DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 74/3
January 1974

Alternatives to the Veneziano Amplitude

by

E. Weimar
I. Mathematisches Institut, Freie Universität Berlin

2 HAMBURG 52 · NOTKESTIEG 1

To be sure that your preprints are promptly included in the
HIGH ENERGY PHYSICS INDEX ,
send them to the following address (if possible by air mail) :

DESY
Bibliothek
2 Hamburg 52
Notkestieg 1
Germany

ALTERNATIVES TO THE VENEZIANO AMPLITUDE

by

Evelyn Weimar

1. Mathematisches Institut, Freie Universität Berlin

Abstract:

We study alternatives to the Veneziano amplitude from a mathematical point of view within the framework of dual resonance models in the zero-width approximation for scalar particles without internal quantum numbers.

Introduction

We want to study dual resonance models in the zero-width approximation for scalar particles without any internal quantum numbers. A dual resonance model means an analytic expression for the scattering amplitude $M(s,t,u)$ which has the following three properties ([1])

a) all its singularities are due to resonance exchange

b) good asymptotic behaviour

c) exact crossing symmetry

Furthermore, we want to restrict ourselves to amplitudes without u-channel poles or - more generally - to request planar duality which means a decomposition of M into three terms

$$(1) \qquad M(s,t,u) = V(s,t) + V(s,u) + V(t,u)$$

each of which possesses besides the properties a), b), c) only singularities in the variables written down explicitly.

The simplest model of this kind certainly is the Veneziano model ([2]) which describes the four-point fuction according to the beta function

$$(2) \qquad V(s,t) = V(t,s) = B(-\alpha_s,-\alpha_t) = \int_0^1 dx\, x^{-\alpha_s-1}\cdot(1-x)^{-\alpha_t-1}$$

In the last years a lot of alternatives to (2) - won by modifications of the integrand - have been proposed and discussed by several authors. There have been several physical motivations for these modifications, such as taking into account absorptive effects to explain the appearance of dips in many differential cross-sections ([3]), or reducing the drastic violation of unitarity. In this connection, the development of DAMA ([4]) seems to be very fruitful, where the resonance poles are shifted from the positive real axis into the unphysical sheet, where a branch point corresponding to the elastic threshold is introduced, and where the amplitudes satisfy a Mandelstam representation.

We want to look for alternatives from a mathematical point of view without changing the physical (or unphysical) situation underlying the Veneziano amplitude:

In part I we look for all fuctions $V(s,t)$ admitting a Mittag-Leffler expansion with polynomial residues. (In [5] this expansion was used to show that even in the case of non-identical channels the properties of the residues in s and t are related to each other.) At the end of this examination a theorem tells us that the general solution is a countably infinite superposition of certain "base" functions - which we can choose, e.g., to be B-functions - multiplied by polynomials of an appropriate degree.

In part II the operator approach to the Veneziano model ([6]) is studied and modified to give an operator approach to a subclass of solutions we found in part I.

Part I :

V(s,t) certainly has to fulfil the following two assumptions:

(3) symmetry : $$V(s,t) = V(t,s)$$

pole spectrum:

V(s,t) is an analytic function the only singularities in s of which (t being fixed) are simple poles lying at

(4) $$s = s_i \geqslant 0 \qquad \text{i.e.} \qquad 0 \leqslant s_1 < s_2 < s_3 < \cdots$$

By changing the variable s into $\alpha_s := \alpha(s)$ we can transfer the poles to the non-negative integers:

V(s,t) has simple poles at

(5) $$\alpha_s = j \qquad j = 0,1,2\ldots$$

An open question that remains, is the interpretation of the term "good" asymptotic behaviour.

If one demands V(s,t) to reproduce the asymptotic behaviour originating from a Sommerfeld-Watson formula of an amplitude with an infinite series of parallel Regge trajectories, V(s,t) has to be a series of Veneziano type amplitudes ([7]). The same result can be won by assuming that V(s,t) admits a dispersion relation with an infinite number of superconvergence relations ([8]).

We assume polynomial boundedness:

There exists an integer $K \geqslant 0$ such that

(6a) $$\lim_{|\alpha_s| \to \infty} \alpha_s^{-K} \cdot V(s,t) = 0$$

apart from the poles, with t being fixed with

(6b) $$\mathrm{Re}\ \alpha_t < 0$$

If one has in mind analytic Regge behaviour

(7) $$V(s,t) \underset{\substack{|s| \to \infty \\ t \text{ fixed}}}{\widetilde{\longrightarrow}} \alpha_s^{\alpha_t}$$

apart from the poles, (6) is fulfilled for K = 0. This coincides with the interpretation of "good" asymptotic behaviour by Veneziano himself ([1]). Therefore, we shall study at first the case K = 0 and only then give our modifications for K > 0.

1) K = 0; no subtractions

If V(s,t) vanishes asymptotically in s, it admits an unsubtracted Mittag-Leffler expansion

(8) $$V(s,t) = \sum_{j=0}^{\infty} \frac{P_j(\alpha_t)}{j - \alpha_s} \qquad \mathrm{Re}\ \alpha_t < 0 \qquad \alpha_s \neq 0,1,2\ldots$$

converging absolutely and uniformly apart from the poles ([9]).
Symmetry of V then means

(9) $$V(s,t) = \sum_{j=0}^{\infty} \frac{P_j(\alpha_t)}{j - \alpha_s} = \sum_{j=0}^{\infty} \frac{P_j(\alpha_s)}{j - \alpha_t} \qquad \mathrm{Re}\ \alpha_t < 0,\ \mathrm{Re}\ \alpha_s < 0$$

This identity is often used as a definition of duality.

For $\mathrm{Re}\ \alpha_s < 0$ we can transform (8) into an integral by means of the identity

(10) $$(j - \alpha_s)^{-1} = \int_0^1 dx \cdot x^{j - \alpha_s - 1} \qquad \mathrm{Re}\ \alpha_s < 0$$

For interchanging summation and integration we have to make sure of the convergence of the power series

(11) $$F(x, \alpha_t) := \sum_{j=0}^{\infty} P_j(\alpha_t) \cdot x^j$$

In appendix A it is shown that F converges uniformly in general only for $0 \leq x < 1$ and may diverge for $x = 1$, but only in such a way that it still remains integrable. Since the integrand in (10) behaves nicely for $x \to 1$, we are thus allowed to interchange summation and integration which leads to

(12) $$V(s,t) = \int_0^1 dx\, x^{-\alpha_s - 1} \cdot F(x, \alpha_t)$$

Symmetry of V now demands

(13) $$\int_0^1 dx\, x^{-\alpha_s - 1} \cdot F(x, \alpha_t) = \int_0^1 dy\, y^{-\alpha_t - 1} \cdot F(y, \alpha_s) \qquad \begin{matrix} \mathrm{Re}\,\alpha_s < 0 \\ \mathrm{Re}\,\alpha_t < 0 \end{matrix}$$

- - -

At first, we want to restrict ourselves to the case where there exists a substitution $y = y(x)$ $x \in [0,1]$, $y \in [0,1]$, which transfers one integrand into the other:

(14) $$x^{-\alpha_s - 1} \cdot F(x, \alpha_t) = y(x)^{-\alpha_t - 1} \cdot F(y(x), \alpha_s) \cdot |y'(x)|$$

Since the integrand is an analytic function in α_s and α_t (apart from the poles), the real function of substitution $y(x)$ cannot depend on α_s and α_t.
(14) yields

$$F(x, \alpha_t) \cdot y(x)^{\alpha_t + 1} = F(y(x), \alpha_s) \cdot x^{\alpha_s + 1} \cdot |y'(x)| = f(x)$$

Combining these equations finally gives

(15) $$F(x, \alpha_t) = \sum_{j=0}^{\infty} P_j(\alpha_t) \cdot x^j = f(x) \cdot y(x)^{-\alpha_t - 1}$$

with

(16) $$y(y(x)) = x \qquad \text{i.e.} \qquad y = y^{-1}$$

and

(17) $$f(y(x)) \cdot |y'(x)| = f(x)$$

Since the identity $y(x) = x$ would destroy the assumed pole spectrum, $y(x)$ can be any monotonically decreasing function with $y = y^{-1}$, $y(0) = 1$,

$y(1) = 0$.
The simplest candidate $y(x) = 1-x$ is realized by the Veneziano amplitude, and additional satellite terms are the result of choosing $f(x) = x^l \cdot (1-x)^l$.

Before constructing other examples, we deduce general properties of our solution

$$(18) \qquad V(s,t) = \int_0^1 dx\, x^{-\alpha_s - 1} \cdot y(x)^{-\alpha_t - 1} \cdot f(x)$$

According to (15), the residues $P_j(\alpha_t)$ are

$$(19) \qquad P_j(\alpha_t) = \frac{1}{j!} \frac{\partial^j}{\partial x^j} \left\{ f(x) \cdot y(x)^{-\alpha_t - 1} \right\}_{x=0}$$

polynomials in α_t of degree $\leq j$ starting with

$$(20) \qquad P_j(\alpha_t) = \frac{1}{j!} \cdot f(0) \cdot |y'(0)|^j \cdot \alpha_t^j + \dots$$

If $f(0) = f'(0) = \dots = f^{(l-1)}(0) = 0$, $f^{(l)}(0) \neq 0$, then the degree of this polynomial is reduced to $j-l$:

$$(21) \qquad P_0 = P_1 = \dots = P_{l-1} = 0 \qquad P_l(\alpha_t) = \frac{1}{l!} \cdot f^{(l)}(0) \neq 0$$

so that the pole spectrum only starts with l.
The degree of the following polynomials $P_j(\alpha_t)$ $j > l$ can be reduced further if $y'(0) = y''(0) = \dots = y^{(k-1)}(0) = 0$, $y^{(k)}(0) \neq 0$ without affecting the pole spectrum.

Since the main contribution asymptotically is due to the neighbourhood of $x = 1$, (18) gives for $|s| \to \infty$, $\mathrm{Re}\,\alpha_s \to -\infty$ the following Regge behaviour ([10]):

$$(22) \qquad V(s,t) \longrightarrow f(0) \cdot \Gamma(-\alpha_t) \left[(\alpha_s + 1)\, y'(0)\right]^{\alpha_t} \qquad \begin{array}{l} f(0) \neq 0 \\ y'(0) \neq 0 \end{array}$$

If $y'(0) = y''(0) = \dots = y^{(k-1)}(0) = 0$, $y^{(k)}(0) \neq 0$; $f(0) = f'(0) = \dots = f^{(l-1)}(0) = 0$, $f^{(l)}(0) \neq 0$, the trajectory is shifted by l and modified by a factor $1/k$:

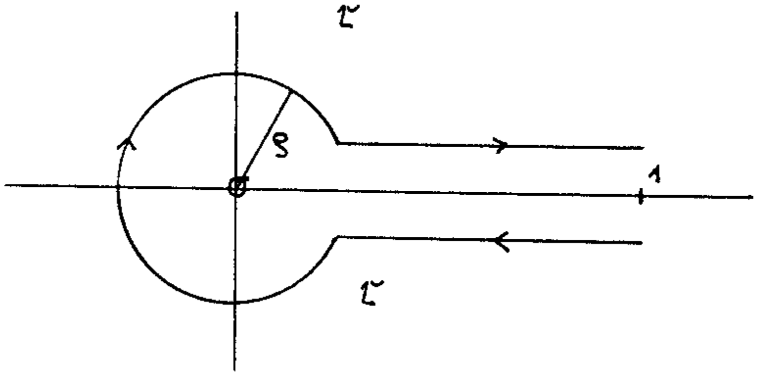
$$(24) \quad V(s,t) \sim \frac{(k-1)!}{l!} f^{(l)}(0) \cdot (k!)^{\frac{l-\alpha_t}{k} - 1} \cdot \Gamma\left(\frac{l-\alpha_t}{k}\right) \left[(\alpha_s + 1)\, y^{(k)}(0)\right]^{\frac{\alpha_t - l}{k}}$$

which means

$$|\alpha_s|^{1/k} \cdot V(s,t) \sim 0 \qquad \text{Re}\ \alpha_t < 0 \tag{24}$$

For Re $\alpha_s \geqslant 0$, Re $\alpha_t < 0$ we can use the same method of contour integration to get an analytic continuation for $V(s,t)$ as it is well known for the B-function ([11]), since $F(x, \alpha_t)$ is a regular function for $|x| < 1$. This yields

$$V(s,t) = \frac{1}{1-e^{-2\pi i \alpha_s}} \int_{\mathcal{C}} dx\ x^{-\alpha_s - 1} \cdot F(x, \alpha_t) \qquad \begin{array}{l}\text{Re}\ \alpha_t < 0 \\ \alpha_s \neq j = 0,1,2\ldots\end{array} \tag{25}$$



For Re $\alpha_s \longrightarrow +\infty$ apart from the real axis, integration along the circle results for all $0 < \rho < 1$ ($F(x,\alpha_t)$ being bounded) in an exponentially decreasing expression. Thus, if we choose ρ very close to 1, the main contribution is again due to the neighbourhood of $x = 1$ of the integral $\int^{1} dx\ x^{-\alpha_s - 1} \cdot F(x, \alpha_t)$ which gives us the same Regge behaviour as expressed by the formulas (22) and (23) for Re $\alpha_s \longrightarrow +\infty$, too.

- - -

Now we select a family $V^{(l)}(s,t)$ $l = 0,1,2\ldots$ of our solutions (18) obeying

$$|\alpha_s|^{l} \cdot V^{(l)}(s,t) \xrightarrow[|s| \to \infty]{} 0 \qquad \text{Re}\ \alpha_t < 0 \tag{26}$$

$$\begin{cases} P_0^{(l)} = P_1^{(l)} = \quad = P_{l-1}^{(l)} = 0 \qquad P_l^{(l)}(\alpha_t) = \text{const} \neq 0 \\ P_j^{(l)}(\alpha_t) = \text{polynomial of degree} \leq j-l \qquad j > l \end{cases} \tag{27}$$

The simplest choice of these "base" functions $V^{(l)}(s,t)$ is certainly that of the B-functions

(28) $y(x) = 1 - x \qquad f^{(1)}(x) = x^{1} \cdot (1-x)^{1}$

This basic system of Veneziano amplitudes including satellite terms was used by several authors for the construction of physical amplitudes (cf. [7], [8], [12]). In all those cases where one is obliged to take into account a few satellite terms in order to fit the data ([12]), another possibility would be to look for another "base" better suited to the special problem. Therefore a competitive family of "base" functions is studied in appendix B.

Every solution $V(s,t)$, the residues $P_j(\alpha_t)$ of which are polynomials of degree j, can be expressed as a convergent linear combination of these "base" functions $V^{(1)}(s,t)$ with polynomials $R_1(\alpha_s+\alpha_t)$ of degree 1

(29) $$V(s,t) = \sum_{j=\sigma}^{\infty} \frac{P_j(\alpha_t)}{j-\alpha_s} = \sum_{\ell=\sigma}^{\infty} R_1(\alpha_s+\alpha_t)\, V^{(1)}(s,t)$$

This can been shown in the following successive way:
The pole $\alpha_s = 0$ only occurs in the first term of the right-hand side. Its residue can be made to be $P_o(\alpha_t)$ by choosing

(30) $$R_o(\alpha_s+\alpha_t) = P_o(\alpha_t) \,/\, P_o^{(0)}(\alpha_t)$$

The identity of the residues of the next pole at $\alpha_s = 1$ – where the first two terms of the right-hand side contribute – can be achieved by choosing $R_1(\alpha_s+\alpha_t)$ such that

(31) $$R_1(1+\alpha_t) = \left\{ P_1(\alpha_t) - R_o(1+\alpha_t) \cdot P_1^{(0)}(\alpha_t) \right\} / \, P_1^{(1)}(\alpha_t)$$

and so on. The rearrangement of the series on the right-hand side is allowed because $V(s,t)$ converges absolutely.

– – –

If we assume linear trajectories, the requirement for P_j to be a polynomial of degree j guarantees physically the absence of ancestors.

In order to understand the meaning of this additional restriction a little bit better from the mathematical point of view, we want to relate this property to the asymptotic behaviour of the amplitude. We shall find out: Regge behaviour e.g. gives rise to polynomial residues.
From (9) we get for n = 1,2,3...

$$(32) \qquad \frac{\partial^n}{\partial \alpha_t^n} V(s,t) = n! \cdot \sum_{j=0}^{\infty} \frac{P_j(\alpha_s)}{(j-\alpha_t)^{n+1}} \qquad \alpha_t \neq j=0,1,2...$$

valid at least for Re $\alpha_s < 0$ plus the additional region of convergence of the sum in the right-hand α_s-plane, which is growing larger and larger when n increases.

In any case, this sum converges uniformely in α_s for Re $\alpha_s < 0$ which yields convergence for Re $\alpha_s = 0$, too. Thus, $\frac{\partial^n}{\partial \alpha_t^n} V(s,t)$ n=1,2,3... has no longer a pole at $\alpha_s = 0$.
(9) gives in the neighbourhood of $\alpha_s = 0$, Re $\alpha_t < 0$

$$(33) \qquad V(s,t) = - \frac{P_o(\alpha_t)}{\alpha_s} + \text{regular}\ (\alpha_s, \alpha_t)$$

which then leads together with the above result to

$$(34) \qquad P_o'(\alpha_t) = 0 \quad \curvearrowright \quad P_o(\alpha_t) = \text{const}$$

Unfortunately, the further steps ($P_1'' = 0$ etc.) do not follow automatically.
We can only state

<u>Lemma 1 :</u>

If $(n+1)! \cdot \sum_{j=0}^{\infty} \frac{P_j(\alpha_s)}{(j-\alpha_t)^{n+2}}$ still converges for $\alpha_s = n$, i.e. if $\frac{\partial^{n+1}}{\partial \alpha_t^{n+1}} V(s,t)$ admits an unsubtracted Mittag-Leffler expansion for $\alpha_s = n$ which holds if $\left| \frac{\partial^{n+1}}{\partial \alpha_t^{n+1}} V(s,t) \right| \xrightarrow[|\alpha_t| \to \infty]{} 0$ $\alpha_s = n$, then P_n is a polynomial of degree n.

Now this assumption is fulfilled, if the amplitude possesses analytic Regge behaviour which means

$$(35) \qquad V(s,t) \underset{|\alpha_t| \to \infty}{\sim} \alpha_t^{\alpha_s} \qquad \alpha_s \text{ fixed}$$

and thus

$$(36) \qquad \frac{\partial^{n+1}}{\partial \alpha_t^{n+1}} V(s,t) \underset{|\alpha_t| \to \infty}{\sim} \alpha_t^{\alpha_s - n - 1}$$

On the other hand, polynomial residues do not assure Regge behaviour ([8], reference 5).

- - -

The results of this part I 1) can be collected in the following theorem:

Theorem:

Every symmetric meromorphic function $V(s,t)$ vanishing asymptotically in s for Re $\alpha_t < 0$ with simple poles at $s = s_i \quad i=1,2\ldots$ $0 \leq s_1 < s_2 < \ldots$, respectively $\alpha_s = \alpha(s) = j = 0,1,2\ldots$ (t being fixed), the residues $P_j(\alpha_t)$ of which are polynomials of degree j, admits the following expansion

$$V(s,t) = \sum_{l=0}^{\infty} R_l(\alpha_s + \alpha_t) \cdot V^{(l)}(s,t)$$

where $R_l(\alpha_s + \alpha_t)$ are polynomials of degree l and $\{V^{(l)}(s,t)\}$ is a family of special solutions with steadily improving asymptotic behaviour

$$|\alpha_s|^l \cdot V^{(l)}(s,t) \underset{|s| \to \infty}{\longrightarrow} 0 \qquad \text{Re } \alpha_t < 0$$

accompanied by a reduced pole spectrum and lowered degree of the residue polynomials

$$P_o^{(l)} = \ldots = P_{l-1}^{(l)} = 0 \qquad P_l^{(l)}(\alpha_t) = \text{const} \neq 0$$

$$P_j^{(l)}(\alpha_t) = \text{polynomial of degree} \leq j-l \qquad j > l$$

The simplest choice of the "base" functions $V^{(l)}(s,t)$ are the Veneziano amplitudes including satellite terms:

$$V^{(l)}(s,t) = B\,(l-\alpha_s,\; l-\alpha_t)$$

2) K > 0; K subtractions

Now our starting point is a K-times subtracted Mittag-Leffler expansion

$$V(s,t) =$$

(37) $$= \sum_{k=0}^{K-1} \frac{(\alpha_s - \alpha)^k}{k!} V^{(k)}(\alpha, \alpha_t) + (\alpha_s - \alpha)^K \cdot \sum_{j=0}^{\infty} \frac{P_j(\alpha_t)}{(j-\alpha)^K \cdot (j-\alpha_s)}$$
$$= \sum_{j=0}^{\infty} \frac{P_j(\alpha_t, \alpha_s)}{j - \alpha_s}$$

converging absolutely for Re $\alpha_t < 0$ apart from the poles, where $P_j(\alpha_t, \alpha_s)$ is a polynomial of degree K in the second argument.

As in part 1) we get

(38) $$V(s,t) = \int_0^1 dx\, x^{-\alpha_s - 1} \cdot F(x, \alpha_t, \alpha_s) \qquad \text{Re}\, \alpha_s < 0, \ \text{Re}\, \alpha_t < 0$$

with

(39) $$F(x, \alpha_t, \alpha_s) = \sum_{j=0}^{\infty} P_j(\alpha_t, \alpha_s) \cdot x^j$$

a polynomial of order K in the last argument.
Symmetry can be achieved by a substitution y(x) with

(40) $$x^{-\alpha_s - 1} \cdot F(x, \alpha_t, \alpha_s) = y(x)^{-\alpha_t - 1} \cdot F(y(x), \alpha_s, \alpha_t) \cdot |y'(x)|$$

This equation can only be fulfilled when

(41) $$F(x, \alpha_t, \alpha_s) = Q(x, \alpha_t, \alpha_s) \cdot y(x)^{-\alpha_t - 1}$$

and

(42) $$y(y(x)) = x$$

Q, which replaces the function f in 1), is a polynomial of degree K in both variables α_t and α_s obeying

(43) $$|y'(x)| \cdot Q(y(x), \alpha_s, \alpha_t) = Q(x, \alpha_t, \alpha_s)$$

If we put

(44) $$Q(x, \alpha_t, \alpha_s) = \sum_{k,l=0}^{K} b_{kl}(x) \cdot \alpha_t^{\,k} \cdot \alpha_s^{\,l}$$

the coefficients are subjected to the condition

(45) $$|y'(x)| \cdot b_{kl}(y(x)) = b_{lk}(x)$$

One possible choice is

(46) $$b_{kl}(x) = b_k \cdot b_l \cdot f(x) \cdot (y(x))^k \cdot x^l$$

with

(47) $$|y'(x)| \cdot f(y(x)) = f(x)$$

leading to

(48) $$Q(x, \alpha_t, \alpha_s) = f(x) \cdot Q_K(\alpha_t \cdot y(x)) \cdot Q_K(\alpha_s \cdot x)$$

with

(49) $$Q_K(x) = \sum_{l=0}^{K} b_l \cdot x^l$$

Another one would be

(50) $$b_{kl}(x) = b_{kl} \cdot f(x) \qquad\qquad b_{kl} = b_{lk}$$

leading to

(51) $$Q(x, \alpha_t, \alpha_s) = f(x) \cdot \sum_{k,l=0}^{K} b_{kl} \cdot \alpha_t^{\,k} \cdot \alpha_s^{\,l}$$

This choice covers the satellite terms $V^{(1)}(s,t)$ examined in I 1), if $f(x)$ has the appropriate improved asymptotic behaviour.

Altogether, we have in general

(52) $$V(s,t) = \int_0^1 dx\, x^{-\alpha_s - 1} \cdot (y(x))^{-\alpha_t - 1} \cdot Q(x, \alpha_t, \alpha_s)$$

or specially

(53) $$V(s,t) = \int_0^1 dx\, x^{-\alpha_s - 1} \cdot (y(x))^{-\alpha_t - 1} \cdot f(x) \cdot Q_K(\alpha_t \cdot y(x)) \cdot Q_K(\alpha_s \cdot x)$$

– – –

Of course, the number of subtractions affects the general properties of the solution.

Thus, the Regge behaviour as expected from (22) is

$$(54)\qquad V(s,t) \underset{|s|\to\infty}{\sim} \sum_{k=0}^{K} b_{kK}(0)\cdot \alpha_t^{K}\cdot \Gamma(-\alpha_t)\,[(\alpha_s+1)\cdot y'(0)]^{\alpha_t}\cdot \alpha_s^{K}$$

$$\sim |\alpha_s|^{\alpha_t+K} \qquad y'(0) \neq 0$$

$$b_{kK} \neq 0 \text{ at least for one } k$$

Because of

$$(55)\qquad P_j(\alpha_t,\alpha_s) = \frac{1}{j!}\frac{\partial^j}{\partial x^j}\left\{y(x)^{-\alpha_t-1}\cdot Q(x,\alpha_t,\alpha_s)\right\}_{x=0}$$

the "generalized" residues $P_j(\alpha_t,\alpha_s)$ turn out to be polynomials of degree $\leq j+K$ in α_t.

In [3] the polynomials Q_K in (53) are replaced by entire functions, so that this alternative neither admits a Mittag-Leffler expansion with a finite number of subtractions nor does it exhibit analytic Regge behaviour (only for $s \to -\infty$).

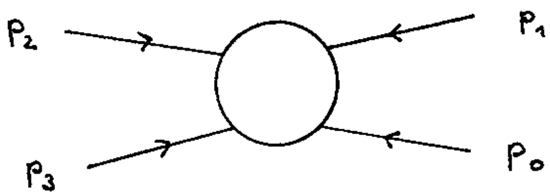
– – –

The further proceeding of looking for "base" functions and forming linear combinations with symmetric polynomials is exactly the same as in part I1) and can be omitted.

Part II :

In this section we want to study an operator approach to dual resonance models. Therefore, we start with a reproduction of the operator approach to the Veneziano amplitude ([6]), but in a slightly generalized form.

We deal with the scattering of scalar particles



with Euclidean momenta $\vec{p}_j = (p_{j1}, p_{j2}, \dots p_{jN})$, $j = 0,1,2,3$ after having performed a Wick rotation

(56) $$s = - (\vec{p}_0 + \vec{p}_1)^2 \qquad t = - (\vec{p}_1 + \vec{p}_2)^2$$

We define the operators $a_\nu^{(r)}$, $a_\mu^{(s)+}$ $\nu, \mu = 1,2,\dots N$; $r,s = 1,2,\dots \infty$ according to

(57) $$[a_\nu^{(r)}, a_\mu^{(s)+}] = \delta_{rs} \cdot \delta_{\mu\nu}$$

and

(58) $$H = \sum_{r=1}^{\infty} r \cdot H_r \quad \text{with} \quad H_r = \sum_{\nu=1}^{N} a_\nu^{(r)+} \cdot a_\nu^{(r)}$$

We define the vector $|0\rangle$ by

$$a_\nu^{(r)} |0\rangle = 0 \quad \text{for all } r,$$

and a 1-particle state by

(59) $$|\{l\}\rangle := \prod_{r=1}^{\infty} (l_1^{(r)}! \dots l_N^{(r)}!)^{-1/2} (a_1^{(r)+})^{l_1^{(r)}} \dots (a_N^{(r)+})^{l_N^{(r)}} |0\rangle$$

with

(60) $$l = \sum_{r=1}^{\infty} r \cdot l^{(r)} \qquad l^{(r)} = \sum_{\nu=1}^{N} l_\nu^{(r)}$$

and

(61) $$H |\{l\}\rangle = l |\{l\}\rangle$$

If we define the "physical" states by

(62) $$|\vec{p}\rangle := \exp\left\{ - \sum_{r=1}^{\infty} b_r \cdot \vec{p} \cdot \vec{a}^{(r)+} \right\} \cdot \exp\left\{ \sum_{r=1}^{\infty} b_r^* \cdot \vec{p} \cdot \vec{a}^{(r)} \right\} |0\rangle$$

with $b_r \in C$, we can construct a four-point fuction by

$$(63) \qquad M(s,t) = \langle \vec{p}_2 \mid D(\alpha_s, H) \mid \vec{p}_1 \rangle$$

with

$$(64) \qquad D(\alpha_s, H) = \sum_{l_o=0}^{\infty} c_{l_o} \cdot \frac{1}{H + l_o - \alpha_s} = \int_0^1 dx\, x^{-\alpha_s + H - 1} \left(\sum_{l_o=0}^{\infty} c_{l_o} \cdot x^{l_o} \right)$$

where a linear trajectory

$$(65) \qquad \alpha_s = \alpha_o + \frac{1}{2} s$$

has been assumed. Because of this definition of the propagator the amplitude has the desired pole spectrum in α_s. If we choose a distribution of the different oscillators in $|\vec{p}\rangle$ according to

$$(66) \qquad b_r = \frac{1}{\sqrt{r}}$$

and form the average of the propagators with

$$(67) \qquad c_{l_o} = \binom{l_o - a}{l_o} \qquad\qquad a = -\alpha_t - \vec{p}_1 \cdot \vec{p}_2$$

then $M(s,t)$ can be identified as Veneziano amplitude ([6]).

- - -

In the following we would like to examine the influence of the choice of b_r and c_{l_o} upon the analytic behaviour of the amplitude, and we will especially study the question which oscillator model of the vertex (described by b_r) leads to a dual amplitude.

Inserting a complete system of eigenstates of H into (63) gives

$$(68) \qquad M(s,t) = \sum_{l=0}^{\infty} F_l(\vec{p}_1 \cdot \vec{p}_2)\, D(\alpha_s, l) = \int_0^1 dx\, x^{-\alpha_s - 1} \left(\sum_{l_o=0}^{\infty} c_{l_o} \cdot x^{l_o} \right) \cdot \left(\sum_{l=0}^{\infty} F_l(\vec{p}_1 \cdot \vec{p}_2) \cdot x^{l} \right)$$

with

$$(69) \qquad F_l(\vec{p}_1 \cdot \vec{p}_2) = \sum_{\sum_{r=1}^{\infty} \sum_{\nu=1}^{N} r \cdot l_\nu^{(r)} = l} \langle \vec{p}_2 \mid \{l\} \rangle \langle \{l\} \mid \vec{p}_1 \rangle$$

or

$$(70) \qquad M(s,t) = \sum_{j=0}^{\infty} \frac{1}{j-\alpha_s} \sum_{l+l_0=j} c_{l_0} \cdot F_l(\vec{p}_1 \cdot \vec{p}_2)$$

where the residues turn out (cf. (77)) to be polynomials of degree j in $\vec{p}_1 \cdot \vec{p}_2$ resp. α_t.

As (68) shows, averaging with c_{l_0} only gives a power series in x whose radius of convergence has to be ≥ 1. In the Veneziano case this series is needed to guarantee full symmetry in s and t.
The complete t-dependence comes via the functions F_l building up the residues from the oscillator excitation of the vertices. We therefore look for a closed expression for

$$\sum_{l=0}^{\infty} F_l(\vec{p}_1 \cdot \vec{p}_2) \cdot x^l$$

We have

$$(71) \quad \langle \{l\} | \vec{p} \rangle = \langle 0 | \prod_{r=1}^{\infty} \prod_{\nu=1}^{N} (l_\nu^{(r)}!)^{-1/2} (a_\nu^{(r)})^{l_\nu^{(r)}} \cdot \exp\left\{ -\sum_{s=1}^{\infty} b_s \cdot \vec{p} \cdot \vec{a}^{(s)+} \right\} | 0 \rangle$$

Developing the exponential function into a series and realizing that all matrix elements vanish which do not possess exactly the same creation and annihilation operators leads to

$$(72) \quad \langle \{l\} | \vec{p} \rangle = \prod_{r=1}^{\infty} \prod_{\nu=1}^{N} (-1)^{l_\nu^{(r)}} \cdot b_r^{l_\nu^{(r)}} \cdot (l_\nu^{(r)}!)^{-1/2} \cdot (p_\nu)^{l_\nu^{(r)}}$$

and therefore

$$(73) \quad \langle \vec{p}_2 | \{l\} \rangle \langle \{l\} | \vec{p}_1 \rangle = \prod_{r=1}^{\infty} \prod_{\nu=1}^{N} \frac{(-1)^{l_\nu^{(r)}}}{l_\nu^{(r)}!} \cdot |b_r|^{2 l_\nu^{(r)}} \cdot (p_{2\nu} \cdot p_{1\nu})^{l_\nu^{(r)}}$$

If we put

$$(74) \qquad F_l(\vec{p}_1 \cdot \vec{p}_2) = \sum_{\sum_{r=1}^{\infty} r \cdot l^{(r)} = l} \; \prod_{r=1}^{\infty} F_{l^{(r)}}^{(r)}(\vec{p}_1 \cdot \vec{p}_2)$$

it follows

$$\left\{ F_{l^{(r)}}^{(r)}(\vec{p}_1 \cdot \vec{p}_2) = \sum_{\sum_{\nu=1}^{N} l_\nu^{(r)} = l^{(r)}} \; \prod_{\nu=1}^{N} \frac{(-1)^{l_\nu^{(r)}}}{l_\nu^{(r)}!} |b_r|^{2 l_\nu^{(r)}} (p_{2\nu} \cdot p_{1\nu})^{l_\nu^{(r)}} = \right.$$

(75) $$\left\{ = \frac{(-1)^{l^{(r)}}}{l^{(r)}!} \cdot |b_r|^{2l^{(r)}} \cdot (\vec{p}_1 \cdot \vec{p}_2)^{l^{(r)}} \right.$$

This gives

(76) $$F_l(\vec{p}_1 \cdot \vec{p}_2) = \sum_{\sum_{r=1}^{\infty} r \cdot l^{(r)} = l} \prod_{r=1}^{\infty} \frac{(-1)^{l^{(r)}}}{l^{(r)}!} \cdot |b_r|^{2l^{(r)}} \cdot (\vec{p}_1 \cdot \vec{p}_2)^{l^{(r)}}$$

Because of this weighted summation F_l becomes a polynomial of degree l in $\vec{p}_1 \cdot \vec{p}_2$:

(77) $$F_o = 1, \qquad F_l(p_1\ p_2) = \sum_{m=1}^{l} a_m^{(l)} \cdot (\vec{p}_1 \cdot \vec{p}_2)^m \qquad l=1,2,\ldots$$

with

(78) $$\left\{ \begin{aligned} a_m^{(l)} &= (-1)^m \sum_{\left\{ \begin{smallmatrix} \sum_{r=1}^{\infty} l^{(r)} = m \\ \sum_{r=1}^{\infty} r \cdot l^{(r)} = l \end{smallmatrix} \right\}} \prod_{r=1}^{\infty} \frac{|b_r|^{2l^{(r)}}}{l^{(r)}!} = \\ &= \frac{(-1)^m}{m!} \sum_{\nu_1+\nu_2+\cdots+\nu_m = l} |b_{\nu_1}|^2 \cdot |b_{\nu_2}|^2 \cdot \cdots \cdot |b_{\nu_m}|^2 \end{aligned} \right.$$

In getting equ. (78) we have used the fact that because of $\sum_{r=1}^{\infty} l^{(r)} = m$ we can write

$$\prod_{r=1}^{\infty} |b_r|^{2l^{(r)}} = |b_{\nu_1}|^2 \cdot |b_{\nu_2}|^2 \cdots |b_{\nu_m}|^2 \qquad \nu_i = 1,2,\ldots$$

where $l^{(r)}$ is the number of ν_i's equal to r. The condition $\sum_{r=1}^{\infty} r \cdot l^{(r)} = l$ leads to $\nu_1 + \nu_2 + \cdots + \nu_m = l$. At last, doing the summation, we have to realize that the contributing r-values can be distributed on $\nu_1, \nu_2, \ldots \nu_m$ in $\frac{m!}{\prod_{r=1}^{\infty} l^{(r)}!}$ different ways.

Thus we get

(79) $$\sum_{l=0}^{\infty} F_l(\vec{p}_1 \cdot \vec{p}_2) \cdot x^l = 1 + \sum_{l=1}^{\infty} x^l \cdot \sum_{m=1}^{l} \frac{(-1)^m}{m!} (\vec{p}_1 \cdot \vec{p}_2)^m \cdot \sum_{\nu_1+\nu_2+\cdots+\nu_m = l} |b_{\nu_1}|^2 \cdot |b_{\nu_2}|^2 \cdot \ldots \cdot |b_{\nu_m}|^2$$

or, by interchanging the order of summation

$$\left\{ \sum_{l=0}^{\infty} F_l(\vec{p}_1 \cdot \vec{p}_2) \cdot x^l = \right.$$

$$(80)\quad \begin{cases} = 1 + \sum\limits_{m=1}^{\infty} \frac{(-1)^m}{m!} (\vec{p}_1 \cdot \vec{p}_2)^m \sum\limits_{l=m}^{\infty} x^l \sum\limits_{\substack{\nu_1,\dots,\nu_m = 1,2,\dots \\ \nu_1+\dots+\nu_m = l}} |b_{\nu_1}|^2 \cdot \ldots \cdot |b_{\nu_m}|^2 = \\ = 1 + \sum\limits_{m=1}^{\infty} \frac{(-1)^m}{m!} (\vec{p}_1 \cdot \vec{p}_2)^m \left(\sum\limits_{l=1}^{\infty} |b_l|^2 \cdot x^l \right)^m = \exp\left(-\vec{p}_1 \cdot \vec{p}_2 \sum\limits_{l=1}^{\infty} |b_l|^2 \cdot x^l\right) \end{cases}$$

Thus the final structure of the amplitude is

$$(81)\qquad M(s,t) = \int_0^1 dx\, x^{-\alpha_s - 1} \left(\sum_{l_o=0}^{\infty} c_{l_o} \cdot x^{l_o} \right) \exp\left\{ (\alpha_t + a) \sum_{l=1}^{\infty} |b_l|^2 \cdot x^l \right\}$$

which reveals to us immediately the analytic behaviour in α_t or t, respectively:

If only a finite number of oscillators is excited in the vertex, i.e. only a finite number of $|b_l|$ is different from zero, we always have an exponential behaviour in α_t - a drastic violation of duality.

This result becomes interesting in connection with the attempt to desribe mesons as strongly bound quark-antiquark systems whose interaction is approximated by a single harmonic oscillator potential ([13]).

Singularities in t can only appear if an infinite number of oscillators is excited and if the power series

$$\sum_{l=1}^{\infty} |b_l|^2 \cdot x^l$$

possesses a radius of convergence equal to 1 and diverges at x = 1.

(The operator approach to "DAMA" ([4]) needs infinitely many oscillators, too.)

A comparison with part I 1) gives us more accurate conditions for duality to hold:

From (18) and (81) we conclude

$$(82)\qquad \sum_{l=1}^{\infty} |b_l|^2 \cdot x^l \stackrel{!}{=} -\ln y(x)$$

$$(83)\qquad \sum_{l_o=0}^{\infty} c_{l_o} \cdot x^{l_o} \stackrel{!}{=} y(x)^{a-1} \cdot f(x)$$

Certainly not all allowed y(x) can be written as a power series (82) with equal signs. Thus we have constructed an operator approach only to those Veneziano alternatives for which equ. (82) holds.

((83) can always be fulfilled.)

By looking carefully at the signs of each term in the derivatives of ln y(x) we find:

Lemma 2 :

All solutions (18)

$$V(s,t) = \int_0^1 dx\; x^{-\alpha_s - 1} \cdot y(x)^{-\alpha_t - 1} \cdot f(x)$$

with

$$y^{(k)}(0) \leq 0 \qquad k = 1,2,\ldots$$

admitting an operator approach of the above sort.

I want to thank Prof. H. Joos for a long and fruitful discussion.

APPENDIX A :

The absolute convergence of

$$\sum_{j=0}^{\infty} \frac{P_j(\alpha_t)}{j-\alpha_s} \qquad \mathrm{Re}\,\alpha_t < 0; \quad \alpha_s \neq 0,1,2,\ldots$$

tells us that

$$\text{(A1)} \qquad q_j(\alpha_s,\alpha_t) := \left|\frac{P_{j+1}(\alpha_t)}{P_j(\alpha_t)}\right| \cdot \left|\frac{j-\alpha_s}{j+1-\alpha_s}\right| \xrightarrow[j\to\infty]{} q(\alpha_t) \leq 1$$

if we assume the identity of $\overline{\lim}$ and $\underline{\lim}$ in the quotient criterion.
Defining

$$\text{(A2)} \qquad p_j(\alpha_t) := \left|\frac{P_{j+1}(\alpha_t)}{P_j(\alpha_t)}\right|$$

we get

$$\text{(A3)} \qquad q_j = p_j \left(1 - \frac{1}{j} + 0\left(\frac{1}{j^2}\right)\right) \xrightarrow[j\to\infty]{} q \leq 1$$

and thus

$$\text{(A4)} \qquad p_j \cdot x \xrightarrow[j\to\infty]{} q\cdot x < 1 \qquad \text{if} \begin{cases} 0 \leq x < 1 & q \leq 1 \\ & \text{or} \\ 0 \leq x \leq 1 & q < 1 \end{cases}$$

Therefore, absolute convergence of

$$\sum_{j=0}^{\infty} P_j(\alpha_t)\cdot x^j \qquad 0 \leq x \leq 1$$

is established except in the case $q = 1$, $x = 1$.
If we write in the case of $q = 1$

$$\text{(A5)} \qquad q_j(\alpha_s,\alpha_t) = 1 - \frac{\rho(\alpha_t)}{j} + 0\left(\frac{1}{j^2}\right)$$

the criterion of Gauss demands

$$\text{(A6)} \qquad \rho > 1$$

as otherwise

$$\sum_{j=0}^{\infty} \frac{P_j(\alpha_t)}{j-\alpha_s}$$

would diverge. Because of (A3), this leads to

(A7) $$P_j(\alpha_t) = 1 - \frac{\varrho(\alpha_t) - 1}{j} + O(\frac{1}{j^2})$$

and we can conclude, again by the criterion of Gauss:

In the case of $\varrho > 2$ the convergence of the original series has to be strong enough to guarantee absolute convergence of

$$\sum_{j=0}^{\infty} P_j(\alpha_t)\cdot x^j$$

at $x = 1$ too, but for $1 < \varrho \leq 2$ this mechanism fails to work,

$$\sum_{j=\sigma}^{\infty} P_j(\alpha_t)\cdot x^j$$

diverges at $x = 1$.

Since a Gaussian coefficient $\tilde{\varrho}$ with $0 < \tilde{\varrho} < 1$ describes the divergence at $x = 1$ of a pole term $(1-x)^m$ with $-1 < \mathrm{Re}\, m = \tilde{\varrho} - 1 \leq 0$ and $\tilde{\varrho} = 1$ the divergence of a logarithmic term $\ln(1-x)$, we can conclude that (even in the case of divergence at $x = 1$)

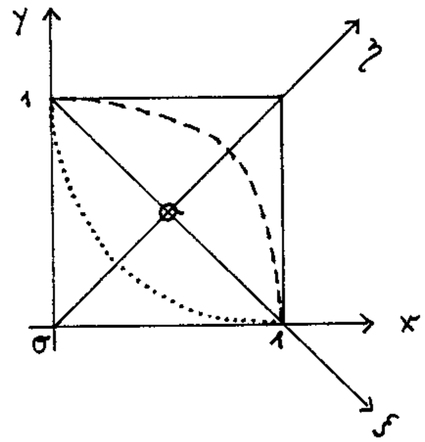
$$F(x, \alpha_t) = \sum_{j=\sigma}^{\infty} P_j(\alpha_t)\cdot x^j$$

still remains integrable.

APPENDIX B :

Now we want to construct "base" systems competitive to the B-functions: y(x) has to be a smooth monotonically decreasing function ($y(0) = 1$, $y(1) = 0$) with $y = y^{-1}$, i.e. won by reflection at the line $g(x) = x$. Since the only polynomial in x to do the job is $y(x) = 1 - x$, we choose as an example the parables upon the line $g(x) = 1 - x$

(B1)



$$\eta(\xi) = \frac{a}{2\sqrt{2}}\left(1 - 2\xi^2\right)$$

$$-1 \leq a \leq +1$$

which gives

(B2)
$$y(x) = \begin{cases} x - \frac{1}{a} + \frac{1}{a}\cdot\sqrt{(1+a)^2 - 4ax} & a \neq 0 \\ 1 - x & a = 0 \end{cases}$$

The accompanying function f(x) has to obey (17) :

(B3)
$$- y'(x)\cdot f(y(x)) = f(x)$$

Candidates always are real linear combinations of

(B4)
$$f_n(x) = [y'(x)]^n \cdot [1 - y'(x)]^{1-2n} \qquad n = 0, \pm 1, \pm 2, \ldots$$

reducing to constants in the Veneziano case $a = 0$.
Since

$$F(x, \alpha_t) = f(x)\cdot y(x)^{-\alpha_t - 1}$$

has to converge absolutely for $|x| < 1$, we have to avoid branch points in this interval by restricting a to

(B5)
$$-3 + 2\sqrt{2} \leq a \leq 1$$

The complete pole spectrum is reproduced if $f(0) \neq 0$. In this case,

(B2) demands that possible divergencies of f and y' for $x \to 1$ have to happen in the same way:
If e.g. y runs like

(B6) $$y(x) \underset{x \to 1}{\longrightarrow} (1 - x)^m \qquad 0 < m \leq 1$$

this has to coincide with

(B7) $$f(x) \underset{x \to 1}{\longrightarrow} (1 - x)^{m-1}$$

Thus, the Gaussian coefficient (cf. appendix A) will be equal to $-m \text{ Re } \alpha_t$.
For (B2) we find

(B8) $$m = \begin{cases} 1/2 & a = 1 \\ 1 & \text{otherwise} \end{cases}$$

\- - -

In the following we want to study the simplest candidate for $f(x)$, namely

(B9) $$f(x) = 1 - y'(x)$$

with the nice property (cf. (20), (22))

(B10) $$f(0) \neq 0$$

Residues and Regge behaviour can be calculated by means of the derivatives

(B11) $$\begin{cases} y(0) = 1 \\ y'(0) = -\frac{1-a}{1+a} \\ y^{(k)}(0) = -2^k \cdot 1 \cdot 3 \cdots (2k-3) \cdot a^{k-1} \cdot (1+a)^{-2k+1} & k \geq 2 \end{cases}$$

(B12) $$\begin{cases} f_o(0) = 1 - y'(0) \\ f_o^{(k)}(0) = -y^{(k+1)}(0) & k \geq 1 \end{cases}$$

The case $a = 1$ is the only one with $y'(0) = 0$ which yields a Regge behaviour (cf. (23)) modified with respect to that of the Veneziano amplitude.
According to (19), the residues are

$$\text{(B13)} \quad P_j(\alpha_t) = \frac{1}{j!}\frac{\partial^j}{\partial x^j}\left\{y(x)^{-\alpha_t-1}\right\}_{x=0} + \frac{1}{\alpha_t \cdot j!}\frac{\partial^{j+1}}{\partial x^{j+1}}\left\{y(x)^{-\alpha_t}\right\}_{x=0}$$

It is easy to show that in the case of $y^{(k)}(0) \leq 0 \quad k = 1,2,\ldots$ i.e. $a \geq 0$ both terms in (B13) separately lead to polynomials with coefficients of only equal (positive) sign. This property reflects a characteristic of the elastic scattering of two spinless particles for all poles above threshold.

The choice $a < 0$, however, leads in general to nonuniform signs.

In part II it is shown that the amplitudes belonging to $a \geq 0$ are distinguished in another respect too: They admit an operator approach.

– – –

Now we are able to build a competitive family of "base" functions $V^{(1)}(s,t)$ by choosing

$$\text{(B14)} \qquad y(x) = x - \frac{1}{a} + \frac{1}{a}\sqrt{(1+a)^2 - 4ax} \qquad -3 + 2\sqrt{2} \leq a < 1$$

and putting

$$\text{(B15)} \qquad f^{(1)}(x) = x^1 \cdot [y(x)]^1 \cdot (1 - y'(x))$$

References :

[1] : G. Veneziano, in Elementary Processes at High Energy, part A Erice 1970

[2] : G. Veneziano, Nuov. Cim. 57A, 190 (1968)

[3] : A. Hosoya, N. Tokuda, Nuov. Cim. 13A, 813 (1973)

[4] : A. Bugrij, G. Cohen-Tannoudji, L. Jenciszky, N. Kobylinsky, Review on DAMA, preprint Kiev 1972

[5] : D.D. Coon, D.A. Geffen, Phys. Rev. Lett. 27, 627 (1971) Slac-Pub- 1090 (1972)

[6] : V. Alessandrini, D. Amati, M. Le Bellac, D. Olive, Phys. Rep. 1C,6 (1971)

[7] : N. Khuri, Phys. Rev. 185, 1876 (1969)

[8] : S. Matsuda, preprint CALT-68-188 (1969)

[9] : E.T. Whittaker, G.N. Watson, Modern Analysis Cambridge University Press (1963)

[10] : V. Alessandrini, D. Amati, B. Morel, Nuov. Cim. 7A,797 (1972) appendix

[11] : A. Erdély ed., Higher Transcendental Functions Vol I, p. 14 (1953)

[12] : G. Altarelli, H.R. Rubinstein, Phys. Rev. 183, 1469 (1969)

[13] : G. Berendt, E. Weimar, On Scattering Amplitudes of Quark-Anti-quark Bound States, preprint Feb. 1973