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Asymptotic Behaviour at Exceptional Momenta

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Abstract

We develop a method for investigating the asymptotic behaviour of vertex functions at certain Minkowskian exceptional momenta. It is a direct generalization of earlier treatments of Euclidean exceptional momenta. It makes use of formal expansions in momentum space, closely related to light cone expansions in position space. Our expansions have to be performed in those channels which carry finite total momentum squared and admit (in A^4-theory) two-particle intermediate states.

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1. Introduction

It is an old aim of field theory to determine the asymptotic behaviour of Greensfunctions for large momenta. One can, for example, investigate their behaviour under scaling of all momenta

$$p \longrightarrow \lambda p \qquad \text{for} \qquad \lambda \longrightarrow \infty \; .$$

Dimensional reasons imply [1)]

$$\Gamma(\lambda p_1 \cdots \lambda p_n; m, g) = \lambda^{n-4}\, \Gamma(p_1 \cdots p_n; \frac{m}{\lambda}, g)\,, \tag{1.1}$$

i.e. the asymptotic behaviour for large momenta at fixed mass is expressed by that of vanishing mass at fixed momenta.

An important tool for investigating these limits is the technique of mass vertex insertions, which gives rise to the Callan-Symanzik (CS) equations [2,3)]

$$\begin{aligned}\left[m \frac{\partial}{\partial m} + \beta(g) \frac{\partial}{\partial g} - n\, \gamma(g)\right] \Gamma(p_1 \cdots p_n; m, g) &= \\ &= \Delta \Gamma(p_1 \cdots p_n; m, g)\end{aligned} \tag{1.2}$$

where $\Delta\Gamma$ is obtained from Γ by insertion of the soft [4)] mass vertex operator

$$\Delta = -2m^2\varphi(g)\,\Delta_0$$

and Δ_0 is one of Lowenstein's differential vertex operations [5]

$$\Delta_0 := \frac{i}{2}\int dx\, N_2[A^2](x) = \frac{i}{2}\, N_2\widetilde{[A^2]}(0) .$$

The parametric functions β , γ, and φ can be calculated in perturbation theory [3,6],

$$\beta(g) = \frac{3}{16\pi^2}\, g^2 + O(g^3) ,$$

$$\gamma(g) = \frac{1}{3\cdot 2^{10}\cdot\pi^4}\, g^2 + O(g^3) ,$$

$$\varphi(g) = 1 + O(g^2) .$$

These partial differential equations (PDEs) relate a change of the mass to a wavefunction and coupling constant renormalization apart from the extra term on the r.h.s. From (1.1) and (1.2) we have

$$[\mathcal{D} - n\gamma(g)]\,\Gamma(p_1\cdots p_n;\frac{m}{\lambda},g) = \Delta\,\Gamma(p_1\cdots p_n;\frac{m}{\lambda},g) \qquad (1.3)$$

with

$$\Delta\Gamma(p_1\cdots p_n;\frac{m}{\lambda},g) = -2\frac{m^2}{\lambda^2}\varphi(g)\,\Delta_0\Gamma(p_1\cdots p_n;\frac{m}{\lambda},g) \qquad (1.4)$$

and the differential operator

$$\mathcal{D} := -\lambda\frac{\partial}{\partial\lambda} + \beta(g)\frac{\partial}{\partial g} \quad .$$

In perturbation theory $\Gamma(p_1\cdots p_n;\frac{m}{\lambda},g)$ can (at least for Euclidean momenta) be expanded in a double power series in λ^{-1} and $\ln\lambda$. The formal sum obtained by discarding all terms which for large λ are smaller by powers of λ^{-1} than the leading ones is called the asymptotic form of Γ [3,7,8]. It can be most easily obtained from (1.3) if for large λ $\Delta\Gamma$ is smaller than Γ (in every order of perturbation theory) by a positive power of λ^{-1}, such that $\Delta\Gamma$ can be asymptotically neglected. Such momenta are called nonexceptional [7,8]. It is easy to give examples of nonexceptional momenta, whereas the question of which momenta sets are exceptional usually requires a somewhat deeper investigation. Euclidean momenta are nonexceptional if no (in A^4-theory in 4 dimensions even) partial sum of momenta vanishes. This can be verified by application of Weinberg's power counting theorem [9].

Conversely a momentum configuration is exceptional only if in (1.3) $\Delta\Gamma$ is not negligible in comparison to Γ. Then $\Gamma(p_1\cdots p_n;\frac{m}{\lambda},g)$ must develop a sufficiently strong infrared (IR) singularity such that the explicit factor of λ^{-2}, see (1.4), is not sufficient to suppress, for $\lambda \to \infty$, $\Delta\Gamma$ relative to Γ. For example, if $\Gamma(\cdots;\frac{m}{\lambda},g)$ diverges logarithmically in that limit then $\Delta_0\Gamma(\cdots;\frac{m}{\lambda},g)$ must diverge at least quadratically, or if Γ behaves like [10] $OL(\lambda^2)$ for large λ then $\Delta_0\Gamma$ must behave like $OL(\lambda^4)$ [11].

The asymptotic forms at nonexceptional and exceptional momenta are denoted by Γ_{as} and $\Gamma_{\underline{as}}$, respectively [7,8]. It was shown that the Γ_{as} are vertex functions of a theory with massless particles, the so-called preasymptotic theory [12]. They obey homogeneous CS equations (equivalent to the usual renormalization group equations [13] for a massless theory) which can be integrated by standard methods and yield the transformation laws [8]

$$\Gamma_{as}(p_1\cdots p_n;\tfrac{m}{\lambda},g) = \tag{1.5}$$
$$= a^n(g,\bar{g}(\lambda,g))\;\;\Gamma_{as}(p_1\cdots p_n;m,\bar{g}(\lambda,g))$$

with an effective coupling constant $\bar{g}(\lambda,g)$ implicitly defined by

$$\ln\lambda = \int_g^{\bar{g}(\lambda,g)} dg'\,\frac{1}{\beta(g')} \tag{1.6a}$$

or equivalently by

$$\lambda \frac{\partial}{\partial \lambda} \bar{g}(\lambda, g) = \beta(\bar{g}(\lambda, g)) \quad , \qquad \bar{g}(1, g) = g \tag{1.6b}$$

such that

$$\mathcal{D} \, \bar{g}(\lambda, g) = [-\lambda \frac{\partial}{\partial \lambda} + \beta(g) \frac{\partial}{\partial g}] \, \bar{g}(\lambda, g) = 0 \quad ,$$

and an effective wavefunction renormalization

$$\begin{aligned} a(g, \bar{g}(\lambda, g)) &= \exp \int_{\bar{g}(\lambda, g)}^{g} dg' \, \frac{\gamma(g')}{\beta(g')} \\ &= \exp - \int_{1}^{\lambda} \frac{d\lambda'}{\lambda'} \, \gamma(\bar{g}(\lambda', g)) \end{aligned} \tag{1.7}$$

such that

$$\begin{aligned} \mathcal{D} \, a(g, \bar{g}(\lambda, g)) &= \gamma(g) \, a(g, \bar{g}(\lambda, g)) \quad , \\ a(g, g') \, a(g', g'') &= a(g, g'') \quad , \qquad a(g, g) = 1 \ . \end{aligned} \tag{1.8}$$

Equation (1.5) effects a resummation of perturbation theory. Apart from a g-dependent factor the r.h.s. of (1.5) can be calculated by an expansion in terms of $\bar{g}(\lambda, g)$. If one knows from (1.6) that $\bar{g}(\lambda, g)$ approaches zero for $\lambda \longrightarrow \infty$ (which is the case in asymptotically free theories [14]) then $a^{n}(0, \bar{g}(\lambda, g))$.

$\cdot\, \Gamma_{as}(p_1 \cdots p_n; m, \bar{g}(\lambda, g))$ can be calculated to arbitrary accuracy. Otherwise one has to resort to assumptions outside of perturbation theory. For example one may assume [8] the existence of an eigenvalue g_∞ with

i) $\beta(g_\infty) = 0 \quad , \qquad \beta'(g_\infty) < 0$ (1.9a)

such that $\bar{g}(\lambda, g) \longrightarrow g_\infty$ for $\lambda \rightarrow \infty$,

ii) $\Gamma_{as}(\cdots ; m, g)$ left continuous at g_∞ , (1.9b)

iii) $\gamma(g)$ left continuous at g_∞ . (1.9c)

Under these conditions one finds (for nonexceptional momenta)

$$\Gamma(\lambda p_1 \cdots \lambda p_n; m, g) = \lambda^{4-n}\, \Gamma(p_1 \cdots p_n; \frac{m}{\lambda}, g) \simeq$$

$$\simeq \lambda^{4-n(1+\gamma(g_\infty))} \cdot \exp(-n\, r(\lambda, g)) \cdot \Gamma_{as}(p_1 \cdots p_n; m, g_\infty)$$

where

$$r(\lambda, g) := \int_1^\lambda \frac{d\lambda'}{\lambda'} \left[\gamma(\bar{g}(\lambda', g)) - \gamma(g_\infty) \right] = o(\ln \lambda) ,$$

i.e. the field A has the (anomalous) dimension

$$d = 1 + \gamma(g_\infty)$$

in the sense of Wilson [15].

The vertex functions of the preasymptotic theory are singular (in perturbation theory infinite) at exceptional momenta. This indicates that for such momenta the (true) asymptotic form $\Gamma_{\underline{as}}$ of Γ cannot be taken directly from the preasymptotic theory - one must determine it from the CS equation (1.3) without neglecting $\Delta\Gamma$. Actually, the $\Gamma_{\underline{as}}$ are certain IR finite parts of the Γ_{as} [16].

The ultimate aim is of course to calculate the asymptotic behaviour of physical amplitudes where usually only a few momentum invariants go to infinity (energies or some subenergies) while most of them stay fixed (masses, momentum transfers). Such cases can be realized as a one parameter limit if the appropriate momenta are "stretched along hyperboloids" in the sense

$$p(\lambda) = \lambda a + b + \lambda^{-1} c$$

with (1.10)

$$a^2 = c^2 = ab = bc = 0$$

such that

$$\left[p(\lambda)\right]^2 = 2\,ac + b^2$$

remains fixed for $\lambda \longrightarrow \infty$. With the choice $a = \frac{m}{2}(1,0,0,1)$, $b = (0\ ,b^1,\ b^2, 0)$, and $c = \frac{m}{2}(1,0,0,-1)$ such parametrizations are equivalent to the introduction of infinite momentum variables $p_+ := p^0 + p^3 = m\,\lambda$, $p_- := p^0 - p^3 = m\,\lambda^{-1}$, $p_\perp := (b^1, b^2)$ [17].
Dimensional analysis then yields

$$\Gamma(\{p(\lambda)\}\,; m, g) = \lambda^{4-n}\,\Gamma(\{a + \lambda^{-1}b + \lambda^{-2}c\}\,; \tfrac{m}{\lambda}, g)\,,$$

i.e. one has to study IR singularities near exceptional momenta [18].
We shall use such momenta only in section 3 and appendix B where we deal with a simplified version of deep inelastic scattering.

The outline of this paper is the following. In section 2 we review Symanzik's treatment of some vertex functions with Euclidean exceptional momenta [8]. In section 3 we repeat Christ, Hasslacher, and Mueller's [19] discussion of deep inelastic scattering. We give a version simplified to A^4-theory and scalar currents. In our terminology it is a mixed exceptional configuration containing both Euclidean (zero) and Minkowskian (essentially lightlike) momenta. We give both an x-space and a p-space discussion. By the direct momentum space analysis we circumvent the assumption of light cone dominance. Section 4 is devoted to the discussion of two configurations with lightlike exceptional momenta. We obtain results that are straightforward generalizations of Symanzik's corresponding Euclidean ones in the sense that his equations now have to be interpreted as (infinite dimensional) matrix equations. Section 5 contains a summary and the conclusions. In appendix A we recall the perturbation theoretical derivation of the Wilson expansion [20] and explain the notion of normal product (NP) expansions [21] in momentum space. Appendix B contains the derivation of formal expansions of vertex-functions involving lightlike momenta. This is the basis of our momentum space analyses in sections 3 and 4. Finally in appendix C some asymptotic estimates are derived which serve to simplify the CS equations.

2. Euclidean Exceptional Momenta

In this section we want to briefly describe the method for obtaining the asymptotic forms at Euclidean exceptional momenta. We take Symanzik's examples [8])

$$\Gamma((-p)p\,00;\tfrac{m}{\lambda},g) \qquad \text{with} \quad p^2<0 \tag{2.1}$$

$$\Gamma((-p)p\,p'(-p');\tfrac{m}{\lambda},g) \qquad \text{with} \quad p^2,p'^2,(p\pm p')^2<0 \tag{2.2}$$

$$\Gamma(p(-p),0;\tfrac{m}{\lambda},g) \qquad \text{with} \quad p^2<0 \tag{2.3}$$

where in the latter case the entry "0" denotes the insertion of a composite operator [21]) $B := \frac{1}{2} N_2 \left[A^2\right]$ with zero momentum.

The CS equation for the function (2.1) reads

$$\left[\mathcal{D}-4\gamma\right]\Gamma((-p)p\,00;\tfrac{m}{\lambda},g) = \Delta\Gamma((-p)p\,00;\tfrac{m}{\lambda},g)\ . \tag{2.4}$$

It is easily seen that Γ on the l.h.s. of (2.4) is logarithmically divergent for $\lambda \rightarrow \infty$. Namely, logarithms of λ arise from the self-energy and vertex corrections, as is also the case at nonexceptional momenta. But here one finds additional singularities. They originate from two-particle intermediate states with vanishing total momentum, cf. fig. 1a, such that the integrand behaves like $\sim\left(k^2-\frac{m^2}{\lambda^2}\right)^{-2}$ for small loop momentum K . On the other hand $\Delta_0 \Gamma$ is quadratically IR divergent. Such singularities arise e.g. from graphs with two-particle intermediate states with the vertex insertion in one of the connecting lines, cf. fig. 1b. Then the integrand is $\sim \left(K^2 - \frac{m^2}{\lambda^2} \right)^{-3}$ for small K . Hence $\Delta \Gamma$ cannot be neglected. The trick now is to extract from $\Delta \Gamma$ only those contributions which for large λ have the same growth properties as Γ itself. These parts are given by the first term in Zimmermann's [21] normal product (NP) expansion, the momentum space analog of Wilson's short distance (SD) expansion [20], cf. appendix A,

$$\Delta \Gamma ((-p) p\, 0 0 ; \tfrac{m}{\lambda}, g) =$$

$$= \Gamma ((-p) p\, 0 0 ; \tfrac{m}{\lambda}, g) \cdot \eta(g) \quad + \quad OL(\lambda^{-2}) \tag{2.5}$$

with

$$\eta(g) := \langle T\, B(0)\, \Delta\, \tilde{A}(0) \tilde{A}(0) \rangle^{PROP}$$

$$= \frac{1}{16\pi^2}\, g \quad + O(g^2) \quad . \tag{2.6}$$

From (2.4) and (2.5) one obtains the homogeneous asymptotic CS equation

$$[\mathcal{D} - 2\gamma_1 - \gamma_2]\, \Gamma_{as} ((-p) p\, 0 0 ; \tfrac{m}{\lambda}, g) = 0 \tag{2.7}$$

with

$$\gamma_1 := \gamma \quad , \quad \gamma_2 := 2\gamma_1 + \eta \quad . \tag{2.8}$$

It is equivalent to the transformation law

$$\Gamma_{\underline{as}}((-p)\,p\,0\,0;\ \tfrac{m}{\lambda},g) =$$
$$= a_1^2(g,\bar{g}(\lambda,g))\ a_2(g,\bar{g}(\lambda,g))\ \ \Gamma_{\underline{as}}((-p)\,p\,0\,0;\ m,\bar{g}(\lambda,g)) \qquad (2.9)$$

which differs from (1.5) insofar as two factors of a_1 are replaced by a_2 . The wavefunction renormalization constants a_1 and a_2 are related to γ_1 and γ_2 as in (1.7). Under assumptions analogous to (1.9) the transformation law (2.9) would give rise to a changed power behaviour

$$\lambda^{4-2(1+\gamma_1(g_\infty))-(2+\gamma_2(g_\infty))}$$

i.e. $2\,\gamma_1(g_\infty)$ is replaced by $\gamma_2(g_\infty) = 2\,\gamma_1(g_\infty) + \eta(g_\infty)$.

The second example of exceptional momenta, the function

$$\Gamma((-p)\,p\,p'(-p');\ \tfrac{m}{\lambda},g) \qquad \text{with} \qquad p^2,\ p'^2;\ (p\pm p')^2 < 0 \qquad (2.2)$$

is more complicated to deal with. After a similar NP expansion of $\Delta\Gamma$ one does not yet get factorization into an IR singular part times a (momentum independent) parametric function as in the former case. Here it is necessary to perform a second NP expansion in order to obtain the asymptotic CS equation

$$[\mathcal{D}-4\gamma_1]\ \Gamma_{\underline{as}}((-p)p\,p'(-p');\tfrac{m}{\lambda},g) =$$

$$= \Gamma_{\underline{as}}((-p)p\,00;\tfrac{m}{\lambda},g)\cdot i\,\gamma_{22}(g)\cdot\ \Gamma_{\underline{as}}(00\,p'(-p');\tfrac{m}{\lambda},g) \tag{2.10}$$

with

$$\gamma_{22}(g) := -i\ \langle T\, B(0)\, \Delta\, \tilde{B}(0)\rangle$$

$$= \frac{1}{16\pi^2} + O(g^2)\ . \tag{2.11}$$

This time we end up with an inhomogeneous PDE. It can be integrated since the λ-dependence of the vertex functions on the r.h.s. is known from (2.9). It yields the transformation law

$$\Gamma_{\underline{as}}((-p)p\,p'(-p');\tfrac{m}{\lambda},g) =$$

$$= a_1^4(g,\bar{g}(\lambda,g))\Big\{\Gamma_{\underline{as}}((-p)p\,p'(-p');m,\bar{g}(\lambda,g)) + \tag{2.12}$$

$$+\ i\,\Gamma_{\underline{as}}((-p)p\,00;m,\bar{g}(\lambda,g))\ a_{22}(g,\bar{g}(\lambda,g);\bar{g}(\lambda,g))\cdot$$

$$\cdot\,\Gamma_{\underline{as}}(00\,p'(-p');m,\bar{g}(\lambda,g))\Big\}$$

where another renormalization constant a_{22} is introduced,

$$a_{22}(g, g_1; g_2) = \int_{g_1}^{g} \frac{dg'}{\beta(g')} \, a_2(g', g_2) \, \gamma_{22}(g) \, a_2(g', g_2) \tag{2.13}$$

such that

$$\mathcal{D} \, a_{22} = a_2 \, \gamma_{22} \, a_2 \quad , \quad a_{22}(g_1, g_1; g_2) = 0 \, .$$

This constant is related to the subtractive renormalization of the vertexfunction of two B-operators, $\langle T \, B(0) \, \hat{B}(q) \rangle$.
In (2.12) the term proportional to a_{22} is a specific solution of the inhomogeneous PDE (2.10), the remaining solution of the corresponding homogeneous equation is adjusted such that for $\lambda = 1$ (2.12) reduces to an identity. In view of (2.9) and (2.13) equation (2.12) can be rewritten as

$$\Gamma_{\underline{as}}((-p)\,p\,p'(-p');\, \tfrac{m}{\lambda}, g) =$$

$$= a_1^4(g, \bar{g}(\lambda, g)) \, \Gamma_{\underline{as}}((-p)\,p\,p'(-p');\, m, \bar{g}(\lambda, g)) +$$

$$+ \, i \, \Gamma_{\underline{as}}((-p)\,p\,0\,0;\, \tfrac{m}{\lambda}, g) \;\; a_{22}(g, \bar{g}(\lambda, g); g) \;\; \Gamma_{\underline{as}}(0\,0\,p'(-p');\, \tfrac{m}{\lambda}, g) \, .$$

From (2.9), (2.12) and (1.5) one sees that at exceptional momenta the transformation laws of the asymptotic forms differ from the one at nonexceptional momenta which is the reason for the notation $\Gamma_{\underline{as}}$ instead of Γ_{as}. They then depend on the actual momentum configuration. This can be regarded as an alternative criterion of exceptionality. In general each exceptional configuration necessitates a distinct investigation [22].

The treatment of $\Gamma(p(-p), 0; \frac{m}{\lambda}, g)$ is closely related to that of $\Gamma((-p)p\,00; \frac{m}{\lambda}, g)$, the reason being that they are "adjoint" in the sense that in a NP expansion the latter is the coefficient function of the $\frac{1}{2}N_2[A^2]$-operator of the former [23]. The CS equation now reads

$$[\mathcal{D} - 2\gamma_1 + \gamma_2]\,\Gamma(p(-p), 0; \tfrac{m}{\lambda}, g) =$$

$$= \Delta\Gamma(p(-p), 0; \tfrac{m}{\lambda}, g) = \tag{2.14}$$

$$= i\,\gamma_{22}(g)\;\Gamma(00\,p(-p); \tfrac{m}{\lambda}, g) \quad + \quad O\!L(\lambda^{-2})$$

which entails for the asymptotic form the transformation law

$$\Gamma_{\underline{as}}(p(-p), 0; \tfrac{m}{\lambda}, g) =$$

$$= i\,a_{22}(g, \bar{g}(\lambda, g); g)\;\Gamma_{\underline{as}}(00\,p(-p); \tfrac{m}{\lambda}, g) \; + \tag{2.15}$$

$$+ \; a_1^2(g, \bar{g}(\lambda, g))\;\; a_2(\bar{g}(\lambda, g), g)\;\; \Gamma_{\underline{as}}(p(-p), 0; m, \bar{g}(\lambda, g)) .$$

The above method can also be applied to higher Euclidean vertex functions with more elementary fields A and/or composite operators $B = \frac{1}{2}N_2[A^2]$. The generalization is straightforward if two-particle intermediate states with zero total momentum arise in one channel only. If more than one even partial sum of momenta vanishes then a more detailed investigation has to be performed. Such configurations will in general give still more complicated transformation laws.

3. Deep Inelastic Scattering

From section 2 we know, in principle, how to deal with Euclidean exceptional momenta. An important contribution to the analysis of the more complicated Minkowskian configurations is due to Christ, Hasslacher, and Mueller [19]. Following them we now discuss the asymptotic behaviour in deep inelastic scattering. First we give an x-space version along their lines, simplified, however, to A^4-theory and scalar currents. Then we give a p-space version, which is not directly obtained from the former by Fourier transformation. The difference between the two methods is that CHM first use the CS equation and only then make light cone (LC) expansions [24], whereas we prefer to perform the appropriate expansion first and use the CS equations for the coefficient functions only afterwards. Our method is also applicable to more general configurations of exceptional momenta, as we will demonstrate in the following section.

The inclusive cross section of lepton hadron scattering is related by unitarity to the absorptive part of the forward Compton scattering amplitude

$$T(q,p) := \int dx\, e^{iqx} < T\, j(x)\, j(0)\, \tilde{A}(p)\, \hat{A}(-p) >^{PROP} . \qquad (3.1)$$

The Bjorken limit [25] is

$$q^2 \to -\infty$$

$$\nu := p \cdot q \to \infty$$

but $\omega := -\frac{2\nu}{q^2}$ and $p^2 = m^2$

fixed. CHM insert into (3.1) a formal LC expansion [24]

$$j(\tfrac{x}{2})\, j(-\tfrac{x}{2}) \simeq \sum_n C_n(x^2)\, x^{\mu_1} \dots x^{\mu_n}\, O_{\mu_1 \cdots \mu_n}(0)$$

where the symmetric traceless operators $O_{\mu_1 \cdots \mu_n}(0)$ have matrix elements

$$< T\, O_{\mu_1 \cdots \mu_n}(0)\, \tilde{A}(p)\, \tilde{A}(-p) >^{PROP} = \\ = [\, p_{\mu_1} \cdots p_{\mu_n} - \text{traces}\,]\; b_n\left(\frac{p^2}{m^2}, g\right) \qquad (3.2)$$

for any momentum p , and are normalized such that [26]

$$b_n(0, g) = 1 \; . \qquad (3.3)$$

Fourier transformation yields

$$\int dx\, e^{iqx}\, C_n(x^2)\, x^{\mu_1} \cdots x^{\mu_n} =$$

$$= \tilde{C}_n(q^2) \left(\frac{-2q}{q^2}\right)^{\mu_1} \cdots \left(\frac{-2q}{q^2}\right)^{\mu_n} \quad + \quad g^{\mu_i \mu_j}\text{-terms} ,$$

so that one ends up with

$$T(q,p) \simeq \sum_n \tilde{C}_n(q^2) \quad \omega^n \quad b_n\left(\frac{p^2}{m^2}, g\right) . \tag{3.4}$$

Proceeding in the same way with the r.h.s. of the CS equation for $T(q,p)$ they obtain

$$\Delta T(q,p) \simeq \sum_n \tilde{C}_n(q^2) \quad \omega^n \quad a_n\left(\frac{p^2}{m^2}, g\right) , \tag{3.5}$$

where the invariant functions [26] $a_n\left(\frac{p^2}{m^2}, g\right)$ arise in the forward matrix element of $O_{\mu_1 \cdots \mu_n}(0)$ and the mass vertex operator Δ,

$$\langle T\, O_{\mu_1 \cdots \mu_n}(0)\; \Delta\; \hat{A}(p)\, \hat{A}(-p) \rangle^{PROP} =$$

$$= [\, p_{\mu_1} \cdots p_{\mu_n} - \text{traces} \,]\; a_n\left(\frac{p^2}{m^2}, g\right) .$$

Comparison of coefficients of ω^n at $p^2 = 0$ yields, in view of the normalization conditions (3.3), homogeneous CS equations for $\tilde{C}_n(q^2)$ alone. These can be integrated and may give rise to power behaviour. The coefficients $b_n(1, g)$ for use of (3.4) on the mass shell must be computed separately.

In the above way of deriving asymptotic CS equations it is not clear a priori why the insertion of a LC expansion is asymptotically relevant. It seems to rest on the assumption that in the Bjorken limit the contributions from the strongest singularities of the operator product $j(x)\, j(0)$ actually are those from the LC. Furthermore the existence of a LC expansion (at least in the sense of an asymptotic expansion) has not yet been demonstrated in perturbation theory [27]. While we cannot improve the situation with respect to the second remark we believe to be able to give better arguments for the first one by directly investigating the large momentum (respectively small internal mass) behaviour of the relevant vertex functions.

We shall demonstrate our method not with the function $T(q,p)$ but simplify the algebra by considering the twice amputated connected Greensfunction

$$E(\lambda q, 0, p(\lambda); m, g) := \\ = \langle T\, N_0[A(0)\tilde{A}(\lambda q)]\, \tilde{A}(p(\lambda))\, \tilde{A}(-p(\lambda)) \rangle^{PROP'} \tag{3.6}$$

where $p(\lambda)$ is a momentum of the type (1.10),

$$p(\lambda) = \lambda \ell + \lambda^{-1} \ell'$$

with

$$\ell^2 = \ell'^2 = 0 \,, \quad 2\ell\ell' = M^2 \,,$$

and

$$q^2 < 0 \,, \quad -2\frac{q\ell}{q^2} = \omega \neq 0 \,.$$

$E(\lambda q, 0, p(\lambda); m, g)$ is a special case of the function $E(\lambda q, p_+(\lambda), p_-(\lambda); m, g)$ discussed in appendix B. With the above parametrization of the momenta $\lambda \rightarrow \infty$ corresponds to the Bjorken limit in deep inelastic scattering since both

$$\lambda^2 q^2$$

and

$$\nu(\lambda) = \lambda q \cdot p(\lambda) = \lambda^2 (q\ell + \lambda^{-2} q \ell')$$

become large with asymptotically fixed ratio

$$\omega(\lambda) = -2 \frac{\nu(\lambda)}{\lambda^2 q^2} \longrightarrow \omega$$

and the momentum $p(\lambda)$ is on the mass shell

$$[p(\lambda)]^2 = M^2 .$$

In appendix B we give arguments for what we call a "lightlike momentum (LLM) expansion". It is obtained by formal iteration of Zimmermann's identities [21]. At each step only contributions from two-particle operators need be kept. The others are smaller by powers of λ. From (B.18) we get an expansion, which is the analog of (3.4),

$$E(\lambda q, 0, p(\lambda); m, g) =$$

$$= \sum_{\substack{n \\ \text{even}}} C_{nn}(\lambda q, \lambda \ell; m, g) \cdot b_n\left(\frac{p^2}{m^2}, g\right) \quad + \; OL(\lambda^{-6}) \tag{3.7}$$

with

$$\begin{aligned} C_{nn}(q, \ell; m, g) &= \\ &= \frac{1}{n!} \, \partial_\beta^n \, \langle T N_0 [A(0) \tilde{A}(q)] \, \tilde{A}(\beta\ell) \, \tilde{A}(-\beta\ell) \rangle^{PROP'} \Big|_{\beta=0} \\ &= \frac{1}{n!} \, \partial_\beta^n \, E(q, 0, \beta\ell; m, g) \Big|_{\beta=0} \end{aligned} \tag{3.8}$$

and

$$b_n\left(\frac{p^2}{m^2}, g\right) = b_n\left(0, 1; \frac{p^2}{m^2}, 0, g\right) . \tag{3.9}$$

Since C_{nn} is of order n in (the components of) the light-like vector ℓ it is clear that it could be written in the form $\hat{C}_n(q^2) \cdot \omega^n$, but we prefer to use (3.8). It is well known [19] that the coefficient functions in deep inelastic scattering can be isolated by calculating Callan-Gross integrals [28], i.e. moments with respect to the variable $x = \omega^{-1}$.

We remark that in our direct momentum space analysis we did not refer to LC dominance. However, it was necessary to assume that the formal expansion obtained by infinite iteration of Zimmermann identities in the limit of large λ indeed contains all non negligible terms.

In contrast to CHM we can derive CS equations directly for the functions $C_{nn}(q, \ell ; m, g)$ since by (3.8) they are expressed in terms of derivatives of a Greensfunction. We again express the large momentum behaviour by that of small mass,

$$C_{nn}(\lambda q, \lambda \ell ; m, g) = \lambda^{-4} C_{nn}(q, \ell ; \tfrac{m}{\lambda}, g) \; . \tag{3.10}$$

The CS equations read

$$\mathcal{D} C_{nn}(q, \ell ; \tfrac{m}{\lambda}, g) = \Delta C_{nn}(q, \ell ; \tfrac{m}{\lambda}, g) \tag{3.11}$$

and with (C.9b) we obtain the homogeneous asymptotic equations

$$\begin{aligned} \left[\mathcal{D} + 2\gamma_1\right] C^{as}_{nn}(q, \ell ; \tfrac{m}{\lambda}, g) = \\ = \gamma_{2\,nn}\, C^{as}_{nn}(q, \ell ; \tfrac{m}{\lambda}, g) \, , \end{aligned} \tag{3.12}$$

where $\gamma_{2\,nn} := 2\gamma_1 + \eta_{nn}$ (3.13)

and, cf. (C.6), $\eta_{nn}(g) = \delta_{n0} \dfrac{1}{1+n} \dfrac{1}{16\pi^2} g + O(g^2)$.

They are equivalent to

$$\begin{aligned} C^{as}_{nn}(q, \ell ; \tfrac{m}{\lambda}, g) = \\ = a_1^2(\bar{g}(\lambda, g), g)\, a_2(g, \bar{g}(\lambda, g))_{nn}\, C^{as}_{nn}(q, \ell ; m, \bar{g}(\lambda, g)) \end{aligned} \tag{3.14}$$

where again $\alpha_{2\,nn}$ is defined from $\gamma_{2\,nn}$ as in (1.7).

We thus see that the asymptotic form of the Minkowskian exceptional function $E(q, 0, \beta \ell; \frac{m}{\lambda}, g)$ does not have a simple transformation law which relates its mass and coupling constant dependence only, as it is the case for the Euclidean analog $\Gamma((-q)q\,00; \frac{m}{\lambda}, g)$, cf. (2.9). But (3.14) implies that different derivatives with respect to β, cf. (3.8), in general transform distinctively.

The above discussion of asymptotic behaviour directly in momentum space can be generalized to functions like $T(q,p)$ which involve the product of two current operators. This, however, would require in appendix B the use of bilocal normal products of operators which themselves are local normal products. If spin is included the same considerations hold for the invariant structure functions.

4. Lightlike Exceptional Momenta

In this section we apply the method of LLM expansions to more complicated cases of Minkowskian exceptional momenta. We are able to derive the asymptotic form of a vertex function whenever only one lightlike momentum vector is involved, e.g. for the functions

$$E(\lambda q, \alpha\lambda \ell, \beta\lambda \ell; m, g) := \tag{4.1}$$
$$= \langle T N_0[A(0)\tilde{A}(\lambda q - \alpha\lambda\ell)]\, \tilde{A}((\alpha+\beta)\lambda\ell)\, \tilde{A}((\alpha-\beta)\lambda\ell)\rangle^{PROP'}$$

with $\quad q^2 < 0, \quad \ell^2 = 0$

and

$$F(\lambda p, \lambda\ell, \lambda p'; m, g) := \tag{4.2}$$
$$= \langle T N_0[A(0)\tilde{A}(\lambda p - \lambda\ell)]\, N_0[\tilde{A}(\lambda p' + \lambda\ell)\, \tilde{A}(-\lambda p' + \lambda\ell)\rangle^{PROP'}$$

with $\quad p^2,\ p'^2,\ (p \pm p')^2 < 0 \quad, \quad \ell^2 = 0 \quad .$

For dimensional reasons we have

$$E(\lambda q, \alpha\lambda\ell, \beta\lambda\ell; m, g) = \lambda^{-4}\, E(q, \alpha\ell, \beta\ell; \tfrac{m}{\lambda}, g)\ , \tag{4.1a}$$

$$F(\lambda p, \lambda\ell, \lambda p'; m, g) = \lambda^{-8}\, F(p, \ell, p'; \tfrac{m}{\lambda}, g)\ . \tag{4.2a}$$

In these more academic examples we confine ourselves to exactly lightlike momenta, but it should be clear from the previous section that momenta of the type (1.10) could be used, too.

The above functions are generalizations of previous ones in the sense that lightlike momenta replace zero ones. We therefore expect related results. Indeed, (4.1) and (4.2) are generalizations of (3.6) and (2.1) and of (2.2), respectively. Namely, for $\alpha = 0$, $\beta = 1$ we obtain from (4.1) the function $E(\lambda q, 0, \lambda \ell; m, g)$. It is equal to (3.6) except for the replacement $\lambda \ell \to \lambda \ell + \lambda^{-1} \ell'$. If both α and β are set equal to zero the function on the r.h.s. of (4.1a) reduces to $E(q, 0, 0; \frac{m}{\lambda}, g)$. Apart from amputation this equals the Euclidean exceptional vertex function $\Gamma((-q) q\ 0 0; \frac{m}{\lambda}, g)$ of (2.1). Similarly $F(p, 0, p'; \frac{m}{\lambda}, g)$ is related to $\Gamma((-p)\ p\ p'(-p'); \frac{m}{\lambda}, g)$, cf. (2.2).

In section 3 we already dealt with the special case $\alpha = 0$ in (4.1) which corresponds to forward scattering. The general case of nonforward scattering (with, however, $q^2 > 0$) is related to deep inelastic annihilation [29]. Its discussion is along the same lines. We first expand $E(\lambda q, \alpha \lambda \ell, \beta \lambda \ell; m, g)$ in terms of Euclidean exceptional functions by use of iterated Zimmermann identities. This yields the formal LLM expansion, cf. (B.16,18), which expresses E in terms of its derivatives at Euclidean exceptional momenta,

$$E(\lambda q, \alpha\lambda\ell, \beta\lambda\ell; m, g) = \\ = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \sum_{r=0}^{n} C_{rn}(\lambda q, \lambda\ell; m, g)\ \alpha^{n-r}\beta^{r} \tag{4.3}$$

with

$$C_{rn}(q, \ell; m, g) = \\ = \frac{1}{(n-r)!}\frac{1}{r!}\partial_\alpha^{n-r}\partial_\beta^{r} \langle TN_0[A(0)\tilde{A}(q)]\,\tilde{A}((\alpha+\beta)\ell)\tilde{A}((\alpha-\beta)\ell)\rangle^{PROP'}\Big|_{\alpha=\beta=0} \tag{4.4}$$

$$= \frac{1}{(n-r)!}\frac{1}{r!}\partial_\alpha^{n-r}\partial_\beta^{r}\, E(q, \alpha\ell, \beta\ell; m, g)\Big|_{\alpha=\beta=0} \quad .$$

From the CS equations

$$\mathcal{D}\, C_{rn}(q, \ell; \tfrac{m}{\lambda}, g) = \Delta C_{rn}(q, \ell; \tfrac{m}{\lambda}, g) \tag{4.5}$$

and an expansion of $\Delta C_{rn}(q, \ell; \frac{m}{\lambda}, g)$, cf. (C.9b), we obtain the system of asymptotic equations

$$[\mathcal{D} + 2\gamma_1]\, C_{rn}^{as}(q, \ell; \tfrac{m}{\lambda}, g) = \\ = \sum_{\substack{r \le r' \le n \\ \text{even}}} \gamma_{2\,rr'}\, C_{r'n}^{as}(q, \ell; \tfrac{m}{\lambda}, g) \tag{4.6}$$

with, cf. (C.5,6), $\gamma_{2\,rr'} := 2\gamma_1 + \eta_{r'r}$. (4.7)

Similar equations are implicitly contained in the paper by Mason [29]. At the level of the leading logarithm approximation, were only the lowest order terms in the parametric functions

$\beta(g)$, $\gamma(g)$, ... are kept, he constructs certain integral operators. They are equivalent to the PDEs (4.5) plus boundary conditions which are taken from lowest order graphs. Using these operators Mason is able to derive the Gribov-Lipatov reciprocity relation [30] in the pseudo-scalar theory. It connects the scaling functions of deep inelastic scattering and annihilation.

Equation (4.6) can be read as a homogeneous partial differential equation for infinite dimensional triangular matrices

$$C^{\underline{as}} = \left(C^{\underline{as}}_{rn} \right) \quad , \quad C^{\underline{as}}_{rn} = 0 \quad \text{for } r > n$$

$$\gamma_2 = \left(\gamma_{2\,rn} \right) \quad , \quad \gamma_{2\,rn} = 0 \quad \text{for } r > n \; .$$

The transformation law then is of the familiar form

$$C^{\underline{as}}(q, \ell\, ; \tfrac{m}{\lambda}, g) = \tag{4.8}$$

$$= a_1^2(\bar g(\lambda, g), g) \;\; a_2(g, \bar g(\lambda, g)) \;\; C^{\underline{as}}(q, \ell\, ; m, \bar g(\lambda, g))$$

where a_2 now has to be understood as a triangular matrix, too. It is the solution of the differential equation

$$\beta(g) \frac{\partial}{\partial g} \, a_2(g, g') = \gamma_2(g) \;\; a_2(g, g') \tag{4.9a}$$

normalized such that

$$a_2(g,g) = 1 \tag{4.9b}$$

and obeys the (matrix) multiplication law

$$a_2(g,g')\, a_2(g',g'') = a_2(g,g'') \ . \tag{4.9c}$$

It can be obtained by iteration from the integral equation

$$\begin{aligned} a_2(g,g_0) &= \delta a_2(g,g_0) + \\ &+ \int_{g_0}^{g} \frac{dg'}{\beta(g')}\, \delta a_2(g,g')\, \Delta\gamma_2(g')\, a_2(g',g_0) \end{aligned} \tag{4.10a}$$

where δa_2 is the diagonal of the matrix a_2 given by

$$\delta a_2(g,g_0) = \exp \int_{g_0}^{g} \frac{dg'}{\beta(g')}\, \delta\gamma_2(g') \tag{4.10b}$$

and

$$\Delta\gamma_2 = \gamma_2 - \delta\gamma_2 \tag{4.10c}$$

is a matrix with nonvanishing elements only above its diagonal. For the calculation of any matrix element $a_{2\,rn}$ - with r and n even, $r \leq n$ - only a finite number of iterations is necessary, namely $\frac{n-r}{2}$.

It is clear that (4.8) generalizes (3.14) and (2.9). The equations (3.14) are contained in the diagonal of the matrix equation (4.8) while (2.9) corresponds to the first term in the diagonal. The α_1 -factors are different in (2.9) and (4.8) because $\Gamma_{\underline{as}}((-p)p\,00;\frac{m}{\lambda},g)$ is totally amputated whereas $C^{\underline{as}}$ includes two external propagators, cf. (4.4).

If in (4.1) α is different from zero it may be set equal to one since the normalization of the lightlike vector ℓ is free. For fixed r the sum over n can be performed in (4.3), and one formally obtains

$$E(\lambda q,\lambda\ell,\beta\lambda\ell;m,g)=\sum_{\substack{r\geq 0\\ \text{even}}}\hat{C}_r(\lambda q,\lambda\ell;m,g)\;\beta^r \qquad (4.3')$$

with

$$\begin{aligned}\hat{C}_r(q,\ell;m,g) &= \sum_{\substack{n\geq r\\ \text{even}}} C_{rn}(q,\ell;m,g) = \\ &= \frac{1}{r!}\,\partial_\beta^r\,E(q,\ell,\beta\ell;m,g)\,.\end{aligned} \qquad (4.4')$$

From (4.5,6,8) one finds in the same way the differential equations

$$\mathcal{D}\;\hat{C}_r(q,\ell;\tfrac{m}{\lambda},g)=\Delta\,\hat{C}_r(q,\ell;\tfrac{m}{\lambda},g)\quad, \qquad (4.5')$$

$$[\mathcal{D}+2\gamma_1]\ \hat{C}_r^{\underline{as}}(q,\ell;\tfrac{m}{\lambda},g) = \sum_{\substack{r'\geq r\\ \text{even}}} \gamma_{2\,rr'}\ \hat{C}_{r'}^{\underline{as}}(q,\ell;\tfrac{m}{\lambda},g) \tag{4.6'}$$

and the transformation law

$$\begin{aligned}&\hat{C}^{\underline{as}}(q,\ell;\tfrac{m}{\lambda},g) = \\ &= a_1^2(\bar{g}(\lambda,g),g)\ a_2(g,\bar{g}(\lambda,g))\ \ \hat{C}^{\underline{as}}(q,\ell;\ m,\ \bar{g}(\lambda,g))\end{aligned} \tag{4.8'}$$

where now $\hat{C}^{\underline{as}} = \left(C_r^{\underline{as}}\right)$ is an infinite component vector which, in view of (4.4'), is formally obtained by summing the column vectors of the matrix $C^{\underline{as}}$.

In our second example of lightlike exceptional momenta (4.2) a direct LLM expansion, e.g. for the first N_0-product, turns out not to be useful since the formfactors of the composite operators depend on λ through the invariants $\lambda^2\ \frac{p'^2}{m^2}$ and $\lambda^2\ \frac{p'\cdot\ell}{m^2}$. If, however, in the CS equation

$$[\mathcal{D}+4\gamma_1]\ F(p,\ell,p';\tfrac{m}{\lambda},g) = \Delta F(p,\ell,p';\tfrac{m}{\lambda},g) \tag{4.11}$$

we perform two LLM expansions on the r.h.s. we obtain the asymptotic equation, cf. (C.21),

$$\begin{aligned}&[\mathcal{D}+4\gamma_1]\ F^{\underline{as}}(p,\ell,p';\tfrac{m}{\lambda},g) = \\ &= \sum_{\substack{n\geq r\geq 0\\ \text{even}}} \sum_{n'\geq r'\geq 0} C_{rn}^{\underline{as}}(p,\ell;\tfrac{m}{\lambda},g)\ \ (i\,\gamma_{22}(g))_{rr'}\ \ C_{r'n'}^{\underline{as}}(p',\ell;\tfrac{m}{\lambda},g).\end{aligned} \tag{4.12}$$

Here again the sums over n and n' can be performed. In terms of the vector $\hat{C}^{\underline{as}}$ and a symmetric matrix γ_{22} (4.12) then reads

$$[\mathcal{D} + 4\gamma_1]\, F^{\underline{as}}(p, \ell, p'; \tfrac{m}{\lambda}, g) = \tag{4.12'}$$
$$= \hat{C}^{\underline{as}\,T}(p, \ell; \tfrac{m}{\lambda}, g) \quad i\gamma_{22}(g) \quad \hat{C}^{\underline{as}}(p', \ell; \tfrac{m}{\lambda}, g) .$$

Using (4.8') we find the transformation law of the asymptotic form

$$F^{\underline{as}}(p, \ell, p'; \tfrac{m}{\lambda}, g) =$$
$$= a_1^4(\bar{g}(\lambda, g), g) \{ F^{\underline{as}}(p, \ell, p'; m, \bar{g}(\lambda, g)) + \tag{4.13}$$
$$+ i\, \hat{C}^{\underline{as}\,T}(p, \ell; m, \bar{g}(\lambda, g))\, a_{22}(g, \bar{g}(\lambda, g); \bar{g}(\lambda, g))\, \hat{C}^{\underline{as}}(p', \ell; m, \bar{g}(\lambda, g)) \} =$$
$$= a_1^4(\bar{g}(\lambda, g), g)\; F^{\underline{as}}(p, \ell, p'; m, \bar{g}(\lambda, g)) +$$
$$+ i\; \hat{C}^{\underline{as}\,T}(p, \ell; \tfrac{m}{\lambda}, g)\;\; a_{22}(g, \bar{g}(\lambda, g); g)\; \hat{C}^{\underline{as}}(p', \ell; \tfrac{m}{\lambda}, g)$$

where a symmetric matrix of renormalization constants has been introduced,

$$a_{22}(g, g_1; g_2) = \int_{g_1}^{g} \frac{dg'}{\beta(g')} \, a_2^T(g', g_2) \, \gamma_{22}(g') \, a_2(g', g_2) \,, \qquad (4.14)$$

i.e.

$$\beta(g) \frac{\partial}{\partial g} \, a_{22}(g, g_1; g_2) = a_2^T(g, g_2) \, \gamma_{22}(g) \, a_2(g, g_2) \,,$$

$$a_{22}(g, g; g_2) = 0 \quad .$$

(4.13) and (4.14) are obviously matrix generalizations of (2.12) and (2.13).

We close with the remark that the formal resummations in (4.4', 5', 6', 8', 12') indicate that the expansions in the throughgoing momentum might be superfluous from the beginning. We have, however, not succeeded in deriving (4.3', 6', 12') directly by expansions in the relative momentum variables only. Apparently this would require the use of normal products which are subtracted at a throughgoing lightlike momentum. But then asymptotic estimates which replace (A.7) are not available. They are necessary to single out the asymptotically leading contributions from two-particle intermediate states.

5. Summary and Conclusions

The Callan-Symanzik (CS) equations provide, in conjunction with dimensional arguments, for a tool of investigating large momenta limits. They give quick results, however, only in the nonexceptional cases where, loosely spoken, all momenta go to infinity far off the mass shells and all subenergies become large, too. With this understanding the physically interesting high energy limits are exceptional. It requires detailed analyses to see whether nevertheless any useful information can be gotten from the CS equations for these limits.

The asymptotic behaviour at exceptional momenta is related to the infrared singularities of a massless theory. In A^4-theory in four dimensions the singularities at Euclidean exceptional momenta originate from (the iteration of) two-particle intermediate states in those channels which have zero throughgoing momentum. We have reviewed how they can be extracted by the application of a normal product expansion in momentum space.

We generalized this idea to some Minkowskian exceptional situations where large lightlike momenta are involved. This led us to formal infinite normal product expansions, closely related to the light cone (LC) expansions in position space. In our special examples of (nearly) lightlike momenta (LLM) they again reduce asymptotically to the contributions from two-particle intermediate states. Assuming the applicability of such formal expansions we obtained asymptotic CS equations of the same form as in the corresponding Euclidean cases, with now, however, some expressions to be understood as infinite dimensional matrices or vectors.

With this direct momentum space analysis we were able to rederive Christ, Hasslacher, and Mueller's results on deep inelastic scattering in a simplified model. We did not use LC dominance in position space. Instead we had to assume that the formal LLM expansion actually carries the asymptotically leading behaviour.

Maybe our method can be generalized to other physically interesting cases. But it is hard to believe that these momentum space expansions are adequate for more than a few situations. Probably it is necessary to find expansions (or integral transforms) in different variables [31] depending on the process under consideration. In section 4 we already encountered the possibility that for the asymptotically leading terms our expansions could be partly resummed. This indicates the relevance of operators which are different from Zimmermann's composite fields.

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Appendix

A. Normal Product Expansions

In this appendix we review the perturbation theoretical derivation of the small distance (SD) or Wilson expansion and explain what is meant by the notion of a Zimmermann or normal product (NP) expansion in momentum space. The starting point of the derivation of the SD expansion are the Zimmermann identities which relate bilocal normal products of different degree [21]. They read in operator form
(a, b even, $a > b \geq 0$)

$$N_b[A(x+\xi)A(x-\xi)] = \sum_{\{\mu\}}{}_b^a \; G^{\{\mu\}}_{(b)}(\xi) \; B^{(a)}_{\{\mu\}}(x) + \\ + N_a[A(x+\xi)A(x-\xi)] \quad . \tag{A.1}$$

The composite operators and corresponding coefficient functions are defined, respectively, by [32]

$$B^{(a)}_{\{\mu\}}(x) := \frac{(-i)^{\sum \#(\mu)_j}}{m!} \; N_a[A_{(\mu)_1} \cdots A_{(\mu)_m}](x) \\ \text{for} \quad a \geq m + \sum \#(\mu)_j \tag{A.2}$$

$$G^{\{\mu\}}_{(b)}(\xi) := \frac{1}{\prod \#(\mu)_j!} \langle T N_b[A(\xi)A(-\xi)] \, \hat{A}^{(\mu)_1}(0) \cdots \hat{A}^{(\mu)_m}(0) \rangle^{PROP'} \tag{A.3}$$

with the notation

$$A_{(\mu)_j}(x) = \partial_{(\mu)_j} A(x) = \left(\frac{\partial}{\partial x}\right)_{(\mu)_j} A(x)$$

$$\tilde{A}^{(\mu)_j}(p) = \partial^{(\mu)_j} \tilde{A}(p) = \left(\frac{\partial}{\partial p}\right)^{(\mu)_j} \tilde{A}(p)$$

$$(\mu)_j = \mu_{j_1} \cdots \mu_{j m(j)} \quad , \quad m(j) = \#(\mu)_j \geqslant 0 \tag{A.4}$$

$$\partial_{(\mu)_j} = \partial_{\mu_{j_1}} \cdots \partial_{\mu_{j m(j)}} \quad , \quad \partial_{(\mu)_j} = 1 \quad \text{if} \quad m(j) = 0$$

$$\{\mu\} = (\mu)_1 \cdots (\mu)_m \quad , \quad m \geqslant 2$$

The summation in (A.1) is over all sets $\{\mu\}$ with $b < m + \sum \#(\mu)_j \leq a$. Parity and Bose statistics restrict m and $\sum \#(\mu)_j$ to be even in A^4-theory. This is also why we need consider normal products of even degree only. The Zimmermann identity (A.1) relates a bilocal operator of degree b to one of higher degree a plus a sum of local composite operators multiplied by c-number coefficient functions.

The bilocal NPs are defined through the matrix elements of their Fourier transforms, e.g.

$$\langle T N_a [A(x+\xi) A(x-\xi)] \tilde{A}(q_1) \cdots \tilde{A}(q_n) \rangle =$$

$$= \int \frac{dk}{(2\pi)^4} e^{-i2k\xi} \langle T N_a [\tilde{A}(\tfrac{p}{2}+k) \tilde{A}(\tfrac{p}{2}-k)] A(0) \tilde{A}(q_2) \cdots \tilde{A}(q_n) \rangle , \tag{A.5}$$

$$p = -\sum_{i=1}^{n} q_i \quad .$$

$$\langle T N_a [\tilde{A}(\tfrac{p}{2}+k)\tilde{A}(\tfrac{p}{2}-k)] A(0)\tilde{A}(q_2)\cdots\tilde{A}(q_n)\rangle = \tag{A.6}$$
$$= \sum_{\Delta} \frac{1}{\gamma(\Delta)} \int \frac{dk_1}{(2\pi)^4}\cdots\frac{dk_s}{(2\pi)^4} R_{\Delta}^{(a)}(\tfrac{p}{2}+k, \tfrac{p}{2}-k; k_1\cdots k_s; q_1\cdots q_n)$$

is the appropriately subtracted integrand integrated over all internal loop momenta $k_1 \cdots k_s$ (for details we refer to Zimmermann's original papers [21]). The subtractions made in $R_{\Delta}^{(a)}$ imply the large momentum behaviour

$$\langle T N_a [\tilde{A}(\tfrac{p}{2}+\lambda k)\tilde{A}(\tfrac{p}{2}-\lambda k)] A(0)\tilde{A}(q_2)\cdots\tilde{A}(q_n)\rangle^{\text{PROP}'} = \tag{A.7}$$
$$= O_L(\lambda^{-a-4}) \qquad \text{for } \lambda\to\infty,\; k^2<0$$

at least for Euclidean momenta. The same estimate holds if a mass vertex Δ is inserted in (A.7).

Setting a = b + 2 and iterating (A.1) one obtains an expansion

$$N_0[A(x+\xi)A(x-\xi)] = \sum_{\{\mu\}}{}_0^{a} G^{\{\mu\}}(\xi)\, B_{\{\mu\}}(x) +$$
$$+ N_a[A(x+\xi)A(x-\xi)] \tag{A.8}$$

in terms of minimally subtracted composite operators and corresponding coefficient functions

$$B_{\{\mu\}}(x) \quad := \quad B^{(a)}_{\{\mu\}}(x) \qquad \text{for} \quad a = m + \Sigma\#(\mu)_j \tag{A.9}$$

$$G^{\{\mu\}}(\xi) \quad := \quad G^{\{\mu\}}_{(b)}(\xi) \qquad \text{for} \quad b = m + \Sigma\#(\mu)_j - 2 \tag{A.10}$$

(A.8) is not yet an asymptotic expansion since for $\xi \to 0$ the remainder approaches $N_a[A^2](x)$ which in general does not vanish. This, however, can be remedied by introducing the so-called M-products [21])

$$M_a[A(x+\xi)A(x-\xi)] = (1 - t^{a-2}_{\xi})\ N_a[A(x+\xi)A(x-\xi)] \tag{A.11}$$

which have the small distance behaviour

$$M_a[A(x+\varrho\eta)A(x-\varrho\eta)] = o(\varrho^{a-2}) \tag{A.12}$$

$$\text{for} \quad \varrho \to 0 \;, \quad \eta^2 \neq 0 \;, \quad a > 2 \;.$$

In terms of them (A.8) can be rewritten as

$$\begin{aligned} M_0[A(x+\xi)A(x-\xi)] &= \sum_{\{\mu\}\,0}^{a} H^{\{\mu\}}(\xi)\ B_{\{\mu\}}(x)\ + \\ &\quad + M_a[A(x+\xi)A(x-\xi)] \end{aligned} \tag{A.13}$$

with modified coefficient functions [21]

$$H^{\{\mu\}}(\xi) = \langle T\, M_b [A(\xi)\, A(-\xi)]\, \tilde{A}^{(\mu)_1}(0) \cdots \tilde{A}^{(\mu)_m}(0) \rangle^{PROP}$$

$$b = m + \Sigma \#(\mu)_i - 2 \quad . \tag{A.14}$$

In view of (A.12) equation (A.13) is an asymptotic SD expansion.

Since we are interested in the asymptotic behaviour for large momenta we take the Fourier transform of matrix elements of (A.8),

$$\langle T\, N_0 [\tilde{A}(\tfrac{p}{2}+k)\tilde{A}(\tfrac{p}{2}-k)]\, A(0)\, \tilde{A}(q_2) \cdots \tilde{A}(q_n) \rangle^{PROP'} =$$

$$= \sum_{\{\mu\}}{}_0^a\ \tilde{G}^{\{\mu\}}(k)\ \langle T\, B_{\{\mu\}}(0)\, \tilde{A}(q_1)\, \tilde{A}(q_2) \cdots \tilde{A}(q_n) \rangle^{PROP} + \tag{A.15}$$

$$+ \langle T N_a [\tilde{A}(\tfrac{p}{2}+k)\tilde{A}(\tfrac{p}{2}-k)]\, A(0)\tilde{A}(q_2) \cdots \tilde{A}(q_n) \rangle^{PROP'} .$$

In view of (A.7) this is an asymptotic expansion for large spacelike k. As distinguished from the Wilson expansion in x-space we call it Zimmermann or NP expansion. It is not identical to the Fourier transform of the SD expansion. The Riemann-Lebesgue lemma [33] relates the large-k-behaviour to the strongest singularities in ξ , not necessarily to the small- ξ -behaviour. However, according to (A.11)

$M_a[A(x+\xi)\, A(x-\xi)]$ differs (for $a \geqslant 2$) from

$N_a[A(x+\xi)\, A(x-\xi)]$ by a polynomial in ξ only.

Therefore the Fourier transform of the former differs from

$N_a[\tilde{A}(\frac{p}{2}+k)\,\tilde{A}(\frac{p}{2}-k)]$ by terms which are proportional to derivatives of a $\delta(k)$ -function. We stress that for momentum space considerations the expansion (A.15) is the adequate one.

B. Formal Expansions at Lightlike Momenta

We first recall how in position space the light cone (LC) expansion is related to the SD expansion. The main topic of this appendix is, however, to find a similar formal expansion in momentum space, which we call "lightlike momentum (LLM) expansion".

The LC expansion is formally derived [24] from the SD expansion (A.13) by reordering the terms according to the singularities of the coefficient functions $H^{\{r\}}(\xi)$ on the light cone $\xi^2 = 0$. This cannot be done rigorously since "in principle it is conceivable that none of the leading terms of the Wilson expansion carries the leading light cone singularity" [27]. In a SD expansion up to any finite degree the remainder may be more singular on the LC than the other terms.

In the following we want to make use of the NP expansion (A.15) in the case when the relative momentum k becomes large in spacelike region and the momenta q_i become large, too. This seems to be possible if the latter are essentially lightlike and parallel, such that their components become large but the invariants $q_i \cdot q_j$ stay fixed.

We are interested in the large momentum behaviour of a function which is used in sections 3 and 4,

$$E(\lambda q, p_+(\lambda), p_-(\lambda); m, g) :=$$
$$= \langle T N_0 [A(0) \hat{A}(p_2(\lambda))] \hat{A}(p_3(\lambda)) \hat{A}(p_4(\lambda)) \rangle^{PROP'} \tag{B.1a}$$

where

$$p_{1,2}(\lambda) = \mp \lambda q - p_+(\lambda)$$
$$p_{3,4}(\lambda) = p_+(\lambda) \pm p_-(\lambda) \tag{B.1b}$$

and $p_\pm(\lambda)$ are momenta of the type (1.10),

$$p_+(\lambda) = \alpha_+ (\lambda \ell - \lambda^{-1} \ell')$$
$$p_-(\lambda) = \alpha_- (\lambda \ell + \lambda^{-1} \ell') \tag{B.1c}$$

with

$$\ell^2 = \ell'^2 = 0 , \quad (\alpha_-^2 - \alpha_+^2) 2 \ell \ell' = M^2 , \quad q^2 < 0 .$$

The mass squares

$$p_3^2 = p_4^2 = M^2$$

and the momentum transfer

$$t = (p_3 + p_4)^2 = -8\,\alpha_+^2\,\ell\ell'$$

stay fixed in the limit $\lambda \to \infty$. The NP expansion (A.15) reads in this special case

$$E = \sum_{\{\sigma\}}{}^{\circ\,\alpha} \; \tilde{G}^{\{\sigma\}}(\lambda q) \langle T B_{\{\sigma\}}(0)\,\tilde{A}(p_3(\lambda))\,\tilde{A}(p_4(\lambda))\rangle^{PROP} + \tag{B.2}$$

$$+ \langle T N_a [A(0)\,\tilde{A}(p_2(\lambda))]\,\tilde{A}(p_3(\lambda))\,\tilde{A}(p_4(\lambda))\rangle^{PROP'}$$

with

$$\tilde{G}^{\{\sigma\}}(\lambda q) = \frac{1}{\prod \#(\sigma)_j!} \langle T N_n [A(0)\tilde{A}(\lambda q)]\; \tilde{A}^{(\sigma)_1}(0) \cdots \tilde{A}^{(\sigma)_s}(0)\rangle^{PROP'}$$

$$n = s + \sum \#(\sigma)_j - 2 \tag{B.3}$$

It can be seen that asymptotically only the operators which are composed of two elementary fields contribute to the sum in (B.2). This comes about as follows. From (A.7) we know that the coefficient functions have the asymptotic behaviour

$$\tilde{G}^{\{\sigma\}}(\lambda q) = OL(\lambda^{-n-4}) \quad . \tag{B.4}$$

The matrix elements of the composite operators in (B.2) obtain contributions both from the trivial and from non-trivial diagrams (recall PROP = PROP' + TRIV). The normalization conditions [21,34] of the NPs imply that the nontrivial parts

$$< T\, B_{\{\sigma\}}(0)\, \tilde{A}(\varkappa\, p_3(\lambda))\, \tilde{A}(\varkappa\, p_4(\lambda)) >^{PROP'} \tag{B.5}$$

vanish like $O(\varkappa^{n+2})$ for small $\varkappa$ since they must be even in $\varkappa$ and all terms up to order n have been subtracted. Now in the tensor decomposition of (B.5) the $\varkappa$-dependence can arise from explicit factors $(\varkappa \cdot p_{3,4}(\lambda))_\sigma$ and from the invariants $\varkappa^2 M^2$, $\varkappa^2 t$. Since the latter are λ-independent the maximal power of λ's, namely $\sum \#(\sigma)_j = n+2-s$, arises only if all indices are generated by the large light-like vector $\lambda \ell$. So we obtain

$$< T\, B_{\{\sigma\}}(0)\, \tilde{A}(p_3(\lambda))\, \tilde{A}(p_4(\lambda)) >^{PROP'} = \delta_{s2}\, \lambda^n\, \ell_{\{\sigma\}}\, f \;\; + O(\lambda^{b-2}) \tag{B.6}$$

where we introduced a shorthand notation similar to that in (A.4)

$$\ell_{\{\sigma\}} = \ell_{(\sigma)_1 \cdots (\sigma)_s} = \ell_{\sigma_{11}}\, \ell_{\sigma_{12}} \cdots\; \ell_{\sigma_{s1}} \cdots \tag{B.7}$$

The "formfactor" f depends on $\alpha_\pm$, M , and t. Clearly the contribution from the trivial diagram has the same λ- and ℓ-dependence as in (B.6). Combining (B.4) and (B.6) we see that the sum in (B.2) reduces asymptotically to contributions from operators which are composed of two elementary fields as was stated above.

Furthermore only the symmetric and traceless part of their matrix elements is needed which is proportional to the large lightlike momentum $\lambda \ell$.

At this point it is convenient to reparametrize the relevant set of operators in terms of total and relative momentum variables,

$$O_{(\varsigma)_-}(x) \;:=\; \frac{1}{2}\left(-\frac{i}{2}\right)^{r} N_{r+2}\,[A\,\overleftrightarrow{\partial}_{(\varsigma)_-}A\,](x) \tag{B.8}$$

with $\quad r = \#(\varsigma)_-$.

According to Lowenstein's differentiation rule [35] we have

$$\left(\frac{-i}{2}\right)^{n-r} \partial_{(\varsigma)_+} O_{(\varsigma)_-}(x) = \frac{1}{2}\left(\frac{-i}{2}\right)^{n} N_{n+2}\left[\partial_{(\varsigma)_+}(A\,\overleftrightarrow{\partial}_{(\varsigma)_-}A)\right](x) \tag{B.9}$$

with $\quad \#(\varsigma)_- = r \;, \quad \#(\varsigma)_+ = n-r$

The corresponding coefficient functions are

$$C_{rn}^{(\varsigma)_+(\varsigma)_-}(\lambda q; m, g) = \left\langle T\, N_n[\,A(0)\tilde{A}(\lambda q)]\; X_{rn}^{(\varsigma)_+(\varsigma)_-} \right\rangle^{\mathrm{prop}'} \tag{B.10}$$

with

$$X_{rn}^{(\varsigma)_+(\varsigma)_-} \equiv \frac{1}{(n-r)!}\,\frac{1}{r!}\;\partial^{(\varsigma)_+}\left(\hat{A}(0)\,\overleftrightarrow{\partial}^{(\varsigma)_-}\hat{A}(0)\right) . \tag{B.11}$$

The matrix elements of the operators (B.9) are

$$\left(-\frac{i}{2}\right)^{n-r} \langle T\, \partial_{(\mathfrak{s})_+} O_{(\mathfrak{s})_-}(0)\ \tilde{A}(p_3(\lambda))\,\tilde{A}(p_4(\lambda))\rangle^{PROP} =$$

$$= (p_+(\lambda))_{(\mathfrak{s})_+} \langle T\, O_{(\mathfrak{s})_-}(0)\ \tilde{A}(p_3(\lambda))\,\tilde{A}(p_4(\lambda))\rangle^{PROP} = \tag{B.12}$$

$$= \lambda^n\, \ell_{(\mathfrak{s})_+(\mathfrak{s})_-}\, \alpha_+^{n-r}\, b_r\left(\alpha_+, \alpha_-; \frac{M^2}{m^2}, \frac{t}{m^2}, g\right) + O(\lambda^{n-2}).$$

The "formfactor" $b_r(\cdots\cdot)$ is a homogeneous polynomial in α_+ and α_- of degree r . It is easily verified from the discussion after (B.5) that for exactly lightlike and parallel momenta $p_{3,4}(\lambda)$, i.e. $\ell' = 0$ in (B.1c), only the trivial diagram contributes and yields

$$b_r(\alpha_+, \alpha_-; 0, 0, g) = \alpha_-^{\,r} . \tag{B.13}$$

In order to get rid of the indices we define scalar coefficient functions by contracting $C_{rn}^{(\mathfrak{s})_+(\mathfrak{s})_-}(\lambda q; m, g)$ with the large lightlike momentum $\lambda\ell$, i.e.

$$C_{rn}(\lambda q, \lambda\ell; m, g) := C_{rn}^{(\mathfrak{s})_+(\mathfrak{s})_-}(\lambda q; m, g)\, \ell_{(\mathfrak{s})_+(\mathfrak{s})_-}\, \lambda^n \tag{B.14}$$

Now in the Zimmermann identity

$$\langle T\, N_0[A(0)\tilde{A}(\lambda q)]\ X_{rn}^{(\mathfrak{s})_+(\mathfrak{s})_-}\rangle^{PROP'} =$$

$$= \sum_{\{\sigma'\}}^{n}{}_{0}\ \tilde{G}_{(0)}^{\{\sigma'\}}(\lambda q) \langle T\, B_{\{\sigma'\}}^{(n)}(0)\ X_{rn}^{(\mathfrak{s})_+(\mathfrak{s})_-}\rangle^{PROP} + \tag{B.15}$$

$$+ \langle T\, N_n[A(0)\tilde{A}(\lambda q)]\ X_{rn}^{(\mathfrak{s})_+(\mathfrak{s})_-}\rangle^{PROP'}$$

the matrix elements of the composite operators $B^{(n)}_{\{\sigma'\}}(0)$ cannot be traceless in $(\varsigma)_+(\varsigma)_-$ since $s' + \sum \#(\sigma')_j \leq n = \#(\varsigma)_+ + \#(\varsigma)_-$, $s' \geq 2$.

Therefore in C_{rn} the N_n-product may be replaced by N_0 [36),]

$$C_{rn}(\lambda q, \lambda \ell; m, g) =$$

$$= \langle T N_0[A(0)\tilde{A}(\lambda q)] \; X_{rn}^{(\varsigma)_+(\varsigma)_-} \rangle^{PROP'} \quad \ell_{(\varsigma)_+(\varsigma)_-} \lambda^n = \tag{B.16}$$

$$= \frac{1}{(n-r)!} \frac{1}{r!} \partial_\alpha^{n-r} \partial_\beta^{r} \langle T N_0[A(0)\hat{A}(\lambda q)] \; \tilde{A}((\alpha+\beta)\lambda\ell)\tilde{A}((\alpha-\beta)\lambda\ell) \rangle^{PROP'}\Big|_{\alpha-\beta=0}$$

So we finally obtain from (B.2) the simplified Zimmermann expansion

$$E(\lambda q, p_+(\lambda), p_-(\lambda); m, g) =$$

$$\begin{aligned} &= \sum_{\substack{n=0 \\ even}}^{a-2} \sum_{r=0}^{n} C_{rn}(\lambda q, \lambda \ell; m, g) \, \alpha_+^{n-r} \, b_r\left(\alpha_+, \alpha_-; \frac{M^2}{m^2}, \frac{t}{m^2}, g\right) + \\ &\quad + \langle T N_a[A(0)\hat{A}(p_2(\lambda))] \, \hat{A}(p_3(\lambda)) \, \tilde{A}(p_4(\lambda)) \rangle^{PROP'} + \\ &\quad + OL(\lambda^{-6}) \;. \end{aligned} \tag{B.17}$$

Letting herein a go to infinity does not yield an asymptotic expansion in λ since for every finite a the remainder is $OL(\lambda^{-4})$ like the separate terms in the sum. The expansion (B.17) suggests, however, that all terms of order $OL(\lambda^{-4})$

are contained in the formal series which is obtained by letting the degree a go to infinity and discarding the remainder. This assumption apparently is similar to that of the existence of LC expansion, which has not yet been proved in perturbation theory either [27]. We therefore write a formal "lightlike momentum (LLM) expansion"

$$E(\lambda q, p_+(\lambda), p_-(\lambda); m, g) =$$

$$= \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \sum_{r=0}^{n} C_{rn}(\lambda q, \lambda \ell; m, g)\, \alpha_+^{n-r}\, b_r\left(\alpha_+, \alpha_-; \frac{M^2}{m^2}, \frac{t}{m^2}, g\right) + \tag{B.18}$$

$$+ OL(\lambda^{-6}).$$

For exactly lightlike and parallel momenta $p_\pm(\lambda)$ it reduces, in view of (B.13), to a formal power series in α_+ and α_- . Then the $OL(\lambda^{-6})$ terms vanish identically. The point of the formal series (B.18) is that it expresses a function with Minkowskian exceptional momenta by its derivatives at Euclidean exceptional momenta.

C. Asymptotic Estimates

We now make use of the previously derived NP and LLM expansions in order to extract the asymptotically leading terms from Greensfunctions with mass vertex insertions. Our first example is the function $\Delta C_{rn}(q, \ell; \frac{m}{\lambda}, g)$

which occurs on the r.h.s. of the CS equations (3.11) and (4.5).

$$\Delta C_{rn}(q,\ell;m,g) = \langle T N_0[A(0)\tilde{A}(q)]\ \Delta\ \ell\cdot X_{rn}\rangle^{PROP} \qquad (C.1)$$

$$q^2 < 0 \quad , \quad \ell^2 = 0$$

with

$$\ell\cdot X_{rn} := \ell_{(\xi)_+(\xi)_-}\ X_{rn}^{(\xi)_+(\xi)_-} \qquad (C.2)$$

and $X_{rn}^{(\xi)_+(\xi)_-}$ is defined in (B.11). We use a Zimmermann identity to express the N_0 -product in terms of N_{n+2} [37]

$$\langle T N_0[A(0)\tilde{A}(q)]\ \Delta\ X_{rn}^{(\xi)_+(\xi)_-}\rangle^{PROP} =$$

$$= \sum_{\{\sigma'\}}{}_0^{n+2}\ \tilde{G}_{(0)}^{\{\sigma'\}}(q)\ \langle T B_{\{\sigma'\}}^{(n+2)}(0)\ \Delta\ X_{rn}^{(\xi)_+(\xi)_-}\rangle^{PROP} +$$

$$+ \sum_{\{\sigma''\}}{}_0^{n}\ \Delta\tilde{G}_{(0)}^{\{\sigma''\}}(q)\ \langle T B_{\{\sigma''\}}^{(n)}(0)\ X_{rn}^{(\xi)_+(\xi)_-}\rangle^{PROP} + \qquad (C.3)$$

$$+ \langle T N_{n+2}[A(0)\tilde{A}(q)]\ \Delta\ X_{rn}^{(\xi)_+(\xi)_-}\rangle^{PROP}$$

where the second sum arises from renormalization parts which contain the mass vertex Δ . When contracted with $\ell_{(\xi)_+(\xi)_-}$ the second sum in (C.3) does not contribute because of the argument which was given after (B.15), namely the matrix elements of the composite operators $B_{\{\sigma''\}}^{(n)}$ cannot be trace-

less in $(\varsigma)_+ (\varsigma)_-$ since $s'' + \sum \#(\sigma'')_j \leq n = \#(\varsigma)_+ + \#(\varsigma)_-$,
i.e. $\sum \#(\sigma'') < \sum \#(\varsigma)$.

For the same reason it is necessary to have $s' = 2$ and $\sum \#(\sigma')_j = n$ in the first sum. Transforming again to total and relative variables, with $r' = \#(\varsigma')_-$ and $n = \#(\varsigma')_+ + \#(\varsigma')_-$, one sees furthermore that in fact only terms with $r \leq r' \leq n$ give nonvanishing contributions,

$$\left(-\frac{i}{2}\right)^{n-r'} \langle T\, \partial_{(\varsigma')_+} O_{(\varsigma')_-}(0)\, \Delta\, \ell \cdot X_{rn} \rangle^{PROP} =$$

$$= \ell_{(\varsigma')_+} \langle T\, O_{(\varsigma')_-}(0)\, \Delta\, \ell \cdot X_{rr'} \rangle^{PROP} = \qquad (C.4)$$

$$=: \ell_{(\varsigma')_+ (\varsigma')_-}\, \eta_{r'r}(q) .$$

The functions $\eta_{r'r}(q)$ can be isolated from (C.4) e.g. by introducing an auxiliary lightlike momentum $\bar{\ell}$,

$$\eta_{r'r}(q) = \bar{\ell}^{(\varsigma')} \langle T\, O_{(\varsigma')}(0)\, \Delta\, \ell \cdot X_{rr'} \rangle^{PROP} =$$

$$= \bar{\ell}^{(\varsigma')} \frac{1}{(r'-r)!} \frac{1}{r!} \partial_\alpha^{r'-r} \partial_\beta^{r} \langle T\, O_{(\varsigma')}(0)\, \Delta\, \hat{A}((\alpha+\beta)\ell)\, \hat{A}((\alpha-\beta)\ell) \rangle^{PROP} \Big|_{\alpha=\beta=0} \qquad (C.5)$$

with $\bar{\ell}^2 = \ell^2 = 0$, $\bar{\ell} \cdot \ell = 1$; $r' = \#(\varsigma')$ and r even.

An explicit calculation of $\eta_{r'r}(g)$ in perturbation theory gives, cf. (2.6),

$$\eta_{r'r}(g) = \delta_{r0} \frac{1}{1+r'} \frac{1}{16\pi^2} g + O(g^2) . \tag{C.6}$$

Since the lowest order graph for

$$\langle T\, O_{(s')}(0)\, \Delta\, \tilde{A}((\alpha+\beta)\ell)\, \tilde{A}((\alpha-\beta)\ell) \rangle^{PROP}$$

is independent of the relative momentum $\beta \cdot \ell$ it contributes to $\eta_{r'0}$ only.

So (C.1) can be written as

$$\Delta C_{rn}(q,\ell;m,g) =$$

$$= \sum_{\substack{r\le r'\le n\\ \text{even}}} C_{r'n}(q,\ell;m,g)\, \eta_{r'r}(g) + \tag{C.7}$$

$$+ \langle T\, N_{n+2}[A(0)\tilde{A}(q)]\, \Delta\, \ell \cdot X_{rn} \rangle^{PROP} .$$

The last term in (C.7) is negligible under overall momentum scaling or, equivalently, if the mass m is replaced by $\frac{m}{\lambda}$. This is seen as follows. We replace in (C.7) the momenta by λq and $\lambda\ell$. According to the definitions (B.14) and (C.2) the dependence on the vector $\lambda\ell$ can be factored off. The large-λ-behaviour of the remaining functions is, in view of (B.10) and (A.7),

$$C_{rn}^{(3)_+(3)_-}(\lambda q; m, g) \quad = \quad OL(\lambda^{-n-4}) \; , \tag{C.8a}$$

$$\langle T\, N_{n+2}[A(0)\tilde{A}(\lambda q)]\, \Delta X_{rn}^{(3)_+(3)_- \; PROP} \rangle = OL(\lambda^{-n-6}) \; , \tag{C.8b}$$

since $q^2 < 0$. Therefore we obtain the asymptotic behaviour

$$\begin{aligned} &\Delta C_{rn}(\lambda q, \lambda \ell; m, g) = \\ &= \sum_{\substack{r \le r' \le n \\ \text{even}}} C_{r'n}(\lambda q, \lambda \ell; m, g)\, \eta_{r'r}(g) \quad + OL(\lambda^{-6}) \end{aligned} \tag{C.9a}$$

or for dimensional reasons

$$\begin{aligned} &\Delta C_{rn}(q, \ell; \tfrac{m}{\lambda}, g) = \\ &= \sum_{\substack{r \le r' \le n \\ \text{even}}} C_{r'n}(q, \ell; \tfrac{m}{\lambda}, g)\, \eta_{r'r}(g) \quad + OL(\lambda^{-2}) \, . \end{aligned} \tag{C.9b}$$

We remark that the above extraction of the asymptotically leading terms from $\Delta C_{rn}(q, \ell; \frac{m}{\lambda}, g)$ makes use of the Zimmermann identity (C.3) which relates NPs of finite degree. It does not contain formal infinite sums.

In contrast to the previous example we need the formal LLM expansions in our second one. The function $\Delta F(p,\ell,p';\frac{m}{\lambda},g)$ is the r.h.s. of the CS equation (4.5).

$$\Delta F(p,\ell,p';m,g) = \langle T N_0[A(0)\hat{A}(p_2)]\ \Delta\ N_0[p_3,p_4]\rangle \tag{C.10}$$

with
$$p_{1,2} = \mp p - \ell \quad , \quad p_{3,4} = \pm p' + \ell$$

$$\ell^2 = 0 \quad ; \qquad p^2, p'^2, (p \pm p')^2 < 0$$

and the shorthand notation

$$N_a[p_3,p_4] \equiv N_a[\hat{A}(p_3)\hat{A}(p_4)] \ . \tag{C.11}$$

The formal expansion of the first N_0-product in (C.10) is

$$\begin{aligned}\Delta F(p,\ell,p';m,g) = \\ = \sum_{\substack{n\geq 0\\ \text{even}}} \Big\{ &\sum_{\{\sigma\}} \tilde{G}^{\{\sigma\}}_{(n)}(p) \langle T B^{(n+2)}_{\{\sigma\}}(0)\ \Delta\ N_0[p_3,p_4]\rangle + \\ &+ \sum_{\{\tau\}} \Delta\tilde{G}^{\{\tau\}}_{(n)}(p) \langle T B^{(n)}_{\{\tau\}}(0)\ N_0[p_3,p_4]\rangle + \\ &+ T_\ell^{n-4} \langle T N_n[A(0)\hat{A}(p_2)]\Delta N_0[p_3,p_4]\rangle \Big\}\end{aligned} \tag{C.12}$$

The summation is over sets $\{\sigma\}$, $\{\tau\}$ such that

$$s + \sum \#(\sigma)_j = n+2 \quad , \qquad t + \sum \#(\tau)_j = n \ . \tag{C.13}$$

The "overall subtraction terms" are defined by

$$T_\ell^{a} \langle \cdots \rangle := \frac{1}{a!}\, \ell^{(\alpha)} \left[\left(\frac{\partial}{\partial \ell}\right)_{(\alpha)} \langle \cdots \rangle \right]_{\ell = 0}$$

$$a = \#(\alpha) \tag{C.14}$$

$$T_\ell^{a} \langle \cdots \rangle = 0 \qquad \text{if} \quad a < 0 \quad .$$

They arise for $n \geqslant 4$ since ΔF is the vacuum expectation value of NPs only (as opposed to both NPs and A-fields). They would be absent if e.g. the second N_0-product in (C.10) were replaced by a Wick product. This is an example that the N_0-prescription in general is not identical to the Wick product. Further expansions of the remaining N_0-products in the two sums of (C.12) give

$$\langle T\, B^{(n+2)}_{\{\sigma\}}(0)\ \Delta\ N_0[p_3, p_4] \rangle =$$

$$= \sum_{\substack{n' \geqslant 0 \\ \text{even}}} \Big\{ \sum_{\{\sigma'\}} \langle T\, B^{(n+2)}_{\{\sigma\}}(0)\, \Delta\, \widetilde{B}^{(n'+2)}_{\{\sigma'\}}(2\ell) \rangle\ \widetilde{G}^{\{\sigma'\}}_{(n')}(p') \; + \tag{C.15}$$

$$+ \sum_{\{\tau'\}} \langle T\ B^{(n+2)}_{\{\sigma\}}(0)\ B^{(n')}_{\{\tau'\}}(2\ell) \rangle\ \Delta\, \widetilde{G}^{\{\tau'\}}_{(n')}(p') \; +$$

$$+ \; T_\ell^{n+n'-2} \langle T\, B^{(n+2)}_{\{\sigma\}}(0)\ \Delta\ N_{n'}[p_3, p_4] \rangle \Big\} \quad ,$$

$$\langle T\ B^{(n)}_{\{\tau\}}(0)\ N_0[p_3, p_4] \rangle =$$

$$= \sum_{\substack{n' \geq 0 \\ \text{even}}} \Big\{ \sum_{\{\sigma'\}} \langle T B^{(n)}_{\{\tau\}}(0)\ \tilde{B}^{(n'+2)}_{\{\sigma'\}}(2\ell) \rangle\ \tilde{G}^{\{\sigma'\}}_{(n')}(p') + \tag{C.16}$$

$$+ T_\ell^{n+n'-2} \langle T B^{(n)}_{\{\tau\}}(0)\ N_{n'}[p_3, p_4] \rangle \Big\}$$

with $\quad s' + \sum \#(\sigma')_j = n'+2 \quad , \quad t' + \sum \#(\tau')_j = n \quad .$ (C.17)

In the following we show that in the formal expansion of $\Delta F(\lambda p, \lambda \ell, \lambda p'; m, g)$ again only two-particle contributions must be kept in the limit $\lambda \to \infty$. We list the asymptotic behaviour of the various terms in (C.12,15,16) if the momenta are scaled with the common factor λ.

1. From (A.7) we have

$$\tilde{G}^{\{\sigma\}}_{(n)}(\lambda p; m, g) = OL(\lambda^{-n-4}) \quad , \tag{C.18a}$$

$$\Delta \tilde{G}^{\{\tau\}}_{(n)}(\lambda p; m, g) = OL(\lambda^{-n-4}) \quad . \tag{C.18b}$$

2. The three matrix elements which contain two B-operators, cf. (C.15,16), depend on λ through the momentum $\lambda\ell$ only. By definition they are subtracted at zero throughgoing momentum up to the order $n+n'-2$. The argumentation used after (B.5) then implies, in view of (C.13,17),

$$\langle T B^{(n+2)}_{\{\sigma\}}(0)\ \Delta\ B^{(n'+2)}_{\{\sigma'\}}(2\lambda\ell) \rangle =$$

$$= \delta_{s2}\, \delta_{s'2}\, \lambda^{n+n'}\, \ell_{\{\sigma\}\{\sigma'\}}\ f(g) \quad , \tag{C.19a}$$

$$\langle T\, B^{(n+2)}_{\{\sigma\}}(0)\ \tilde{B}^{(n')}_{\{\tau'\}}(2\ell\lambda)\rangle = 0 \quad , \tag{C.19b}$$

$$\langle T\, B^{(n)}_{\{\tau\}}(0)\ \tilde{B}^{(n'+2)}_{\{\sigma'\}}(2\lambda\ell)\rangle = 0 \quad . \tag{C.19c}$$

We remark that a throughgoing momentum $\lambda\ell + \lambda^{-1}\ell'$ gives additional terms of order $O(\lambda^{n+n'-2})$ on the r.h.s. of (C.19) which do not invalidate our reasoning.

3. Finally the overall subtraction terms have an asymptotic behaviour as expected from naive dimensional analysis ,

$$T^{n-4}_{\ell}\langle T N_n[A(0)\tilde{A}(\lambda p_2)]\ \Delta N_0[\lambda p_3,\lambda p_4]\rangle = OL(\lambda^{-10}) \quad , \tag{C.20a}$$

$$T^{n+n'-2}_{\ell}\langle T\, B^{(n+2)}_{\{\sigma\}}(0)\ \Delta\ N_{n'}[\lambda p_3,\lambda p_4]\rangle = OL(\lambda^{n-6}) \quad , \tag{C.20b}$$

$$T^{n+n'-2}_{\ell}\langle T\, B^{(n)}_{\{\tau\}}(0)\ \ N_{n'}[\lambda p_3,\lambda p_4]\rangle = OL(\lambda^{n-6}) \quad . \tag{C.20c}$$

We postpone the proof of these latter estimates to the end of this appendix.

It is now straightforward to verify that the asymptotically leading terms of $\Delta F(\lambda p, \lambda \ell, \lambda p'; m, g)$ come from the composite operators $B_{(\sigma),(\sigma)_2}$. Transforming again to relative and total momentum variables we find after some algebra

$$\Delta F(\lambda p, \lambda \ell, \lambda p'; m, g) = \sum_{\substack{n \geq r \geq 0 \\ \text{even}}} \sum_{n' \geq r' \geq 0} C_{rn}(\lambda p, \lambda \ell; m, g) \cdot$$

$$\cdot (i \gamma_{22}(g))_{rr'} \cdot C_{r'n'}(\lambda p', \lambda \ell; m, g) + \tag{C.21}$$

$$+ OL(\lambda^{-10}) .$$

The functions $(\gamma_{22}(g))_{rr'}$ are given by

$$(\gamma_{22}(g))_{rr'} = -i \, \bar{\ell}^{(s)} \langle T \, O_{(s)}(0) \, \Delta \, \tilde{O}_{(s')}(2\ell) \rangle \, \bar{\ell}^{(s')} \tag{C.22}$$

with $\bar{\ell}^2 = \ell^2 = 0 \quad , \quad \bar{\ell} \cdot \ell = 1$

$r = \#(s)$ and $r' = \#(s')$ even.

They are symmetric in r and r', and from the lowest order graph we obtain, cf. (2.11),

$$(\gamma_{22}(g))_{rr'} = \frac{1}{1+r+r'} \cdot \frac{1}{16\pi^2} + O(g^2) . \tag{C.23}$$

We still have to verify the estimates (C.20). By the usual dimensional argument the l.h.s. of (C.20a) can be rewritten as

$$\text{l.h.s. of (C.20a)} = \\ = \lambda^{-10} (-2m^2\varphi) \frac{1}{\#(\alpha)!} \, \ell^{(\alpha)} \, \Delta_0 f_{(\alpha)} (p, p'; \tfrac{m}{\lambda}) \tag{C.24a}$$

with

$$\Delta_0 f_{(\alpha)} (p, p'; \tfrac{m}{\lambda}) = \\ = \left(\frac{\partial}{\partial \ell}\right)_{(\alpha)} \langle T N_n [A(0) \hat{A}(p-\ell)] \, \Delta_0 \, N_0 [p'+\ell, -p'+\ell] \rangle (\tfrac{m}{\lambda}) \Big|_{\ell=0} \tag{C.24b}$$

$$\#(\alpha) = n-4$$

The estimate will be established if one can show that $\Delta_0 f_{(\alpha)}(p, p'; \frac{m}{\lambda})$ diverges at most logarithmically for $\lambda \rightarrow \infty$. To this end we investigate the IR divergence of the mass zero function $\Delta_0 f_{(\alpha)} (p, p'; 0)$.

Zimmermann's renormalization scheme is strictly speaking not suitable for zero mass theories, since the subtractions are performed at zero momenta. The "finite" counterterms which appear in his Lagrangian are (at most logarithmically) divergent for $m \rightarrow 0$. As long as such logarithmic divergences do not add up to positive powers they do not spoil the following argumentation [38,39].

We consider a given diagram $\Delta_0 \Gamma$ which contributes to $\Delta_0 f_{(\alpha)}(p,p';0)$ and a given routing of p, p', and the internal integration momenta. The corresponding diagram without the vertex insertion is denoted by Γ. We identify the end points of the external lines with momenta $\mp p$ and $\pm p'$ and denote them by V_n and V_0, respectively, cf. fig. 2. Since p and p' are spacelike all integration momenta can be Wick rotated. Then IR divergences can arise only from lines with vanishing throughgoing momenta. We consider the contribution from a given part of the integration region where some (or all) integration momenta are small and define a degree ω of IR convergence which counts the powers of small momenta. An integral will superficially converge if $\omega > 0$ for all possible subintegrations.

The lines with nonvanishing momenta form a set $\{\gamma\}$ of mutually disjoint proper subgraphs and two situations may arise, cf. fig. 3a,b,

a) V_n and V_0 are in different subgraphs γ_n, γ_0 or

b) V_n and V_0 are in the same subgraph γ_{n0}.

In addition to γ_n, γ_0 or γ_{n0} there may exist other subgraphs which do not contain the momenta p and p'. The reduced diagram X is obtained from Γ by shrinking all subgraphs of the set $\{\gamma\}$ to a point. By definition it consists of lines with small momenta only. Let r_i be the number of external lines of γ_i. $\#(\alpha)_i$ and $\#(\alpha)_X$ denote the numbers of derivatives acting on lines in γ_i and X, respectively.

We find the following values of ω:

Case a:

$$\omega(X) = 4 + (r_n - 4) + (r_o - 4) + \sum_{i=1}^{c} (r_i - 4) - \#(\alpha)_X \tag{C.25a}$$

follows from power counting,

$$\omega(\gamma_n) = \max\,(n - r_n - \#(\alpha)_n + 1\,,\ 0) \tag{C.25b}$$

is implied by the N_n -prescription,

$$\omega(\gamma_o) = 0 \tag{C.25c}$$

since the N_o -prescription does not imply overall subtractions,

$$\omega(\gamma_i) = \max\,(4 - r_i - \#(\alpha)_i\,,\ 0) \qquad i = 1, 2, \dots, c \tag{C.25d}$$

since the renormalization conditions of a massless theory imply that selfenergy insertions vanish of second order in the throughgoing momentum. Adding up the values (C.25) we find a total degree for the diagram Γ

$$\begin{aligned} \omega_a(\Gamma) &= \omega(X) + \omega(\gamma_n) + \omega(\gamma_o) + \sum_{i=1}^{c} \omega(\gamma_i) \geq \\ &\geq 4 + (r_n - 4) + (r_o - 4) + \sum_{i=1}^{c} (r_i - 4) - \#(\alpha)_X + \\ &\quad + n - r_n - \#(\alpha)_n + 1 + \sum_{i=1}^{c} (4 - r_i - \#(\alpha)_i) = \\ &= \Big[n - 4 - \#(\alpha)_X - \#(\alpha)_n - \sum_i \#(\alpha)_i\Big] + r_o + 1 \\ &\geq 3 \end{aligned} \tag{C.26}$$

The value of the square bracket is nonnegative since the total number of derivatives is $n-4$, and $r_0 \geq 2$.

Case b:

$$\omega(X) = r_{no} + \sum_{i=1}^{c} (r_i - 4) - \#(\alpha)_X \qquad \text{(C.27a)}$$

$$\omega(\gamma_{no}) = \max\,(n-4-r_{no} - \#(\alpha)_{no} + 1\,,\,0) \qquad \text{(C.27b)}$$

$$\omega(\gamma_i) = \max\,(\,4 - r_i - \#(\alpha)_i\,,\,0\,) \qquad i=1,2,\cdots,c \qquad \text{(C.27c)}$$

These values add up to

$$\omega_b(\Gamma) \geq [\,n-4-\#(\alpha)_X - \#(\alpha)_{no} - \sum_{i=1}^{c} \#(\alpha)_i\,] + 1 = 1\,. \qquad \text{(C.28)}$$

So far we have neglected the operator Δ_o which may be inserted in X or any proper subgraph of the set $\{\gamma\}$. It is easy to see that for all possible insertions Δ_o can lower the values (C.25), (C.27) at most by 2. Therefore we obtain

$$\omega_a\,(\Delta_o\Gamma) \geq +1\quad ,$$

$$\omega_b\,(\Delta_o\Gamma) \geq -1\quad ,$$

i.e. the worst IR divergences of $\Delta_o f_{(\alpha)}(p,p';0)$ are linear ones, but symmetric integration reduces them to logarithmic ones. Thus (C.20a) is verified. The proofs of (C.20b) and (C.20c) go through similarly.

References and Footnotes

1. We are dealing with a massive $g \cdot A^4$ – theory in D = 4 dimensions, $g > 0$. $\Gamma(p_1 \cdots p_n ; m, g)$ denotes the n - point vertex function, i.e. the one-particle-irreducible amputated part of the corresponding connected Greensfunction

   $$G(p_1 \cdots p_n ; m, g) = \langle T \tilde{A}(p_1) \cdots \tilde{A}(p_{n-1}) A(0) \rangle_{conn} ,$$

   momentum conservation $\sum_{i=1}^{n} p_i = 0$ is always understood.

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6. Note that we use the differential operator $m \frac{\partial}{\partial m}$ therefore our functions β and γ differ from Symanzik's by a factor of 2.

7. K. Symanzik, in: Springer Tracts in Modern Physics 57, 222 (1971)

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10. By the symbol $OL(\lambda^n)$ we mean "$O(\lambda^n)$ apart from logarithms of λ", i.e. $O(\lambda^n \ln^c \lambda)$ with an unspecified but in any finite order of perturbation theory finite power c of logarithms which may vary in different estimates.

11. We would like to stress that according the above definition exceptionality is not a geometrical criterion but a dynamical concept which depends on the theory considered, e.g. the momentum set $((-p)\, p\;\, p'(-p'))$ with $p^2, p'^2, (p \pm p')^2 < 0$ of a 4-point vertex function is exceptional in the A^4-theory in D = 4 dimensions, cf. sect. 2, whereas it is not for the A^3-theory in D = 6 dimensions.
See also G.C. Marques, Univ. of Maryland, Technical Report No. 74-024, September 1973

12. The vertex functions of the preasymptotic theory depend nontrivially on a mass parameter. This can be seen by relating them to a directly constructed zero mass theory which is normalized at some spacelike point. Cf. refs. 8 and 16.

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22. We remark in passing that Weinberg's renormalization group equations (Phys. Rev. D8, 3497 (1973)) do not allow to treat the exceptional momentum problem since at those momenta the first term in Weinberg's series (expansion in mass vertex insertions in a massless theory) already does not exist in perturbation theory.

23. Cf. e.g. C.G. Callan, Phys. Rev. D5, 3202 (1972)

24. R.A. Brandt, G. Preparata, Nucl. Phys. B27, 541 (1971), cf. also Y. Frishman, Phys. Rev. Lett. 25, 966 (1970)

25. J.D. Bjorken, Phys. Rev. 179, 1547 (1969)

26. $b_n(1,q)$, $b_n(0,q)$, and $a_n(0,q)$ correspond, respectively, to c_n, b_n, and a_n of CHM.

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38. It is possible to introduce modified subtraction schemes such that the renormalization functions are subtracted at a space-like momentum, cf. e.g. M. Gomes, R. Köberle, University of Sao Paulo preprint IFUSP/P-13, November 1973. For our purpose it then would be necessary to use different subtraction prescriptions for different graphs. Subgraphs which do not

contain the external lines of $\Delta_0 f_{(\alpha)}(p,p';0)$ are subtracted at a spacelike momentum. In contrast we need Zimmermann's N-products with subtractions at zero momenta for graphs which do contain the external lines.

39. The following proof is a generalization of a similar one in: K. Johnson, R. Willey, M. Baker, Phys. Rev. 163, 1699 (1967)

Figure Captions

Fig. 1a A diagram which gives rise to a logarithmic IR divergence in $\Gamma((-p)\,p\,0\,0\,;\,\frac{m}{\lambda},g)$.

Fig. 1b A diagram which gives rise to a quadratic IR divergence in $\Delta_0\,\Gamma((-p)\,p\,0\,0\,;\,\frac{m}{\lambda},g)$. The cross marks the insertion of the vertex operator Δ_0 .

Fig. 2 Diagram contributing to $f_{(\alpha)}(p,p';0) =$

$$= \left(\frac{\partial}{\partial \ell}\right)_{(\alpha)} \langle T\, N_n[A(0)\,\tilde{A}(p-\ell)]\, N_0[p'+\ell, -p'+\ell] \rangle^{PROP'}\Big|_{\ell=0} \quad (m=0)$$

Fig. 3a,b Reduced diagrams which are obtained from fig. 2 by contracting subgraphs with nonvanishing momenta. The vertices V_n and V_0 are in γ_n, γ_0 (a) or in γ_{n0} (b).

Fig.1a

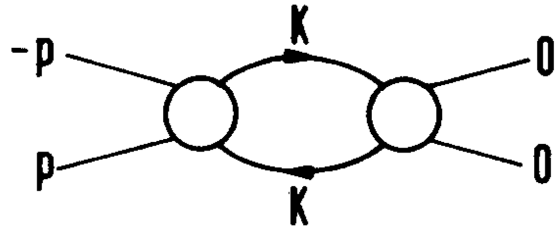

Fig.1b

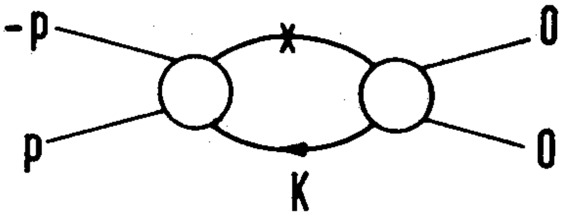

Fig.2

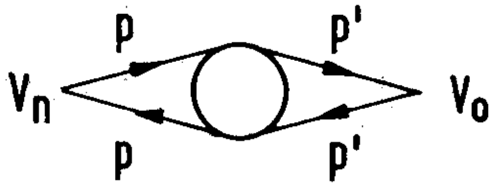

Fig.3a

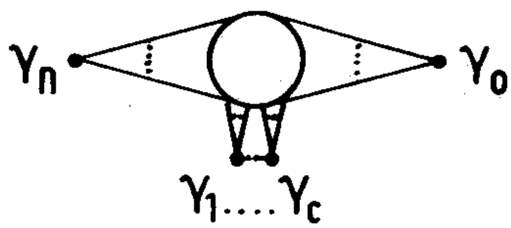

Fig.3b