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BFKL approach and six-particle MHV amplitude in $\mathcal{N} = 4$ super Yang-Mills

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Abstract

We consider the planar MHV amplitude in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory for $2 \to 4$ particle scattering at two and three loops in the Regge kinematics. We perform an analytic continuation of two-loop result for the remainder function found by Goncharov, Spradlin, Vergu and Volovich to the physical region, where the remainder function does not vanish in the Regge limit. After the continuation both the leading and the subleading in the logarithm of the energy terms are extracted and analyzed. Using this result we calculate the next-to-leading corrections to the impact factors required in the BFKL approach. The BFKL technique was used to find the leading imaginary and real parts of the remainder function at three loops.

1 Introduction

Recently we have witnessed revolutionary developments in studying scattering amplitudes in supersymmetric theories. The present progress is traced back to the work of Parke and Taylor [1], who showed that the tree-level gluon scattering amplitude can be written in a very compact form for some particular helicities of the external particles, namely the maximally helicity violating (MHV) amplitude. The simplicity of the Parke-Taylor tree-level formula raised a hope that the quantum corrections could be also compactly encoded in the MHV gluon amplitudes. A great effort in this direction led to formulation by Anastasiou, Bern, Dixon and Kosower (ABDK) [2] and then by Bern, Dixon and Smirnov (BDS) [3] ansatz for multi-loop planar gluon MHV amplitude in $\mathcal{N}=4$ super Yang-Mills theory.

The BDS formula was tested by Alday and Maldacena [4] from strong coupling side using conjectured AdS/CFT correspondence in the limit of large number of external legs. They argued that the BDS ansatz is probably to be violated starting at six external gluons. This violation was established by Bartels, Sabio Vera and one of the authors (BLS) [5] analyzing the analytic structure of the BDS amplitude. It was shown that the BDS ansatz for six-particle amplitude at two loops is not compatible with the Steinmann relations [6], that impose the absence of the simultaneous singularities in the overlapping channels. They also showed [7] that the BDS violating piece originates from the so-called Mandelstam cuts, which are the moving Regge singularities in the complex angular momenta plane. The BDS violating term in the multi-Regge kinematics was explicitly calculated [7] with logarithmic accuracy in the physical region, where it gives a non-vanishing and pure imaginary contribution. We call this region the Mandelstam region (channel). The BDS violating term for the six-point planar MHV amplitude was found using the BFKL approach [8] and for an arbitrary number of external gluons it contains contributions of Mandelstam cuts constructed from an arbitrary number of reggeized gluons for the Bartels-Kwiecinski-Praszalowicz (BKP) state [9, 10] with the local Hamiltonian of an integrable open Heisenberg spin chain [11]. Other interesting limits of MHV amplitudes were studied in the Regge kinematics by Brower, Nastase, Schnitzer and Chung-I Tan [12].

Drummond, Henn, Korchemsky and Sokatchev [13] analyzed the conformal properties of polygon Wilson loops in $\mathcal{N}=4$ SYM and showed that anomalous conformal Ward identities uniquely fix the form of the all-loop 4- and 5-point amplitudes, so that any relative correction to the BDS ansatz starting at six external particles should be a function of conformal invariants (cross ratios of dual coordinates). The relative correction to the BDS formula was named the remainder function $R_n^{(L)}$ for an amplitude with L loops and n external legs, and the first non-trivial remainder function is $R_6^{(2)}$.

It was suggested [14, 15, 16, 17, 18, 19] that $R_n^{(L)}$ can be obtained from the expectation value of the light-like polygonal Wilson loops. Del Duca, Duhr and Smirnov [20, 21] expressed $R_6^{(2)}$ in terms of generalized polylogarithms, which was greatly simplified by Goncharov, Spradlin, Vergu and Volovich (GSVV) [22], and then written only in terms of Li_k functions with arguments depending on three dual conformal cross ratios.

The two-loop remainder function $R_8^{(2)}$ for the scattering of eight gluons was calculated by Del Duca, Duhr and Smirnov [23] and its diagrammatic structure was analyzed by Alday [24]. The form of $R_8^{(2)}$ is remarkably simple and it is constructed only of a product of some logarithms plus a constant term.

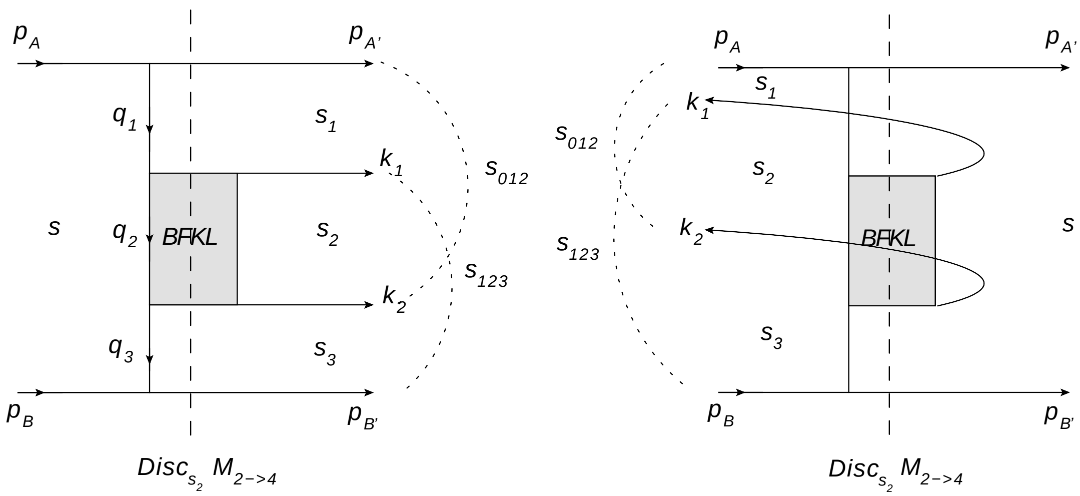
Earlier we performed [25] an analytic continuation of the GSVV formula to a physical region considered in refs. [5, 7]. The continuation showed a full agreement between the BLS formula and the Wilson loop calculations at the leading logarithmic level and allowed to extract the terms subleading in the logarithm of the energy. Numerically, an agreement between the two approaches was demonstrated by Schabinger [26]. The analytic continuation to the mentioned above physical region in the regime of the strong coupling constant was performed by Bartels, Kotanski and Schomerus [27]. They found the leading singularity, which governs the high energy behavior of the scattering amplitude, so-called reggeon intercept. At weak coupling constant the corresponding intercept is determined by the BFKL equation [7].

In this study we present some details of the analytic continuation performed by the authors in ref. [25]. Based on the obtained result we calculate the next-to-leading (NLO) impact factors for the color octet states in the BFKL approach. In the BFKL technique we also find the three-loop contribution to the remainder function of planar six-point MHV amplitude in the leading logarithmic approximation (LLA) as well as the real part of the subleading corrections in the next-to-leading logarithmic approximation.

2 BFKL approach

In this section we briefly outline the results of the BFKL approach to the planar MHV amplitudes in $\mathcal{N}=4$ SYM.

We consider the six-point MHV amplitude for production of two gluons with momenta k_1 and k_2 in small angle scattering of the particles with momenta p_A and p_B as depicted in Fig. 1.



(a) $u_1 = e^{i0}|u_1|$ $(s, s_2, s_1, s_3, s_{012}, s_{123} > 0)$ (b) $u_1 = e^{-i2\pi}|u_1|$ $(s, s_2 > 0;\ s_1, s_3, s_{012}, s_{123} < 0)$

Figure 1: The BDS violating contribution appears in the region $s, s_2 > 0$; $s_1, s_3 < 0$.

All energy invariants are shown in Fig. 1 and are related to the dual conformal cross ratios by

$$u_1 = \frac{s\ s_2}{s_{012}\ s_{123}}, \quad u_2 = \frac{s_1\ t_3}{s_{012}\ t_2}, \quad u_3 = \frac{s_3\ t_1}{s_{123}\ t_2}. \tag{1}$$

The multi-Regge kinematics is equivalent to having $s \gg s_{012}, s_{123} \gg s_1, s_2, s_3 \gg t_1, t_2, t_3$, which in the terms of the cross ratios reads

$$1-u_1 \to +0, \ \ u_2 \to +0, \ \ u_3 \to +0, \ \ \frac{u_2}{1-u_1} \simeq \mathcal{O}(1), \ \ \frac{u_3}{1-u_1} \simeq \mathcal{O}(1). \tag{2}$$

In this kinematics the remainder function of the MHV amplitude goes to zero in direct channel in Fig. 1a, while in the Mandelstam channel Fig. 1b grows with $\ln s_2$ and becomes pure imaginary. In the Mandelstam channel the gluon momenta k_1 and k_2 are flipped and the cross ratio u_1 possesses a phase

$$u_{1b} = u_{1a} e^{-i2\pi}, \tag{3}$$

leading to necessity of an analytic continuation of R_6. It was demonstrated by Bartels, Sabio Vera and one of the authors [5, 7] that the BDS violating piece comes from the Mandelstam cut state propagating in the crossing channel between the produced particles k_1 and k_2, and denoted by the dark box in Fig. 1. In $\mathcal{N} = 4$ SYM (as well in QCD) for a large number of colors this state is described by the color octet BFKL evolution equation [8]. The BFKL equation can be formulated as the Schrödinger equation with a Hamiltonian equivalent to that of a completely integrable open Heisenberg spin chain model [11], which made it possible to solve it analytically [7]. In the direct channel of the multi-Regge kinematics (see Fig. 1a) given by Eq. 2 the remainder function vanishes due to the Mandelstam cancellation of cut contributions as was shown in ref. [5].

The BDS violating piece in the Mandelstam channel is given by [7]

$$M_{2\to 4} = M_{2\to 4}^{BDS}\,(1 + i\Delta_{2\to 4}), \tag{4}$$

where $M_{2\to 4}^{BDS}$ is the BDS amplitude [3] and the correction $\Delta_{2\to 4}$ was calculated in all orders with a leading logarithmic accuracy using the solution to the octet BFKL equation. The all-orders LLA expression for $\Delta_{2\to 4}$ reads

$$\begin{aligned}\Delta_{2\to 4} &= \frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^* k_1^*}{k_2^* q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3 k_1}{k_2 q_1}\right)^{i\nu+\frac{n}{2}}\left(s_2^{\omega(\nu,n)}-1\right) \\ &\simeq \frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^* k_1^*}{k_2^* q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3 k_1}{k_2 q_1}\right)^{i\nu+\frac{n}{2}}\left((1-u_1)^{-\omega(\nu,n)}-1\right)\end{aligned} \tag{5}$$

Here k_1, k_2 are transverse components of produced gluon momenta, q_1, q_2, q_3 are the momenta of reggeons in the corresponding crossing channels and

$$\omega(\nu, n) = -aE_{\nu,n}. \tag{6}$$

The perturbation theory parameter a and the eigenvalue of the color octet BFKL $E_{\nu,n}$ are given by

$$a = \frac{g^2 N_c}{8\pi^2}\left(4\pi e^{-\gamma}\right)^{\epsilon} \Rightarrow \frac{\alpha_s N_c}{2\pi} \tag{7}$$

and

$$E_{\nu,n} = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1 + i\nu + \frac{|n|}{2}\right) + \psi\left(1 - i\nu + \frac{|n|}{2}\right) - 2\psi(1), \tag{8}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$, γ is the Euler constant $\gamma = -\psi(1)$ and the dimensional regularization parameter ϵ is defined by $d = 4 - 2\epsilon$.

The second line of Eq. 5 follows from the fact that in the Regge kinematics the energy invariant s_2 is related to the cross ratio u_1 by

$$1 - u_1 \simeq \frac{(\mathbf{k}_1 + \mathbf{k}_2)^2}{s_2} = \frac{s_0}{s_2}. \tag{9}$$

This way we set $s_0 = (\mathbf{k}_1 + \mathbf{k}_2)^2$ to be an energy scale, which becomes relevant only beyond leading logarithmic approximation. The choice of the energy scale s_0 is natural in the Regge kinematics because it reflects the smallness of the transverse components with respect to the longitudinal components of the particle momenta. The expression in Eq. 5 is a function of the dual conformal cross ratios u_i as discussed in section 4. The BDS violating piece at two loops found in ref. [7] can be written in terms of the reduced cross ratios as

$$a^2 R^{(2)\ LLA} = i\Delta_{2\to4} = -ia^2\frac{\pi}{2}\ln s_2 \ln\left(\frac{|\mathbf{k}_2+\mathbf{k}_1|^2|\mathbf{q}_2|^2}{|\mathbf{k}_2|^2|\mathbf{q}_1|^2}\right)\ln\left(\frac{|\mathbf{k}_2+\mathbf{k}_1|^2|\mathbf{q}_2|^2}{|\mathbf{k}_1|^2|\mathbf{q}_3|^2}\right) \tag{10}$$
$$\simeq ia^2\frac{\pi}{2}\ln(1-u_1)\ln\tilde{u}_2\ln\tilde{u}_3$$

using Eq. 9 and the fact that the reduced cross ratios

$$\tilde{u}_2 = \frac{u_2}{1-u_1}, \quad \tilde{u}_3 = \frac{u_3}{1-u_1} \tag{11}$$

in the multi-Regge kinematics are given by

$$\tilde{u}_2 \simeq \frac{|\mathbf{k}_2|^2|\mathbf{q}_1|^2}{|\mathbf{k}_2+\mathbf{k}_1|^2|\mathbf{q}_2|^2}, \quad \tilde{u}_3 \simeq \frac{|\mathbf{k}_1|^2|\mathbf{q}_3|^2}{|\mathbf{k}_2+\mathbf{k}_1|^2|\mathbf{q}_2|^2}. \tag{12}$$

Surprisingly, the expression in Eq. 10 can be obtained from the BDS formula using only general analytic properties of the scattering amplitudes and the factorization hypothesis (proposed by Alday and Maldacena [14]) as it was shown by one of the authors [28]. In this technique it is enough to know the form of the BDS amplitude at one loop to obtain the leading logarithmic imaginary term at two loops. Unfortunately, for the three loops the knowledge of only the BDS formula is not enough and some extra information is to be included in the analysis. This may come from the full analytic form of the remainder function at two loops $R_6^{(2)}$. The function $R_6^{(2)}$ was calculated by Drummond, Henn, Korchemsky and Sokatchev [29] using the duality between the light-like Wilson loops and the MHV amplitudes, and then greatly simplified by Goncharov, Spradlin, Vergu and Volovich [22] using the integral representation of Del Duca, Duhr and Smirnov [20, 21]. In the next section we perform the analytic continuation of the two-loop remainder function calculated by Goncharov, Spradlin, Vergu and Volovich to the region where $u_1 = |u_1|e^{-i2\pi}$, which corresponds to the Mandelstam channel in Fig. 1b in the multi-Regge kinematics.

3 Analytic continuation

In this section we discuss some details of the analytic continuation to the Mandelstam channel in the Regge kinematics. We also show how the kinematics determines the physical region of the cross ratios and establish the match between our picture and the one drawn by Alday, Gaiotto and Maldacena [30].

The result of Goncharov, Spradlin, Vergu and Volovich [22] for the two-loop remainder function reads [1]

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \chi\frac{\pi^2}{12}\left(J^2+\zeta(2)\right), \tag{13}$$

[1] When the present manuscript was already at the last stage of the preparation, a new version of ref. [22] appeared. The non-analytic term χ was eliminated in the new version. This fact does not affect our result so that here we use the initial version of the GSVV formula.

where

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3}, \tag{14}$$

and $\Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$.

The function $L_4(x^+,x^-)$ is defined by

$$L_4(x^+,x^-) = \sum_{m=0}^{3} \frac{(-1)^m}{(2m)!!} \log(x^+x^-)^m (\ell_{4-m}(x^+) + \ell_{4-m}(x^-)) + \frac{1}{8!!}\log(x^+x^-)^4, \tag{15}$$

together with

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n \mathrm{Li}_n(1/x)\right), \tag{16}$$

as well as the quantities

$$J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-)), \tag{17}$$

and

$$\chi = \begin{cases} -2 & \Delta > 0 \text{ and } u_1+u_2+u_3 > 1, \\ +1 & \text{otherwise.} \end{cases} \tag{18}$$

The result of the analytic continuation to the Mandelstam channel illustrated in Fig. 1b, where $u_1 = |u_1|e^{-i2\pi}$, was presented by the us in ref. [25] and reads (see Appendices D and E for more details)

$$\begin{aligned} &R_6^{(2)\ LLA+NLLA}(|u_1|e^{-i2\pi}, |z|^2(1-u_1), |1-z|^2(1-u_1)) \simeq \frac{i\pi}{2}\ln(1-u_1)\ln|z|^2\ln|1-z|^2 \\ &+\frac{i\pi}{2}\ln\left(|z|^2|1-z|^2\right)\left(\ln z \ln(1-z) + \ln z^* \ln(1-z^*) - 2\zeta_2\right) \\ &+\frac{i\pi}{2}\ln\frac{|1-z|^2}{|z|^2}\left(\mathrm{Li}_2(z) + \mathrm{Li}_2(z^*) - \mathrm{Li}_2(1-z) - \mathrm{Li}_2(1-z^*)\right) \\ &+i2\pi\left(\mathrm{Li}_3(z) + \mathrm{Li}_3(z^*) + \mathrm{Li}_3(1-z) + \mathrm{Li}_3(1-z^*) - 2\zeta_3\right). \end{aligned} \tag{19}$$

In Eq. 19 we introduced complex variables

$$z = \sqrt{\frac{u_2}{1-u_1}}e^{i\phi_2} = \sqrt{\tilde{u}_2}e^{i\phi_2}, \quad 1-z = \sqrt{\frac{u_3}{1-u_1}}e^{-i\phi_3} = \sqrt{\tilde{u}_3}e^{-i\phi_3} \tag{20}$$

to remove some square roots in the arguments of the polylogarithms (see Eq. 14). The phases ϕ_2 and ϕ_3 can be easily expressed in terms of the cross ratios u_i and have a meaning of the angles of the "unitarity" triangle illustrated in Fig. 2. More details on this parametrization are presented in the appendix E.

The first term on RHS of Eq. 19 reproduces the leading logarithm term found by Bartels, Sabio Vera and one of the authors [7] in the BFKL approach as explained below. It is easy to see from Eq. 20 that

$$\ln|z|^2 \ln|1-z|^2 = \ln\tilde{u}_2 \ln\tilde{u}_3 \tag{21}$$

and thus the first term on RHS in Eq. 19 equals to $R^{(2)\ LLA}$ in Eq. 10.

Other terms correspond to the next-to-leading logarithmic approximation (NLLA) and they present a new result, which is yet to be calculated using the BFKL technique. This analysis

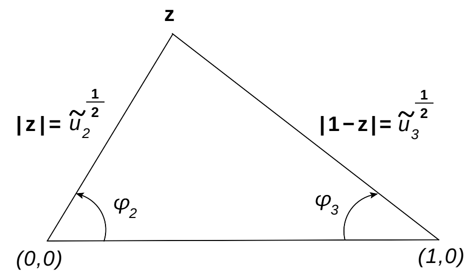


Figure 2: The "unitarity" triangle.

shows an agreement between the conjectured duality between the light-like Wilson loops and the MHV amplitudes, and the BFKL approach at the leading logarithmic level.

The remainder function of Eq. 19 in this channel in the multi-Regge limit is pure imaginary and symmetric under the substitution $z \leftrightarrow 1-z$, which corresponds to the target-projectile symmetry in Fig. 1. Eq. 19 vanishes for $z \to 1$ or $z \to 0$, when the momentum of one of the produced particles k_i in Fig. 1 goes to zero, in an accordance to the expectation that in the collinear limit the six-point amplitude reduces to the five-point amplitude.

Another useful form of the remainder function in Eq. 19 can be written as

$$R_6^{(2)\,LLA+NLLA}\left(|u_1|e^{-i2\pi}, \frac{1}{|1+w|^2}, \frac{|w|^2}{|1+w|^2}\right) \simeq \frac{i\pi}{2}\ln(1-u_1)\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2 \quad (22)$$
$$+\frac{i\pi}{2}\ln|w|^2\ln^2|1+w|^2 - \frac{i\pi}{3}\ln^3|1+w|^2 + i\pi\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right)$$
$$-i2\pi\left(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\right),$$

where the complex variable w is expressed in terms of the reduced cross ratios of Eq. 11 as

$$w = \frac{1-z}{z} = \frac{B^+}{\tilde{u}_2}, \quad w^* = \frac{1-z^*}{z^*} = \frac{B^-}{\tilde{u}_2} \quad (23)$$

for $B^\pm$ defined in Eq. D.4 by

$$B^\pm = \frac{1-\tilde{u}_2-\tilde{u}_3 \pm \sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2 - 4\tilde{u}_2\tilde{u}_3}}{2}. \quad (24)$$

The complete discussion on the details of the analytic continuation and the z and w representations of the remainder function is presented in the appendices A-E, and here we only want to emphasize some important points. The analytic continuation was performed under an assumption that $u_1 + u_2 + u_3 < 1$ to avoid a difficulty related to non-analyticity of χ in Eq. 18. We made sure that after the continuation the remainder function does not have any singularities on the border of this region and therefore it is valid in the whole physical region.

The variables $\tilde{u}_2$ and $\tilde{u}_3$ in Eq. 11 are also cross ratios in the transverse momentum space as can be seen from Eq. 12 defining the dual coordinates in the transverse momenta space as illustrated in Fig. 3.

In terms of the dual coordinates the reduced cross ratios in Eq. 12 read

$$\tilde{u}_2 = \frac{|x_{0B}|^2|x_{0'A}|^2}{|x_{AB}|^2|x_{00'}|^2}, \quad \tilde{u}_3 = \frac{|x_{0A}|^2|x_{0'B}|^2}{|x_{AB}|^2|x_{00'}|^2}. \quad (25)$$

Due to the Möbius invariance we can put

$$x_A = 1, \quad x_B = 0, \quad x_{0'} = \infty, \quad x_0 = z, \quad (26)$$

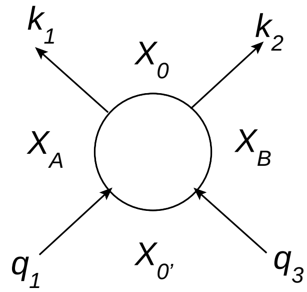


Figure 3: The dual coordinates of the transverse momenta.

then

$$\tilde{u}_2 = |z|^2, \quad \tilde{u}_3 = |1 - z|^2, \tag{27}$$

for z given by Eq. 20. This imposes a restriction on the possible values of the reduced cross ratios as illustrated in Fig. 4 (see ref. [25] for more details).

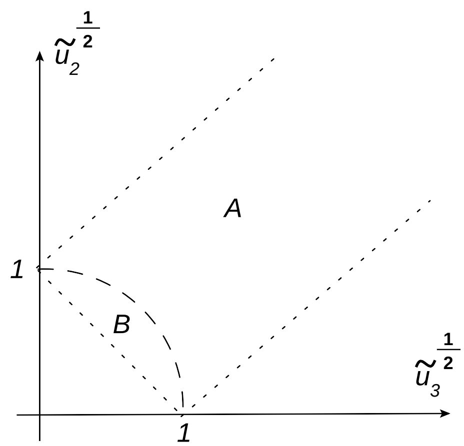


Figure 4: The region of the reduced cross ratios where the analytic continuation is performed.

The region **A** in Fig. 4 is the region of possible values of the reduced cross ratios, which correspond to the particle momentum parametrization. Its subregion **B** is the region, where the analytic continuation is performed. As it was already mentioned we made sure that our result is valid in the whole region **A**.

The same conclusion concerning the physical region of the cross ratios can be reached using the parametrization of the cross ratios introduced by Goncharov, Spradlin, Vergu and Volovich [22]. One can parametrize the cross ratios by six complex variables, namely

$$u_1 = \frac{z_{23} z_{56}}{z_{25} z_{36}}, \quad u_2 = \frac{z_{16} z_{34}}{z_{14} z_{36}}, \quad u_3 = \frac{z_{12} z_{45}}{z_{14} z_{25}}, \tag{28}$$

where $z_{ij} = z_i - z_j$. In this parametrization the square roots in the arguments of the remainder function Eq. 13 disappear and $x_i^{\pm}$ are rational functions. Exploiting the conformal invariance we can set

$$z_4 = 0, \quad z_5 = 1, \quad z_6 = \infty. \tag{29}$$

Then Eq. 28 reads

$$u_1 = \frac{z_3 - z_2}{1 - z_2}, \quad u_2 = \frac{z_3}{z_2}, \quad u_3 = \frac{z_1 - z_2}{z_1(1 - z_2)}. \tag{30}$$

Solving Eq. 30 for z_i we obtain some square roots that determine the physical region. For example, one of the solutions is given by

$$z_1 = \frac{-1 + u_1 + u_2 + u_3 \pm \sqrt{(1 - u_1 - u_2 - u_3)^2 - 4u_1 u_2 u_3}}{2u_2 u_3} \tag{31}$$

The argument of the square root coincides with $\Delta = (1 - u_1 - u_2 - u_3)^2 - 4u_1u_2u_3$ defined in Eq. 14. The surface $\Delta = 0$ determines the boundary of the space of the physical values of the cross ratios. This surface is depicted in Fig. 5.

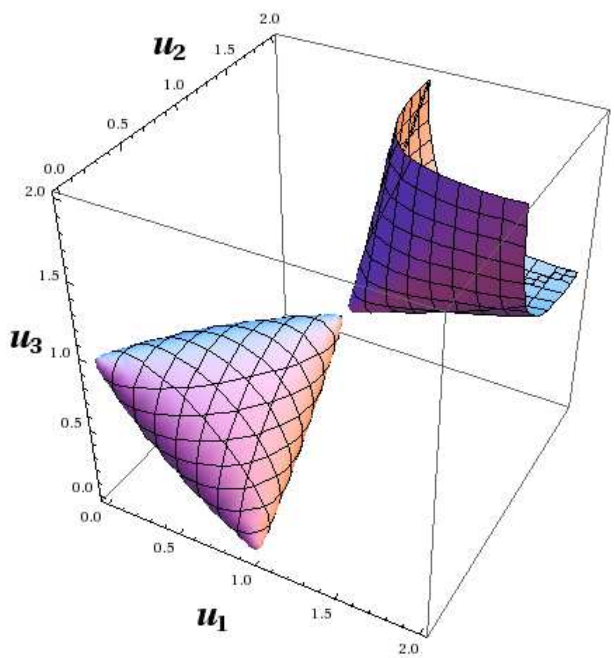
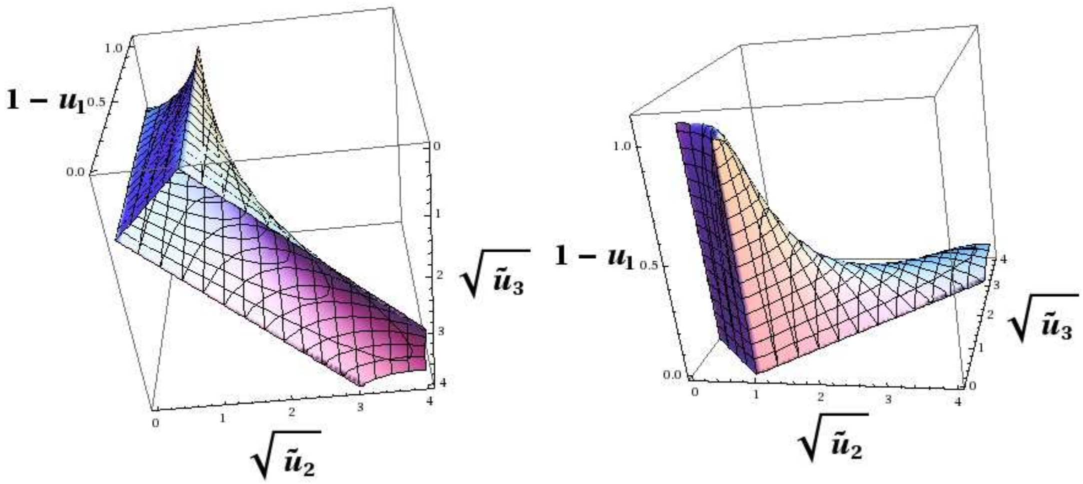


Figure 5: The surface $\Delta = 0$ is the boundary of the physical values of cross ratios.

The same plot, but for AdS_4 surface was obtained by Alday, Gaiotto and Maldacena [30] introducing momenta parametrization of the cross ratios. The space of the physical cross ratios in the unit cube is inside the "bag" in Fig. 5. To find a match between our picture of the physical region depicted in Fig. 4 and the "bag" in Fig. 5 we draw the surface $\Delta = 0$ in the coordinates $1 - u_1$, $\sqrt{\tilde{u}_2}$ and $\sqrt{\tilde{u}_3}$ as illustrated in Fig. 6.

It is clear from Fig. 6a that in the Regge limit, when $(\mathbf{k}_1+\mathbf{k}_2)^2/s_2 \simeq 1-u_1 \to 0$ the boundary of the surface becomes a semi-infinite strip in accordance with Fig. 4. If one relaxes the Regge kinematics this region becomes a closed cigar shaped region as follows from the geodesics in Fig. 6a.

In the next section we calculate the next-to-leading-order impact factor from our result of the analytic continuation of the remainder function in Eq. 19.



(a) (b)

Figure 6: The surface $\Delta = 0$ in the coordinates $1 - u_1$, $\sqrt{\tilde{u}_2}$ and $\sqrt{\tilde{u}_3}$. In the Regge limit, when $(\mathbf{k}_1 + \mathbf{k}_2)^2/s_2 \simeq 1 - u_1 \to 0$ we obtain the region **A** in Fig. 4. Once the Regge kinematics is relaxed we get a physical region of a cigar shape instead of the semi-infinite strip in Fig. 4. The figures (**a**) and (**b**) depicts a different view of the same surface $\Delta = 0$.

4 NLO impact factor

In this section we calculate the next-to-leading (NLO) impact factor appearing in the BFKL approach. We begin with Eq. 4 and Eq. 5, which define the imaginary part of the BDS violating term with logarithmic accuracy. We are interested in generalizing Eq. 5 to include also next-to-leading in logarithm of the energy (NLLA) terms. Taking into account NLLA corrections corresponds to relaxing multi-Regge kinematics to quasi-multi-Regge kinematics (QMRK) for the intermediate particles in the unitarity relation for the amplitude.

The next-to-leading corrections to Eq. 5 are of two kinds: the NLO corrections to the impact factors of the BFKL ladder and to the kernel of the BFKL equation. The later was calculated in QCD by Fadin and Fiore [31, 32] and in $\mathcal{N} = 4$ SYM it can be extracted from QCD calculations applying a principle of the maximal transcendentality proposed by Kotikov and one of the authors [33]. The maximal transcendentality principle was successfully used, for example, to predict the anomalous dimension up to six loops in $\mathcal{N} = 4$ SYM [34, 35, 36, 37, 38, 39, 40]. Another sort of NLO corrections to Eq. 5 is the corrections to the Reggeon-Reggeon-Particles (RRP) impact factor of the BFKL ladder, which were never calculated before for the octet channel[2]. It worth mentioning that there is some ambiguity in the higher order terms, namely, the corrections can be redistributed between the impact factor and the BFKL Kernel, which does not affect the form of the amplitude provided the corrections are included in a consistent way.

We are interested in the NLO contribution to the impact factor. We write Eq. 5 for the leading logarithmic contribution as

[2]The NLO corrections to the impact factor in the singlet channel were found by Balitsky and Chirilli [41, 42].

$$\Delta_{2\to 4} = \frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^*k_1^*}{k_2^*q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3k_1}{k_2q_1}\right)^{i\nu+\frac{n}{2}}\left(s_2^{\omega(\nu,n)}-1\right)$$

$$= \frac{a}{2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}d\nu\,(2{\chi_1}^{LLA})(2{\chi_2}^{LLA})\,\left((1-u_1)^{-\omega(\nu,n)}-1\right), \quad (32)$$

where the leading-log impact factors are given by

$$\chi_1^{LLA} = \frac{1}{2}\frac{1}{\left(i\nu+\frac{n}{2}\right)}\left(-\frac{q_1}{k_1}\right)^{-i\nu-\frac{n}{2}}\left(-\frac{q_1^*}{k_1^*}\right)^{-i\nu+\frac{n}{2}},$$

$$\chi_2^{LLA} = -\frac{1}{2}\frac{1}{\left(i\nu-\frac{n}{2}\right)}\left(\frac{q_3^*}{k_2^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3}{k_2}\right)^{i\nu+\frac{n}{2}}. \quad (33)$$

The functions χ_1^{LLA} and χ_2^{LLA} are a convolution of the octet BFKL eigenfunction and the corresponding impact factor, but for the purpose of our discussion we call them impact factors in the ν, n representation. For more details regarding the rigorous definition of the impact factors the reader is referred to ref. [7]. The factor of two accompanying χ_i^{LLA} in Eq. 32 is introduced to match the notation in Eqs. 90-93 of ref. [7].

In the appendix **F** we found that the NLLA term of the remainder function at two loops in Eq. 22 can be written as

$$R_6^{(2)\,NLLA}\left(|u_1|e^{-i2\pi},\frac{1}{|1+w|^2},\frac{|w|^2}{|1+w|^2}\right) \quad (34)$$

$$= \sum_{n=-\infty}^{\infty}\int d\nu\frac{i}{2}\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\left(E_{\nu,n}^2-\frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right)(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}$$

$$= \sum_{n=-\infty}^{\infty}\int d\nu\frac{i}{2}\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\left(E_{\nu,n}^2-\frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right)\left(\frac{q_3^*k_1^*}{k_2^*q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3k_1}{k_2q_1}\right)^{i\nu+\frac{n}{2}}.$$

This allows to modify the leading logarithmic expression Eq. 32 to account for the next-to-leading in $\ln s_2$ (NLLA) corrections at all orders of the perturbation theory.

Before we proceed there is one fine point to be clarified. According to the factorization hypothesis [14] the all-order remainder function R is defined by

$$M_{2\to 4} = M_{2\to 4}^{BDS}R \quad (35)$$

It was argued by one of the authors [28] that provided the factorization hypothesis holds, a significant information about the remainder function can be obtained from the analytic properties of the BDS formula. In particular, in the region under consideration, where $s, s_2 > 0$ and $s_1, s_3, s_{012}, s_{123} < 0$ in the Regge kinematics the remainder function at all orders of the perturbation theory is given by the dispersion relation (see[3] Eq. 50 of ref. [28])

$$R\,e^{i\pi\,\delta} = \cos\pi\omega_{ab} + i\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\,f(\omega)\,e^{-i\pi\omega}\,(1-u_1)^{-\omega}\,, \quad (36)$$

where the phases δ and ω_{ab} represent contribution of Regge poles obtained directly from the BDS formula and are given by

$$\delta = \frac{\gamma_K}{8}\,\ln(\tilde{u}_2\tilde{u}_3)\,,\ \omega_{ab} = \frac{\gamma_K}{8}\,\ln\frac{\tilde{u}_3}{\tilde{u}_2}. \quad (37)$$

[3] The function R is denoted by c, and the reduced cross ration $\tilde{u}_2$ and $\tilde{u}_3$ are ϕ_2 and ϕ_3 in the notation of ref. [28].

The coefficient γ_K is the cusp anomalous dimension known to any order of the perturbation theory. The only unknown piece in Eq. 36 is the real function $f(\omega)$, which has the Mandelstam cut in ω and depends only on the transverse momenta and has no energy dependence. In the leading logarithmic approximation $f(\omega)$ is given by

$$f^{LLA}(\omega) = \frac{a}{2} \sum_{n=-\infty}^{\infty} \int d\nu \frac{1}{\omega - \omega(\nu, n)} \frac{(-1)^n}{\nu^2 + \frac{n^2}{4}} \left(\frac{q_3^* k_1^*}{k_2^* q_1^*}\right)^{i\nu - \frac{n}{2}} \left(\frac{q_3 k_1}{k_2 q_1}\right)^{i\nu + \frac{n}{2}} \tag{38}$$

$$= \frac{a}{2} \sum_{n=-\infty}^{\infty} \int d\nu \frac{1}{\omega - \omega(\nu, n)} (2\chi_1^{LLA})(2\chi_2^{LLA}),$$

where $\omega(\nu, n)$ is defined in Eq. 6. Therefore in RHS of Eq. 36 the integral over ω gives correctly the leading asymptotics of imaginary and real parts of the amplitudes (see Eq. 5).

The expression for $f^{LLA}(\omega)$ is read out from Eq. 32 and can be generalized to include subleading contributions. By analogy with Eq. 38 we write

$$f^{NLLA}(\omega) = \frac{a}{2} \sum_{n=-\infty}^{\infty} \int d\nu \frac{1}{\omega - \tilde{\omega}(\nu, n)} (2\tilde{\chi}_1)(2\tilde{\chi}_2), \tag{39}$$

where both the impact factors $\tilde{\chi}_i$ and the BFKL energy $\tilde{\omega}(\nu, n)$ include the next-to-leading corrections. They are defined by

$$\tilde{\chi}_i = \chi_i^{LLA} + \chi_i^{NLO} \tag{40}$$

and

$$\tilde{\omega}(\nu, n) = \omega(\nu, n) + \omega^{NLO}(\nu, n). \tag{41}$$

The expression for $\omega^{NLO}(\nu, n)$ can be found from the next-to-leading corrections to the octet BFKL Kernel calculated by Fadin and Fiore [31, 32], and the missing NLO impact factors we can readily extract from Eq. 34. In the appendix **F** we find that the next-to-leading impact factors χ_i^{NLO} are given by

$$\chi_1^{NLO} = \frac{a}{4} \frac{1}{\left(i\nu + \frac{n}{2}\right)} \left(E_{\nu,n}^2 - \frac{1}{4} \frac{n^2}{\left(\nu^2 + \frac{n^2}{4}\right)^2}\right) \left(-\frac{q_1}{k_1}\right)^{-i\nu - \frac{n}{2}} \left(-\frac{q_1^*}{k_1^*}\right)^{-i\nu + \frac{n}{2}} \tag{42}$$

and

$$\chi_2^{NLO} = -\frac{a}{4} \frac{1}{\left(i\nu - \frac{n}{2}\right)} \left(E_{\nu,n}^2 - \frac{1}{4} \frac{n^2}{\left(\nu^2 + \frac{n^2}{4}\right)^2}\right) \left(\frac{q_3}{k_2}\right)^{i\nu - \frac{n}{2}} \left(\frac{q_3^*}{k_2^*}\right)^{i\nu + \frac{n}{2}}, \tag{43}$$

where $E_{\nu,n}$ is defined in Eq. 8. An important feature of χ_i^{NLO} is that both Eq. 42 and Eq. 43 do not have holomorphic separability, namely either $i\nu + n/2$ or $-i\nu + n/2$ cannot be assigned only to one of the projectiles. It is worth emphasizing that the NLO impact factors χ_i^{NLO} are factorized in the product of the Born impact factors in Eq. 33 and a term expressed through the eigenvalue $E_{\nu,n}$ of the BFKL equation in LLA. The form of the NLO impact factor in the ν, n representation resembles the three-loop remainder function in LLA, emphasizing the intimate relation between the two as discussed in the next section.

5 Three loops in LLA and NLLA

In this section we calculate the three-loop leading logarithmic (LLA) contribution to the remainder function of the six-point MHV amplitude and find the real part of the subleading in

$\ln s_2/s_0$ (NLLA) term. In the leading logarithm approximation (LLA) one neglects all terms, which are not enhanced by the logarithm of energy. In our case the main contribution comes from the Mandelstam cut in the variable ω canonically conjugated to $\ln s_2$ (see Fig. 1) and the leading terms are those which have each power of the coupling constant accompanied by the same power of the logarithm of the energy $\ln s_2/s_0$ or equivalently by $\ln(1-u_1)$, because in the Regge limit $1-u_1 \simeq (\mathbf{k}_1+\mathbf{k}_2)^2/s_2$. The all-loop LLA contribution to the remainder function in the Mandelstam channel (see Fig. 1b) is given by Eq. 4 and Eq. 5. The three-loop term is obtained expanding $s_2^{\omega(\nu,n)}$ in Eq. 5 in powers of the coupling a and reads

$$R_6^{(3)\ LLA} = \frac{i\Delta_{2\to4}^{(3)}}{a^3} = \frac{i}{4}\ln^2(1-u_1)\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}}E_{\nu,n}^2\left(\frac{q_3^*k_1^*}{k_2^*q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3k_1}{k_2q_1}\right)^{i\nu+\frac{n}{2}} \tag{44}$$
$$= \frac{i}{4}\ln^2(1-u_1)\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}}E_{\nu,n}^2\,(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}\,,$$

where the complex variable w is defined by

$$w = \frac{q_3k_1}{k_2q_1}, \quad w^* = \frac{q_3^*k_1^*}{k_2^*q_1^*}. \tag{45}$$

The expression in Eq. 44 is calculated in the appendix G and has the form

$$R_6^{(3)\ LLA} = i\Delta_{2\to4}^{(3)}/a^3 = i\pi\frac{1}{4}\ln^2(1-u_1)\Big(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2 \tag{46}$$
$$-\frac{1}{4}\ln^2|w|^2\ln|1+w|^2 + \frac{1}{2}\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) - \mathrm{Li}_3(-w) - \mathrm{Li}_3(-w^*)\Big)\,.$$

The LLA three-loop remainder function in Eq. 46 is pure imaginary and vanishes at $w\to 0$ and $w\to\infty$. It is invariant under $w\to 1/w$ transformation, which corresponds to the target-projectile symmetry.

The next-to-leading in the logarithm of the energy $\ln s_2/s_0 \simeq -\ln(1-u_1)$ contribution can be obtained from Eq. 36 and Eq. 39. Unfortunately we do not have an explicit expression for $\omega(\nu,n)$ beyond the leading order, which is necessary for this calculation. However it turns out that we can find asymptotic behavior of the real part of the NLLA remainder function at three loops, because it does not require any knowledge of the higher order corrections to $\omega(\nu,n)$. $\Re\left(R_6^{(3)\ NLLA}\right)$is calculated expanding Eq. 36 in powers of a. The details of this analysis are presented in the appendix H and the result is given by

$$\Re(R_6^{(3)\ NLLA}) = \frac{\pi^2}{4}\ln(1-u_1)\Big(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2 \tag{47}$$
$$-\frac{1}{2}\ln^2|w|^2\ln|1+w|^2 - \ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) + 2\mathrm{Li}_3(-w) + 2\mathrm{Li}_3(-w^*)\Big)\,.$$

Note that $\Re(R_6^{(3)\ NLLA})$ resembles very much the form of $R^{(3)\ LLA}$ in Eq. G.18 as one could expect from Eq. H.4. The complex variables w is expressed in terms of the reduced cross ratios $\tilde{u}_2 = u_2/(1-u_1)$ and $\tilde{u}_3 = u_3/(1-u_1)$ as

$$w = \frac{1-z}{z} = \frac{B^+}{\tilde{u}_2}, \quad w^* = \frac{1-z^*}{z^*} = \frac{B^-}{\tilde{u}_2} \tag{48}$$

for $B^\pm$ defined in Eq. D.4 by

$$B^\pm = \frac{1-\tilde{u}_2-\tilde{u}_3 \pm \sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2 - 4\tilde{u}_2\tilde{u}_3}}{2}. \tag{49}$$

6 Conclusion

In this paper we present some details of the analytic continuation of the GSVV [22] formula for the two-loop remainder function $R_6^{(2)}$ to a physical region of the $2 \to 4$ particle scattering, where $R_6^{(2)}$ gives a non-vanishing contribution in the Regge limit. We find that after the analytic continuation the remainder function reproduces the leading logarithmic (LLA) term calculated by Bartels, Sabio Vera and one of the authors [7] in the BFKL approach. We also find a term subleading in the logarithm of the energy (NLLA) and extract the next-to-leading (NLO) impact factor used in the BFKL technique. The BFKL approach allows to calculate the LLA contribution to the remainder function at any order of the perturbation theory. The three loop LLA remainder function as well as the real part of the three-loop NLLA remainder function are calculated in the BFKL technique and presented in this study. We find that the all-loop LLA and the two-loop NLLA terms of the remainder function are pure imaginary in the Regge limit, while starting at three loops the NLLA remainder function develops a non-vanishing real part in the Regge limit after the analytic continuation to the relevant physical region.

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A Polylogarithms of u_i

We perform an analytic continuation of the GSVV expression Eq. 13 in the variable u_1. The functions which do not depend on u_1 remain the same after the continuation. In the multi Regge kinematics given by Eq. 2, they can be simplified as follows

$$\mathrm{Li}_2\left(1-\frac{1}{u_2}\right) \simeq -\zeta_2 - \frac{1}{2}\ln^2(1-u_1) - \ln(1-u_1)\ln\tilde{u}_2 - \frac{1}{2}\ln^2\tilde{u}_2 \tag{A.1}$$

and

$$\mathrm{Li}_4\left(1-\frac{1}{u_2}\right) \simeq -\frac{7\pi^4}{360} - \frac{\zeta_2}{2}\ln^2(1-u_1) - \frac{1}{24}\ln^4(1-u_1) - \zeta_2\ln(1-u_1)\ln\tilde{u}_2 \tag{A.2}$$
$$-\frac{1}{6}\ln^3(1-u_1)\ln\tilde{u}_2 - \frac{\zeta_2}{2}\ln^2\tilde{u}_2 - \frac{1}{4}\ln^2(1-u_1)\ln^2\tilde{u}_2 - \frac{1}{6}\ln(1-u_1)\ln^3\tilde{u}_2 - \frac{1}{24}\ln^4\tilde{u}_2.$$

in terms of the reduced cross ratios defined in Eq. 11.

The expressions for $\mathrm{Li}_2\left(1-\frac{1}{u_3}\right)$ and $\mathrm{Li}_4\left(1-\frac{1}{u_3}\right)$ are obtained from Eq. A.1 and Eq. A.2 replacing $\tilde{u}_2$ by $\tilde{u}_3$ in the argument.

The polylogarithms $\mathrm{Li}_n\left(1-\frac{1}{u_1}\right)$ should be analytically continued and in the multi Regge kinematics are given by

$$\mathrm{Li}_2\left(1-\frac{1}{u_1}\right) = -\int_0^{1-\frac{1}{u_1}}\frac{dt}{t}\ln(1-t) \simeq -i2\pi\int_1^{1-\frac{1}{|u_1|}}\frac{dt}{t} = -i2\pi\ln\left(1-\frac{1}{|u_1|}\right) \tag{A.3}$$
$$= -i2\pi(\ln(1-u_1)+i\pi) = -i2\pi\ln(1-u_1) + 2\pi^2.$$

Note that we assign a phase $\ln(-1) = +i\pi$ since the argument $1-\frac{1}{u_1}$ goes counterclockwise around the origin as we continue from $u_1 = e^{i0}|u_1|$ to $u_1 = e^{-i2\pi}|u_1|$ through $u_1 = e^{-i\pi}|u_1| = -|u_1|$. In a similar way we find

$$\mathrm{Li}_4\left(1-\frac{1}{u_1}\right) = -\int_0^{1-\frac{1}{u_1}}\frac{dt}{t}\int_0^t\frac{dt'}{t'}\int_0^{t'}\frac{dt''}{t''}\ln(1-t'') \simeq -i2\pi\frac{1}{6}\ln^3\left(1-\frac{1}{|u_1|}\right) \tag{A.4}$$
$$= -\frac{\pi^4}{3} + i\pi^3\ln(1-u_1) + \pi^2\ln^2(1-u_1) - \frac{i\pi}{3}\ln^3(1-u_1).$$

B Polylogarithms of $x^{\pm}$ and $x_i^{\pm}$

The variables $x^{\pm}$ are defined as follows

$$x^{\pm} = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3}, \quad x_i^{\pm} = u_i x^{\pm}, \tag{B.1}$$

where $\Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$.

First we consider the logarithm of x^-/x^+. This variable goes from the second quadrant of the complex plane (negative real and positive imaginary part) in the counterclockwise direction during the continuation crossing the real axis for the large negative values of the argument, then it again crosses the real axis for small negative values of the variable. Thus the argument crosses the branch cut twice and $\ln x^-/x^+$ remains on the same Riemann sheet and does not acquire any imaginary part after the continuation. Namely,

$$\ln\frac{x^-}{x^+} = \ln\frac{|x^-|}{|x^+|} \simeq \frac{1-\tilde{u}_2-\tilde{u}_3+\sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2-4\tilde{u}_2\tilde{u}_3}}{1-\tilde{u}_2-\tilde{u}_3-\sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2-4\tilde{u}_2\tilde{u}_3}}, \tag{B.2}$$

where the notation $|x^{\pm}|$ denotes the fact that u_1 in $x^{\pm}$ is replaced by $|u_1|$.

For our purposes it is useful to note that $\ln(1-z)$ has the same cut structure as $\mathrm{Li}_n(z)$. This can be seen from the series representation

$$\mathrm{Li}_n(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^n} \tag{B.3}$$

and thus

$$\mathrm{Li}_1(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k} = -\ln(1-z). \tag{B.4}$$

B.1 Logarithms of $x_i^+ x_i^-$

The variables $x_i^{\pm}$ are defined in Eq. B.1. The functions of these variable that are present in the result of Goncharov et al. are $\mathrm{Li}_n(x_i^{\pm})$ and $\mathrm{Li}_n(1/x_i^{\pm})$ $(n = 1...4)$.

The logarithm of the product $x_i^+ x_i^-$ can be easily analytically continued noting that

$$x_i^+ x_i^- = \frac{u_i}{u_{i+1}u_{i+2}}, \quad i = 1...3. \tag{B.5}$$

We readily calculate

$$\ln x_1^+ x_1^- = -i2\pi - \ln \tilde{u}_2 - \ln \tilde{u}_3 - 2\ln(1-|u_1|), \tag{B.6}$$

$$\ln x_2^+ x_2^- = i2\pi + \ln \tilde{u}_2 - \ln \tilde{u}_3, \tag{B.7}$$

$$\ln x_3^+ x_3^- = i2\pi - \ln \tilde{u}_2 + \ln \tilde{u}_3, \tag{B.8}$$

because $\ln u_1 \simeq -i2\pi$ after the continuation in the limit Eq. 2.

B.2 Polylogarithms of $x_1^{\pm}$

We start with $x_1^{\pm}$. During the analytic continuation x_1^+ goes from the second quadrant of the complex plane (negative real and positive imaginary part) in the clockwise direction and then crosses the real axis between 0 and 1. Thus $\mathrm{Li}_n(x_1^+)$ are not changed after continuation and can be simplified as follows.

First we simplify the argument in the limit Eq. 2 separating the "longitudinal"($1-u_1$) and the "transverse" ($\tilde{u}_2$ and $\tilde{u}_3$) cross ratios. We write

$$|x_1^{\pm}| \simeq -\frac{A_1^{\pm}}{1-|u_1|}, \tag{B.9}$$

where $A_1^{\pm}$ is a function of only $\tilde{u}_2$ and $\tilde{u}_3$ and is given by

$$A_1^{\pm} = \frac{1-\tilde{u}_2-\tilde{u}_3 \mp \sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2 - 4\tilde{u}_2\tilde{u}_3}}{2\tilde{u}_2\tilde{u}_3}. \tag{B.10}$$

Using this notation we can expand the polylogarithms as follows

$$\mathrm{Li}_1(x_1^+) = \mathrm{Li}_1(|x_1^+|) = -\ln(1-|x_1^+|) \simeq -\ln A_1^+ + \ln(1-u_1), \tag{B.11}$$

$$\mathrm{Li}_2(x_1^+) = \mathrm{Li}_2(|x_1^+|) \simeq -\frac{\pi^2}{6} - \frac{1}{2}\ln^2 A_1^+ + \ln A_1^+ \ln(1-u_1) - \frac{1}{2}\ln^2(1-u_1), \tag{B.12}$$

$$\begin{aligned}\mathrm{Li}_3(x_1^+) = \mathrm{Li}_3(|x_1^+|) \simeq & -\frac{\pi^2}{6}\ln A_1^+ - \frac{1}{6}\ln^3 A_1^+ + \frac{\pi^2}{6}\ln(1-u_1) + \frac{1}{2}\ln^2 A_1^+ \ln(1-u_1) \\ & -\frac{1}{2}\ln A_1^+ \ln^2(1-u_1) + \frac{1}{6}\ln^3(1-u_1),\end{aligned} \tag{B.13}$$

$$\begin{aligned}\mathrm{Li}_4(x_1^+) = \mathrm{Li}_4(|x_1^+|) \simeq & -\frac{7\pi^4}{360} - \frac{\pi^2}{12}\ln^2 A_1^+ - \frac{1}{24}\ln^4 A_1^+ + \frac{\pi^2}{6}\ln A_1^+ \ln(1-u_1) \\ & +\frac{1}{6}\ln^3 A_1^+ \ln(1-u_1) - \frac{\pi^2}{12}\ln^2(1-u_1) - \frac{1}{4}\ln^2 A_1^+ \ln^2(1-u_1) \\ & +\frac{1}{6}\ln A_1^+ \ln^3(1-u_1) - \frac{1}{24}\ln^4(1-u_1).\end{aligned} \tag{B.14}$$

The variable x_1^- also does not cross the branch cut of the polylogarithm (from 1 to $+\infty$) since during the continuation it goes from the third quadrant of the complex plane (both the real and the imaginary parts are negative) in the clockwise direction and then crosses the real axis for negative values (never crosses the imaginary axis). The expansion of the polylogarithms of x_1^- is obtained from that of x_1^+ replacing A_1^+ by A_1^-.

The polylogarithms of $1/x_1^+$ are analytically continued since the variable goes from the third quadrant of the complex plane in the counterclockwise direction, crosses the imaginary axis and then the real axis behind 1 when we go from $u_1 = e^{i0}|u_1|$ to $u_1 = e^{-i2\pi}|u_1|$. The argument crosses the branch cut of $\mathrm{Li}_n(1/x_1^+)$, and the direction of the rotation determines the sign of the phase of $\ln(1 - 1/x_1^+)$ as positive ($+i2\pi$).

Using this notation we can expand the polylogarithms as follows

$$\mathrm{Li}_1\left(\frac{1}{x_1^+}\right) \simeq -i2\pi, \tag{B.15}$$

$$\begin{aligned}\mathrm{Li}_2\left(\frac{1}{x_1^+}\right) &= -\int_0^{\frac{1}{x_1^+}} \frac{dt}{t}\ln(1-t) = -\int_0^{-\frac{1-|u_1|}{A_1^+}} \frac{dt}{t}\ln(1-t) - i2\pi \int_1^{-\frac{1-|u_1|}{A_1^+}} \frac{dt}{t} \\ &\simeq -i2\pi(\ln(1-u_1) - \ln A_1^+ + i\pi) = -i2\pi\ln(1-u_1) + i2\pi\ln A_1^+ + 2\pi^2.\end{aligned} \tag{B.16}$$

In the second line of Eq. B.16 we used the fact that $\ln(-1) = +i\pi$ because the argument $1/x_1^+$ rotates in the counterclockwise direction around the origin. In a similar way we continue the rest of the polylogarithms. In general we can write

$$\mathrm{Li}_n\left(\frac{1}{x_1^+}\right) \simeq -i2\pi\frac{1}{(n-1)!}\left(\ln(1-u_1) - \ln A_1^+ + i\pi\right)^{n-1}. \tag{B.17}$$

The polylogarithms of $1/x_1^-$ are not continued because the argument goes in the counterclockwise direction in the left complex semi plane and never crosses the imaginary axis and thus also the branch cut of the polylogarithms. In our limit $1/x_1^- \to 0$ and thus

$$\mathrm{Li}_n\left(\frac{1}{x_1^-}\right) \simeq 0. \tag{B.18}$$

From Eq. B.15 and Eq. B.18 we calculate

$$\mathrm{Li}_1\left(\frac{1}{x_1^+}\right) + \mathrm{Li}_1\left(\frac{1}{x_1^-}\right) \simeq -i2\pi, \tag{B.19}$$

which can be checked by the direct calculation eliminating the square roots before the analytic continuation

$$\mathrm{Li}_1\left(\frac{1}{x_1^+}\right)+\mathrm{Li}_1\left(\frac{1}{x_1^-}\right)=-\ln\left(1-\frac{1}{x_1^+}\right)-\ln\left(1-\frac{1}{x_1^-}\right)=-\ln\frac{(1-u_2)(1-u_3)}{u_1}\simeq -i2\pi. \quad \text{(B.20)}$$

This shows that we fix correctly the phase of the argument according to the direction of its rotation during the analytic continuation despite the presence of the square roots.

B.3 Polylogarithms of $x_2^\pm$

The variable x_2^+ rotates from the second quadrant of the complex plane (negative real and positive imaginary parts) in the clockwise direction during the continuation, crossing the real axis in the left semi plane and never crossing the imaginary axis. This means that the polylogarithms of x_2^+ remain the same after the analytic continuation since its argument never crosses the branch cut of the polylogarithm (from real values of 1 to $+\infty$).

On contrary, x_2^- rotates from the third quadrant in the counterclockwise direction crosses the imaginary axis and then the real axis beyond the value of 1, crossing the branch cut of the polylogarithms. Thus the polylogarithms of this argument are to be analytically continued.

We start with simplifying $\mathrm{Li}_n(x_2^+)$. For the sake of convenience, similarly to the previous discussion we write it as

$$|x_2^\pm| = A_2^\pm = -A_1^\pm \tilde{u}_2, \quad \text{(B.21)}$$

where

$$A_2^\pm = \frac{\tilde{u}_2+\tilde{u}_3-1\pm\sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2-4\tilde{u}_2\tilde{u}_3}}{2\tilde{u}_3}, \quad \text{(B.22)}$$

and $A_1^\pm$ was defined in Eq. B.10. We use redundant definitions $A_n^\pm$, which are expressible through each other solely for the sake of transparency of the intermediate calculations. The final result will be simplified using the relations between them.

As we have already mentioned the polylogarithms of x_2^+ remain the same after the continuation and are given by

$$\mathrm{Li}_n\left(x_2^+\right)=\mathrm{Li}_n\left(|x_2^+|\right)\simeq \mathrm{Li}_n\left(A_2^+\right). \quad \text{(B.23)}$$

The polylogarithms of x_2^- are to be continued as follows

$$\mathrm{Li}_1\left(x_2^-\right)=-\ln(1-x_2^-)=-\ln(1-A_2^-)-i2\pi \quad \text{(B.24)}$$

and

$$\mathrm{Li}_2\left(x_2^-\right)=-\int_0^{x_2^-}\frac{dt}{t}\ln(1-t)\simeq \mathrm{Li}_2\left(A_2^-\right)-i2\pi\ln A_2^-. \quad \text{(B.25)}$$

In general

$$\mathrm{Li}_n\left(x_2^-\right)\simeq \mathrm{Li}_n\left(A_2^-\right)-i2\pi\frac{1}{(n-1)!}\ln^{n-1}A_2^-. \quad \text{(B.26)}$$

Using Eq. B.23 and Eq. B.24 we write

$$\mathrm{Li}_1\left(x_2^+\right)+\mathrm{Li}_1\left(x_2^-\right)\simeq -\ln(1-A_2^+)-\ln(1-A_2^-)-i2\pi=-i2\pi+\ln\tilde{u}_3, \quad \text{(B.27)}$$

where we used the identity

$$(1 - A_2^+)(1 - A_2^-) = \frac{1}{\tilde{u}_3}. \tag{B.28}$$

On the other hand we can simplify $\mathrm{Li}_1\left(x_2^+\right) + \mathrm{Li}_1\left(x_2^-\right)$ before the continuation and continue it after that

$$\mathrm{Li}_1\left(x_2^+\right) + \mathrm{Li}_1\left(x_2^-\right) = -\ln\left(1 - x_2^+\right) - \ln\left(1 - x_2^-\right) = -\ln\frac{(1-u_1)(1-u_3)}{u_1 u_3} \simeq -i2\pi + \ln\tilde{u}_3. \tag{B.29}$$

This confirms our choice of the phase sign according to the direction of the rotation of the argument.

The variable $1/x_2^+$ goes from the third quadrant of the complex plane in the clockwise direction during the continuation. It crosses the real axis in the left complex semi plane and thus never crosses the branch cut of the polylogarithms. The argument $1/x_2^-$ goes from the second quadrant of the complex plane in the clockwise direction during the continuation. It crosses the imaginary axis and the real axis, but never reaches the branch cut in our limit. Thus all polylogarithms of $1/x_2^\pm$ remain the same after the continuation and can be written as follows

$$\mathrm{Li}_n\left(\frac{1}{x_2^\pm}\right) \simeq \mathrm{Li}_n\left(\frac{1}{A_2^\pm}\right). \tag{B.30}$$

As an example we calculate

$$\mathrm{Li}_1\left(\frac{1}{x_2^+}\right) + \mathrm{Li}_1\left(\frac{1}{x_2^-}\right) = -\ln\left(1 - \frac{1}{A_2^+}\right) - \ln\left(1 - \frac{1}{A_2^-}\right) \simeq \ln\tilde{u}_2. \tag{B.31}$$

We can check directly the validity of this result by eliminating the square roots before the continuation

$$\mathrm{Li}_1\left(\frac{1}{x_2^+}\right) + \mathrm{Li}_1\left(\frac{1}{x_2^-}\right) = -\ln\left(1 - \frac{1}{x_2^+}\right) - \ln\left(1 - \frac{1}{x_2^-}\right) = -\ln\frac{(1-u_1)(1-u_3)}{u_2} \simeq \ln\tilde{u}_2. \tag{B.32}$$

B.4 Polylogarithms of $x_3^\pm$

The polylogarithms of $x_3^\pm$ are analytically continued exactly in the same way as corresponding polylogarithms of $x_2^\pm$. This can be shown by numerical calculations as well as explained on general grounds by the symmetry $\tilde{u}_2 \Leftrightarrow \tilde{u}_3$, which corresponds to target-projectile symmetry of the scattering amplitude.

The polylogarithms of $x_3^\pm$ are obtained from that of $x_2^\pm$ by making a change $\tilde{u}_2 \Leftrightarrow \tilde{u}_3$ as well as replacing $A_2^\pm$ by $A_3^\pm$, where

$$A_3^\pm = \frac{\tilde{u}_3}{\tilde{u}_2} A_2^\pm = -A_1^\pm \tilde{u}_3. \tag{B.33}$$

B.5 Continuation of J

The function J is defined in Eq. 17 through the sum of the polylogarithm of $x_i^\pm$. Using this definition we can readily find

$$J = \frac{1}{2}\ln\frac{|x^-|}{|x^+|} + i\pi \simeq \frac{1}{2}\ln\frac{A_1^+}{A_1^-} + i\pi. \tag{B.34}$$

As it was anticipated in ref. [22] this contradicts the result of the continuation of the function

$$\ln \frac{x^-}{x^+} = \ln \frac{|x^-|}{|x^+|} \tag{B.35}$$

found in the previous section (see Eq. B.2) due to the difference in the cut structure between the two. The correct analytic continuation was done using the definition of J in terms of $\mathrm{Li}_1(x_i^\pm)$ in Eq. 17.

C Leading Logarithmic Approximation (LLA)

We want to extract the leading logarithmic term from the expression of the remainder function after the continuation. The leading logarithm of the energy $\ln s_2$ is related to $\ln(1-u_1)$ through

$$1 - u_1 \propto \frac{|\mathbf{k}_1 + \mathbf{k}_2|^2}{s_2} \tag{C.1}$$

The reduced cross ratios $\tilde{u}_2$ and $\tilde{u}_3$ depend only on the transverse components of the momenta of the external particles and are of the order of unity. It should be emphasized that only terms of the order of $\ln(1-u_1)$ can contribute to the imaginary part of the remainder function at two loops. Higher order terms $\ln^n(1-u_1)$ $(n > 1)$ would contradict the unitarity of the scattering matrix. These terms do appear at the intermediate steps of the calculations, but they all must cancel out in the final expression.

From our previous discussions we see that only a few terms can have contributions to the LLA result

$$-\frac{1}{2}\mathrm{Li}_4\left(1 - \frac{1}{u_1}\right) + \mathrm{L}_4(x_1^+, x_1^-) - \frac{1}{8}\left(\sum_{i=1}^{3} \mathrm{Li}_2\left(1 - \frac{1}{u_i}\right)\right)^2. \tag{C.2}$$

We consider them in separate. For our purposes it is convenient to single out only LLA contributions produced in the process of the continuation. We use the fact that the remainder function vanishes before the continuation

$$R_6^{(2)} \to 0 \tag{C.3}$$

in the limit $u_1 \to 1$, $u_2 \to 0$ and $u_3 \to 0$ for $\tilde{u}_2 = u_2/(1-u_1)$ and $\tilde{u}_3 = u_3/(1-u_1)$ being kept fixed and of the order of unity. We obtain the imaginary part of the remainder function in LLA by subtracting the contribution before the continuation from those obtained continuing the functions in the physical region where $u_1 \to e^{-i2\pi}|u_1|$, keeping only terms accompanied by the power of $\ln(1-u_1)$. We start with the first term in Eq. C.2

$$-\frac{1}{2}\mathrm{Li}_4\left(1 - \frac{1}{u_1}\right) + \frac{1}{2}\mathrm{Li}_4\left(1 - \frac{1}{|u_1|}\right) \overset{LLA}{\simeq} -\frac{i\pi^3}{2}\ln(1-u_1) - \frac{\pi^2}{2}\ln^2(1-u_1) + \frac{i\pi}{6}\ln^3(1-u_1), \tag{C.4}$$

where we used Eq. A.4 and the fact that

$$\lim_{u_1 \to 1} \mathrm{Li}_n\left(1 - \frac{1}{|u_1|}\right) \simeq 0. \tag{C.5}$$

In an analogous way we calculate the second term of Eq. C.2

$$\begin{aligned}
\mathrm{L}_4(x_1^+, x_1^-) - \mathrm{L}_4(|x_1^+|, |x_1^-|) &\overset{LLA}{\simeq} -\frac{i\pi^3}{3}\ln(1-u_1) - \frac{\pi^2}{2}\ln^2(1-u_1) + \frac{i\pi}{3}\ln^3(1-u_1) \\
&-\frac{\pi^2}{2}\ln(1-u_1)\ln\tilde{u}_2 + \frac{i\pi}{2}\ln^2(1-u_1)\ln\tilde{u}_2 + \frac{i\pi}{4}\ln(1-u_1)\ln^2\tilde{u}_2 - \frac{\pi^2}{2}\ln(1-u_1)\ln\tilde{u}_3 \\
&+\frac{i\pi}{2}\ln^2(1-u_1)\ln\tilde{u}_3 + \frac{i\pi}{2}\ln(1-u_1)\ln\tilde{u}_2\ln\tilde{u}_3 + \frac{i\pi}{4}\ln(1-u_1)\ln^2\tilde{u}_3.
\end{aligned} \tag{C.6}$$

The notation $|x_1^{\pm}|$ denotes that fact that u_1 is replaced by $|u_1|$ in the argument.

Finally, the last term in Eq. C.2 is given by

$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{u_i}\right)\right)^2-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{|u_i|}\right)\right)^2\overset{LLA}{\simeq}\frac{i5\pi^3}{6}\ln(1-u_1)+\pi^2\ln^2(1-u_1)\tag{C.7}$$
$$-\frac{i\pi}{2}\ln^3(1-u_1)+\frac{\pi^2}{2}\ln(1-u_1)\ln\tilde{u}_2-\frac{i\pi}{2}\ln^2(1-u_1)\ln\tilde{u}_2-\frac{i\pi}{4}\ln(1-u_1)\ln^2\tilde{u}_2$$
$$+\frac{\pi^2}{2}\ln(1-u_1)\ln\tilde{u}_3-\frac{i\pi}{2}\ln^2(1-u_1)\ln\tilde{u}_3-\frac{i\pi}{4}\ln(1-u_1)\ln^2\tilde{u}_3.$$

Adding up Eq. C.4, Eq. C.6 and Eq. C.7 we get

$$+\frac{i\pi}{2}\ln(1-u_1)\ln\tilde{u}_2\ln\tilde{u}_3.\tag{C.8}$$

This expression coincides with LLA term obtained by one of the authors [7] in the BFKL approach.

D Next-to-leading logarithmic (NLLA) terms

We have extracted the leading order term in the logarithm of the energy $\ln s_2\simeq-\ln(1-u_1)$ of the imaginary part of the remainder function $R_6^{(2)}$. The term we obtained from Eq. 13 after analytic continuation to the physical region of $u_1\to e^{-i2\pi}|u_1|$ reproduces the term calculated by one of the authors in the BFKL formalism. Unfortunately, due complexity of the calculations, the sub-leading terms in $\ln(1-u_1)$ were not yet calculated in the BFKL approach. The comparison between the two approaches, the BFKL formalism and the Wilson Loop/Scattering Amplitude duality, is not full without matching the NLLA terms. In this section we calculated the NLLA terms from the analytically continued expression of Goncharov et al. given in Eq. 13 that can be further confronted with the BFKL result once it is available. We follow the logic of the LLA calculations outlined above, and extract only the NLLA terms that appeared in the course of the continuation (subtracting the relevant values before they were analytically continued). As it was already mentioned this is possible to do because the remainder function vanishes in the limit of Eq. 2 before the analytic continuation.

The expression of the remainder function is given in Eq. 13. We calculate all contributions in separate leaving only the NLLA terms, i.e. those that are not accompanied by any power of $\ln(1-u_1)$

$$-\frac{1}{2}\sum_{i=1}^{3}\mathrm{Li}_4\left(1-\frac{1}{u_i}\right)+\frac{1}{2}\sum_{i=1}^{3}\mathrm{Li}_4\left(1-\frac{1}{|u_i|}\right)\overset{NLO}{\simeq}\frac{\pi^4}{6},\tag{D.1}$$

$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{u_i}\right)\right)^2+\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{|u_i|}\right)\right)^2\overset{NLO}{\simeq}-\frac{\pi^4}{3}+\frac{\pi^2}{4}\ln^2\tilde{u}_2+\frac{\pi^2}{4}\ln^2\tilde{u}_3,\tag{D.2}$$

$$\mathrm{L}_4(x_1^+,x_1^-)-\mathrm{L}_4(|x_1^+|,|x_1^-|)\overset{NLO}{\simeq}\frac{\pi^4}{8}+\frac{\pi^2}{4}\ln^2 B^++\frac{i\pi}{6}\ln^3 B^+-\frac{i\pi^3}{6}\ln(\tilde{u}_2\tilde{u}_3)\tag{D.3}$$
$$-\frac{\pi^2}{4}\ln B^+\ln(\tilde{u}_2\tilde{u}_3)-\frac{i\pi}{4}\ln^2 B^+\ln(\tilde{u}_2\tilde{u}_3)-\frac{\pi^2}{16}\ln^2(\tilde{u}_2\tilde{u}_3)+\frac{i\pi}{8}\ln B^+\ln^2(\tilde{u}_2\tilde{u}_3)+\frac{i\pi}{48}\ln^3(\tilde{u}_2\tilde{u}_3),$$

where we introduced

$$B^{\pm} = \frac{1 - \tilde{u}_2 - \tilde{u}_3 \pm \sqrt{(1 - \tilde{u}_2 - \tilde{u}_3)^2 - 4\tilde{u}_2\tilde{u}_3}}{2} \tag{D.4}$$

and used its property

$$B^+B^- = \tilde{u}_2\tilde{u}_3 \tag{D.5}$$

to eliminate B^-.

Before we calculate the contributions from $\mathrm{L}_4(x_2^+, x_2^-)$ and $\mathrm{L}_4(x_3^+, x_3^-)$ we find the function J.

$$J \simeq \frac{1}{2} \ln \frac{B^+}{B^-} + i\pi. \tag{D.6}$$

The expression in Eq. D.6 depends only on the "transverse" cross ratios $\tilde{u}_2$ and $\tilde{u}_3$ as one can see from Eq. D.4. The function χ defined in Eq. 18 has the same value $\chi = 1$ in all points on the circle $u_1 = |u_1|e^{i\phi}$ for $|u_1| + u_2 + u_3 < 1$ and thus does not posses any additional terms in the analytic continuation when the phase ϕ changes from 0 to $-i2\pi$. Now we can readily find the contribution from all terms that include J

$$\begin{aligned} \frac{J^4}{24} + \chi\frac{\pi^2}{12}J^2 - \left(\frac{J^4}{24} + \chi\frac{\pi^2}{12}J^2\right)|_{u_1=|u_1|} \simeq& -\frac{\pi^4}{24} - \frac{\pi^2}{4}\ln^2 B^+ + \frac{i\pi}{6}\ln^3 B^+ \\ &+\frac{\pi^2}{4}\ln B^+ \ln(\tilde{u}_2\tilde{u}_3) - \frac{i\pi}{4}\ln^2 B^+ \ln(\tilde{u}_2\tilde{u}_3) - \frac{\pi^2}{16}\ln^2(\tilde{u}_2\tilde{u}_3) \\ &+\frac{i\pi}{8}\ln B^+ \ln^2(\tilde{u}_2\tilde{u}_3) - \frac{i\pi}{48}\ln^3(\tilde{u}_2\tilde{u}_3). \end{aligned} \tag{D.7}$$

Note that

$$\chi\frac{\pi^2}{12}\zeta_2 - \left(\chi\frac{\pi^2}{12}\zeta_2\right)|_{u_1=|u_1|} = 0. \tag{D.8}$$

Summing up Eq. D.1-D.3, Eq. D.7 and Eq. D.8 we obtain a compact expression

$$-\frac{\pi^4}{12} + \frac{i\pi}{3}\ln^3 B^+ + \frac{\pi^2}{8}\ln^2\left(\frac{\tilde{u}_2}{\tilde{u}_3}\right) - \frac{i\pi^3}{6}\ln(\tilde{u}_2\tilde{u}_3) - \frac{i\pi}{2}\ln^2 B^+ \ln(\tilde{u}_2\tilde{u}_3) + \frac{i\pi}{4}\ln B^+ \ln^2(\tilde{u}_2\tilde{u}_3) \tag{D.9}$$

As a last step in our analysis we calculate the contributions of $\mathrm{L}_4(x_2^+, x_2^-)$ and $\mathrm{L}_4(x_3^+, x_3^-)$. Namely,

$$\mathrm{L}_4(x_2^+,x_2^-)-\mathrm{L}_4(|x_2^+|,|x_2^-|)\overset{NLO}{\simeq}\frac{\pi^4}{24}-\frac{i\pi}{3}\ln^3 B^+-\frac{i\pi^3}{6}\ln^3(B^++\tilde u_2)+\frac{i\pi^3}{6}\ln\tilde u_3 \tag{D.10}$$
$$+\frac{\pi^2}{2}\ln(B^++\tilde u_2)\ln\tilde u_3-\frac{\pi^2}{4}\ln^2\tilde u_3+\frac{i\pi}{2}\ln(B^++\tilde u_2)\ln^2\tilde u_3-\frac{i\pi}{6}\ln^3\tilde u_3+\frac{i\pi^3}{12}\ln(\tilde u_2\tilde u_3)$$
$$+\frac{i\pi}{2}\ln^2 B^+\ln(\tilde u_2\tilde u_3)-\frac{\pi^2}{4}\ln(B^++\tilde u_2)\ln(\tilde u_2\tilde u_3)-\frac{\pi^2}{4}\ln\tilde u_3\ln(\tilde u_2\tilde u_3)$$
$$-\frac{i\pi}{2}\ln(B^++\tilde u_2)\ln\tilde u_3\ln(\tilde u_2\tilde u_3)-\frac{i\pi}{4}\ln^2\tilde u_3\ln(\tilde u_2\tilde u_3)+\frac{3\pi^2}{16}\ln^2(\tilde u_2\tilde u_3)-\frac{i\pi}{4}\ln B^+\ln^2(\tilde u_2\tilde u_3)$$
$$+\frac{i\pi}{8}\ln(B^++\tilde u_2)\ln^2(\tilde u_2\tilde u_3)+\frac{i3\pi}{8}\ln\tilde u_3\ln^2(\tilde u_2\tilde u_3)-\frac{i\pi}{16}\ln^3(\tilde u_2\tilde u_3)-\frac{i\pi^3}{6}\ln(B^++\tilde u_3)$$
$$+\frac{\pi^2}{2}\ln\tilde u_3\ln(B^++\tilde u_3)+\frac{i\pi}{2}\ln^2\tilde u_3\ln(B^++\tilde u_3)-\frac{\pi^2}{4}\ln(\tilde u_2\tilde u_3)\ln(B^++\tilde u_3)$$
$$-\frac{i\pi}{2}\ln\tilde u_3\ln(\tilde u_2\tilde u_3)\ln(B^++\tilde u_3)+\frac{i\pi}{8}\ln^2(\tilde u_2\tilde u_3)\ln(B^++\tilde u_3)+\frac{\pi^2}{2}\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_2}\right)$$
$$+i\pi\ln\tilde u_3\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_2}\right)-\frac{i\pi}{2}\ln(\tilde u_2\tilde u_3)\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_2}\right)-\frac{\pi^2}{2}\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_3}\right)-i\pi\ln\tilde u_3\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_3}\right)$$
$$+\frac{i\pi}{2}\ln(\tilde u_2\tilde u_3)\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_3}\right)-i\pi\mathrm{Li}_3\left(-\frac{B^+}{\tilde u_2}\right)-i\pi\mathrm{Li}_3\left(-\frac{B^+}{\tilde u_3}\right).$$

The contribution of $\mathrm{L}_4(x_3^+,x_3^-)$ is readily obtained from Eq. D.10 by changing variables $\tilde u_2\leftrightarrow\tilde u_3$.

Summing up Eq. D.9 and Eq. D.10 (together with $\tilde u_2\leftrightarrow\tilde u_3$) we get the NLLA part of the remainder function after the analytic continuation

$$-\frac{i\pi^3}{3}\ln B^+-\frac{i\pi}{3}\ln^3 B^+-\frac{i\pi}{3}\ln^3\tilde u_2-i\pi\ln B^+\ln\tilde u_2\ln\tilde u_3-\frac{i\pi}{3}\ln^3\tilde u_3+\frac{i\pi^3}{6}\ln(\tilde u_2\tilde u_3) \tag{D.11}$$
$$+\frac{i\pi}{2}\ln^2 B^+\ln(\tilde u_2\tilde u_3)+\frac{i\pi}{6}\ln^3(\tilde u_2\tilde u_3)-i\pi\ln\left(\frac{\tilde u_2}{\tilde u_3}\right)\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_2}\right)+i\pi\ln\left(\frac{\tilde u_2}{\tilde u_3}\right)\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_3}\right)$$
$$-i2\pi\mathrm{Li}_3\left(-\frac{B^+}{\tilde u_2}\right)-i2\pi\mathrm{Li}_3\left(-\frac{B^+}{\tilde u_3}\right).$$

The expression of Eq. D.11 is pure imaginary in the limit $\tilde u_{2,3}>0$ and $\tilde u_2+\tilde u_3<1$ despite the fact that it contains a square root in its arguments through B^+ defined in Eq. D.4. It is also symmetrical with respect to the exchange of $\tilde u_2$ and $\tilde u_3$.

Adding to Eq. D.11 the Leading Order result calculated in the previous section and given by Eq. C.8 we obtain the final result

$$+\frac{i\pi}{2}\ln(1-u_1)\ln\tilde u_2\ln\tilde u_3 \tag{D.12}$$
$$-\frac{i\pi^3}{3}\ln B^+-\frac{i\pi}{3}\ln^3 B^+-\frac{i\pi}{3}\ln^3\tilde u_2-i\pi\ln B^+\ln\tilde u_2\ln\tilde u_3-\frac{i\pi}{3}\ln^3\tilde u_3+\frac{i\pi^3}{6}\ln(\tilde u_2\tilde u_3)$$
$$+\frac{i\pi}{2}\ln^2 B^+\ln(\tilde u_2\tilde u_3)+\frac{i\pi}{6}\ln^3(\tilde u_2\tilde u_3)-i\pi\ln\left(\frac{\tilde u_2}{\tilde u_3}\right)\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_2}\right)+i\pi\ln\left(\frac{\tilde u_2}{\tilde u_3}\right)\mathrm{Li}_2\left(-\frac{B^+}{\tilde u_3}\right)$$
$$-i2\pi\mathrm{Li}_3\left(-\frac{B^+}{\tilde u_2}\right)-i2\pi\mathrm{Li}_3\left(-\frac{B^+}{\tilde u_3}\right).$$

An important remark is in order. The last expression was calculated in the region $\tilde u_2+\tilde u_3<1$ and, in principle, should be analytically continued to any other physical region. The first term

$+\frac{i\pi}{2}\ln(1-u_1)\ln\tilde{u}_2\ln\tilde{u}_3$, which corresponds to the Leading Logarithmic Approximation is a smooth function also outside the region $\tilde{u}_2+\tilde{u}_3<1$ since it does not have any singularities on the boundary of the region. This is not obvious for the rest of the NLLA terms, where individual terms do have branch points on the boundary of $\tilde{u}_2+\tilde{u}_3<1$. However the singularities are canceled in the sum as can be shown introducing back B^-

$$\begin{aligned}&R(|u_1|e^{-i2\pi},\tilde{u}_2(1-u_1),\tilde{u}_3(1-u_1))\simeq+\frac{i\pi}{2}\ln(1-u_1)\ln\tilde{u}_2\ln\tilde{u}_3+\frac{i\pi}{3}\ln^3\tilde{u}_2\\&-\frac{i\pi}{2}\ln^2\tilde{u}_2\ln\left(\frac{\tilde{u}_2}{\tilde{u}_3}\right)-i\pi\ln\left(\frac{\tilde{u}_2}{\tilde{u}_3}\right)\left(\mathrm{Li}_2\left(-\frac{B^+}{\tilde{u}_2}\right)+\mathrm{Li}_2\left(-\frac{B^-}{\tilde{u}_2}\right)\right)\\&-i2\pi\left(\mathrm{Li}_3\left(-\frac{B^+}{\tilde{u}_2}\right)+\mathrm{Li}_3\left(-\frac{B^-}{\tilde{u}_2}\right)\right).\end{aligned}\tag{D.13}$$

It can be easily seen from the series representation of the polylogarithms that all square roots in the argument cancel out, and the expression in Eq. D.13 is also valid for $\tilde{u}_2+\tilde{u}_3\geq 1$, but only in the region **A** of the multi Regge kinematics shown in Fig. 4. Eq. D.13 is the main result of this study.

E $R_6^{(2)}$ in complex variables

In this section we eliminate the square roots in the arguments of the remainder function of Eq. D.13 introducing complex variables

$$z=\sqrt{\tilde{u}_2}e^{i\phi_2},\quad 1-z=\sqrt{\tilde{u}_3}e^{-i\phi_3}.\tag{E.1}$$

It is useful to calculate $\cos(\phi_2-\phi_3)$ and $\sin(\phi_2-\phi_3)$ from

$$|z+(1-z)|^2=1=\tilde{u}_2+\tilde{u}_3+2\sqrt{\tilde{u}_2}\sqrt{\tilde{u}_3}\cos(\phi_2-\phi_3).\tag{E.2}$$

We readily find

$$\cos(\phi_2-\phi_3)=\frac{1-\tilde{u}_2-\tilde{u}_3}{2\sqrt{\tilde{u}_2}\sqrt{\tilde{u}_3}}\tag{E.3}$$

and

$$\sin(\phi_2-\phi_3)=\frac{\sqrt{4\tilde{u}_2\tilde{u}_3-(1-\tilde{u}_2-\tilde{u}_3)^2}}{2\sqrt{\tilde{u}_2}\sqrt{\tilde{u}_3}}\tag{E.4}$$

as well as

$$i\sin(\phi_2-\phi_3)=-\frac{\sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2-4\tilde{u}_2\tilde{u}_3}}{2\sqrt{\tilde{u}_2}\sqrt{\tilde{u}_3}}.\tag{E.5}$$

With the help of Eq. E.3 an Eq. E.5 the function $B^{\pm}$, defined in Eq. D.4, can be written as

$$B^{\pm}=e^{\mp i\phi_2}e^{\mp i\phi_3}|z|\,|1-z|\tag{E.6}$$

and thus

$$\frac{B^+}{\tilde{u}_2}=\frac{1-z}{z},\quad\frac{B^-}{\tilde{u}_2}=\frac{1-z^*}{z^*},\quad\frac{B^+}{\tilde{u}_3}=\frac{z^*}{1-z^*},\quad\frac{B^-}{\tilde{u}_3}=\frac{z}{1-z}.\tag{E.7}$$

Using Eq. E.7 and the identities between Li_k of different arguments we write the expression in Eq. D.13 as follows

$$R(|u_1|e^{-i2\pi}, |z|^2, |1-z|^2) \simeq \frac{i\pi}{2}\ln(1-u_1)\ln|z|^2\ln|1-z|^2 \tag{E.8}$$
$$-i4\pi\zeta_3 + \frac{i\pi}{2}\ln|z|^2|1-z|^2\left(\ln z\ln(1-z)+\ln z^*\ln(1-z^*)-2\zeta_2\right)$$
$$+\frac{i\pi}{2}\ln\frac{|1-z|^2}{|z|^2}\left(\mathrm{Li}_2(z)+\mathrm{Li}_2(z^*)-\mathrm{Li}_2(1-z)-\mathrm{Li}_2(1-z^*)\right)$$
$$+i2\pi\left(\mathrm{Li}_3(z)+\mathrm{Li}_3(z^*)+\mathrm{Li}_3(1-z)+\mathrm{Li}_3(1-z^*)\right).$$

From Eq. E.8 we see that the square roots present in $B^{\pm}$ disappear and the remainder function is manifestly pure imaginary in LLA. The target-projectile symmetry $\tilde{u}_2 \leftrightarrow \tilde{u}_3$, which is $z \leftrightarrow 1-z$ symmetry in terms of the variables Eq. E.1 is also obvious in Eq. E.8.

Because of the holomorphic factorization of the impact factor in Eq. 5 it is more natural to express the final answer in complex variables

$$w = \frac{1-z}{z}, \quad w^* = \frac{1-z^*}{z^*}. \tag{E.9}$$

Noting that the reduced crossed ratios $\tilde{u}_2$ and $\tilde{u}_3$ are related to the transverse momenta (see Eq. 12) we can write

$$\frac{q_3 k_1}{k_2 q_1} = \frac{\sqrt{\tilde{u}_3}e^{i\phi_3}}{\sqrt{\tilde{u}_2}e^{i\phi_2}} = \frac{1-z}{z} = w \tag{E.10}$$

and thus Eq. 5 reads

$$\Delta_{2\to 4} = \frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{1-z^*}{z^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{1-z}{z}\right)^{i\nu+\frac{n}{2}}\left(s_2^{\omega(\nu,n)}-1\right)$$
$$= \frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(w^*\right)^{i\nu-\frac{n}{2}}\left(w\right)^{i\nu+\frac{n}{2}}\left(s_2^{\omega(\nu,n)}-1\right). \tag{E.11}$$

Eq. E.11 is explicitly symmetric in $z \leftrightarrow 1-z$ ($w \leftrightarrow 1/w$). We recast Eq. E.8 in terms of the variables w and w^* as follows

$$R\left(|u_1|e^{-i2\pi}, \frac{1}{|1+w|^2}, \frac{|w|^2}{|1+w|^2}\right) \simeq \frac{i\pi}{2}\ln(1-u_1)\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2 \tag{E.12}$$
$$+\frac{i\pi}{2}\ln|w|^2\ln^2|1+w|^2 - \frac{i\pi}{3}\ln^3|1+w|^2 + i\pi\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right)$$
$$-i2\pi\left(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\right).$$

F NLO impact factor

We wish to calculate inverse Mellin and Fourier transforms of the next-to-leading contribution to the remainder function. The form of the direct transforms in the complex variable w can be read out from the last line of Eq. E.11 and is given by

$$\tilde{f}(w, w^*) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}d\nu\left(w^*\right)^{i\nu-\frac{n}{2}}\left(w\right)^{i\nu+\frac{n}{2}}f(\nu,n) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}d\nu\rho^{2i\nu}e^{i\phi n}f(\nu,n), \tag{F.1}$$

where $w = \rho e^{i\phi}$. The inverse transform thus reads

$$f(\nu, n) = \frac{1}{(2\pi)^2} \int_0^\infty d\rho^2 \int_0^{2\pi} d\phi \rho^{-2i\nu-2} e^{-i\phi n} \tilde{f}(\rho, \phi), \tag{F.2}$$

which can be written as

$$f(\nu, n) = \frac{2}{(2\pi)^2} \int d^2\vec{w}(w)^{-i\nu-\frac{n}{2}-\frac{1}{2}} (w^*)^{-i\nu+\frac{n}{2}-\frac{1}{2}} \tilde{f}(w, w^*). \tag{F.3}$$

The integration in Eq. F.3 is performed on the two dimensional plane in Cartesian coordinates w_1 and w_2 defined by $w = w_1 + iw_2$. We start with the logarithmic terms appearing in Eq. E.12. The relevant logarithms $\ln^k |1+w|^2$ can be obtained by differentiation of the power function $(|1+w|^2)^a$ and thus it is convenient to consider

$$\begin{aligned} &\frac{1}{(2\pi)^2} \int d^2w \, |w|^{2b} w^{*\,n} |1+w|^{2a} \\ &= \frac{1}{(2\pi)^2} \frac{\Gamma(a+n+1)}{\Gamma(a+1)} \frac{\partial^n}{\partial z^n} \int d^2w \, |w|^{2b} |z+w|^{2a+2n} |_{z=z^*=1} \end{aligned} \tag{F.4}$$

We introduce the master integral

$$g(a, b; z) = \frac{1}{\pi} \int d^2x |x|^{2a} |z-x|^{2b}, \tag{F.5}$$

which corresponds to the one loop diagram with z being a momentum of the external particles.

This integral is found by using the well-known formula of the momentum integration

$$\int \frac{d^d\mathbf{k}}{(\mathbf{k}^2)^{\lambda_1}((\mathbf{q}-\mathbf{k})^2)^{\lambda_2}} = \pi^{d/2} \frac{\Gamma(d/2-\lambda_1)\Gamma(d/2-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(d-\lambda_1-\lambda_2)} \frac{\Gamma(\lambda_1+\lambda_2-d/2)}{(\mathbf{q}^2)^{\lambda_1+\lambda_2-d/2}}, \tag{F.6}$$

and it reads

$$g(a, b; z) = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(-a)\Gamma(-b)\Gamma(2+a+b)} \frac{\Gamma(-1-a-b)}{|z|^{2(-1-a-b)}}. \tag{F.7}$$

Using the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ this can be written as

$$g(a, b; z) = \frac{\Gamma^2(1+a)\Gamma^2(1+b)}{\Gamma^2(2+a+b)} \frac{\sin \pi a \sin \pi b}{\pi \sin \pi(a+b)} |z|^{2(1+a+b).} \tag{F.8}$$

From Eq. F.4 and Eq. F.5 we find a general expression for the inverse transform of the logarithms

$$\mathcal{G}_{(k,m)}(\nu, n) = \frac{(-1)^n}{2\pi} \frac{\partial^k}{\partial a^k} \frac{\partial^m}{\partial b^m} \frac{\Gamma^2(1+a)\Gamma(1+b)\Gamma(1+b-n)}{\Gamma(2+b+a)\Gamma(2+a+b-n)} \frac{\sin \pi a \sin \pi b}{\pi \sin \pi(a+b)} |_{a=0, b=-i\nu-1+n/2} \tag{F.9}$$

Applying Eq. F.9 for $k = 1$ and $m = 0$ we get (for $n \neq 0$)

$$2\pi \ln |1+w|^2 \Rightarrow -\frac{(-1)^n}{\nu^2 + \frac{n^2}{4}} \tag{F.10}$$

in full agreement with the Born term of the six-point amplitude.

For $k = 2$ and $m = 0$ we get from Eq. F.9

$$2\pi \ln^2 |1+w|^2 \Rightarrow 2\frac{(-1)^n}{\nu^2 + \frac{n^2}{4}} \left(\psi\left(i\nu + \frac{|n|}{2}\right) + \psi\left(1 - i\nu + \frac{|n|}{2}\right) - 2\psi(1) \right), \tag{F.11}$$

where we used the identity $\psi(z) = \psi(1-z) - \pi\cot\pi z$.

Plugging $k=3$ and $m=0$ in Eq. F.9 we get

$$2\pi \ln^3 |1+w|^2 \Rightarrow -\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\left(-\pi^2 + 3\left(\psi\left(i\nu+\frac{|n|}{2}\right)+\psi\left(1-i\nu+\frac{|n|}{2}\right)-2\psi(1)\right)^2 \right.$$
$$\left. +3\left(\psi'\left(i\nu+\frac{|n|}{2}\right)-\psi'\left(1-i\nu+\frac{|n|}{2}\right)+2\psi(1)\right)\right) \tag{F.12}$$

and for $k=1$ and $m=1$ one obtains

$$2\pi \ln|w|^2 \ln|1+w|^2 \Rightarrow (-1)^n \frac{2i\nu}{\left(\nu^2+\frac{n^2}{4}\right)^2}. \tag{F.13}$$

Therefore from Eq. F.13 and Eq. F.11 we get

$$2\pi \ln|1+w|^2 \ln\left|1+\frac{1}{w}\right|^2 = 2\pi\ln^2|1+w|^2 - 2\pi \ln|w|^2\ln|1+w|^2 \Rightarrow \tag{F.14}$$
$$2\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\left(-\frac{|n|}{2}\frac{1}{\nu^2+\frac{n^2}{4}}+\psi\left(1+i\nu+\frac{|n|}{2}\right)+\psi\left(1-i\nu+\frac{|n|}{2}\right)-2\psi(1)\right) = 2\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}E_{\nu,n}$$

in an agreement with the analysis of ref. [7].

In a similar way we calculate

$$2\pi \ln|w|^2\ln^2|1+w|^2 \Rightarrow \tag{F.15}$$
$$= -(-1)^n\frac{2}{\nu^2+\frac{n^2}{4}}\left(\frac{-i2\nu}{\nu^2+\frac{n^2}{4}}\left(2\psi(1)-\psi\left(i\nu+\frac{|n|}{2}\right)-\psi\left(1-i\nu+\frac{|n|}{2}\right)\right)\right.$$
$$\left.-\psi'\left(1-i\nu+\frac{|n|}{2}\right)+\psi'\left(i\nu+\frac{|n|}{2}\right)\right).$$

The rest of the terms in Eq. E.12 are found by noting that

$$2\pi \ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) - 4\pi\left(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\right) \Rightarrow -\frac{(-1)^n}{\left(\nu^2+\frac{n^2}{4}\right)^2}. \tag{F.16}$$

This can be easily checked by calculating residue at $\nu = \pm i|n|/2$ and summing over n. Both of the terms on LHS of Eq. F.16 have poles double poles at $\nu = 0$ for $n \neq 0$ (for $n = 0$ the remainder function vanishes), which correspond to the infrared divergencies absent in the remainder function. Due to the special coefficients of these terms appearing in Eq. E.12 the infrared divergency is canceled in the final expression. This fact suggests that the relative coefficients of the individual terms can be fixed demanding the absence of the poles at $\nu = 0$ for $n \neq 0$, together with $w \to 1/w$ symmetry.

Gathering together the inverse transform of all terms in Eq. E.12 we finally obtain

$$R^{NLLA}\left(|u_1|e^{-i2\pi}, \frac{1}{|1+w|^2}, \frac{|w|^2}{|1+w|^2}\right) \Rightarrow \frac{i}{2}\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\left(E^2_{\nu,n} - \frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right), \tag{F.17}$$

where $E_{\nu,n}$ is given by Eq. 8. This expression vanishes for $n=0$, which corresponds to absence of the infrared divergencies in the remainder function; it is symmetric in $n \to -n$ and $\nu \to -\nu$, which implied by the target-projectile symmetry $w \to 1/w$.

From the definition of the impact factors in ν, n representation given by Eq. 33 we read out the form of the next-to-leading-order (NLO) impact factor

$$\chi_1^{NLO} = \frac{a}{4} \frac{1}{\left(i\nu + \frac{n}{2}\right)} \left(E_{\nu,n}^2 - \frac{1}{4} \frac{n^2}{\left(\nu^2 + \frac{n^2}{4}\right)^2} \right) \left(-\frac{q_1}{k_1} \right)^{-i\nu - \frac{n}{2}} \left(-\frac{q_1^*}{k_1^*} \right)^{-i\nu + \frac{n}{2}}, \tag{F.18}$$

$$\chi_2^{NLO} = -\frac{a}{4} \frac{1}{\left(i\nu - \frac{n}{2}\right)} \left(E_{\nu,n}^2 - \frac{1}{4} \frac{n^2}{\left(\nu^2 + \frac{n^2}{4}\right)^2} \right) \left(\frac{q_3}{k_2} \right)^{i\nu - \frac{n}{2}} \left(\frac{q_3^*}{k_2^*} \right)^{i\nu + \frac{n}{2}}. \tag{F.19}$$

In the next section we use Eq. F.17 to calculate the three loop leading-log contribution to the six-point amplitude.

G Three loop contribution in LLA

The general expression for the leading logarithmic contribution to the imaginary part of the remainder function at any number of loops is given by

$$\Delta_{2\to 4} = \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \left(\frac{q_3^* k_1^*}{k_2^* q_1^*} \right)^{i\nu - \frac{n}{2}} \left(\frac{q_3 k_1}{k_2 q_1} \right)^{i\nu + \frac{n}{2}} \left(s_2^{\omega(\nu,n)} - 1 \right) \tag{G.1}$$

$$= \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \left(w^* \right)^{i\nu - \frac{n}{2}} \left(w \right)^{i\nu + \frac{n}{2}} \left(s_2^{\omega(\nu,n)} - 1 \right),$$

where the $\omega(\nu, n)$ is related to the eigenvalue of the octet BFKL Hamiltonian $E_{\nu,n}$ by

$$\omega(\nu, n) = -a E_{\nu,n} \tag{G.2}$$

with

$$E_{\nu,n} = -\frac{1}{2} \frac{|n|}{\nu^2 + \frac{n^2}{4}} + \psi\left(1 + i\nu + \frac{|n|}{2}\right) + \psi\left(1 - i\nu + \frac{|n|}{2}\right) - 2\psi(1). \tag{G.3}$$

The complex variable w is defined through

$$w = \frac{q_3 k_1}{k_2 q_1} \text{ and } w^* = \frac{q_3^* k_1^*}{k_2^* q_1^*}. \tag{G.4}$$

According to the discussion presented in appendix **E** in the Regge limit w can be written as

$$w = \sqrt{\frac{\tilde{u}_3}{\tilde{u}_2}} e^{i(\phi_3 - \phi_2)} \tag{G.5}$$

with

$$\cos(\phi_2 - \phi_3) = \frac{1 - \tilde{u}_2 - \tilde{u}_3}{2\sqrt{\tilde{u}_2 \tilde{u}_3}} \tag{G.6}$$

and

$$\sin(\phi_2 - \phi_3) = \frac{\sqrt{4\tilde{u}_2 \tilde{u}_3 - (1 - \tilde{u}_2 - \tilde{u}_3)^2}}{2\sqrt{\tilde{u}_2 \tilde{u}_3}} \tag{G.7}$$

for

$$\tilde{u}_2 = \frac{u_2}{1-u_1}, \quad \tilde{u}_3 = \frac{u_3}{1-u_1}. \tag{G.8}$$

The LLA three-loop contribution to the remainder function $R_6^{(3)}$ is obtained from Eq. G.2 by expanding in powers of the coupling constant a as follows

$$\frac{a^3 R_6^{(3)\ LLA}}{i} = \Delta^{(3)}_{2\to 4} = \frac{a^3}{4}\ln^2 s_2 \sum_{n=-\infty}^{\infty}(-1)^n \int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}\,E^2_{\nu,n} \tag{G.9}$$
$$\simeq \frac{a^3}{4}\ln^2(1-u_1)\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}\,E^2_{\nu,n},$$

where we used the fact that in the Regge limit $1-u_1 \simeq (\mathbf{k}_1+\mathbf{k}_2)^2/s_2$.

The integral in RHS of Eq. G.9 can be easily obtained using the calculations of the previous section. Namely, we have shown that Eq. F.17 gives the ν, n representation of the NLLA contribution to the remainder function at two loops, namely

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\left(E^2_{\nu,n}-\frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right)(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}. \tag{G.10}$$

We calculate in separate the transform of the second term in the brackets in Eq. G.10

$$\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\left(-\frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right)(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}. \tag{G.11}$$

The integral in Eq. G.11 can be calculated using the residue theorem closing the contour either in the upper semiplane for poles $\nu = i|n|/2$ and multiplying the residue by $i2\pi$, or in the lower semiplane for poles $\nu = -i|n|/2$ and then multiplying the residue by $-i2\pi$. The result has $w \leftrightarrow 1/w$ symmetry so that it is enough to consider only contributions for $|w|<1$. The residue at $\nu = -i|n|/2$ for $|w|<1$ reads

$$-i2\pi \mathrm{Res}\left(\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\frac{1}{2}\left(-\frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right)(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}, -i|n|/2\right) \tag{G.12}$$
$$= -\frac{3}{2}\frac{(-1)^n\pi(w^*)^{|n|}}{|n|^3}+\frac{3(-1)^n\pi(w^*)^{|n|}\ln|w|^2}{4n^2}-\frac{(-1)^n\pi(w^*)^{|n|}\ln^2|w|^2}{8|n|}.$$

The summation over n (for $n>0$) is readily performed using the series representation of the polylogarithms $\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} x^k/k^n$ and we get

$$-i2\pi\sum_{n=1}^{\infty}\mathrm{Res}\left(\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\frac{1}{2}\left(-\frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right)(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}, -i|n|/2\right) \tag{G.13}$$
$$= \frac{1}{8}\pi\ln^2|w|^2\ln(1+w^*)+\frac{3}{4}\pi\ln|w|^2\mathrm{Li}_2(-w^*)-\frac{3}{2}\pi\mathrm{Li}_3(-w^*).$$

The contribution from the sum over negative n is added by substitution $w^* \to w$ and we obtain

$$\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\frac{1}{2}\left(-\frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right)(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}} \tag{G.14}$$
$$= \frac{1}{8}\pi\ln^2|w|^2\ln|1+w|^2+\frac{3}{4}\pi\ln|w|^2\left(\mathrm{Li}_2(-w)-\mathrm{Li}_2(-w^*)\right)-\frac{3}{2}\pi\mathrm{Li}_3(-w)-\frac{3}{2}\pi\mathrm{Li}_3(-w^*).$$

From Eq. G.9 it follows that for the three-loop LLA contribution we need to calculate the following expression

$$\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}} E_{\nu,n}^2 \left(w^*\right)^{i\nu-\frac{n}{2}} \left(w\right)^{i\nu+\frac{n}{2}} . \tag{G.15}$$

This can be obtained by subtracting Eq. G.14 from Eq. G.10. In the previous section we found that

$$\begin{aligned}&\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}} \left(E_{\nu,n}^2 - \frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right) \left(w^*\right)^{i\nu-\frac{n}{2}} \left(w\right)^{i\nu+\frac{n}{2}} \\ &= \frac{\pi}{2}\ln|w|^2\ln^2|1+w|^2 - \frac{\pi}{3}\ln^3|1+w|^2 + \pi\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) \\ &\quad -2\pi\left(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\right)\end{aligned} \tag{G.16}$$

and thus we write

$$\begin{aligned}&\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}} E_{\nu,n}^2 \left(w^*\right)^{i\nu-\frac{n}{2}} \left(w\right)^{i\nu+\frac{n}{2}} = \frac{\pi}{2}\ln|w|^2\ln^2|1+w|^2 - \frac{\pi}{3}\ln^3|1+w|^2 \\ &-\frac{\pi}{8}\ln^2|w|^2\ln|1+w|^2 + \frac{\pi}{4}\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) - \frac{\pi}{2}\left(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\right).\end{aligned} \tag{G.17}$$

Finally from Eq. G.9 and Eq. G.17 we obtain the remainder function at three loops in the leading logarithm approximation (LLA), namely

$$\begin{aligned}&a^3 R_6^{(3)\ LLA} = i\Delta_{2\to 4}^{(3)} = i\pi\frac{a^3}{4}\ln^2(1-u_1)\Bigg(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2 \\ &-\frac{1}{4}\ln^2|w|^2\ln|1+w|^2 + \frac{1}{2}\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) - \mathrm{Li}_3(-w) - \mathrm{Li}_3(-w^*)\Bigg).\end{aligned} \tag{G.18}$$

The complex variables w is expressed in terms of the reduced cross ratios of Eq. 11 as

$$w = \frac{1-z}{z} = \frac{B^+}{\tilde{u}_2}, \quad w^* = \frac{1-z^*}{z^*} = \frac{B^-}{\tilde{u}_2} \tag{G.19}$$

for $B^\pm$ defined in Eq. D.4 by

$$B^\pm = \frac{1-\tilde{u}_2-\tilde{u}_3 \pm \sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2 - 4\tilde{u}_2\tilde{u}_3}}{2}. \tag{G.20}$$

H The real part of the remainder function at three loops in NLLA

In this section we calculate the real part of the remainder function at three loops in the next-to-leading logarithmic approximation (NLLA). The expression for $\Re(R_6^{(3)NLLA})$ is obtained expanding the dispersion relation Eq. 36 in powers of the perturbation expansion parameter a. It is worth emphasizing that the calculation of $\Re(R_6^{(3)NLLA})$ does not require the knowledge of currently unavailable subleading corrections to the BFKL eigenvalue $\omega(\nu,n)$. We plug the LLA function $f^{LLA}(\omega)$ of Eq. 38 in Eq. 36 and expand it in a to the third order

$$i\pi\delta a^2 R_6^{(2)} + a^3 R_6^{(3)} - \frac{i\pi\delta^3}{6} \simeq \frac{ia^3}{4}(\ln(1-u_1)+i\pi)^2 \sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}} E_{\nu,n}^2 \left(w^*\right)^{i\nu-\frac{n}{2}} \left(w\right)^{i\nu+\frac{n}{2}} , \tag{H.1}$$

where w and $E_{\nu,n}$ are given by Eq. G.4 and Eq. 8 respectively. The phases δ and ω_{ab} of Eq. 37 can be written as

$$\delta = \frac{a}{2}\ln(\tilde{u}_2\tilde{u}_3) = \frac{a}{2}\ln\frac{|w|^2}{|1+w|^4}, \ \ \omega_{ab} = \frac{a}{2}\ln\frac{\tilde{u}_3}{\tilde{u}_2} = \frac{a}{2}\ln|w|^2 \tag{H.2}$$

using the leading order term for the cusp anomalous dimension $\gamma_K \simeq 4a$. The equation Eq. H.1 is valid only for the LLA term and the real part of the NLLA term of the remainder function at three loops. Solving it for the LLA term we get

$$R_6^{(3)\ LLA} = \frac{i}{4}\ln^2(1-u_1)\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\,(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}\,E_{\nu,n}^2 \tag{H.3}$$

in full agreement with Eq. G.9.

Next we solve Eq. H.1 for the real part of the NLLA remainder function

$$\begin{aligned}\Re(R_6^{(3)\ NLLA}) &= -\frac{i\pi\delta R_6^{(2)\ LLA}}{a} - \frac{\pi}{2}\ln(1-u_1)\sum_{n=-\infty}^{\infty}\int d\nu \frac{(-1)^n E_{\nu,n}^2}{\nu^2+\frac{n^2}{4}}\,(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}} \\ &= -\frac{i\pi\delta R_6^{(2)\ LLA}}{a} - \frac{2\pi R_6^{(3)\ LLA}}{i\ln(1-u_1)}. \end{aligned}\tag{H.4}$$

From Eq. H.4 we see that the next-to-leading logarithmic contribution is related to the leading logarithmic terms at two and three loops. The function $R^{(2)\ LLA}$ was found using BFKL approach in ref. [7] and given by the first term on RHS of Eq. 22

$$R_6^{(2)\ LLA} = \frac{i\pi}{2}\ln(1-u_1)\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2. \tag{H.5}$$

The LLA remainder function at three loops $R^{(3)\ LLA}$ was calculated in the appendix G and is given by Eq. G.18. Summing up all terms in Eq. H.4 we readily obtain

$$\begin{aligned}\Re(R_6^{(3)\ NLLA}) &= \frac{\pi^2}{4}\ln(1-u_1)\Big(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2 \\ &-\frac{1}{2}\ln^2|w|^2\ln|1+w|^2 - \ln|w|^2\left(\text{Li}_2(-w)+\text{Li}_2(-w^*)\right) + 2\text{Li}_3(-w) + 2\text{Li}_3(-w^*)\Big). \end{aligned}\tag{H.6}$$

Note that $\Re(R_6^{(3)\ NLLA})$ resembles very much the form of $R^{(3)\ LLA}$ in Eq. G.18 as one could expect from Eq. H.4. The complex variables w is expressed in terms of the reduced cross ratios of Eq. 11 as

$$w = \frac{1-z}{z} = \frac{B^+}{\tilde{u}_2}, \quad w^* = \frac{1-z^*}{z^*} = \frac{B^-}{\tilde{u}_2} \tag{H.7}$$

for $B^{\pm}$ defined in Eq. D.4 by

$$B^{\pm} = \frac{1-\tilde{u}_2-\tilde{u}_3 \pm \sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2 - 4\tilde{u}_2\tilde{u}_3}}{2}. \tag{H.8}$$

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