy.

We now review certain features of the path integral. Let us start by summarizing the construction of Itô, who defines an integral over a Hilbert space $\mathcal{H}$ in terms of a generalized invariant measure $\mathcal{D}(\cdot)$ [8]. Consider the Gaussian measures on $\mathcal{H}$. We denote such a measure by $d\mu_{T,\alpha}$ if it has the covariance operator T (strictly positive, symmetric, and trace-class) and the mean vector α. We then set

$$\int \mathcal{D}(\gamma)\, e^{\frac{1}{2}i\langle\gamma,\gamma\rangle} f(\gamma) = \lim_{T\to\infty} c_T^{-1} \int d\mu_{T,\alpha}(\gamma)\, e^{\frac{1}{2}i\langle\gamma,\gamma\rangle} f(\gamma), \tag{1.14a}$$

where

$$c_T = \int d\mu_{T,0}(\gamma)\, e^{\frac{1}{2}i\langle\gamma,\gamma\rangle}. \tag{1.14b}$$

The limit $T\to\infty$ is to be taken in a suitable way, and is to be independent of α.

The limit of the integrals in (1.14), if it exists, is a linear functional of f. The generalized measure $\mathcal{D}(\cdot)$ is then defined by this functional, while heuristically it is $\lim_{T\to\infty} c_T^{-1} d\mu_{T,\alpha}$. More generally, one can define generalized measures as suitable limits of measures, or in terms of limits of integrals [9].

The generalized measure $\mathcal{D}(\cdot)$ of (1.14) is invariant under rotations and translations. This fact allows us to extend the definition of the integral to a <u>hilbertian space</u>, i.e. to a metric space which can be mapped isometrically onto a Hilbert space.

The foregoing ideas can be used as follows. Take the real Hilbert space of functions (or paths) $\eta : R^1 \to R^n$, $\eta = (\eta^j)$, such that

$$\eta(0) = \eta(t) = 0, \quad \langle \dot{\eta}, \dot{\eta} \rangle = \int_0^t d\tau \sum_j \left(\frac{d\eta^j}{d\tau}\right)^2 < \infty. \tag{1.15}$$

This space can be mapped isometrically onto the space of functions satisfying

$$\eta(0) = u, \quad \eta(t) = v, \quad \langle \dot{\eta}, \dot{\eta} \rangle < \infty. \tag{1.16}$$

The Green's function for a Schroedinger particle with mass m in an external potential V is now given by

$$G(t, v; 0, u) = \left(\frac{m}{2\pi i t}\right)^{\frac{1}{2}} \int_{\eta(0)=u,\, \eta(t)=v} \mathcal{D}(\eta)\, e^{iA(\eta)}, \tag{1.17}$$

where

$$A(\eta) = \int_0^t d\tau\, L = \int_0^t d\tau \left(H^{(0)} - V\right) = \frac{1}{2} m \langle \dot{\eta}, \dot{\eta} \rangle - \int_0^t d\tau\, V(\eta(\tau)). \tag{1.18}$$

The quantity A is the action, and we should like to make this comment. The time integral of $H^{(0)}$ serves to define the path integral, while V is treated like a perturbation. One can say that V is coarser than $H^{(0)}$, in the following sense: If a path integral is defined with the help of $H^{(0)}$ as above, then a potential will be continuous on the space of integration, but not vice-versa. (We do not attempt to give a general definition at this point.)

Let us also review the usual heuristic construction of the path integral. One breaks the interval $[0, t]$ into small subintervals $[t_j, t_{j+1}]$ where

$$0 = t_0 < t_1 < \ldots < t_n = t. \tag{1.19}$$

We assume now free propagation from t_j to t_{j+1}, and let the potential $V(\eta)$ be computed along the resulting polygonal paths. Moreover, a complete set of states is inserted at each t_k. The limit $\max(t_{j+1} - t_j) \to 0$ is then taken. Various details can of course be modified in this procedure.

We consider degenerate Lagrangians next. These are the Lagrangians where the quadratic form giving the kinetic energy is of less than the maximal rank [10]. A situation of this general kind arises in the case of spin, and it is instructive therefore to examine a more elementary example of degeneracy, namely that of (scalar) particles in the static limit, $m \to \infty$.

To study this limit, approximate the integral by assuming free propagation from t_j to t_{j+1}, as above. The integral then contains the factors

$$(m/2\pi i t)^{\frac{1}{2}} \exp\left\{ im\left[\eta(t_{j+1}) - \eta(t_j)\right]^2/2t \right\}, \tag{1.20a}$$

and when $m \to \infty$, one finds factors proportional to

$$\delta\left(\eta(t_{j+1}) - \eta(t_j)\right), \tag{1.20b}$$

so that the paths become static. (In particular, $m\langle\dot\eta, \dot\eta\rangle \to 0$

and not ∞). The static paths are of course in accord with the solution to the Schrödinger equation,

$$-i^{-1}\partial_t \psi = V\psi \quad \Rightarrow \quad \psi(t,\underset{\sim}{x}) = e^{-itV(\underset{\sim}{x})}\psi(0,\underset{\sim}{x}). \tag{1.21}$$

We close this section with an example of another form of the path integral [11]. This example corresponds to the polarization (1.4), and at each t_j of the subdivision (1.19) we insert the eigenstates of z. (For simplicity we take one degree of freedom here). Since z can be interpreted as an annihilation operator, the eigenstates are the coherent states, which we can index by $u = u_r + i u_i$. To get completeness, one integrates over C^1 with the measure

$$d\nu(u) = \pi^{-1} du_r du_i e^{-|u|^2}. \tag{1.22}$$

The expectation values of z and z^* are then given (heuristically) by

$$\langle (F(z^*,z))_+ \rangle_0 = \int_{\alpha(\pm\infty)=0} \mathcal{D}(\alpha^*,\alpha) e^{iA(\alpha^*,\alpha)} F(\alpha^*,\alpha), \tag{1.23}$$

where

$$A(\alpha^*,\alpha) = \int_{-\infty}^{\infty} d\tau \left[\tfrac{1}{2} i \left(\alpha^* \dot{\alpha} - (\alpha^*)^{\cdot} \alpha\right) - H(\alpha^*,\alpha)\right]. \tag{1.24}$$

The $\alpha(\cdot)$ are paths in the complex plane, and $\mathcal{D}(\alpha^*,\alpha)$ is formally an infinite product of measures $\pi^{-1} du_r du_i$. Presumably $\mathcal{D}(\cdot)$ can be given a precise meaning as a generalized measure. The Gaussian in (1.22) yields the first part of the action A. The values $\alpha(\pm\infty) = 0$ correspond to having the vacuum initially and finally.

The operators z and z^* do not commute and so $z^*(\tau)$ cannot be represented multiplicatively together with $z(\tau)$. However, z^* acts on a coherent vector at left (in a scalar product) by multiplication, and this implies that $z^*(\tau)$ can be represented by one of the $\lim_{\varepsilon\downarrow 0} \alpha^*(\tau\pm\varepsilon)$. In [11] the analogue to the commutation rule $[z, z^*] = 1$ was in fact established:

$$\lim_{\varepsilon\downarrow 0} \int \mathcal{D}(\alpha^*, \alpha)\, e^{iA} \left[\alpha(\tau)\alpha^*(\tau-\varepsilon) - \alpha^*(\tau+\varepsilon)\alpha(\tau) - 1\right] = 0. \quad (1.25)$$

In view of this equation, eq. (1.24) for A is ambiguous. A satisfactory way of defining the products in (1.24) is to replace

$$\alpha^* \to \alpha^*(\tau + 0), \quad \alpha \to \alpha(\tau). \quad (1.26)$$

One can, moreover, verify the following. If the earliest time τ_e that enters explicitly in a product is an argument of α, or if the latest τ_l is an argument of α^*, then the path integral (1.23) vanishes:

$$\text{occurrance of } \alpha(\tau_e) \text{ or of } \alpha^*(\tau_l) \Rightarrow \int \mathcal{D}(\alpha, \alpha^*) e^{iA} F = 0. \quad (1.27)$$

2. Nonrelativistic systems with spin

2A. The path integral for a particle

In the nonrelativistic theory the translational and the spin degrees of freedom are kinematically independent, and this fact allows us to treat separately the respective contributions to the path integral. Explicitly, let $\check{q} = (q,y)$, where q is the coordinate, and y refers to the spin. I.e., y can be a pair (j, m), or a point of a sphere, etc. In place of q one can also have here p, or the z^j of (1.4). The path integral should then be defined in terms of a generalized measure $\mathcal{D}(\check{q})$ over paths $\check{q}(\cdot)$, and in our construction this measure factorizes,

$$\mathcal{D}(\check{q}) = \mathcal{D}(q)\,\mathcal{D}(y)\,. \tag{2.1}$$

The integration with respect to q is as usual, and we will now consider only the integral with respect to y.

Let us take for definiteness an electron free to flip its spin but otherwise stationary. Then the construction of the path integral for spin is suggested by the arguments given for a (scalar) static particle, cf. eqs. (1.20), and by the polarization $\partial/\partial z^*$ [see the discussion following eq. (1.9)].

The (free) Lagrangian associated with spin vanishes. In other words, one has a degeneracy, and the example of a static scalar particle suggests that we should let only the static paths contribute. If we give the weight $\frac{1}{2}$ to each of the two possibilities, $b(\tau) = \pm\frac{1}{2}$ for $\forall\tau$, then the path integral for the Green's function takes the form

$$2\int_{b(0)=k,\ b(t)=l} \mathcal{D}_0(b) = G_{kl}(t,0) = \delta_{kl}\,, \qquad (2.2)$$

where $k, l = \pm\frac{1}{2}$. This equation presupposes the same sign for the spinor at $\tau = 0$ and $\tau = t$. Note also that the generalized measure $\mathcal{D}_0(\cdot)$ is in fact a measure.
One can perhaps make the analogy between spin and a static particle more striking by the following example. Consider paths $b(\cdot)$ on $(0,t)$ with a finite number of jumps (i.e. where $\pm\frac{1}{2}$ changes to $\mp\frac{1}{2}$). Let $\|b\|$ indicate this number. Let us give the relative weight $e^{-K\|b\|}$ to the totality of paths with a given $\|b\|$, and let us introduce the measure $\mathcal{D}(b)$ that assigns to a set of paths the Lebesque measure, in $R^{\|b\|}$, of their discontinuity points. For the two constant paths, take the measure $\frac{1}{2}$ for each. Then one has the limit

$$\lim_{K\to\infty} \mathcal{D}(b)\, e^{-K\|b\|}\, t^{-\|b\|} \Big/ \Big(\textstyle\sum_{n=0}^{\infty} e^{-Kn}\Big) = \mathcal{D}_0(b). \qquad (2.3)$$

Here the parameter K indicates the reluctance, or inertia, of the spin against free-movement flipping, and the usual theory corresponds to $K \to \infty$.

One may be tempted to say that there is no particular content to the formula (2.2). We include it for completeness, and as background for the history integrals for fields which are more interesting.

However, we should like to make two further observations about (2.2). First, the restriction to constant paths makes it impossible to incorporate the effect of e.g. a magnetic field in the x-direction, which induces a mixing of the two states. But one should note that an interaction potential like $h_x \psi_{+\frac{1}{2}} \psi_{-\frac{1}{2}}$ is in fact analogous to a nonlocal potential in the coordinate space, i.e. to a term like

$$H_{int} = \int dq\, dq' \, \sigma(q, q') \, \psi^*(q) \, \psi(q') . \tag{2.4}$$

As far as we know, such interactions have not yet been considered in the context of path integrals (even though a temporal nonlocality can be readily handled, cf. [12]). We will return to the problem of the mixing of spins in Sec. 2B.

Second, the possibility of handling all spins simultaneously was indicated in [13] . This can also be done in our set-up, where one would take static paths on the linear bundle $(G \times C^1)/G$, see eq. (1.7). Then by projecting at initial and final times one can obtain the restriction to a definite spin.

2B. History integrals for fields.

We recall the basic conclusions concerning the history integral for the relativistic free scalar field $\varphi^{(o)}$ [14]. There the integration is over a space of real-valued functions on R^4 (or on M^4). The action functional is the following,

$$A^{(o)}(\eta) = \frac{1}{2}\int d^4u\,[(\partial_t\eta)^2 - (\nabla\eta)^2 - \mu^2\eta^2]. \qquad (2.5)$$

Then the functional integral can be defined by the weight $\exp(iA^{(o)})$ and yields the time-ordered vacuum expectation values,

$$\langle (F(\varphi^{(o)}))_+\rangle_o = \int \mathcal{D}(\eta)\exp[iA^{(o)}(\eta)]\,F(\eta). \qquad (2.6)$$

This formula has been established for a very limited class of functionals F .

As with the path integral, other forms of the history integral can be readily given. One alternative is given in [14] . Another can be obtained by first approximating the field by a large number of oscillators, each having a definite value of $\underset{\sim}{k}$. One then uses (1.23) for each oscillator and passes (heuristically) to the limit of a continuous distribution of the $\underset{\sim}{k}$'s . The result is

$$\langle (F(a^*,a))_+\rangle_o = \int \mathcal{D}(\alpha^*,\alpha)\, e^{i\bar{A}^{(o)}(\alpha^*,\alpha)}\, F(\alpha^*,\alpha). \qquad (2.7)$$

The entities $a(\underset{\sim}{k})$ are the annihilation operators (rather, distributions), satisfying

$$[a(\underset{\sim}{k}),\ a^*(\underset{\sim}{k}')] = \delta(\underset{\sim}{k} - \underset{\sim}{k}'). \tag{2.8}$$

The action functional can be represented in terms of the function α (which represents a as a multiplicative operator) and α^* as follows, cf. (1.24), (1.26),

$$\bar{A}^{(0)}(\alpha^*, \alpha) = \int_{-\infty}^{\infty} d\tau \int d^3\underset{\sim}{k} \left(\tfrac{1}{2} i\, [\alpha^*(\tau+0, \underset{\sim}{k})\dot{\alpha} - (\alpha^*)^{\cdot}\alpha] - H^{(0)}(\alpha^*, \alpha)\right). \tag{2.9}$$

We remark that if one makes the restriction to a fixed time, then both (2.6) and (2.7) reduce to measure-theoretic integrals which are related by the duality theorem of Segal [15].

The form (2.7) of the integral can be adapted at once to the nonrelativistic scalar field ψ. Indeed, this field and ψ^* are the Fourier transforms of a and a^* respectively. Thus the formulas (2.7 - 2.9) can be transcribed directly to the coordinate space. In particular,

$$\bar{A}_{nr} = \int_{-\infty}^{\infty} d\tau \int d^3\underset{\sim}{x} \left(\tfrac{1}{2} i\, [\psi^*(\tau+0, \underset{\sim}{x})\dot{\psi} - (\psi^*)^{\cdot}\psi] - H(\psi^*, \psi)\right). \tag{2.10}$$

The Hamiltonian need not be the free-field one.

An analogue to (2.5 - 2.6) for a nonrelativistic scalar field is also possible. Let $\varphi = \psi + \psi^*$. Then φ satisfies

$$\left[-\partial_t^2 - (2\mu)^{-2}\Delta^2\right]\varphi = 0, \tag{2.11}$$

and the corresponding action functional is given by

$$A^{(0)}_{nr}(\eta) = \frac{1}{2}\int d^4k\left[(k^0)^2 - (2\mu)^{-2}(\underset{\sim}{k}^2)^2\,|\tilde{\eta}(\underset{\sim}{k})|^2\right] \tag{2.12}$$

One can then obtain (heuristically) the analogue to (2.6). We will not consider this form further.

Let us now treat the case of a nonrelativistic field with spin. For the basic kinematical variables we take $\check{q}=(\underset{\sim}{q},m)$ and t, as in sec. 2A, and the field ψ with spin s has the components $\psi(t, \underset{\sim}{q}, m)$ or $\psi_m(t, \underset{\sim}{q})$, where $m = \pm s, \pm(s-1),\ldots$ This field can be represented by section-valued functions. This corresponds to the representation of $\varphi^{(0)}$ by η in (2.6), in view of the use of sections for spin. Following sec. 1B, we make the correspondence

$$\psi \leftrightarrow (\psi_s,\ldots,\psi_{-s}) \leftrightarrow (\beta \text{ or } \pm\beta) \leftrightarrow (\beta_s,\ldots,\beta_{-s} \text{ or } \pm\beta_s,\ldots), \tag{2.13a}$$

$$(\beta_m \text{ or } \pm\beta_m) \leftrightarrow \{\tau_\sigma(\gamma, (\beta_m \text{ or } \pm\beta_m)\,\hat{\sigma}_{s,m}(\gamma))\}, \tag{2.13b}$$

where β_m refers to integer spin, and $\pm\beta_m$, to half-odd-integer. [See eq. (1.12)].

We want next to integrate functionally over a space of functions β_m or of pairs $\pm\beta_m$. The elementary Gaussian integrals and generalized measures over a space of pairs $\pm\beta_m$ are briefly discussed in the Appendix.

We will assign a generalized measure $\mathcal{D}(\beta)$ to the field ψ. The kinematical independence of the translational modes and of spin, and the degeneracy associated with the latter, imply the factorization

$$\mathcal{D}(\beta) = \mathcal{D}(\beta_s)\mathcal{D}(\beta_{s-1})\dots\mathcal{D}(\beta_{-s}) \qquad (2.14)$$

where $\mathcal{D}(\beta_m)$ should more properly be written $\mathcal{D}((\beta_m, -\beta_m))$ in case of half-odd-integer spin. (Same for β .)

We now look for a formula of the kind

$$\langle (F(\psi^*, \psi))_+ \rangle_0 = \int \mathcal{D}(\beta^*, \beta)\, e^{iA(\beta^*, \beta)} F(\beta^*, \beta). \qquad (2.15)$$

Here the functional A involves β and β^* in a way which corresponds to scalar combinations of ψ and ψ^* (and of perhaps other fields). We presuppose for this functional appropriate ordering and counter terms, as needed. [E.g., $\beta^*(t+0)$ and $\beta(t-0)$; the counter terms arise in particular in relativistic interactions.]
The product $\mathcal{D}(\beta)e^{iA}$ will specify a generalized measure and a weight.

For F , we consider first polynomial functionals, with all the arbitrary functions restricted to nonoverlapping time intervals. In the integrand, ψ and ψ^* will be represented respectively by functions β and β^* in case of integral spin, and by $\pm\,\beta$ and $\pm\,\beta^*$ for half-odd-integer spin. We will determine in sec. 2C the corresponding rules of integration, in particular the choice of signs, and the correlation between spin and statistics. Once these rules are determined, one can envisage various limiting cases like equal times and nonpolynomial functionals.

We close this section with the example of a static electron quantized along the z-axis, in a magnetic field in the x-direction. The Green's function can be represented by

$$G_{jk}(t) = \int \mathcal{D}(b_+)\mathcal{D}(b_-) \exp\left(i A^{(o)}_{stat}\right) b_j(t)\, b_k^*(0) \tag{2.17a}$$
$$\times \exp\left(-ih \int_{-\infty}^{\infty} d\tau \left[b_+^*(\tau) b_-(\tau) + b_-^*(\tau) b_+(\tau)\right]\right),$$

$$A^{(o)}_{stat} = \sum_{j=\pm} \int_{-\infty}^{\infty} d\tau \, \tfrac{1}{2} i \left[b_j^*(\tau)\dot{b}_j - (b_j^*)^{\cdot} b_j\right]. \tag{2.17b}$$

Here $t > 0$; j, k = $\pm \frac{1}{2}$. The $\frac{1}{2}$ in $b_{\pm\frac{1}{2}}$ was suppressed. The limits $b^*(\tau+0)$ are presupposed.

We deal here essentially with a single-particle theory, so the ambiguities of sign are irrelevant. One can evaluate the history integral by expanding the exponential and by using the rules (1.25 - 26), or otherwise. The result,

$$\left(G_{jk}(t)\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(th) - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin(th), \tag{2.18}$$

is in agreement with the solution of the system

$$-\frac{1}{i}\partial_t \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = h \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{2.19}$$

2C. Commutation rules and homotopy.

In the functional framework one works only with ordinary commuting entities. The familiar examples of noncommutation must, however, have their counterparts in the properties of functionals, or of functional integrals. Now the only possible way to order the factors in an integrand is with respect to time (aside from effects due to fermions), and so the familiar relation $[p,q]=i^{-1}$ corresponds to

$$\lim_{\varepsilon\downarrow 0}\int \mathcal{D}(\eta)\, e^{\frac{1}{2}i\langle\dot\eta,\dot\eta\rangle} \exp\left[-i\int_0^t dt'\, V(\eta(t'))\right] \times\left[\dot\eta(\tau+\varepsilon)\eta(\tau)-\eta(\tau)\dot\eta(\tau-\varepsilon)-i^{-1}\right]=0. \tag{2.20}$$

(One assumes here $0<\tau<t$, $\mu = 1$, and that V does not contain derivatives.) This equation is derived heuristically in [12], but it can be adapted, and proved, for the Wiener integral.

This equation for either the path integral or the Wiener integral can be understood as follows. We consider the Hilbert space $\mathcal{H}_W$ defined by the inner product $\langle\dot\eta,\dot\eta\rangle$ and the condition $\eta(0)=0$. This is the space naturally associated with the weight $e^{\frac{1}{2}i\langle\dot\eta,\dot\eta\rangle}$. Every $\eta\in\mathcal{H}_W$ is continuous, but in general does not have a continuous derivative.

The path integral for an oscillator constructed in terms of the creation and the annihilation operators, $z^*(\tau)$ and $z(\tau)$, cf. see 1C, offers another example. The action (1.24) becomes

$$A=\int_0^t d\tau\left[\tfrac{1}{2}iz^*\dot z-\tfrac{1}{2}i(z^*)^{\cdot}z-\omega^2 z^*z\right]. \tag{2.21}$$

The natural space of integration is that of (complex) functions α satisfying

$$\int_{-\infty}^{\infty} d\tau\left(|\alpha|^2+|\alpha\dot\alpha|\right)<\infty,\quad\text{or}\quad \int_{-\infty}^{\infty} d\omega\left(|\omega|+1\right)|\tilde\alpha(\omega)\tilde\alpha(-\omega)|<\infty. \tag{2.22a,b}$$

Such functions are not necessarily continuous as the following example (due to D. Buchholz) shows. If

$$\tilde{\alpha}(\omega) = \left[(1+|\omega|) \log (1+|\omega|) \right]^{-1}, \qquad (2.23)$$

then (2.22b) is fulfilled, but

$$\alpha(0) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} d\omega \, \tilde{\alpha}(\omega) = \infty. \qquad (2.24)$$

The non-continuity of the α's is of course expected, in view of the commutation rule $[z, z^*] = 1$. Cf eq. (1.25), for which the Hamiltonian should not contain derivatives.

The situation with the anticommutativity of fields with half-odd-integer spin is of a different nature. Intuitively, there should be no difference between the two values

$$\beta_m(t+0, \underset{\sim}{u})\beta_m(t-0, \underset{\sim}{u}') \text{ and } \beta_m(t+0, \underset{\sim}{u}')\beta_m(t-0, \underset{\sim}{u}) \qquad (2.25)$$

(for "most" of the functions β_m), if $\underset{\sim}{u}$ and $\underset{\sim}{u}'$ are far apart. This appears particularly evident in the relativistic theory.

The change of sign between the two terms in (2.21) must then come from performing some kind of an exchange operation between the two terms. This possibility depends of course on the homotopic properties of the space of integration, specifically on its double connectivity, and was demonstrated in detail by Finkelstein and Rubinstein [16].

Their results provide a more general context for the following discussion of an exchange process.

Consider two wave functions $f^{(\pm w)}$ for a particle with $s = \frac{1}{2}$, $m = \frac{1}{2}$, at the time t. Assume that these functions are spherically symmetric around the respective points $\pm \mathbf{w}$ located in the x-y plane, and that the supports have a radius $< \|\mathbf{w}\|$. The annihilation parts $\psi_+(f^{(\pm w)})$ contribute to the integrand of the history integral the factor

$$F_o(\beta_+) = \int d^3\underset{\sim}{u}\, f^{(w)}(\underset{\sim}{u})\,\beta_+(t,\underset{\sim}{u}) \int d^3\underset{\sim}{u}'\, f^{(-w)}(\underset{\sim}{u}')\,\beta_+(t,\underset{\sim}{u}'). \qquad (2.22)$$

For reasons of covariance, the functions $f^{(\pm w)}$ as well as β_+ should be associated with sections. For $f^{(\pm w)}$, these are

$$\{\tau_v(\gamma, \pm f^{(w)}(\cdot)\hat{\sigma}_+(\gamma))\} \text{ and } \{\tau_v(\gamma, \pm f^{(-w)}(\cdot)\hat{\sigma}_+(\gamma))\}. \qquad (2.23)$$

We now exchange the two supports by rotating counterclockwise around the z-axis by π. This rotation to be denoted by γ_1, transforms the two sections respectively into [cf. eqs. (1.13)]

$$\{\tau_v(\gamma, \pm i f^{(-w)}(\cdot)\hat{\sigma}_+(\gamma))\} \text{ and } \{\tau_v(\gamma, \pm i f^{(w)}(\cdot)\hat{\sigma}_+(\gamma))\}. \qquad (2.24)$$

The transformation of the functional F_o, induced by γ_1, corresponds to taking the same signs (upper or lower) in both (2.23) and (2.24). Therefore

$$F_o(\beta_+) \to - F_o(\beta_+). \qquad (2.25)$$

Now, it is shown in [16] that the exchange of one pair of identical particles (as e.g. above) is homotopic to the exchange of any other such pair. We will consequently use (2.25) to describe the effect of any such exchange, and moreover, we impose the following requirement on the functionals: If parameters associated with one component of a spinor field are varied so as to invert the relative time ordering of two factors, then an exchange must be made when the two times are equal.

We will return to the problem of exchange in secs. 4B and 4C.

For several independent spinor components, the symmetry under the Klein group allows us to select anticommutation. (This is as in the Wightman theory [17].) Finally, for the two spinor fields ψ_m and ψ_m^* we will determine the commutation law by an algebraic argument, which can also be easily cast into the functional form.

Let us consider only one degree of freedom, that which corresponds to the (real) test function f, and let us set

$$z(\tau) = \int d^4u \, f(\underline{u}) \, \delta(u^0 - \tau) \, \psi_m(u), \tag{2.26a}$$

similarly $z^*(\tau) = \int d^4u \, f \ldots \psi_m^*$, where $\int d^3\underline{u} \, f^2 = 1$. (2.26b,c)

We assume that

$$(z(\tau))^2 = (z^*(\tau))^2 = 0, \tag{2.27}$$

but in place of (1.25), we assume more generally that

$$\lim_{\varepsilon \downarrow 0}\left[\pm z(\tau) z^*(\tau-\varepsilon) \pm z^*(\tau+\varepsilon) z(\tau)\right] = 1, \tag{2.28}$$

where the appropriate signs are to be determined. Upon multiplying both sides by $z^*(\tau + 2\varepsilon)$ and $z(\tau - 2\varepsilon)$ and assuming time-ordering, we get

$$\pm \lim_{\varepsilon \downarrow 0}\left[z^*(\tau+2\varepsilon) z(\tau) z^*(\tau-\varepsilon) z(\tau-2\varepsilon)\right] =$$

$$= \lim_{\varepsilon \downarrow 0}\left[z^*(\tau+2\varepsilon) z(\tau-2\varepsilon)\right], \tag{2.29}$$

from which the + sign in the first term of (2.28) follows directly. The second term can be similarly shown to be plus. The extension to more degrees of freedom is direct.

One can, conversely, assume (2.28) with plus signs, and a similar argument will lead to (2.27).

Finally, we summarize the rules for integrals over section-valued functions, or histories. The problem is primarily that of determing the signs of functionals.

First, on physical grounds we assume that a polynomial functional with an odd number of spinor factors integrates to zero.

Second, the relative signs of β_m and of β_m^* are to be chosen so that

$$\beta_m^*(\tau+0)\beta_m(\tau) \geq 0, \quad \beta_m(\tau)\beta_m^*(\tau-0) \geq 0. \tag{2.30}$$

Third, when time parameters are varied, the sign of a function β_m or of β_m^* should change only when relative time ordering is inverted, as explained above.

Fourth, an interaction can mix different spinor components, cf. (2.17). Then the form of the interaction presupposes relative signs of the components.

We should also like to recall here the possibility that contributions of different homotopy classes to the integral may enter with different phase factors [13 , 18] . This complication does not arise here, since the histories have the same phases as in a bose-field integral.

With regard to calculations, one can evaluate an integral for fermions in the same way as for bosons, provided compensating terms are supplied whenever time ordering is inverted. One can start with factors $\beta_m(\tau)\ \beta_m^*(\tau-0)$, which do not require compensating terms.

3. Aspects of the relativistic field

3A. The canonical structure for fields with spin.

In order to extend the preceding development to relativistic systems, we need the corresponding Lagrangians or actions. We will specify Lagrangians for the free fields by using the momentum space and by choosing a specific Lorentz frame and helicity states. For each $\underset{\sim}{p} \neq 0$ (the neglect of the line $\underset{\sim}{p} = 0$ is of no consequence), a classical, irreducible field η with spin s will have $2s+1$ components $\tilde{\eta}(\underset{\sim}{p})$, where m is the helicity, $m = \pm s$, $\pm(s-1)$, ... We will choose the following form for the action:

$$A^{(0)}(\eta) = \frac{1}{2}\int d^4p\,\left[(p^0)^2 - \underset{\sim}{p}^2 - \mu^2\right]\sum_{m=-s}^{s}\tilde{\eta}_m(p)\,\tilde{\eta}_{-m}(-p). \quad (3.1)$$

We assume here that $\mu > 0$, or that $\mu = 0$, $s = 0$. (Cf. also below.)

As far as we know, this obvious expression has not been explicitly given before. We should therefore like to make several comments concerning it.

a) The form (3.1) shows the kinematical independence of spin and the translational modes, when one uses the momentum variables. This fact allows the discussion of Part 2 to be easily adapted to the present case.

b) It is not at all obvious if the action (3.1) can be put into the form of an integral of a local Lagrangian over d^4x. In fact, only for a few low values of spin are the local Lagrangians available [19]. We will show in sec. 3B how one can relate (3.1) (taken twice) to the usual expression for the Dirac field.

(c) Let P_ν, $M_{\mu\nu}$ be the generators of the Lie algebra $\mathfrak{g}$ of the Poincaré group. Given a particle of mass $\mu>0$ and spin s, the orbit of the corresponding element of $\mathfrak{g}'$ is the manifold M defined by

$$P^\nu P_\nu = \mu^2, \qquad W^\nu W_\nu = \mu^2 s(s+1), \tag{3.2}$$

where $W^\nu = \frac{1}{2}\varepsilon^{\nu\mu\kappa\lambda} P_\mu M_{\kappa\lambda}$ [20]. Equation (1.6) then defines a symplectic structure on M. The group-invariant polarizations on M specify $\underset{\sim}{p}$ and m [in specific Lorentz frames, cf. (3.1)] as the independent variables [3].

(d) If $s = 0$, then m cannot vary, but otherwise nothing is changed. However, the cases $\mu = 0$, $s>0$ have to be considered as exceptional in all the approaches to quantum field theory. Here, in particular, p and m are no longer suitable independent variables. Indeed, if $\mu = 0$ and $s>0$ then the form (3.1) requires additional special tricks, as one knows from quantum electrodynamics.

One could try to proceed in these exceptional cases as follows. Determine a satisfactory set of (three) independent variables from a group-invariant polarization and a satisfactory form for the free-field action. Then the construction of the history integral should be possible.

(e) The canonical structure for fields, or for systems with an infinite number of degrees of freedom, is of course much less developed than the theory for a finite number (e.g. [10]). Discussions of the infinite case can be found e.g. in [21 - 24]. However, the fragment of the theory that is important for us, the passage from a Lagrangian to a Hamiltonian, can be done in various cases by the usual coordinate-dependent manipulations, e.g. as in [25].

The following points should be noted. First, the term $\dot{\eta}^2$ [or $(p^0)^2|\tilde{\eta}|^2$] in the Lagrangian will normally remain in the Hamiltonian. In view of the relation

$$0 = \lim_{\varepsilon \downarrow 0} \int \mathcal{D}(\eta) e^{iA} \left[\dot{\eta}(t+\varepsilon,\underset{\sim}{u})\eta(t,\underset{\sim}{v}) - \eta(t,\underset{\sim}{v})\dot{\eta}(t-\varepsilon,\underset{\sim}{u}) - \delta(\underset{\sim}{u}-\underset{\sim}{v})\right], \tag{3.3a}$$

one can represent $\dot{\eta}$ as follows,

$$\dot{\eta}(t,\underset{\sim}{u}) \rightarrow \delta/\delta\eta(\underset{\sim}{u}), \tag{3.3b}$$

when restriction to the time t is made. Second, the resulting Hamiltonian will in general contain an infinite constant associated with the vacuum, and this constant must be eliminated in order to have a meaningful expression.

(f) The occurrence of the manifold M, Eqs. (3.2), brings to mind the former attempts to construct quantized fields on homogeneous spaces of the Poincaré group [26, 27]. It is therefore good to keep in mind that the manifolds are used in rather different ways in those works and in the present one. In particular, in [27], the free-field creation and annihilation parts are, so to say, transferred to a larger manifold by integrating with a suitable kernel. Consequently, the dimension of the manifold no longer indicates the number of degrees of freedom of a particle.

3B The Dirac field, field strength renormalization, and non-standard analysis.

The presentation of the material of this section will be still more fragmentary than the presentation in Part 2. However, we have a certain optimism that the details could be filled without undue complications, and that these details would not alter our general conclusions.

Let us first take the case of a field with $s = \frac{1}{2}$ and two components $\varphi_\pm$, and with the action given by (3.1). The conclusions of sec. 2C tell us that

$$\int \mathcal{D}(\eta_\pm) \exp\left[i A^{(0)}(\eta_+, \eta_-)\right] \eta_+(x)\eta_-(y) = \pm i\varepsilon(x^0 - y^0)\Delta_c(x-y;\mu), \tag{3.4}$$

where $\varepsilon(\pm |a|) = \pm 1$ (for $a \neq 0$). The generalized measure $\mathcal{D}(\cdot)$ is over complex histories; $\eta_+ = \eta_-^*$. In view of the anticommulativity, a definite time ordering must be specified in the action, i.e. one must take

$$A^{(0)} = \int dt\, d^3\underset{\sim}{u}\; \eta_+(t \pm 0, \underset{\sim}{u}) \left[\overleftarrow{\partial_t}\overrightarrow{\partial_t} - \ldots\right] \eta_-(t \mp 0, \underset{\sim}{u}). \tag{3.5}$$

Then the sign $\pm$ before ε in (3.4) is determined by the choice of the upper or of the lower signs in (3.5)

We see that the histories $\eta_\pm$ correspond to an irreducible field $(\varphi_\pm)$ with $s = \frac{1}{2}$, satisfying only the Klein-Gordon equation:

$$\varphi_\pm(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 \underset{\sim}{k}}{(2\omega)^{\frac{1}{2}}} \left[a^*_\mp(-\underset{\sim}{k}) + a_\pm(\underset{\sim}{k}) \right] e^{i(\underset{\sim}{k}\underset{\sim}{x} - \omega t)}, \qquad (3.6)$$

where

$$[a_i(\underset{\sim}{k}),\ a^*_j(\underset{\sim}{l})] = \delta_{ij}\, \delta(\underset{\sim}{k} - \underset{\sim}{l}). \qquad (3.7)$$

The situation with the Dirac field ψ is somewhat more complicated, and we confine ourselves to brief remarks. There are four components, and two irreducible fields with $s = \frac{1}{2}$ are mixed in ψ. One would therefore like to isolate two hermitian conjugate pairs, $(\varphi_1, \varphi_1^*, \varphi_2, \varphi_2^*)$, in terms of which the components of ψ can be expressed. Then one can integrate with respect to the corresponding (complex) histories ξ_1 and ξ_2.

We recall [25] that the Dirac action, in terms of ψ, is

$$A_1^{(0)} = - \int d^4 p\, \bar{\psi}^{\sim}(-p)\, (\not{p} + \mu)\, \tilde{\psi}(p). \qquad (3.8)$$

This action does not agree in general with (3.1) However, one has $A = 0$ in both (3.1) and in (3.8) for free fields, and one can reproduce the usual results by integrating with e^{iA} as weight, A being as in (3.1) or as in (3.8). Only the correspondence between the variable functions and the different fields must be kept in mind. The difference between (3.1) and (3.8) becomes significant, however, when couplings are introduced.

The Dirac spinors can be used to construct various simple models of coupled fields. We note here in particular the gauge model, discussed on numerous occasions and recently in [28], with the Lagrangian

$$L(\psi, \varphi) = L^{(o)}(\psi) + L^{(o)}(\varphi) + ig \lim_{x_1, x_2 \to x} \times \left[\overline{\psi}(x_1)(\not\partial\varphi)(x_2)\psi(x) - F(x_1, x_2, x)\overline{\psi}(x_1)\psi(x)\right]. \tag{3.9}$$

The variables x, x_j are to have space-like separations. The function F is one whose singularity in the limit is to compensate the singularity of the product of fields. The $L^{(o)}$ are the free field Lagrangians, given in (3.8) and in (2.5) resp. An over-all additive constant, which would result from Wick ordering in $L^{(o)}$, is irrelevant and omitted.

In this and in various other models one finds the formal equal time anticommutators for the renormalized fields (with $\psi^* = \overline{\psi}\gamma^o$),

$$[\psi_\alpha(t, \underset{\sim}{x}), \psi^*_\beta(t, \underset{\sim}{y})]_+ = Z_2^{-1}\delta_{\alpha\beta}\delta(\underset{\sim}{x} - \underset{\sim}{y}), \quad Z_2 \approx 0. \tag{3.10}$$

The usual way of handling the situation is to require the fields to be distributions in both the time and the space variables. However, non-standard (n.s.) analysis provides a more general framework than the theory of distributions, and the usual manipulations, allowed by n.s. analysis but not in distribution theory, appear convenient here.

Non-standard analysis was discussed recently in connection with various physical problems [29,30]. In some of these applications one used the standard number system for space-time variables, and the extended (or n.s.) system for magnitudes of wave functions and the like. For the model in question, it appears convenient to introduce one infinite quantity, Z_2^{-1} of Eq.(3.10), and then Z_2 and also $Z_2^{1/2}$ are infinitesimals, not zero. On the other hand, the singularity of F in (3.9), and the distribution-like dependence on $\underset{\sim}{x}$ and $\underset{\sim}{y}$, can perhaps be best handled by the usual means.

For the gauge model we may now employ the unrenormalized fields ${}^u\psi$, ${}^u\psi^*$ of infinitesimal magnitude,

$$ {}^u\psi = Z_2^{\frac{1}{2}}\psi, \quad {}^u\psi^* = Z_2^{\frac{1}{2}}\psi^*, \tag{3.11a}$$

but satisfying

$$[{}^u\psi_\alpha(t,\underset{\sim}{x}),\ {}^u\psi^*_\beta(t,\underset{\sim}{y})]_+ = \delta_{\alpha\beta}\,\delta(\underset{\sim}{x}-\underset{\sim}{y}). \tag{3.11b}$$

If we now employ the action determined by (3.9) for a history integral, then the commutation rules will be determined by the time derivatives which occur in the action. The examples of Sec.2C then show us that we will recover the time-ordered functions of the unrenormalized fields,

$$\langle (F({}^u\psi,\ {}^u\psi^*,\ \varphi))_+\rangle_0 = \int \mathcal{D}(\zeta)\mathcal{D}(\zeta^*)\mathcal{D}(\eta)\, e^{iA} F(\zeta,\zeta^*,\eta). \tag{3.12}$$

Let us consider in particular the two-point function restricted to a fixed time, and with $\alpha=\beta$:

$$W_2 = \langle {}^{u}\psi_\alpha(t+0,\underset{\sim}{x})\,{}^{u}\psi_\alpha^*(t-0,\underset{\sim}{y})\rangle_0 = Z_2\langle \psi_\alpha(t+0,\underset{\sim}{x})\,\psi_\alpha^*(t-0,\underset{\sim}{y})\rangle_0$$

$$= Z_2\left(\langle \psi_\alpha^{(0)}(t+0,\underset{\sim}{x})\,\psi_\alpha^{(0)*}(t-0,\underset{\sim}{y})\rangle_0 + g^2\langle\cdots\rangle + \cdots\right) \tag{3.13a}$$

$$= \int \mathcal{D}(\zeta)\ldots e^{iA}\,\zeta_\alpha(t+0,\underset{\sim}{x})\,\zeta_\alpha^*(t-0,\underset{\sim}{y}). \tag{3.13b}$$

Familiar manipulations allow us to reduce the integral to one over the functions ζ_α, ζ_α^* at time t, which we denote by ξ, ξ^*. (Cf. [14] for the corresponding handling of the free scalar field.) Then we find the form

$$W_2 = \int \mathcal{D}(\xi)\,\mathcal{D}(\xi^*)\,e^{Q(\xi,\xi^*)}\,\xi(\underset{\sim}{x})\,\xi^*(\underset{\sim}{y}). \tag{3.14a}$$

In view of (3.13a), Q should have a bilinear part B which would yield the free-field function, and a remainder $O(g^2)$. Thus

$$W_2 = \int \mathcal{D}(\xi)\,\mathcal{D}(\xi^*)\,e^{-Z_2^{-1}B(\xi,\xi^*)+g^2(\ldots)+\ldots}\,\xi(\underset{\sim}{x})\,\xi^*(\underset{\sim}{y}) \tag{3.14b}$$

$$= Z_2\int \mathcal{D}(\xi)\,\mathcal{D}(\xi^*)\,e^{-B(\xi,\xi^*)+\ldots}\,\xi(\underset{\sim}{x})\,\xi^*(\underset{\sim}{y}). \tag{3.14c}$$

The point of interest here is that Z_2^{-1} should appear in the exponent when the restriction to a fixed time is made.

4. Attempt at a functional framework

4A. Stochastic quantum fields.

In the functional formalism as here discussed, one has to deal with ordinary (c-number) functionals, and the quantum fields are represented as factors of the functionals, i.e. multiplicatively. Their conventional interpretation, by way of operators (which in general do not commute) is quite extraneous to the present approach. We may observe two conspicuous shortcomings of the functional approach at this point. First, one should like to interpret the different entities which enter, in a way which would be more in line with the formalism as a whole. Second, the framework has to include a particle interpretation, and has to yield (at least in principle) the S-matrix.

Now, the history integral yields the time-ordered vacuum expectation values of fields. From these time-ordered functions one can get the S-matrix via the LSZ formalism [31] (or via a modification thereof, in case of infrared phenomena). A new interpretation of the formalism therefore is not absolutely necessary, even though it remains desirable.

One way to interpret the fields is by adapting the ideas of stochastic quantum mechanics, for the following reason. In stochastic quantum mechanics one represents the operators of position at any given time as random variables (i.e. measurable functions) on a measure space. In the functional formalism for fields, one has a similar action of fields.

There exist several approaches to stochastic quantum mechanics ([32,33] and references given there). That of [32] seems convenient for adaptation, and we give a few details for an interacting scalar field. One considers a state functional $\Psi(t,\xi)$, $\xi : R^3 \to R^1$, satisfying the (formal) Schrödinger equation,

$$-\frac{1}{i}\frac{\partial}{\partial t}\Psi = H\Psi = \frac{1}{2}\int d^3\underline{x}\left[-\left(\frac{\delta}{\delta\xi(\underline{x})}\right)^2 + (\nabla\xi)^2 + m^2\right]\Psi + H_{int}\Psi + (\text{ren.}) \qquad (4.1a)$$

$$= -\frac{1}{2}\int d^3\underline{x}\left(\frac{\delta}{\delta\xi(\underline{x})}\right)^2\Psi + H_1\Psi + (\text{ren.}), \qquad (4.1b)$$

where H_1 is assumed to be a multiplicative operator.

We next introduce the definitions,

$$\operatorname{div} F(\xi,\cdot) = \int d^3\underline{x}\,\frac{\delta}{\delta\xi(\underline{x})}F(\xi,\underline{x}), \qquad [\operatorname{grad} G(\xi)](\underline{x}) = \frac{\delta G}{\delta\xi(\underline{x})}, \qquad (4.2a,b)$$

$$\langle F_1(\xi,\cdot), F_2(\xi,\cdot)\rangle = \int d^3\underline{x}\; F_1^*(\xi,\underline{x})\,F_2(\xi,\underline{x}), \qquad (4.2c)$$

$$\Psi = \exp(R+iS), \quad u = \operatorname{grad} R, \quad v = \operatorname{grad} S. \qquad (4.2d)$$

Then, if one makes a suitable choice of the arbitrary constants in H_1 and in S, Eqs.(4.1) will be equivalent to Eqs.(3) of [32, § 15]:

$$\partial_t u = -\frac{1}{2}\operatorname{grad}\operatorname{div} v - \operatorname{grad}\langle v,u\rangle + (\text{ren.}), \qquad (4.3a)$$

$$\partial_t v = -\operatorname{grad} H_1 - v\operatorname{div} v + u\operatorname{div} u + \frac{1}{2}\operatorname{div}\operatorname{grad} u + (\text{ren.}). \qquad (4.3b)$$

The details here are not really important for us. The point is that the functionals u and v can be interpreted in stochastic terms, describing a kind of diffusion in a function space. One obtains in this way a space on which the (smeared) fields at a given time can be random variables.

(Cf. [32] for a description of the diffusion. One new feature in the present case is that the measure for a given time includes the absolute square $|\Psi^{(o)}|^2$ of the vacuum functional as a weight [34]. We may also note that the stochastic mechanics was discussed in the context of field theory in [35].)

These random variables, besides yielding mean values after integration, are also multiplicative operators on functionals Ψ:

$$(\varphi(t,\underset{\sim}{x})\Psi)(t,\xi) = \xi(\underset{\sim}{x})\Psi(t,\xi). \tag{4.4}$$

If one starts with the vacuum Ψ_0, an extensive space of functionals can be obtained by applying the fields. One can (presumably) then discuss the asymptotic limits of the functionals. But we forgo this here.

4B. A tentative set of axioms.

The work in quantum field theory of the last two decades has shown us several ways in which an axiomatic approach can be useful. We shall therefore present a preliminary axiom scheme for quantum field theory in functional form. We wish to emphasize the tentative nature of the axioms, which we do not state below with full precision. We expect that as more experience is gained, a more satisfactory set of axioms will suggest itself.

A quantized field will be assumed to have a definite spin, while its mass spectrum is to be determined by the totality of the interactions (and by the "bare mass"). This is the usual way of proceeding. The case $m = 0$, $s > 0$ is to be excluded, cf.Sec.3A.

Let us suppose for definiteness a single K-component complex field $\phi = (\phi_j)$ of spin s, with $\phi_j = \phi^*_{K-j}$. The following statements introduce the auxiliary entities needed for the theory.

(i) $\mathcal{T}$ is a linear space of functions, $R^4 \to C^1$, with a given topology. (Test functions)

(ii) $\mathcal{L}$ is a linear space of functions $R^4 \to R^1$, and $\mathcal{L}_q$ is the corresponding space of pairs $\{\eta, -\eta\}$. Explicitly, we can write

$$\mathcal{L}_q = (\mathcal{L} \setminus \{0\})/\{1,-1\}. \tag{4.5}$$

$\mathcal{L}^{XK}$, the K-fold Cartesian product, is the integration space if s is integral, and $\mathcal{L}_q^{XK}$ is the integration space of s is half-odd-integer.

In the following, $\mathcal{L}^{XK}$ will refer to integral spin, and $\mathcal{L}_q^{XK}$, to s half-odd-integer.

(iii) A is a functional (action functional),

$$A : \mathcal{L}^{XK} \to R^1 \quad \text{or} \quad A : \mathcal{L}_q^{XK} \to R^1. \tag{4.6}$$

A choice of temporal ordering in products is to be made.

(iv) D(·) is a generalized measure on $\mathcal{L}^{XK}$ or on $\mathcal{L}_q^{XK}$, invariant under translation, with the weight e^{iA}. (See Appendix).

(v) The above imply a space $\mathcal{F}$ of integrable functionals f, i.e. those for which

$$\int_{(\mathcal{L}^{XK} \text{ or } \mathcal{L}_q^{XK})} \mathcal{D}(\eta) e^{iA(\eta)} f(\eta) \tag{4.7}$$

is defined. In particular, $1 \in \mathcal{F}$ by definition of weight. The space $\mathcal{F}$ can be enlarged to $\mathcal{F}_{gen}$ by admitting distributions and/or nonstandard analysis.

(vi) If $f \in \mathcal{F}$ then one introduces the time-ordered vacuum expectation value of $f(\phi)$ by

$$\langle (f(\varphi))_+ \rangle_0 = \int \mathcal{D}(\eta) e^{iA(\eta)} f(\eta). \tag{4.8}$$

Furthermore, a field component at a point acts on a space $\mathcal{F}_0$ of functionals by multiplication. These functionals are associated with a given time, and we do not try to relate here $\mathcal{F}_0$ to $\mathcal{F}$. If $\Psi \in \mathcal{F}_0$ then

$$(\varphi_j(t,\underset{\sim}{x})\Psi)(\eta) = \eta_j(t,\underset{\sim}{x})\Psi(\eta). \tag{4.9}$$

The functional of η implied by the r.h.s. belongs to the extended space $\mathcal{F}_{o,gen}$.

Equation (4.8) allows us to relate the history integral to other approaches to field theory, while Eq.(4.9) allows the stochastic interpretation of ϕ(if an assumption about the measure at time t is made).

We come finally to the axioms proper.

(A1) (Integrability and continuity.) First of all, we require that

$$\mathcal{L} + i\mathcal{L} \subseteq \mathcal{T}'. \tag{4.10}$$

Next, let $f_1, \ldots f_n \in \mathcal{T}$ be such that their supports with reference to the time axis are nonoverlapping. Then

$$f(\eta) = \prod_{k=1}^{n} \langle f_k, \eta_{\ell_k} \rangle \Rightarrow f \in \mathcal{F}. \tag{4.11}$$

(In case of half-odd-integer s, this functional is defined on pairs, and its integration is to be carried out as indicated in the Appendix.) Furthermore, if the f_k are varied continuously in the given topology of $\mathcal{T}$, and the supports of the f_k with reference to the time axis remain nonoverlapping, then the history integral $\int D(\eta) e^{iA} f$ varies continuously.

(A2) (Relativistic covariance.) The history integral must transform covariantly under a transformation of the proper inhomogeneous Poincaré group, provided the transformation does not alter the relative time ordering of the functions f_k. (We forego a more explicit statement here.)

(A3) (Positivity at a fixed time.) Consider the limit in (4.11) where each f_k is of the form

$$f_k(t, \underset{\sim}{x}) = \delta(t - T - \varepsilon_k)\, g_k(0, \underset{\sim}{x}) \quad (g_k \in \mathcal{T}), \tag{4.12a}$$

so that $f \in \mathcal{F}_{gen}$ [cf.(v) above]. Let

$$\bar{f}_k(t,\underset{\sim}{x}) = \delta(t - T + \varepsilon_k)\, g_k^*(0,\underset{\sim}{x}), \quad \bar{f}(\eta) = \prod \langle \bar{f}_k, \eta_{\ell_k} \rangle. \tag{4.12b}$$

Consider the Hamiltonian operator H determined by the action A and normalized so that (cf. Sec. 3A)

$$\int \mathcal{D}(\eta)\, e^{iA(\eta)} (H \cdot 1)(\eta) = 0. \tag{4.13}$$

We now require that

$$\lim_{\varepsilon_k \downarrow 0,\ \forall k} \int \mathcal{D}(\eta)\, e^{iA(\eta)} \bar{f}(\eta^*) (Hf)(\eta) \geqslant 0, \tag{4.14}$$

$$\lim_{\varepsilon_k \downarrow 0,\ \forall k} \int \mathcal{D}(\eta)\, e^{iA(\eta)} \bar{f}(\eta^*) f(\eta) \geqslant 0. \tag{4.15}$$

These axioms are of course modeled on the familiar systems, such as in [17]. Note that we made no special assumption corresponding to locality. This property corresponds to the representation of the field by multiplicative functionals.

We give two simple consequences of the axioms. First, the arguments of [14] can be easily extended to establish the following.

Lemma 1. The free relativistic scalar field (hermitation or complex) satisfies the above axioms. One can take for $\mathfrak{T}$ the space $\mathcal{S}$, and for $\mathfrak{L}$, the Hilbert space described in [14].

Second, we summarize a part of the discussion of Sec. 2C as follows:

Lemma 2. Take two functions f_j and f_k of (4.11) which refer to the same component of the field (i.e. so that $\ell_j = \ell_k$). Let both approach the limit of sharp time support at t = T, one from above, one from below. If the relative time ordering of the functions is changed, then the product (in the limit) is to be multiplied by -1 for s half-odd-integer, and unchanged for s integral.

Note that his lemma and the positivity axiom (A3) delimit the choice of signs for the functionals f, if s is half-odd-integer. The ideas of Sec. 2C, if developed more fully, should suffice to determine the signs completely.

4C. Concluding remarks.

We conceived this work as providing a certain orientation point for further research in the functional approach to quantum field theory. We hope to have offered a glimpse at some of the possibilities and at some of the complications.

One natural question is the following: In what ways does the functional approach differ from the other familiar and better-established ones (i.e. the algebraic and that of Wightman)? We note here three ways. First, the basic objects of the theory are the time-ordered functions, rather than expectations of products of fields or of operators. One might therefore envisage a theory where the latter do not necessarily exist, but the time-ordered functions do. Second, in the functional framework the dynamics is defined by the action rather than by the Hamiltonian. Third, we have here a spin-statistics correlation also for nonrelativistic fields.

The action is indeed a fundamental quantity, which the two other approaches do not seem to accomodate easily. We may illustrate the significance of the action by recalling the existence of systems of field equations for which the time evolution can be described by a Hamiltonian, but for which there is no corresponding action. The equations have no local solutions [36].

With regard to spin, our work lends further support to the view (expressed in [16] and elsewhere) that the spin-statistics correlation should be topological in nature and that it should not depend on the details of a relativistic dynamics, or of the Poincaré group.

We may also emphasize at this point, that our presentation of spin-statistics (like that of other ideas) is tentative. There are obvious shortcomings: We had to consider separately the cases of the same component, of different components, and of conjugate fields ψ and ψ^*. Furthermore, the requirement of Sec. 4B, that there must be continuity if relative time ordering of fields is not changed, but continutiy is not imposed if time ordering is changed, is somewhat ad hoc.

There are two other negative conclusions that we may draw. First, the present work does not appear to contribute to the more concrete (i.e. calculational) applications of the functional formalism. Second, our system of axioms is rather awkward, with the multiply-connected spaces of integration and with the complications of non-standard analysis. This system does not appear at present useful for investigating the general structure of quantized fields.

Appendix: Gaussian integrals over a quotient space

We wish to adapt the Gaussian integral over a real Hilbert space $\mathcal{H}$ to an integral over the space [cf.(4.5)]

$$\mathcal{H}_q = (\mathcal{H} \setminus \{0\}) / \{1, -1\}. \tag{A.1}$$

For definiteness, we will take the isotropic integral with variance unity. Let μ be the corresponding cylinder set measure on $\mathcal{H}$.

We select an extension $\mathcal{H}'$ of $\mathcal{H}$ on which the extension μ' of μ has the value unity, and which is symmetric under reflection. Let $\mathcal{H}'_q$ be the corresponding extension of $\mathcal{H}_q$. We define a measure μ'_q on subsets $U_q \subseteq \mathcal{H}'_q$ by letting

$$\mu'_q(U_q) = \mu'(U) \quad [\text{if } \mu'(U) \text{ is defined}], \tag{A.2}$$

where U is the inverse image of U_q, with reference to the mapping implied in (A.1). If $\mu'(U)$ is not defined, we do not define $\mu'_q(U_q)$. One verifies that μ'_q is a measure. Integration on $\mathcal{H}'_q$ is therefore possible.

In so far as the original integral is often considered to be over $\mathcal{H}$ rather than $\mathcal{H}'$, we may consider the integral just constructed to be over $\mathcal{H}_q$ rather than over $\mathcal{H}'_q$.

There remain the problem of transcribing the usual functionals from $\mathcal{H}$ to $\mathcal{H}_q$. Consider first the linear functional $\langle a,x \rangle$ for $x \in \mathcal{H}$, and let

$$x = \lambda a + x^{\perp}, \quad \langle a, x^{\perp} \rangle = 0. \tag{A.3a}$$

Then, on H_q, the functional becomes

$$f(\{x,-x\}) = \lambda \|a\| \text{ or } -\lambda \|a\|, \tag{A.3b}$$

and in principle the sign can be chosen independently for each pair $\{x,-x\}$. A requirement of continuity would reduce the value of f to one of the two possibilities $\pm|\lambda|\ ||a||$, the chosen expression being valid for all x.

However, for applications envisaged in this paper, we must require that the integral of f vanish. We may rationalize this prescription by saying that there is no reason to prefer one sign over the other, and the average will give zero. This applies also to products of an odd number of such factors.

For a product of two such linear factors we set

$$g(\{x,-x\}) = \langle a, \pm x \rangle \langle b, \pm x \rangle = \pm \langle a, x \rangle \langle b, x \rangle. \tag{A.4}$$

The two signs in the middle are independent, but the over-all sign cannot change if continuity is required. Sometimes this sign can be determined by a positivity condition of Sec. 4B. The integral of g over $\mathcal{H}_q$ is easily evaluated by integrating over $\mathcal{H}$ and by using the usual rules.

If the integration space is complex (this is the case for spinor fields), then the following functional is of interest:

$$\langle a, x+iy\rangle \langle a, x-iy\rangle = \langle a, x\rangle^2 + \langle a, y\rangle^2 . \tag{A.5}$$

We assume that $a \in \mathcal{H}$ is real, for simplicity. The two terms integrate to 2 < a, a > by the above conventions. On the other hand, if we had x + iy twice on the l.h.s., then the two terms would give contributions which cancel.

Another aspect of integration that is relevant for us is that of factorizing the measure into a weight and a translationally invariant generalized measure. The concept of translational invariance is not <u>a priori</u> given, since $\mathcal{H}_q$ is not a linear space. However, by considering (for a given $y \in \mathcal{H}$) a sufficiently restricted set of functions, one can find a way to give a meaning to the equation

$$\mathcal{D}(\{x, -x\}) = \mathcal{D}(\{x, -x\} + \{y, -y\}) . \tag{A.6}$$

Admittedly, the way in question will be somewhat artificial.

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