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Worldsheet four-point functions in $\mathrm{AdS}_3/\mathrm{CFT}_2$

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Abstract

We calculate some extremal and non-extremal four-point functions on the sphere of certain chiral primary operators for strings on $AdS_3 \times S^3 \times T^4$. The computation is done for small values of the spacetime cross-ratio where global $SL(2)$ and $SU(2)$ descendants may be neglected in the intermediate channel. Ignoring also current algebra descendants, we find that in the non-extremal case the integrated worldsheet correlators factorize into spacetime three-point functions, which is non-trivial due to the integration over the moduli space. We then restrict to the extremal case and compare our results with the four-point correlators recently computed in the dual boundary theory. We also discuss a particular non-extremal correlator involving two chiral and two anti-chiral operators.

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1 Introduction

The AdS$_3$/CFT$_2$ correspondence [1] is one of the most studied and tested dualities. In the last couple of years much progress has been made in identifying correlation functions. In [2, 3, 4] extremal and non-extremal three-point functions of chiral primary operators in the worldsheet theory for string theory on $AdS_3 \times S^3 \times T^4$ were successfully matched to the corresponding correlators in the dual boundary theory [5, 6, 7], see also [8, 9] for correlators involving spectrally-flowed states. Later it was also shown in [10] that the cubic couplings in supergravity [11, 12, 13] can be brought into agreement with the symmetric orbifold correlators when mixings with multi-particle operators are taken into account. The equivalence between string theory/supergravity and field theory correlators was at first quite remarkable since the computations were preformed at different points in the moduli space. A careful analysis of the moduli dependence of the chiral ring eventually showed though that all three-point functions obey a non-renormalization theorem [14]. As a corollary, it followed that also all *extremal* n-point functions ($n > 3$) are protected along the moduli space.

In this paper we compute some extremal and non-extremal four-point functions of chiral primary operators in the worldsheet theory. The general structure of four-point functions for string theory on $AdS_3 \times \mathcal{M}$, where $\mathcal{M}$ is some compact manifold, was studied in [15], see also [16] for related work. Our goal is to apply these techniques to a more concrete case, by specializing $\mathcal{M} = S^3 \times T^4$, and compare the results with expectations from the boundary conformal field theory. In this way we may test and explore the non-renormalization theorem of [14].

Apart from the question of non-renormalization, the computation of worldsheet four-point functions in AdS_3 is also interesting in its own right. As compared to similar computations of worldsheet two- and three-point functions [2, 3, 4], four-point functions are much more involved for the following reasons. First, unlike in two- and three-point functions, one cannot fix all worldsheet coordinates by modular invariance anymore. In general, four-point functions require a true integration over the worldsheet cross-ratio z, *i.e.* an integration over the moduli space. Second, four-point functions on AdS_3 involve also an integration over the locus of the continuous representation of the $SL(2)$ affine algebra, *i.e.* along the line $h = 1/2 + is$ ($s \in \mathbb{R}$). Third, the integration over the $SL(2)$ representation label h in turn requires a careful analysis of the pole structure of the four-point functions [15, 16]. Fourth, in principle there are all sorts of states in the intermediate channel, such as primary states, descendants, single- and multi-particle states *etc.* To simplify the computation, one needs to find selection criteria for these states. All these questions will be addressed in some concrete examples.

We begin by computing some non-extremal worldsheet four-point functions. Here we are interested in the question of their factorization into spacetime three-point functions. Other than in the boundary conformal field theory, this question is non-trivial due to the integration over the moduli space. Next, for comparison with the corresponding boundary correlators, we then restrict the four-point functions to the extremal case and find agreement with the (single-particle contribution to the) boundary correlators, which have previously been found in [17]. We also compute a particular non-extremal worldsheet correlator and compare it with its dual boundary correlator [17], which consists of two chiral and two anti-chiral operators. We summarize our results in the conclusions.

2 Some four-point functions in the symmetric orbifold theory

Before turning to the worldsheet theory, we briefly review some of the results in the boundary conformal field theory. We will later compare our integrated worldsheet correlators with the four-point correlators presented in this section.

The boundary theory is a symmetric product orbifold theory of the type $\mathrm{Sym}(T^4)^N = (T^4)^N/S_N$ with $\mathcal{N} = 4$ supersymmetry, where the coordinates of the product of N copies of T^4 are identified by the action of the permutation S_N. The operators of the theory are associated to conjugacy classes of S_N, which contain single cycles, $(1 \dots n_1)$, double cycles, $(1 \dots n_1)(n_1+1 \dots n_1+n_2)$, etc.

The chiral primary operators are given by the *single-cycle* twist operators

$$O_n^{(0,0)}(x,\bar{x})\,, \qquad O_n^{(a,\bar{a})}(x,\bar{x})\,, \qquad O_n^{(2,2)}(x,\bar{x})\,, \tag{2.1}$$

with $a, \bar{a} = \pm$, and $n = 1, ..., N$ denotes the length of the cycle (For a precise definition see *e.g.* [6, 7, 17]). The corresponding conformal dimensions are

$$h^{(0)} = \frac{n-1}{2}\,, \qquad h^{(a)} = \frac{n}{2}\,, \qquad h^{(2)} = \frac{n+1}{2} \tag{2.2}$$

and similarly for the antiholomorphic sector. For the comparison with string theory computations, we will later use the label $h = (n+1)/2$ instead of n such that

$$h^{(0)} = h - 1\,, \qquad h^{(a)} = h - 1/2\,, \qquad h^{(2)} = h\,. \tag{2.3}$$

The (anti-)chiral operators $O_n^{(A,\bar{A})}$ ($O_n^{(A,\bar{A})\dagger}$) ($A = 0, a, 2$) form a (anti-)chiral ring under an $\mathcal{N} = 2$ subalgebra and satisfy $h^{(A)} = q$ ($h^{(A)} = -q$), where q is the corresponding $U(1)$ charge. The fusion rules of the (c, c) ring are

$$\begin{aligned}
(0,0) \times (0,0) &= (0,0) + (2,2)\,,\\
(0,0) \times (2,2) &= (2,2)\,,\\
(0,0) \times (a,a) &= (a,a)\,,\\
(a,a) \times (a,a) &= (2,2)\,.
\end{aligned} \tag{2.4}$$

Similarly, there are multi-cycle operators associated to conjugacy classes containing multi-cycle group elements of S_N. Most prominent are *double-cycle* operators, which appear in the intermediate channel of extremal four-point functions [17].

Correlators of single-cycle twist operators are computed on covering surfaces of different genera. Quite generally, it can be shown from the Riemann-Hurwitz formula that if the cycle lengths of a p-point correlator satisfy

$$n_p = \sum_{i=1}^{p-1} n_i - p + 2\,, \tag{2.5}$$

the sphere is the only covering surface which contributes to the correlator [17].

Pakman, Rastelli and Razamat [17] computed several correlators satisfying (2.5). Among others, they found the extremal four-point functions

$$\langle O_{n_4}^{(0,0)\dagger}(\infty) O_{n_3}^{(0,0)}(1) O_{n_2}^{(0,0)}(x,\bar{x}) O_{n_1}^{(0,0)}(0)\rangle = F_4(n_i) \frac{n_4^{5/2}}{(n_1 n_2 n_3)^{1/2}}\,, \tag{2.6}$$

$$\langle O_{n_4}^{(2,2)\dagger}(\infty) O_{n_3}^{(2,2)}(1) O_{n_2}^{(0,0)}(x,\bar{x}) O_{n_1}^{(0,0)}(0)\rangle = F_4(n_i) \frac{n_3^{3/2} n_4^{1/2}}{(n_1 n_2)^{1/2}}\,, \tag{2.7}$$

$$\langle O_{n_4}^{(b,\bar{b})\dagger}(\infty) O_{n_3}^{(a,\bar{a})}(1) O_{n_2}^{(0,0)}(x,\bar{x}) O_{n_1}^{(0,0)}(0)\rangle = \delta^{ab}\delta^{\bar{a}\bar{b}} F_4(n_i) \frac{n_4^{3/2} n_3^{1/2}}{(n_1 n_2)^{1/2}}\,, \tag{2.8}$$

$$\langle O_{n_4}^{(2,2)\dagger}(\infty) O_{n_3}^{(a,\bar{a})}(1) O_{n_2}^{(b,\bar{b})}(x,\bar{x}) O_{n_1}^{(0,0)}(0)\rangle = \epsilon^{ab}\epsilon^{\bar{a}\bar{b}} F_4(n_i) \frac{(n_4 n_3 n_2)^{1/2}}{n_1^{1/2}}\,, \tag{2.9}$$

where the function $F_4(n_i)$ is given by

$$F_4(n_i) = \left[\frac{(N-n_1)!(N-n_2)!(N-n_3)!}{(N-n_4)!(N!)^2}\right]^{1/2} . \tag{2.10}$$

Note that $F_4 \approx 1/N$ at large N and that the correlators are independent of the cross-ratio x. The extremality conditions

$$h_4^{(0)} = h_1^{(0)} + h_2^{(0)} + h_3^{(0)} , \quad etc. \tag{2.11}$$

imposed on these correlators imply the condition (2.5), $n_4 = n_1 + n_2 + n_3 - 2$.[1]

There are also some *non-extremal* correlators satisfying (2.5). An example is given by the correlator [17]

$$\langle O_{n+2}^{(0,0)\dagger}(\infty) O_2^{(0,0)}(1) O_2^{(0,0)\dagger}(x,\bar{x}) O_n^{(2,2)}(0)\rangle = G(x,\bar{x}) , \tag{2.12}$$

where for small x

$$G(x,\bar{x}) \approx \frac{(n+2)^{3/2}}{2(n+1)n^{1/2}} \sqrt{\frac{(N-n)(N-n-1)}{N^2(N-1)^2}} |x|^{-2} . \tag{2.13}$$

The correlator scales as $1/N$ at large N. The conformal dimensions are $h_1^{(2)} = h_4^{(0)} = \frac{n+1}{2}$ and $h_2^{(0)} = h_3^{(0)} = \frac{1}{2}$ and similarly for the anti-holomorphic sector. The correlator is clearly non-extremal since

$$h_4^{(0)} = h_1^{(2)} + h_2^{(0)} + h_3^{(0)} - 1 , \tag{2.14}$$

but nevertheless satisfies (2.5). The appearance of two anti-chiral operators in (2.12) ensures charge conservation since

$$\sum_{i=1}^{4} q_i = h_1^{(2)} - h_2^{(0)} + h_3^{(0)} - h_4^{(0)} = 0 . \tag{2.15}$$

Extremal correlators satisfy a non-renormalization theorem [14] and are thus protected along the entire moduli space. They should therefore be reproducible by a string or supergravity computation. The non-extremal correlator (2.12) is not a priori protected by a non-renormalization theorem.

[1] There is also a fifth extremal correlator, $\langle O_{n_4}^{(2,2)\dagger}(\infty) O_{n_3}^{(0,0)}(1) O_{n_2}^{(0,0)}(x,\bar{x}) O_{n_1}^{(0,0)}(0)\rangle$ with $n_4 = n_1 + n_2 + n_3 - 4$ [17], which does not satisfy (2.5).

3 Scaling of chiral primaries in the worldsheet theory

In this section we set our notation by defining the chiral primaries of the worldsheet theory. We also review the computation of their two-point functions. The scaling of the operators will be relevant when worldsheet correlators are compared with the corresponding boundary correlators. The notation follows closely that in [3].

3.1 Chiral primary operators

In the following we summarize the chiral primaries of the worldsheet theory [18, 19, 3]. It is understood that all fields depend on the worldsheet coordinate z, even though this dependence will be suppressed in the notation.

The worldsheet theory is the product of an $\mathcal{N} = 1$ WZW model on H_3^+, an $\mathcal{N} = 1$ WZW model on $S^3 \simeq SU(2)$ and an $\mathcal{N} = 1$ $U(1)^4$ free superconformal field theory. This WZW model has the affine world-sheet symmetry $\widehat{sl}(2)_k \times \widehat{su}(2)_{k'} \times u(1)^4$. Criticality of the fermionic string on $AdS_3 \times S^3$ requires the identification of the levels k and k' [20], $k = k'$. The label k denotes the supersymmetric level of the affine Lie algebras and is identified with the bosonic levels k_b and k'_b as $k = k_b - 2 = k'_b + 2$. The bosonic currents are J^a for $SL(2)$ and K^a for $SU(2)$. The free fermions of $SL(2)$ are denoted by ψ^a, those of $SU(2)$ by χ^a ($a = (+, 0, -)$ in either case). It is convenient to split the bosonic currents as

$$J^a = j^a + \hat{j}^a \,, \qquad \hat{j}^a = -\frac{i}{k}\varepsilon^a{}_{bc}\psi^a\psi^b \,, \tag{3.1}$$

and similarly K^a. Finally the $u(1)^4$ symmetry is described in terms of free bosons as $i\partial Y^i$, and the corresponding free fermions are λ_i ($i = 1, 2, 3, 4$).

The chiral operators are constructed from the dimension zero operators

$$\mathcal{O}_j(x, y) = \Phi_h(x)\Phi'_j(y) \qquad \text{with} \qquad h = j + 1 \,, \quad j = 0, \tfrac{1}{2}, ..., \tfrac{k-2}{2} \,, \tag{3.2}$$

where $\Phi_h(x)$ and $\Phi'_j(y)$ are the primaries of the bosonic $SL(2)$ and $SU(2)$ WZW models. The labels x and y correspond to the $SL(2)$ and $SU(2)$ labels m and m', respectively. Our conventions for these models can be found in appendix A. Since $h = j + 1$, the operators $\mathcal{O}_j(x, y)$ have vanishing conformal dimensions, $\Delta(h) + \Delta(j) = 0$.

3.1.1 NS sector

In the NS sector there are two families of chiral primaries. In the -1 picture they are

$$\mathcal{O}_j^{(0)}(x, y) = e^{-\phi}\psi(x)\mathcal{O}_j(x, y) \,, \tag{3.3}$$
$$\mathcal{O}_j^{(2)}(x, y) = e^{-\phi}\chi(y)\mathcal{O}_j(x, y) \,, \tag{3.4}$$

where the fields $\psi(x)$ and $\chi(y)$ are given by

$$\begin{aligned} \psi(x) &= -\psi^+ + 2x\psi^3 - x^2\psi^- \,, \\ \chi(y) &= -\chi^+ + 2y\chi^3 + y^2\chi^- \,. \end{aligned} \tag{3.5}$$

The bosonized superghost field $e^{-\phi}$ ensures that the operators have ghost number -1.

Sometimes we will also need the corresponding ghost number 0 operators, which are obtained from (3.3) by acting with the picture changing operator Γ_{+1}. These operators will be needed to get the correct ghost number in the correlators. The ghost number 0 operators are [3, 2]

$$\tilde{\mathcal{O}}_j^{(0)}(x,y) = \left((1-h)\hat{\jmath}(x) + j(x) + \tfrac{2}{k}\psi(x)\chi_a P_y^a\right)\mathcal{O}_j(x,y)\,, \tag{3.6}$$

$$\tilde{\mathcal{O}}_j^{(2)}(x,y) = \left(h\hat{k}(y) + k(y) + \tfrac{2}{k}\chi(y)\psi_A D_x^A\right)\mathcal{O}_j(x,y)\,, \tag{3.7}$$

where the operators D_x^A and P_y^a are

$$\begin{aligned} &D_x^- = \partial_x\,, \quad D_x^3 = x\partial_x + h\,, \quad D_x^+ = x^2\partial_x + 2hx\,, \\ &P_y^- = -\partial_y\,, \quad P_y^3 = y\partial_y - j\,, \quad P_y^+ = y^2\partial_y - 2jy\,. \end{aligned} \tag{3.8}$$

Here we used again the compact notation

$$\begin{aligned} \hat{\jmath}(x) &= -\hat{\jmath}^+ + 2x\hat{\jmath}^3 - x^2\hat{\jmath}^-\,, \\ \hat{k}(y) &= -\hat{k}^+ + 2y\hat{k}^3 + y^2\hat{k}^-\,, \quad \textit{etc.} \end{aligned} \tag{3.9}$$

3.1.2 R sector

In the R sector there are also two families of chiral primaries, $\mathcal{O}_j^{(a)}(x,y)$ with $a = 1,2$. For their construction we need the spin operators

$$S_{[\varepsilon_1,\varepsilon_2,\varepsilon_3]} = e^{\frac{i}{2}(\varepsilon_1\hat{H}_1+\varepsilon_2\hat{H}_2+\varepsilon_3\hat{H}_3)}\,, \tag{3.10}$$

where $\varepsilon_I = \pm 1$ and $\hat{H}_i$ $(i = 1,2,3)$ are bosonized fermions related to ψ^a and χ^a $(a = \pm, 0)$, as in [3] (Similarly, $\hat{H}_{4,5}$ will be related to λ^i $(i = 1,2,3,4)$ below). Then, in the $-1/2$ and $-3/2$ picture the chiral primaries are given by

$$\mathcal{O}_j^{(a)}(x,y) = e^{-\frac{\phi}{2}}s_-^a(x,y)\mathcal{O}_j(x,y)\,, \tag{3.11}$$

and

$$\tilde{\mathcal{O}}_j^{(a)}(x,y) = -\sqrt{k}(2h-1)^{-1}e^{-\frac{3\phi}{2}}s_+^a(x,y)\mathcal{O}_j(x,y)\,, \tag{3.12}$$

respectively, where

$$s_\pm^1(x,y) = S_\pm(x,y)e^{+\frac{i}{2}(\hat{H}_4-\hat{H}_5)}\,, \qquad s_\pm^2(x,y) = S_\pm(x,y)e^{-\frac{i}{2}(\hat{H}_4-\hat{H}_5)} \tag{3.13}$$

and

$$S_\pm(x,y) = \mp xyiS_{[--\pm]} \mp xS_{[-+\mp]} + yiS_{[+-\mp]} + S_{[++\pm]}\,. \tag{3.14}$$

3.1.3 Full chiral operators

The full chiral primary operators are given by the product of a holomorphic with an anti-holomorphic operator,

$$\mathcal{O}_j^{(A,\bar{A})}(x,\bar{x},y,\bar{y}) \equiv \mathcal{O}_j^{(A)}(x,y)\bar{\mathcal{O}}_j^{(\bar{A})}(\bar{x},\bar{y})\,, \tag{3.15}$$

where $A = 0, a, 2$ and $\bar{A} = \bar{0}, \bar{a}, \bar{2}$. When integrated over the worldsheet, these operators are dual to the chiral primary operators $O_n^{(A,\bar{A})}$ in the boundary theory ($n = 2j+1$).

3.2 Two-point functions and normalized operators

The two-point functions of the above chiral primary operators are worked out in [2, 3, 8]. In order to set the notation, we briefly review the computation here.

In the NS sector the two-point function is ($h = j+1$)

$$\begin{aligned}\langle \mathcal{O}_j^{(0,0)}(x_1,\bar{x}_1,y_2,\bar{y}_2)\mathcal{O}_j^{(0,0)}(x_2,\bar{x}_2,y_2,\bar{y}_2)\rangle &= g_s^{-2}\left|\langle e^{-\phi_1}e^{-\phi_2}\rangle\langle\psi(x_1)\psi(x_2)\rangle\langle\mathcal{O}_j\mathcal{O}_j\rangle\right|^2 \\ &= \frac{k^2}{g_s^2}\,\frac{B(h)\delta(0)\,|y_{12}|^{4j}}{|z_{12}|^4|x_{12}|^{4(h-1)}}\,,\end{aligned} \tag{3.16}$$

where we defined $\phi_i = \phi(z_i)$ and used

$$\begin{aligned}\langle\psi(x_1)\psi(x_2)\rangle = k\frac{(x_{12})^2}{z_{12}}\,, \qquad \langle e^{-\phi(z_1)}e^{-\phi(z_2)}\rangle = \frac{1}{z_{12}}\,,\\ |\langle\mathcal{O}_j(x_1,y_1)\mathcal{O}_j(x_2,y_2)\rangle|^2 = \frac{B(h)\delta(0)\,|y_{12}|^{4j}}{|x_{12}|^{4h}}\,.\end{aligned} \tag{3.17}$$

The two-point function scales as $|x_{12}|^{-4h^{(0)}}$ with $h^{(0)}$ as in (2.3), which agrees with the scaling of the dual boundary operator.

In the Ramond sector we get the two-point function ($h = j+1$)

$$\begin{aligned}&\langle\tilde{\mathcal{O}}_j^{(a,\bar{a})}(x_1,\bar{x}_1,y_2,\bar{y}_2)\mathcal{O}_j^{(b,\bar{b})}(x_2,\bar{x}_2,y_2,\bar{y}_2)\rangle \\ &\quad = g_s^{-2}\left|\frac{\sqrt{k}\langle\mathcal{O}_j\mathcal{O}_j\rangle}{2h-1}\langle e^{-\frac{3}{2}\phi_1}e^{-\frac{1}{2}\phi_2}\rangle\langle s_+^a(x_1,y_1)s_-^b(x_2,y_2)\rangle\right|^2 \\ &\quad = \frac{1}{g_s^2}\,\frac{k}{(2h-1)^2}\,\frac{B(h)\delta(0)\,|y_{12}|^{4(j+1/2)}}{|z_{12}|^4|x_{12}|^{4(h-1/2)}}\,\delta^{ab}\delta^{\bar{a}\bar{b}}\,,\end{aligned} \tag{3.18}$$

where we used

$$\langle s_+^a(x_1,y_1)s_-^b(x_2,y_2)\rangle = \delta^{ab}\frac{ix_{12}y_{12}}{(z_{12})^{5/4}}\,, \qquad \langle e^{-3\phi(z_1)/2}e^{-\phi(z_2)/2}\rangle = \frac{1}{(z_{12})^{3/4}}\,. \tag{3.19}$$

Note that one primary is in the $-1/2$ picture while the other one is in the $-3/2$ picture such that the total ghost number is -2, as required on the sphere. The two-point function scales as $|x_{12}|^{-4h^{(a)}}$ with $h^{(a)}$ as in (2.3).

In order to obtain the corresponding boundary correlators we need to integrate the above two-point functions over the worldsheet coordinates z_1 and z_2. Equivalently, we may fix $z_1 = 1$ and $z_2 = 0$ and divide the correlator by the volume of the conformal group V_{conf} which keeps the two points fixed. As shown in appendix A in [15], this removes the divergence coming from $\delta(0)$ and introduces the factor[2]

$$-\frac{2h-1}{2\pi\nu k^2\gamma(\frac{k+1}{k})c_\nu} = \frac{2h-1}{2\pi^2 k} \quad \text{for} \quad \nu = \frac{\pi}{c_\nu}\frac{\Gamma(1-\frac{1}{k})}{\Gamma(1+\frac{1}{k})}\,. \tag{3.20}$$

We observe that other than the operators in the boundary conformal field theory, the chiral primaries are not normalized to unity. We therefore rescale the operators as

$$\begin{aligned}
\mathbb{O}_j^{(0,0)}(x,\bar{x}) &= \frac{\sqrt{2\pi^2}}{\sqrt{k\,B(h)(2h-1)}} g_s\,\mathcal{O}_j^{(0,\bar{0})}(x,\bar{x})\,,\\
\mathbb{O}_j^{(a,\bar{a})}(x,\bar{x}) &= \sqrt{\frac{2\pi^2(2h-1)}{B(h)}} g_s \mathcal{O}_j^{(a,\bar{a})}(x,\bar{x})\,.
\end{aligned} \tag{3.21}$$

The operator $\mathcal{O}_j^{(2,2)}(x,\bar{x})$ is rescaled as $\mathcal{O}_j^{(0,0)}(x,\bar{x})$.

4 Four-point function in the NS sector

In this section we compute a four-point correlator which involves only chiral primary operators of the NS sector. In particular we are interested in computing the correlator[3]

$$G_4^{NS}(x,\bar{x}) = g_s^{-2}\int d^2z \left\langle \tilde{\mathcal{O}}^{(0,0)}_{j_4,m_4}(\infty)\mathcal{O}^{(0,0)}_{j_3,m_3}(1)\tilde{\mathcal{O}}^{(0,0)}_{j_2,m_2}(x,\bar{x};z,\bar{z})\mathcal{O}^{(0,0)}_{j_1,m_1}(0)\right\rangle\,, \tag{4.1}$$

where we choose the m-labels as $(d \geq 0)$

$$\begin{aligned}
m_1 &= \bar{m}_1 = j_1\,,\\
m_2 &= \bar{m}_2 = j_2 - d\,,\\
m_3 &= \bar{m}_3 = j_3\,,\\
m_4 &= \bar{m}_4 = -j_4 = -(j_1+j_2+j_3-d)\,.
\end{aligned} \tag{4.2}$$

The worldsheet coordinates are fixed as $z_{1,2,3,4} = 0, z, 1, \infty$, where z is the cross-ratio $z = z_{12}z_{34}/(z_{13}z_{24})$ on the worldsheet. Similarly, the continuous $SL(2)$ representation labels are chosen as $x_{1,2,3,4} = 0, x, 1, \infty$. Later, these labels will be identified with the complex coordinates in the boundary conformal field theory [21] and x becomes the spacetime cross-ratio. The correlator $G_4^{NS}(x,\bar{x})$ involves two ghost number zero and two ghost

[2] $\nu = \nu(k)$ is a free parameter in the H_3^+ model. As in [3], we leave c_ν (and thus ν) undetermined for the moment. c_ν will later be fixed, when we compare the bulk and boundary correlators. Note that $c_\nu = 1$ in [15], cf. our definition of ν with (2.12) in [15].

[3] The operators $\mathcal{O}^{(A,A)}_{j,m}$ $(A = 0, a, 2)$ are related to those in (3.15) by the Fourier transformation (C.11).

number -1 operators, $\tilde{\mathcal{O}}_j^{(0,0)}$ and $\mathcal{O}_j^{(0,0)}$, respectively. Note that the total ghost number of a correlator on a genus-g surface must be $-\chi = -(2-2g)$, which is -2 on the sphere.

The correlator (4.1) is called *extremal*, if the spacetime scalings of the operators in $G_4^{NS}(x,\bar{x})$ satisfy (2.11), $h_4^{(0)} = h_1^{(0)} + h_2^{(0)} + h_3^{(0)}$ (These are the scalings in x, defined as the power of the term $|x_{12}|^{-4h^{(0)}}$ in (3.16)). Using (2.3) and $h_i = j_i + 1$ ($i = 1, ..., 4$), this translates into the condition

$$j_4 = j_1 + j_2 + j_3 \tag{4.3}$$

or $d = 0$. We will first consider the non-extremal case $d > 0$ and come back to the extremal case $d = 0$ in section 4.4.

Substituting the explicit expressions for these operators, as given by (3.3) and (3.6), we get[4]

$$\begin{aligned} G_4^{NS}(x,\bar{x}) = g_s^{-2} \int d^2z \Bigg[&(1-h_2)(1-h_4) \left\langle \psi(0)\hat{\jmath}(x)\psi(1)\hat{\jmath}(\infty)\right\rangle \left\langle \prod_{i=1}^4 \Phi_{h_i} \right\rangle \\ &+(1-h_2) \left\langle \psi(0)\hat{\jmath}(x)\psi(1)\right\rangle \left\langle j(\infty) \prod_{i=1}^4 \Phi_{h_i} \right\rangle \\ &+(1-h_4) \left\langle \psi(0)\psi(1)\hat{\jmath}(\infty)\right\rangle \left\langle j(x) \prod_{i=1}^4 \Phi_{h_i} \right\rangle \\ &+ \left\langle \psi(0)\psi(1)\right\rangle \left\langle j(x)j(\infty) \prod_{i=1}^4 \Phi_{h_i} \right\rangle \Bigg] \left\langle \prod_{i=1}^4 \Phi'_{j_i,m_i} \right\rangle \left\langle e^{-\phi(0)} e^{-\phi(1)} \right\rangle \times c.c. \end{aligned} \tag{4.4}$$

The actual computation of $G_4^{NS}(x,\bar{x})$ will be done along the lines of [15].

4.1 Some correlators inside $G_4^{NS}(x,\bar{x})$

Following [15], we write the $SL(2)$ four-point function

$$\begin{aligned} \left\langle \prod_{i=1}^4 \Phi_{h_i} \right\rangle = &|x_{24}|^{-4h_2} |x_{14}|^{2(h_2+h_3-h_1-h_4)} |x_{34}|^{2(h_1+h_2-h_3-h_4)} |x_{13}|^{2(h_4-h_1-h_2-h_3)} \\ &\times |z_{24}|^{-4\Delta_2} |z_{14}|^{2\nu_1} |z_{34}|^{2\nu_2} |z_{13}|^{2\nu_3} \mathcal{F}_{SL(2)}(x,\bar{x};z,\bar{z})\,, \end{aligned} \tag{4.5}$$

in terms of the factorization ansatz [22]

$$\mathcal{F}_{SL(2)}(x,\bar{x};z,\bar{z}) = \int_{\frac{1}{2}+iR} dh\, \mathcal{C}(h) |\mathcal{F}_h(x;z)|^2\,, \tag{4.6}$$

where the normalization $\mathcal{C}(h)$ is given by $\mathcal{C}(h) = \frac{C(h_1,h_2,h)C(h,h_3,h_4)}{B(h)}$. The functions $B(h)$ and $C(h_1,h_2,h_3)$ are the scaling of the $SL(2)$ two-point function and the $SL(2)$ structure

[4] There is also a non-vanishing term involving the correlator $\langle (\chi_a P^a_{y_4})(\chi_b P^b_{y_2}) \prod_{i=1}^4 \Phi_{h_i} \Phi'_{j_i} \rangle$. This term turns out to be subleading in x and may be neglected in the small x region, see the discussion below.

constants, respectively. They are given by (A.3) and (A.5) in appendix A. As in [15], we change variables from z to u by defining $u = z/x$ and consider the case $|x| < 1$. We may then perform an expansion of $\mathcal{F}_h(x;u)$ in powers of x as

$$\mathcal{F}_h(x;u) = x^{\Delta(h)-\Delta(h_1)-\Delta(h_2)+h-h_1-h_2} u^{\Delta(h)-\Delta(h_1)-\Delta(h_2)} \sum_{m=0}^{\infty} g_m(u) x^m \,. \tag{4.7}$$

Substituting this expansion into the KZ equation for $SL(2)$ [22], one finds that the first term obeys the hypergeometric equation in u, *i.e.*

$$g_0(u) = F(a,b,c|u)\,, \tag{4.8}$$

with $a = h_1 + h_2 - h$, $b = h_3 + h_4 - h$, $c = k - 2h$. We will sometimes use the shorthand notation $F_h(u) \equiv F(a,b,c|u)$. In what follows we will focus on the leading term in the x expansion,

$$\mathcal{F}_h(x;u) = x^{\Delta(h)-\Delta(h_1)-\Delta(h_2)+h-h_1-h_2} u^{\Delta(h)-\Delta(h_1)-\Delta(h_2)} F_h(u) + \ldots\,, \tag{4.9}$$

where the ellipsis represents higher order terms in x. Such terms correspond to descendants under the global $SL(2)$ algebra [15], which do not play a role in the small x region. It is convenient to write $F_h(u)$ as a power series in u,

$$F_h(u) = \sum_{n=0}^{\infty} \mathcal{H}(a,b,c,n) u^n\,, \tag{4.10}$$

with coefficients

$$\mathcal{H}(a,b,c,n) = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)}\,. \tag{4.11}$$

A similar factorization ansatz can be found for the $SU(2)$ four-point function. As shown in appendix C, at small z the $SU(2)$ four-point function with m-values as in (4.2) can be expanded as[5]

$$\left\langle \prod_{i=1}^{4} \Phi'_{j_i,m_i} \right\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{C}'(j)\, |\mathcal{G}_j(z)|^2\,, \tag{4.12}$$

with

$$\begin{aligned} |\mathcal{G}_j(z)|^2 &= \sum_{n'=0}^{\infty} G_{j,n'}\, |z|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}\,, \\ G_{j,n'} &= \delta^2_{j_1+j_2+j_3-j_4,d}\, c_{2j_2}^{j_2+m_2} \mathcal{D}(j_1,j_2,J)\mathcal{D}(J,j_3,j_4) \\ &\quad \times \frac{\Gamma(0)^2}{\Gamma(j+n'-j_1-j_2+1+d)^2\Gamma(j_4-j-n'-j_3)^2}\,. \end{aligned} \tag{4.13}$$

[5]We assume that the level k is large enough. For small k, the upper bound of summation is changed [27].

c_{2j}^{j+m} are the inverse of the binomial coefficients,

$$c_{2j}^{j+m} = \frac{\Gamma(j+m+1)\Gamma(j-m+1)}{\Gamma(2j+1)} \,. \tag{4.14}$$

The δ-function reflects the charge conservation $m_1+m_2+m_3+m_4 = 0$. The normalization $\mathcal{C}'(j)$ is given by $\mathcal{C}'(j) = C'_{j,j_1,j_2} C'_{j,j_3,j_4}$ (no summation over j). The $SU(2)$ structure constants C'_{j_1,j_2,j_3} and the functions $\mathcal{D}(j_1, j_2, J)$ are given by (A.20) and (C.8) in the appendix, respectively.

We will also need some other four-point correlators for $G_4^{NS}(x, \bar{x})$. For the following, it is useful to define the n-point correlators

$$d_k^{(n)} = \left\langle j(x_k) \prod_{i=1}^{n} \Phi_{h_i}(x_i) \right\rangle , \qquad d_{km}^{(n)} = \left\langle j(x_k) j(x_m) \prod_{i=1}^{n} \Phi_{h_i}(x_i) \right\rangle , \tag{4.15}$$

with $k, m = 1, ..., n$, in which one or two bosonic currents $j(x)$ act on the product of n $SL(2)$ functions $\Phi_h(x)$. As shown in appendix B, such correlators can entirely be expressed in terms of derivatives of the $SL(2)$ n-point function. In particular, the functions $d_2^{(4)}$, $d_4^{(4)}$ and $d_{24}^{(4)}$ appearing in (4.4) can be computed by means of (B.6) and (B.7). Using only the first term in the small x expansion (4.9) of the $SL(2)$ four-point function (4.5) (and $x = x_{12}x_{34}/(x_{13}x_{24})$), we find

$$d_k^{(4)} = \langle j(x_k) \prod_{i=1}^{4} \Phi_{h_i}(x_i) \rangle = \int dh \, \mathcal{C}(h) \left| \sum_{n=0}^{\infty} \hat{d}_{k,n}^{(4)} \, \mathbb{S}_n \right|^2 , \tag{4.16}$$

with

$$\begin{aligned} \mathbb{S}_n = & \, (x_{24})^{-2h_2} (x_{14})^{h_2+h_3-h_1-h_4} (x_{34})^{h_1+h_2-h_3-h_4} (x_{13})^{h_4-h_1-h_2-h_3} \\ & \times (z_{24})^{-2\Delta_2} (z_{14})^{\nu_1} (z_{34})^{\nu_2} (z_{13})^{\nu_3} \\ & \times x^{h-h_1-h_2-n} z^{\Delta(h)-\Delta(h_1)-\Delta(h_2)+n} \mathcal{H}(a,b,c,n) \end{aligned} \tag{4.17}$$

and $\mathcal{H}(a,b,c,n)$ as in (4.11). For $k = 4, 2, 1$, the coefficients are given by[6]

$$\begin{aligned} \hat{d}_{4,n}^{(4)} = & - \frac{z_{13}}{z_{34}z_{14}} \frac{x_{34}x_{14}}{x_{13}} (h + h_3 - h_4 - n) \\ & + \frac{z_{12}}{z_{24}z_{14}} \frac{x_{24}x_{14}}{x_{12}} (h - h_1 - h_2 - n) \, , \end{aligned} \tag{4.18}$$

$$\begin{aligned} \hat{d}_{2,n}^{(4)} = & \frac{z_{34}}{z_{24}z_{23}} \frac{x_{24}x_{23}}{x_{34}} (h - h_3 - h_4 - n) \\ & + \frac{z_{14}}{z_{24}z_{12}} \frac{x_{24}x_{12}}{x_{14}} (h_1 - h_2 - h_3 + h_4 - n) \\ & - \frac{z_{13}}{z_{23}z_{12}} \frac{x_{23}x_{12}}{x_{13}} (h + h_3 - h_4 - n) \, , \end{aligned} \tag{4.19}$$

$$\begin{aligned} \hat{d}_{1,n}^{(4)} = & \frac{z_{34}}{z_{14}z_{13}} \frac{x_{14}x_{13}}{x_{34}} (h - h_3 - h_4 - n) \\ & - \frac{z_{24}}{z_{14}z_{12}} \frac{x_{14}x_{12}}{x_{24}} (h - h_1 + h_2 - n) \, . \end{aligned} \tag{4.20}$$

[6] Here we also list the coefficient $\hat{d}_{1,n}^{(4)}$ for later use.

Finally, the correlator $d_{24}^{(4)}$ is given by

$$d_{24}^{(4)} = \int dh\, \mathcal{C}(h) \left| \sum_{n=0}^{\infty} \hat{d}_{24,n}^{(4)}\, \mathbb{S}_n \right|^2 , \tag{4.21}$$

$$\hat{d}_{24,n}^{(4)} = -\frac{(h + h_3 - h_4 - n)(-h - h_1 + h_2 + n)x}{z} + \dots , \tag{4.22}$$

which, for brevity, is expanded around $z = 0$ (the ellipses denote further terms subleading in z). Also the x- and z-dependence is already fixed as above. Note that the above expressions for the $d^{(4)}$ correlators are only valid for small x.

We will also need the fermionic correlators

$$\begin{aligned}
\langle \psi(x_1)\psi(x_2)\rangle &= k\frac{(x_{12})^2}{z_{12}} , \\
\langle \psi(x_1)\psi(x_2)\hat{\jmath}(x_3)\rangle &= 2k\frac{x_{12}x_{23}x_{31}}{z_{31}z_{23}} , \\
\langle \psi(x_1)\hat{\jmath}(x_2)\psi(x_3)\hat{\jmath}(x_4)\rangle &= 2k \left[\frac{z_{13}x_{23}x_{14}}{z_{34}z_{23}z_{14}x_{13}^2}(x_{13}x_{24} + x_{12}x_{34}) \right. \\
&\qquad \left. - \frac{z_{13}x_{34}x_{12}}{z_{34}z_{14}z_{12}x_{13}^2}(x_{14}x_{32} + x_{13}x_{42}) \right] ,
\end{aligned} \tag{4.23}$$

which have been computed using (B.8) in appendix B.

Substituting now the correlators (4.18), (4.19), (4.21) and (4.23) as well as the expansions (4.5) (with (4.9)) and (4.12) for the $SL(2)$ and $SU(2)$ four-point functions into (4.4), for small x we find

$$\begin{aligned}
G_4^{NS}(x, \bar{x}) = g_s^{-2}k^2 \int d^2u \sum_{j,n'} \mathcal{C}'(j) \int dh\, \mathcal{C}(h)\, |x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1+n')} |u|^{2(\Delta(h)+\Delta(j)+n')} \\
\times \left| \sum_{n=0}^{\infty} \left[(1-h_2)(1-h_4)2\frac{(2x-1)z + x(x-2)}{z(z-1)} + (1-h_2)2\frac{(x-1)x}{(z-1)z}\hat{d}_{4,n}^{(4)} \right. \right. \\
\left. \left. + \ (1-h_4)2\, \hat{d}_{2,n}^{(4)} + \hat{d}_{24,n}^{(4)} \right] \mathcal{H}(a, b, c, n)u^n \right|^2 G_{j,n'} ,
\end{aligned} \tag{4.24}$$

where it is understood that z needs to be replaced by $z = ux$. Note also $\Delta(h_i) + \Delta(j_i) = 0$ for the external fields.

4.2 Moduli integration and integral over h

We now perform the integrals over the worldsheet cross-ratio u and the $SL(2)$ representation label h. We wish to do the u-integral before the integral over h but need to be careful about the occurrence of divergences. Following [15, 16], we therefore regularize the u-integral by introducing a cut-off parameter ε and divide the range of u into two regions:

$$\begin{aligned}
\text{region I:} \quad & |u| < \varepsilon \\
\text{region II:} \quad & |u| > \varepsilon \, .
\end{aligned}$$

In region I there are only operators in the intermediate channel whose $SL(2)$ part is associated with short strings with winding number $w = 0$ [15]. In region II there can be long strings with $w = 1$ and two-particle states [15]. The representation theory of $SL(2)$ does not allow any other spectrally-flowed states in the intermediate channel.

An important observation is that "single-cycle" operators in the spacetime CFT arise locally on the worldsheet, *i.e.* in the small u region, while "multi-cycle" operators correspond to non-local contributions coming from the large u region [15, 16].[7] Since at large N multi-particle contributions are suppressed in non-extremal correlators [17], we may restrict to the one-particle contributions to the four-point correlator. We therefore consider only region I and ignore possible two-particle contributions coming from region II.

Formally, the one-particle contributions are taken into account by first integrating over the small u region, $|u| < \varepsilon$, and then taking the limit $\varepsilon \to 0$. This is the limit where the operators approach each other in their worldsheet coordinates. For $|u| < \varepsilon$, we may then expand $G_4^{NS}(x, \bar{x})$ in powers of u as

$$\begin{aligned} &G_4^{NS}(x, \bar{x}) \\ &= g_s^{-2} k^2 \int d^2u \int dh \sum_{j,n'} \mathcal{C}(h)\mathcal{C}'(j) G_{j,n'} \, |x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1+n')} |u|^{2(\Delta(h)+\Delta(j)+n')} \\ &\quad \times \left| \sum_{n=0}^{\infty} \left[-\frac{(h+h_1+h_2-2-n)(h+h_3+h_4-2-n)}{u} + O(u^0) \right] \mathcal{H}(a,b,c,n) u^n \right|^2 , \end{aligned} \tag{4.25}$$

where we display only the most singular term in the square brackets. Subleading terms are summarized in $O(u^0)$.

The relevant u-integral inside $G_4^{NS}(x, \bar{x})$ is

$$\sum_{n,\bar{n}=0}^{\infty} \int_{|u|<\varepsilon} d^2u \, |u|^{2(\lambda-1)} u^n \bar{u}^{\bar{n}} = \sum_{n,\bar{n}=0}^{\infty} \frac{\pi}{\lambda+n} \varepsilon^{2(\lambda+n)} \delta_{n,\bar{n}} \tag{4.26}$$

with $\lambda = \Delta(h) + \Delta(j) + n'$.

We now turn to the integration over h. The h-integral is defined along the line $h = \frac{k-1}{2} + is$ ($s \in \mathbb{R}$), away from the locus of the continuous representation of $SL(2)$, $h = \frac{1}{2} + is$. The reason for the deformation is that only there the integrand is equivalent to a monodromy invariant solution, cf. (4.34) in [15]. It is possible to shift the integration contour back to $h = \frac{1}{2} + is$. However, in general, the integral picks up pole residues when the poles cross the integration contour. At small u there are altogether four types of poles of the h-integral which may contribute to the integral. These are [15]:

$$\begin{aligned} \text{type I:} \quad & \lambda + n = 0 \,, \\ \text{type II:} \quad & h = h_1 + h_2 + n \,, \\ \text{type III:} \quad & h = k - h_1 - h_2 + n \,, \\ \text{type IV:} \quad & h = |h_1 - h_2| - n \,, \qquad n \in \{0, 1, 2, \ldots\} \,. \end{aligned}$$

[7] The "single-cycle" operators (or "single-trace" operators in higher-dimensional CFTs) correspond to one-particle states in the worldsheet theory. Similarly, "multi-cycle" operators correspond to multi-particle states.

The poles of type II-IV are poles in the structure constants $C(h, h_1, h_2)$. As discussed extensively in [16], none of these poles contributes to the integral. Even though naively one might interpret the contributions from the poles of type II as "double-cycle" operators in the spacetime CFT, such contributions go to zero in the $\varepsilon \to 0$ limit [16]. Type III poles do not appear if $h_1 + h_2 < \frac{k+1}{2}$ [15]. The contribution coming from poles of type IV was found to be canceled by the same contribution from crossing the integration contour [16].

We are left with poles of type I. These poles correspond to short string representations (with zero winding number) in the $SL(2)$ WZW model [15]. The condition

$$\lambda + n = \Delta(h) + \Delta(j) + n + n' = 0 \qquad (n, n' \geq 0) \tag{4.27}$$

is solved by ($h > 0$)

$$h = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4k(n + n') + 4j(j+1)}\,. \tag{4.28}$$

A particular solution is $n + n' = 0$ and $h = j + 1$. Since n and n' are both positive, $n = n' = 0$ and we recover the on-shell condition for chiral primaries in the intermediate channel. As such they map to single-cycle chiral primary operators in the spacetime CFT.

For $n+n' \neq 0$, we generically do not get a rational conformal weight h. Substituting the condition (4.27) into (4.25), we find that the correlator depends on x as $x^{h-n-h_1-h_2}$. This should be compared with the x dependence of the corresponding boundary four-point function, which is $x^{H-H_1-H_2}$ (see e.g. (4.2) in [15]), where H denotes the corresponding spacetime conformal weights. Since $H = h - n$ with h as in (4.28), one therefore identifies this contribution as coming from $SL(2)$ short string descendants (of the type $(J^-_{-1})^n(J^-_{-1})^{\bar{n}}|h, m = \bar{m} = h\rangle$) in the intermediate channel [15]. These states have a continuous spectrum for $h > 0$, if one chooses the universal cover of $SL(2)$ as the target space. Since $H = h - n$ is generically irrational, it is not clear to us which boundary states can be identified with the current algebra descendants. In the following we therefore restrict to the case $n = n' = 0$ ($h = j+1$), for which there are only chiral primary operators in the intermediate channel, and ignore possible contributions from current algebra descendants.

This leads to some simplification of the product $\mathcal{C}(h)\mathcal{C}'(j)$. Recall the following relation between the structure constants of $SL(2)$ and $SU(2)$ found in [2, 3],

$$C(h_1, h_2, h_3)C'(j_1, j_2, j_3) = \frac{c_\nu^{1/2}}{2\pi}\prod_{i=1}^{3}\sqrt{B(h_i)}\,, \tag{4.29}$$

which holds for $h_i = j_i + 1$ ($i = 1, 2, 3$) and $k_b = k'_b - 4$. From this we find the identity

$$\mathcal{C}(h)\mathcal{C}'(j) = \frac{c_\nu}{(2\pi)^2}\prod_{i=1}^{4}\sqrt{B(h_i)} \tag{4.30}$$

since $h = j+1$. In other words, the poles of the $SL(2)$ structure constants cancel against the zeros of the $SU(2)$ structure constants.

With these identities, we may now return to $G_4^{NS}(x,\bar{x})$. Applying the residue theorem[8] and taking the limit $\varepsilon \to 0$, we get

$$G_4^{NS}(x,\bar{x}) = g_s^{-2}k^2 \sum_j \prod_{i=1}^{4} \sqrt{B(j_i+1)} G_{j,0} \frac{c_\nu}{(2\pi)^2} \frac{2\pi^2}{\partial_h(\Delta(h))|_{h=j+1}} |x|^{2(j-j_1-j_2)} \times \left((j+j_1+j_2+1)(j+j_3+j_4+1)\right)^2 . \tag{4.31}$$

The factor $\partial_h(\Delta(h))|_{h=j+1}/(2\pi^2) = (2j+1)/(2\pi^2 k)$ in the denominator is precisely the factor (3.20). It is related to the fact that we need to integrate over the conformal group on the worldsheet when comparing two-point functions on the worldsheet to two-point functions in spacetime. Recall that spacetime four-point functions can be considered as a sum over the product of two three-point functions divided by the two-point function.

We must still normalize the four-point function with respect to the scaling of the two-point functions. For the four-point function of the corresponding normalized operators (3.21), we then find

$$\mathbb{G}_4^{NS}(x,\bar{x}) = s(k) \sum_j \frac{(j+j_1+j_2+1)^2(j+j_3+j_4+1)^2}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}} \frac{G_{j,0}}{2j+1} |x|^{2(j-j_1-j_2)} , \tag{4.32}$$

where we introduced the factor

$$s(k) = g_s^{-2}k^2 \left(g_s \sqrt{\frac{2\pi^2}{k}} \right)^4 \frac{c_\nu}{(2\pi)^2} 2\pi^2 \, k \, . \tag{4.33}$$

If we choose $c_\nu = 1/(2\pi^4 k^3)$, then $s(k) = g_s^2/k^2$, which scales as $1/N$ at large N [2, 3].

4.3 Factorization into three-point functions

It is possible to rewrite $\mathbb{G}_4^{NS}(x,\bar{x})$ as the product of two three-point functions. For that, we label the state in the intermediate channel by j and set its m quantum number as $m = j$.[9] Then, the charge conservation $m = m_1 + m_2$ selects the term with

$$j = j_1 + j_2 - d \tag{4.34}$$

[8]Let us denote the r.h.s. of (4.26) by $f(h)$ such that for $n=0$ we have $f(h) \equiv \frac{\pi\varepsilon^{2\lambda(h)}}{\lambda(h)}$. Define also h_0 by $\lambda(h_0) = 0$. Then $\oint dh f(h) = 2\pi i \,\mathrm{Res}(f;h_0)$ with $\mathrm{Res}(f;h_0) = \frac{\pi\varepsilon^{2\lambda(h_0)}}{\lambda'(h_0)}$ such that

$$\int dh \frac{\pi\varepsilon^{2\lambda(h)}}{\lambda(h)} \propto \frac{2\pi^2}{\partial_h\Delta(h_0)}$$

with $h_0 = j+1$.

[9]More generally, one could have set $m = j - \tilde{d}$ with $\tilde{d} \geq 0$. Each term in $\mathbb{G}_4^{NS}(x,\bar{x})$ would then scale as $|x|^{2(j-j_1-j_2)} = |x|^{2(-d+\tilde{d})}$. Since at small x the leading term in the sum over j is that for $\tilde{d}=0$, we may neglect global $SU(2)$ descendants. Note that we have already ignored global $SL(2)$ descendants in (4.9).

in the sum over j. For this particular value of j, or $d = j_1 + j_2 - j$, $G_{j,0}$ reduces to

$$\begin{aligned} G_{j,0} &= c_{2j_2}^{j_2+m_2} \delta^2_{j_1+j_2+j_3-j_4,d} \\ &= \frac{\Gamma(j_2+j-j_1+1)\Gamma(j_1+j_2-j+1)}{\Gamma(2j_2+1)} \delta^2_{j_1+j_2+j_3-j_4,d} \end{aligned} \tag{4.35}$$

and $\mathbb{G}_4^{NS}(x,\bar{x})$ becomes

$$\begin{aligned} \mathbb{G}_4^{NS}(x,\bar{x}) = \frac{g_s^2}{k^2} \frac{\Gamma(j_2+j-j_1+1)\Gamma(j_1+j_2-j+1)}{\Gamma(2j_2+1)} \frac{(j+j_1+j_2+1)^2}{\sqrt{(2j_1+1)(2j_2+1)(2j+1)}} \\ \times \frac{(j+j_3+j_4+1)^2}{\sqrt{(2j+1)(2j_3+1)(2j_4+1)}} |x|^{-2d} + \dots . \end{aligned} \tag{4.36}$$

However, this is nothing but the expected factorization in terms of three-point functions,

$$\mathbb{G}_4^{NS}(x,\bar{x}) = \frac{\left\langle \mathbb{O}_j^{(0,0)}(\infty) \tilde{\mathbb{O}}_{j_2}^{(0,0)}(x,\bar{x}) \mathbb{O}_{j_1}^{(0,0)}(0) \right\rangle \left\langle \tilde{\mathbb{O}}_{j_4}^{(0,0)}(\infty) \mathbb{O}_{j_3}^{(0,0)}(1) \mathcal{O}_j^{(0,0)}(0) \right\rangle}{\left\langle \mathbb{O}_j^{(0,0)}(\infty) \mathbb{O}_j^{(0,0)}(0) \right\rangle} + \dots \tag{4.37}$$

with [2]

$$\left\langle \mathbb{O}_{j_1}^{(0,0)}(\infty) \mathbb{O}_{j_2}^{(0,0)}(1) \tilde{\mathbb{O}}_{j_3}^{(0,0)}(0) \right\rangle = \frac{g_s}{k} \frac{(j_1+j_2+j_3+1)^2}{\prod_i (2j_i+1)^{\frac{1}{2}}} \frac{\Gamma(j_{13}+1)\Gamma(j_{12}+1)}{\Gamma(2j_1+1)} . \tag{4.38}$$

The ellipsis indicates terms subleading in x. The x-dependence $|x|^{-2d}$ is now contained in the left three-point function.

4.4 The extremal case and comparison with the boundary theory

So far, general non-extremal four-point functions have not been considered in the dual symmetric orbifold theory. For comparison with the results in the boundary conformal field theory, we therefore specialize now to the extremal case $j_4 = j_1 + j_2 + j_3$, for which the dual boundary correlator is known [17].

As we can see from (4.35) for $d = 0$ (*i.e.* $j = j_1 + j_2$), $G_{j,0} = \delta^2_{j_1+j_2+j_3,j_4}$, and hence

$$\mathbb{G}_4^{NS}(x,\bar{x}) = \frac{g_s^2}{k^2} \frac{(2j+1)(2j_4+1)^2}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}} . \tag{4.39}$$

The result is independent of the cross-ratio x, as expected for extremal correlators. Changing variables from j to n by setting $n_i = 2j_i + 1$ $(i = 1,2,3,4)$, we get

$$\mathbb{G}_4^{NS}(x,\bar{x}) = \frac{1}{N} \frac{n_4^{5/2}}{(n_1 n_2 n_3)^{1/2}} \frac{\tilde{n}}{n_4} \tag{4.40}$$

with $\tilde{n} = n_1 + n_2 - 1$. In the large N limit, this is in agreement with the single-cycle contribution to the boundary correlator (2.6), which is given by (2.6) times the factor $\tilde{n}/n_4$ [17]. This is the contribution coming from single-cycle operators in the intermediate channel.

As argued in [17], in the extremal case contributions coming from double-cycle operators in the intermediate channel are not suppressed at large N. It was found that the combined effect of single- and double-cycle operators is given by the single-cycle contribution times the factor $n_4/\tilde{n}$, symbolically:

$$\begin{aligned}\text{full extremal correlator} &= \text{single-} + \text{double-cycle contribution} \\ &= \frac{n_4}{\tilde{n}} \cdot (\text{single-cycle contribution})\,.\end{aligned}$$

Clearly, it would be desirable to reproduce this factor in the worldsheet theory. Double-cycle terms in the spacetime OPE arise nonlocally on the worldsheet and are presently not very-well understood.

4.5 Crossing symmetry

We conclude this section with some comments on the crossing symmetry of $\mathbb{G}_4^{NS}(x, \bar{x})$.

An essential part of the correlator is the $SL(2)$ four-point function, which may be denoted by

$$\mathcal{G}_{34}^{12}(x, z) \equiv \left\langle \prod_{i=1}^{4} \Phi_{h_i}(x_i, z_i) \right\rangle . \tag{4.41}$$

On the right hand side we set again $z_{1,2,3,4} = 0, z, 1, \infty$ and $x_{1,2,3,4} = 0, x, 1, \infty$. As shown by Teschner in [23], the $SL(2)$ four-point function is invariant under crossing symmetry, *i.e.* it satisfies the following identity:

$$\mathcal{G}_{34}^{12}(x, z) = \mathcal{G}_{14}^{32}(1 - x, 1 - z)\,. \tag{4.42}$$

This corresponds to the simultaneous exchange

$$x_1 \leftrightarrow x_3\,, \qquad z_1 \leftrightarrow z_3\,, \qquad h_1 \leftrightarrow h_3\,, \tag{4.43}$$

which map the cross-ratios as $x \leftrightarrow 1-x$ and $z \leftrightarrow 1-z$. The operators $\mathcal{O}_{j,m}^{(0,0)}$ are basically $SL(2)$ primaries dressed by some spinors ψ and $e^{-\phi}$ (and currents in case of $\tilde{\mathcal{O}}_{j,m}^{(0,0)}$). We need to show that this dressing does not violate crossing symmetry.

Let us investigate the crossing symmetry of (4.1) (or, equivalently, (4.4)), which follows if each term in (4.4) is invariant under (4.43). For instance, consider the four-point function

$$\begin{aligned} d_2^{(4)} = \left\langle j(x_2) \prod_{i=1}^{4} \Phi_{h_i}(x_i, z_i) \right\rangle \; &= \; \bigg[\frac{x_{21}}{z_{21}}(x_{21}\partial_{x_1} - 2h_1) + \frac{x_{23}}{z_{23}}(x_{23}\partial_{x_3} - 2h_3) \\ &\quad + \frac{x_{24}}{z_{24}}(x_{24}\partial_{x_4} - 2h_4) \bigg] \left\langle \prod_{i=1}^{4} \Phi_{h_i}(x_i, z_i) \right\rangle . \end{aligned} \tag{4.44}$$

Here we used the explicit expression (B.6) in Appendix B. Clearly, due to (4.42), this expression is invariant under the exchange (4.43), and similarly $d_4^{(4)}$ and $d_{24}^{(4)}$ appearing in (4.4). The action of the currents $j(x)$ on the $SL(2)$ four-point function therefore remains crossing invariant. Similarly, we can verify the crossing symmetry of correlators in (4.4) which involve only $SL(2)$ fermions by checking the explicit expressions (4.23).

In summary, assuming the crossing invariance of the $SL(2)$ four-point function $\mathcal{G}_{34}^{12}(x,z)$ (proven in [23]), we find that (4.1) is also invariant under this symmetry. Note however that in the computation of the one-particle contribution we used an approximation for the $SL(2)$ four-point function (Eq. (4.9)), valid at small x and u, which is not crossing invariant. The one-particle contribution computed here is therefore not crossing invariant by itself. The above analysis shows however that it can in principle be made invariant by including the two-particle contributions in the intermediate channel.

5 Mixed NS and R four-point function

The computation of the previous section can easily be adapted to other four-point functions. As a further example, we next compute a four-point function which involves two chiral primaries in the NS sector and two in the R sector. Such a four-point function is given by

$$
\begin{aligned}
G_4^R(x,\bar{x}) &= g_s^{-2}\int d^2z\,\left\langle \mathcal{O}^{(b,\bar{b})}_{j_4,m_4}(\infty)\mathcal{O}^{(a,\bar{a})}_{j_3,m_3}(1)\mathcal{O}^{(0,0)}_{j_2,m_2}(x,\bar{x};z,\bar{z})\tilde{\mathcal{O}}^{(0,0)}_{j_1,m_1}(0)\right\rangle \\
&= g_s^{-2}\int d^2z\,\langle e^{-\frac{\phi(\infty)}{2}}e^{-\frac{\phi(1)}{2}}e^{-\phi(z)}\rangle\left[(1-h_1)\left\langle s_-^a(1)s_-^b(\infty)\psi(x)\hat{\jmath}(0)\right\rangle\left\langle\prod_{i=1}^4\Phi_{h_i}\right\rangle\right. \\
&\quad\left.+\left\langle s_-^a(1)s_-^b(\infty)\psi(x)\right\rangle\left\langle\prod_{i=1}^4 j(0)\Phi_{h_i}\right\rangle\right]\left\langle\prod_{i=1}^4\Phi'_{j_i,m_i}\right\rangle\times c.c. \qquad (5.1)
\end{aligned}
$$

with m-values as in (4.2). The first two operators are Ramond chiral primaries with ghost number $-1/2$. The third and fourth operators are NS chiral primaries with ghost number -1 and 0. The total ghost number is therefore again -2, as required on the sphere.

For the computation, we will need the fermionic correlators

$$
\left\langle s_-^b(x_4)\psi(x_2)s_-^a(x_3)\right\rangle = k^{1/2}\frac{x_{23}x_{24}}{z_{23}^{1/2}z_{24}^{1/2}z_{34}^{3/4}}\,\delta^{ab}\,, \qquad (5.2)
$$

$$
\left\langle s_-^a(x_4)s_-^b(x_3)\psi(x_2)\hat{\jmath}(x_1)\right\rangle = -\left[\frac{x_{14}x_{12}}{x_{24}}\frac{z_{42}}{z_{14}z_{12}}+\frac{x_{13}x_{12}}{x_{23}}\frac{z_{23}}{z_{13}z_{12}}\right]\left\langle s_-^b(x_4)\psi(x_2)s_-^a(x_3)\right\rangle\,. \qquad (5.3)
$$

For simplicity, we neglected the dependence on the y-labels here. The contribution from the ghosts is $\langle e^{-\phi(z_4)/2}e^{-\phi(z_3)/2}e^{-\phi(z_2)}\rangle = z_{23}^{-1/2}z_{24}^{-1/2}z_{34}^{-1/4}$.

Proceeding as before, we use again the factorization ansatz (4.6) and get

$$G_4^R(x,\bar{x}) = g_s^{-2}k \int d^2u \int dh \sum_j \mathcal{C}(h)\mathcal{C}'(j)\, |x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1)}|u|^{2(\Delta(h)+\Delta(j))}$$
$$\times\, \delta^{ab}\delta^{\bar{a}\bar{b}} \left|(1-h_1)\left(\frac{1}{u}+\frac{1}{u}\frac{xu-1}{x-1}\right)+\hat{d}_{1,0}^{(4)}\right|^2 G_{j,0}\,, \tag{5.4}$$

where the first term in the four-point function $d_1^{(4)}$, denoted by $\hat{d}_{1,n}^{(4)}$ with $n=0$, is given by (4.20). As in the previous section, we keep only the terms with $n=n'=0$ (and $F_h(u)\approx 1$). Notice that in the small-u, small-x region, we have

$$\frac{1}{u}+\frac{1}{u}\frac{xu-1}{x-1} \approx \frac{2}{u}\,, \qquad \hat{d}_{1,0}^{(4)}(h,h_i,x,z) \approx -\frac{(h-h_1+h_2)x}{z}\,, \tag{5.5}$$

with $z=ux$, as before. The structure of $G_4^R(x,\bar{x})$ is similar to that of $G_4^{NS}(x,\bar{x})$ as given, for instance, by (4.24). The only change is the terms in the second line.

We now perform the u- and h-integrals. In the region $|u|<\varepsilon$ we expand (5.4) as

$$G_4^R(x,\bar{x}) = g_s^{-2}k \int d^2u \int dh \sum_j \mathcal{C}(h)\mathcal{C}'(j)G_{j,0}\, |x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1)}|u|^{2(\Delta(h)+\Delta(j))}$$
$$\times\, \delta^{ab}\delta^{\bar{a}\bar{b}} \left|-\frac{(h+h_1+h_2-2)}{u}+O(u^0)\right|^2 \tag{5.6}$$

and do the u-integral as in (4.26). Performing also the h-integral and taking the $\varepsilon\to 0$ limit we get

$$G_4^R(x,\bar{x}) = g_s^{-2}k\,\delta^{ab}\delta^{\bar{a}\bar{b}} \sum_j \prod_{i=1}^4 \sqrt{B(j_i+1)}G_{j,0}\frac{c_\nu}{(2\pi)^2}|x|^{2(j-j_1-j_2)}\frac{2\pi^2(j+j_1+j_2+1)^2}{\partial_h(\Delta(h))|_{h=j+1}}\,. \tag{5.7}$$

As argued above, there are only chiral primary states in the intermediate channel (with $h=j+1$), which allows us to use (4.30).

With the above value for c_ν, $c_\nu = 1/(2\pi^4k^3)$, the corresponding rescaled correlator is

$$\mathbb{G}_4^R(x,\bar{x}) = \frac{g_s^2}{k^2}\,\delta^{ab}\delta^{\bar{a}\bar{b}} \sum_j \frac{G_{j,0}}{2j+1}(j+j_1+j_2+1)^2\left[\frac{(2j_3+1)(2j_4+1)}{(2j_1+1)(2j_2+1)}\right]^{1/2}|x|^{2(j-j_1-j_2)}\,. \tag{5.8}$$

Note here the difference in the scaling of R and NS operators. As argued in the previous section, at small x the leading term in the sum over j is that for $j=j_1+j_2-d$. Recalling now (4.35), $\mathbb{G}_4^R(x,\bar{x})$ can be rewritten as

$$\mathbb{G}_4^R(x,\bar{x}) = \frac{g_s^2}{k^2}\,\delta^{ab}\delta^{\bar{a}\bar{b}}$$
$$\times\,\frac{\Gamma(j_2+j-j_1+1)\Gamma(j_1+j_2-j+1)}{\Gamma(2j_2+1)}\,\frac{(j+j_1+j_2+1)^2}{[(2j+1)(2j_1+1)(2j_2+1)]^{1/2}}$$
$$\times\left[\frac{(2j_3+1)(2j_4+1)}{(2j+1)}\right]^{1/2}|x|^{-2d}+\ldots\,, \tag{5.9}$$

with $j = j_1 + j_2 - d = j_4 - j_3$. Ellipses represent again subleading terms in x. After comparing with the three-point functions, we get the factorization

$$\mathbb{G}_4^R(x,\bar{x}) = \frac{\left\langle \mathbb{O}_j^{(0,0)}(\infty)\mathbb{O}_{j_2}^{(0,0)}(x,\bar{x})\tilde{\mathbb{O}}_{j_1}^{(0,0)}(0)\right\rangle \left\langle \mathbb{O}_{j_4}^{(b,\bar{b})}(\infty)\mathbb{O}_{j_3}^{(a,\bar{a})}(1)\mathbb{O}_j^{(0,0)}(0)\right\rangle}{\left\langle \mathbb{O}_j^{(0,0)}(\infty)\mathbb{O}_j^{(0,0)}(0)\right\rangle} + \dots\,, \quad (5.10)$$

with the left three-point function as in (4.38) and the right one given by [3]

$$\left\langle \mathbb{O}_{j_3}^{(b,\bar{b})}(\infty)\mathbb{O}_{j_2}^{(a,\bar{a})}(1)\mathbb{O}_{j_1}^{(0,0)}(0)\right\rangle = \frac{g_s}{k}\delta^{ab}\delta^{\bar{a}\bar{b}}\left[\frac{(2j_2+1)(2j_3+1)}{(2j_1+1)}\right]^{1/2}, \quad j_3 = j_1 + j_2\,. \quad (5.11)$$

For comparison with the corresponding boundary correlator, we restrict again to the extremal case, $d = 0$ or $j_4 = j_1 + j_2 + j_3$. Then, the only non-vanishing term in the sum over j is that for $j = j_1 + j_2$ (with $G_{j,0} = \delta^2_{j_1+j_2+j_3,j_4}$) and $G_4^R(x,\bar{x})$ as given by (5.8) becomes independent of x,

$$\mathbb{G}_4^R(x,\bar{x}) = \delta^{ab}\delta^{\bar{a}\bar{b}}\,\frac{g_s^2}{k^2}\left[\frac{(2j_3+1)(2j_4+1)}{(2j_1+1)(2j_2+1)}\right]^{1/2}(2j+1)\,. \quad (5.12)$$

The result precisely coincides with the one-particle contribution to (2.8) upon identifying $n_i = 2j_i + 1$. At large N it is given by[10]

$$\mathbb{G}_4^R(x,\bar{x}) = \delta^{ab}\delta^{\bar{a}\bar{b}}\frac{1}{N}\frac{(n_4 n_3)^{1/2}}{(n_1 n_2)^{1/2}}\,\tilde{n} \quad (5.13)$$

with $\tilde{n} = n_1 + n_2 - 1$. The result does not include possible contributions from the exchange of two-particle states.

We expect that the remaining extremal spacetime four-point correlators (2.7) and (2.9) can be reproduced by a similar worldsheet computation.

6 A particular non-extremal four-point function

In this section we consider the non-extremal four-point function

$$G_4(x,\bar{x}) = g_s^{-2}\int d^2z\left\langle \tilde{\mathcal{O}}_{j_4}^{(0,0)}(\infty)\mathcal{O}_{j_3}^{(0,0)}(1)\tilde{\mathcal{O}}_{j_2}^{(0,0)}(x,\bar{x};z,\bar{z})\mathcal{O}_{j_1}^{(2,2)}(0)\right\rangle\,, \quad (6.1)$$

for, at first, arbitrary j-values. Later we will fix the j-labels in order to compare the correlator with the corresponding boundary correlator (2.12).

We begin by substituting the explicit expressions for the chiral primary operators,

$$\begin{aligned} G_4(x,\bar{x}) = g_s^{-2}\int d^2z\,\Big\langle &\big((1-h_4)\hat{\jmath}(\infty) + j(\infty) + \tfrac{2}{k}\psi(\infty)\chi_a P^a_{y_4}\big)\,\mathcal{O}_{j_4} \\ &\times e^{-\phi(1)}\psi(1)\mathcal{O}_{j_3} \\ &\times \big((1-h_2)\hat{\jmath}(x) + j(x) + \tfrac{2}{k}\psi(x)\chi_a P^a_{y_2}\big)\,\mathcal{O}_{j_2} \\ &\times e^{-\phi(0)}\chi(0)\mathcal{O}_{j_1}\Big\rangle \times c.c.\,. \end{aligned} \quad (6.2)$$

[10]This is the contribution from single-cycle operators in the intermediate channel. It is given by (2.8) times the factor $\tilde{n}/n_4$ [17].

Keeping only the nonvanishing terms, we get

$$
\begin{aligned}
G_4(x,\bar{x}) = g_s^{-2}k^{-2}\int d^2z\, \Bigg[& (1-h_4)\langle \hat{j}(\infty)\psi(1)\psi(x)\rangle\langle 2\chi_a P^a_{y_2}\chi(0)\prod_{i=1}^{4}\mathcal{O}_{j_i}\rangle \\
& + \langle\psi(1)\psi(x)\rangle\langle 2\chi_a P^a_{y_2}\chi(0) j(\infty)\prod_{i=1}^{4}\mathcal{O}_{j_i}\rangle \\
& + (1-h_2)\langle \hat{j}(x)\psi(\infty)\psi(1)\rangle\langle 2\chi_a P^a_{y_4}\chi(0)\prod_{i=1}^{4}\mathcal{O}_{j_i}\rangle \\
& + \langle\psi(\infty)\psi(1)\rangle\langle 2\chi_a P^a_{y_4}\chi(0) j(x)\prod_{i=1}^{4}\mathcal{O}_{j_i}\rangle \Bigg] \times c.c\,. \qquad (6.3)
\end{aligned}
$$

This can be simplified by means of the identity

$$
2\chi_a P^a_y = \chi(y)\partial_y - j\partial_y\chi(y)\,, \qquad (6.4)
$$

which is obtained from the expansion of χ in the y-basis, Eq. (3.5), and $\chi_\pm = \chi_1 \pm i\chi_2$.

We will also need the correlators

$$
d_2^{(4)} = \langle j(x)\prod_{i=1}^{4}\mathcal{O}_{j_i}(x_i,y_i)\rangle\,, \qquad d_4^{(4)} = \langle j(\infty)\prod_{i=1}^{4}\mathcal{O}_{j_i}(x_i,y_i)\rangle\,, \qquad (6.5)
$$

given by (4.16) with (4.19) and (4.18), and the relations

$$
\begin{aligned}
\langle \hat{j}(\infty)\psi(1)\psi(x)\rangle &= 2\frac{z-1}{x-1}\langle\psi(1)\psi(x)\rangle\,, && (6.6)\\
\langle \hat{j}(x)\psi(1)\psi(\infty)\rangle &= 2\frac{x-1}{z-1}\langle\psi(1)\psi(\infty)\rangle\,, && (6.7)\\
\partial_y\langle\chi(y)\chi(0)\rangle &= \frac{2}{y}\langle\chi(y)\chi(0)\rangle\,, && (6.8)\\
\lim_{y_4\to\infty}\partial_{y_4}\langle\chi(y_4)\chi(0)\rangle &= \lim_{y_4\to\infty}\frac{2}{y_4}\langle\chi(y_4)\chi(0)\rangle\,. && (6.9)
\end{aligned}
$$

Substituting everything back into (6.3), we get

$$
\begin{aligned}
G_4(x,\bar{x}) = g_s^{-2}k^{-2}\int d^2z\, & \left[\left((1-h_4)2\frac{z-1}{x-1} + d_4^{(4)}\right)\frac{2j_2-(j_1+j_2-j)}{y}\right. \\
& \left.\times\langle\psi(1)\psi(x)\rangle\langle\chi(y)\chi(0)\rangle + \ldots\right]\langle\prod_{i=1}^{4}\mathcal{O}_{j_i}(x_i,y_i)\rangle\times c.c\,, \qquad (6.10)
\end{aligned}
$$

where the ellipsis indicates terms subleading in x (In particular, at small x we may neglect the third and fourth term in (6.3)). As before, we use the factorization ansatz (4.6) and change variables, $z = ux$. At small u and small x, we obtain

$$
\begin{aligned}
G_4(x,\bar{x}) = g_s^{-2}k^{2}\int d^2u\int dh\sum_j \mathcal{C}(h)\mathcal{C}'(j)\,|x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1)}|u|^{2(\Delta(h)+\Delta(j))} \\
\times\left|\frac{(2-h-h_3-h_4)(j-j_1+j_2)}{u\,x}\,y\right|^2. \qquad (6.11)
\end{aligned}
$$

At this point we need to specify the chirality of the operators in the dual boundary correlator. For this, we assign labels $a_{1,2,3,4} \in \{0,1\}$ to the boundary operators. The label a_i is zero (one), if the dual operator is chiral (antichiral). Then, $U(1)$ charge conservation,

$$\sum_{i=1}^4 q_i = (-1)^{a_1} h_1^{(2)} + \sum_{i=2}^4 (-1)^{a_i} h_i^{(0)} = 0\,, \tag{6.12}$$

yields the following relation among the j-values,

$$(-1)^{a_1}(j_1+1) + (-1)^{a_2} j_2 + (-1)^{a_2} j_3 + (-1)^{a_4} j_4 = 0\,. \tag{6.13}$$

In view of the boundary correlator (2.12) let us consider the case $a_1 = a_3 = 0$ (chirals) and $a_2 = a_4 = 1$ (antichirals) and fix the j-labels as $j_1 = \frac{n-1}{2}$, $j_2 = j_3 = \frac{1}{2}$ and $j_4 = \frac{n+1}{2}$. These values have been chosen to agree with the conformal dimensions of the dual chiral operators appearing in the correlator (2.12). For instance, the spacetime conformal dimensions of the operators dual to $\mathcal{O}_{j_1}^{(2,2)}$ and $\tilde{\mathcal{O}}_{j_4}^{(0,0)}$ are

$$h_1^{(2)} = h_1 = j_1 + 1 = \tfrac{n+1}{2} \qquad \text{and} \qquad h_4^{(0)} = h_4 - 1 = j_4 = \tfrac{n+1}{2}\,, \tag{6.14}$$

as required in (2.12). Using the relations (2.3) and $h_i = j_i + 1$, we find that the non-extremality condition (2.14) translates into $j_4 = j_1 + j_2 + j_3$. Since $j_2 = 1/2$, this relation is equivalent to the $U(1)$ charge conservation relation $j_4 = j_1 - j_2 + j_3 + 1$.

For the above values of j_i and a_i ($i = 1,2,3,4$), it was found in [17] that in the boundary theory $O_{n+1}^{(0,0)}$ is the only operator running in the intermediate channel. In the worldsheet theory this operator is dual to $\mathcal{O}_j^{(0,0)}$ with $j = j_1 + 1 - j_2 = j_1 + 1/2$. If we assume that the one-to-one correspondence between worldsheet and boundary operators also holds in the intermediate channel, then the sum over j reduces to a single term for which $j = j_1 + 1/2$.

Proceeding as before, we get

$$G_4(x,\bar{x}) = g_s^{-2} k^2 \prod_{i=1}^4 \sqrt{B(j_i+1)} \frac{c_\nu}{(2\pi)^2} (2j_4+1)^2 \frac{2\pi^2}{2j+1} \frac{|y|^2}{|x|^2}\,. \tag{6.15}$$

The corresponding rescaled correlator is[11]

$$\mathbb{G}_4(x,\bar{x}) = \frac{g_s^2}{k^2} \frac{(2j_4+1)^2}{\prod_{i=1}^4 \sqrt{2j_i+1}} \frac{1}{2(j_1+j_2)+1} |x|^{-2} \tag{6.16}$$

or

$$\mathbb{G}_4(x,\bar{x}) = \frac{1}{N} \frac{(n+2)^{3/2}}{2n^{1/2}} \frac{1}{n+1} |x|^{-2}\,. \tag{6.17}$$

At large N this agrees with the non-extremal correlator (2.12).

[11] The operator $\mathcal{O}_{j_2=1/2}^{(0,0)}$ is dual to the *anti*-chiral operator $O_2^{(0,0)\dagger}$. As compared to the corresponding chiral operator, it is rescaled by an additional factor $|y|^{-4j_2}$ [3], which cancels $|y|^2$ in the numerator.

7 Conclusions

We discussed extremal and non-extremal four-point correlators in the worldsheet theory for $AdS_3 \times S^3 \times T^4$. The computations were done at small cross-ratios x where we were allowed to ignore subleading contributions from global $SL(2)$ and $SU(2)$ descendants in the intermediate channel (In the boundary theory this corresponds to neglecting spacetime descendants.) For simplicity, we also ignored possible contributions from current algebra descendants. This is certainly allowed for extremal correlators, for which the $N = 2$ chiral ring structure ensures that there are only chiral primary operators in the intermediate channel. For non-extremal correlators, however, there are in principle further contributions coming from current algebra descendants, which we have not computed, but should be studied in more detail in the future.

We obtain the following results: i) We found that the integrated *non-extremal* correlators $G_4^{NS}(x, \bar{x})$ and $G_4^R(x, \bar{x})$, as defined in (4.1) and (5.1), factorize into the product of two spacetime three-point functions composed out of chiral primaries, see (4.37) and (5.10). Other than in the spacetime CFT, the factorization is non-trivial in the worldsheet theory because of the integration over the moduli space. If there were only chiral primary operators running in the intermediate channel, the factorization property would imply the non-renormalization of the correlator, at least at small x. However, as just stated, there can be additional terms coming from current algebra descendants, which would renormalize the four-point function. ii) We then evaluated $G_4^{NS}(x, \bar{x})$ and $G_4^R(x, \bar{x})$ for the *extremal* case and find agreement with the single-particle contribution to the corresponding extremal boundary correlators computed in [17]. This has been expected from the non-renormalization theorem of [14]. Note that in contrast to their non-extremal cousins, extremal four-point correlators also have two-particle states in the intermediate channel, whose contribution to the correlator is not suppressed at large N. In the boundary theory, the inclusion of the two-particle contribution amounts to multiplying the single-particle contribution by a simple factor, $n_4/\tilde{n}$ [17]. Clearly, it would be desirable to also derive this universal factor in the worldsheet theory by taking into account nonlocal contributions on the worldsheet. Such contributions are presently not very well understood. iii) We also computed a particular non-extremal four-point correlator, defined in (6.1), whose dual correlator in the boundary theory contains two chiral and two anti-chiral operators. This correlator is not covered by the non-renormalization theorem of [14] and therefore need not necessarily agree with its boundary counterpart. Nevertheless, we find exact agreement, cf. our result (6.16) or (6.17) with (2.12), again under the premise that we may ignore possible contributions from current algebra descendants in the intermediate channel.

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Appendix

A Correlators in $SL(2)_k$ and $SU(2)_{k'}$ WZW models

A.1 Two- and three-point functions in the $SL(2)_k$ WZW model

The chiral primaries of the $SL(2)$ WZW model are denoted by[12]

$$\Phi_h(z,\bar z;x,\bar x) = \Phi_h(z,x)\,\bar\Phi_h(\bar z,\bar x) \qquad \text{with} \qquad \Delta(h) = \bar\Delta(h) = -\frac{h(h-1)}{k-2}\ , \tag{A.1}$$

where k is the level of the affine Lie algebra. In the current context only half-integer h will be relevant.

The two- and three-point functions of $\Phi_h(z,\bar z;x,\bar x)$ were computed in [24, 25, 26]. The two-point function is given by

$$\begin{aligned}&\langle\Phi_{h_1}(z_1,\bar z_1;x_1,\bar x_1)\Phi_{h_2}(z_2,\bar z_2;x_2,\bar x_2)\rangle\\ &\qquad = \frac{1}{|z_{12}|^{4\Delta(h_1)}}\left[\frac{1}{(2\pi)^2}\,\delta(x_{12})\,\delta(\bar x_{12})\,\delta(h_1+h_2-1) + \frac{B(h_1)}{|x_{12}|^{4h_1}}\delta(h_1-h_2)\right]\ ,\end{aligned} \tag{A.2}$$

with coefficient

$$B(h) = \frac{k-2}{\pi}\frac{\nu^{1-2h}}{\gamma(\frac{2h-1}{k-2})} \qquad \text{and} \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\ , \qquad \nu = \frac{\pi}{c_\nu}\frac{\Gamma(1-\frac{1}{k-2})}{\Gamma(1+\frac{1}{k-2})}\ . \tag{A.3}$$

The parameter c_ν is free.
The three-point function is

$$\langle\Phi_{h_1}(z_1,\bar z_1;x_1,\bar x_1)\,\Phi_{h_2}(z_2,\bar z_2;x_2,\bar x_2)\,\Phi_{h_3}(z_3,\bar z_3;x_3,\bar x_3)\rangle = C(h_1,h_2,h_3)\prod_{i<j}\frac{1}{|x_{ij}|^{2h_{ij}}|z_{ij}|^{2\Delta_{ij}}}\ , \tag{A.4}$$

with $\Delta_{12} = \Delta(h_1)+\Delta(h_2)-\Delta(h_3)$, $h_{12} = h_1+h_2-h_3$, *etc.* and coefficients

$$C(h_1,h_2,h_3) = \frac{k-2}{2\pi^3}\,\frac{G(1-h_1-h_2-h_3)G(-h_{12})G(-h_{23})G(-h_{31})}{\nu^{h_1+h_2+h_3-2}G(-1)G(1-2h_1)G(1-2h_2)G(1-2h_3)}\ , \tag{A.5}$$

where

$$G(h) = (k-2)^{\frac{h(k-1-h)}{2(k-2)}}\,\Gamma_2(-h|1,k-2)\,\Gamma_2(k-1+h|1,k-2)\ , \tag{A.6}$$

and $\Gamma_2(x|1,\omega)$ is the Barnes double Gamma function. $G(h)$ has poles at $h = n+m(k-2)$ and $h = -n-1-(m+1)(k-2)$ with $n,m = 0,1,\ldots$. In C_{h_1,h_2,h_3} the poles $h_1+h_2+h_3 = n+k$, $n = 0,1,\ldots$ are excluded by the condition

$$h_1+h_2+h_3 \le k-1\,. \tag{A.7}$$

The function $G(h)$ satisfies the recursion relation

$$G(h+1) = \gamma(-\tfrac{h+1}{k-2})\,G(h)\ . \tag{A.8}$$

[12]In this appendix we only deal with the bosonic currents; k and k' therefore refer to the bosonic levels.

A.2 Four-point function in the $SL(2)_k$ WZW model

The four-point function of the $SL(2)$ chiral primary $\Phi_{h_i}(z,\bar{z};x,\bar{x})$ is given by

$$\langle\prod_{i=1}^{4}\Phi_{h_i}(z_i,\bar{z}_i;x_i,\bar{x}_i)\rangle = |x_{24}|^{-4h_2}|x_{14}|^{2(h_2+h_3-h_1-h_4)}|x_{34}|^{2(h_1+h_2-h_3-h_4)}|x_{13}|^{2(h_4-h_1-h_2-h_3)}$$
$$\times\,|z_{24}|^{-4\Delta_2}|z_{14}|^{2\nu_1}|z_{34}|^{2\nu_2}|z_{13}|^{2\nu_3}\,\mathcal{F}_{SL(2)}(z,\bar{z};x,\bar{x}) \qquad \text{(A.9)}$$

with

$$\nu_1=\Delta_2+\Delta_3-\Delta_1-\Delta_4\,,\quad \nu_2=\Delta_1+\Delta_2-\Delta_3-\Delta_4\,,\quad \nu_3=\Delta_4-\Delta_1-\Delta_2-\Delta_3\ ,$$

and

$$z=\frac{z_{12}z_{34}}{z_{14}z_{32}}\ ,\qquad\qquad x=\frac{x_{12}x_{34}}{x_{14}x_{32}}\ . \qquad \text{(A.10)}$$

The function $\mathcal{F}_{SL(2)}(z,\bar{z};x,\bar{x})$ is given by

$$\mathcal{F}_{SL(2)}(z,\bar{z},x,\bar{x})=\mathcal{M}(h_1,h_2,h_3,h_4)\ |z|^{-\frac{4h_1h_2}{k-2}}\ |1-z|^{-\frac{4h_1h_3}{k-2}}\Gamma(2h_1)b^{-1}\mu^{-2h_1}\times \qquad \text{(A.11)}$$
$$\times\int\prod_{i=1}\frac{dt_i d\bar{t}_i}{(2\pi i)}\ |t_i-z|^{-\frac{2\beta_1}{k-2}}\ |t_i|^{-\frac{2\beta_2}{k-2}}\ |t_i-1|^{-\frac{2\beta_3}{k-2}}\ |x-t_i|^2\ |D(t)|^{\frac{-4}{k-2}}\ ,$$

where

$$D(t)=\prod_{i<j}(t_i-t_j)\ , \qquad \text{(A.12)}$$

and

$$\begin{aligned}\beta_1&=h_1+h_2+h_3+h_4-1\,,\\ \beta_2&=h_1+h_2-h_3-h_4-1+k\,,\\ \beta_3&=h_1+h_3-h_2-h_4-1+k\ .\end{aligned} \qquad \text{(A.13)}$$

The normalization is

$$\mathcal{M}=\frac{\pi C_W^2(b)}{b^{5+4b^2}\Upsilon_0^2}\frac{(\nu(b))^s}{(\pi\mu\gamma(b^2)b^4)^{-2h_1}}\,\frac{G(1-h_1-h_2-h_3-h_4)}{G(1-2h_1)}$$
$$\times\prod_{i=2}^{4}\frac{G(-h_2-h_3-h_4+h_1+2h_i)}{G(1-2h_i)}\,, \qquad \text{(A.14)}$$

where $s=1-\sum_{i=1}^4 h_i$, $b^2=\frac{1}{k-2}$, $\gamma(x)=\Gamma(x)/\Gamma(1-x)$ and

$$\nu(b)=-b^2\gamma(-b^2)=\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\,. \qquad \text{(A.15)}$$

A.3 Two- and three-point functions in the $SU(2)_{k'}$ WZW model

The chiral primaries of the $SU(2)_{k'}$ WZW model are denoted by

$$\Phi'_j(z,\bar{z};y,\bar{y}) = \Phi'_j(z,y)\,\bar{\Phi}'_j(\bar{z},\bar{y}) \ , \tag{A.16}$$

and have conformal dimension

$$\Delta(j) = \bar{\Delta}(j) = \frac{j(j+1)}{k'+2} \ , \qquad 0 \le j \le \frac{k'}{2} \ , \tag{A.17}$$

where j is the $SU(2)$ representation label and k' the level of the affine Lie algebra.

The two- and three-point functions of $\Phi'_j(z,\bar{z};y,\bar{y})$ are then [27, 28]

$$\langle \Phi'_{j_1}(z_1,\bar{z}_1;y_1,\bar{y}_1)\Phi'_{j_2}(z_2,\bar{z}_2;y_2,\bar{y}_2)\rangle = \delta_{j_1,j_2}\frac{|y_{12}|^{4j_1}}{|z_{12}|^{4\Delta(j_1)}} \ , \tag{A.18}$$

and

$$\langle \Phi'_{j_1}(z_1,\bar{z}_1;y_1,\bar{y}_1)\Phi'_{j_2}(z_2,\bar{z}_2;y_2,\bar{y}_2)\Phi'_{j_3}(z_3,\bar{z}_3;y_3\bar{y}_3)\rangle = C'_{j_1,j_2,j_3}\prod_{i<j}\frac{|y_{ij}|^{2j_{ij}}}{|z_{ij}|^{2\Delta_{ij}}} \ , \tag{A.19}$$

with $\Delta_{12} = \Delta(j_1)+\Delta(j_2)-\Delta(j_3)$, *etc.* The relevant coefficients are

$$C'_{j_1,j_2,j_3} = \sqrt{\frac{\gamma(\frac{1}{k'+2})}{\gamma(\frac{2j_1+1}{k'+2})\gamma(\frac{2j_2+1}{k'+2})\gamma(\frac{2j_3+1}{k'+2})}}\,\frac{P(j_1+j_2+j_3+1)\,P(j_{12})\,P(j_{23})\,P(j_{31})}{P(2j_1)\,P(2j_2)\,P(2j_3)} \tag{A.20}$$

and

$$P(j) = \prod_{m=1}^{j}\gamma(\tfrac{m}{k'+2}) \ , \qquad P(0) = 1 \ , \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \ . \tag{A.21}$$

The functions $P(j)$ are nonvanishing for $0 \le j \le k'+1$. Therefore, $C'_{j_1,j_2,j_3} \neq 0$, if

$$j_1 + j_2 + j_3 \le k' \ . \tag{A.22}$$

A.4 Four-point function in the $SU(2)_{k'}$ WZW model

The four-point function of the $SU(2)$ chiral primary $\Phi'_{j_i}(z,\bar{z};y,\bar{y})$ is given by

$$\begin{aligned}\langle \prod_{i=1}^{4}\Phi'_{j_i}(z_i,\bar{z}_i;y_i,\bar{y}_i)\ \rangle &= |y_{24}|^{4j_2}|y_{14}|^{2(j_1+j_4-j_2-j_3)}|y_{34}|^{2(j_3+j_4-j_1-j_2)}|y_{13}|^{2(j_1+j_2+j_3-j_4)} \\ &\times |z_{24}|^{-4\Delta'_2}|z_{14}|^{2\nu'_1}|z_{34}|^{2\nu'_2}|z_{13}|^{2\nu'_3}\mathcal{F}_{SU(2)}(z,\bar{z},y,\bar{y}) \ ,\end{aligned} \tag{A.23}$$

with

$$\nu'_1 = \Delta'_1+\Delta'_3-\Delta'_2-\Delta'_4\,, \quad \nu'_2 = \Delta'_1+\Delta'_2-\Delta'_3-\Delta'_4\,, \quad \nu'_3 = \Delta'_4-\Delta'_1-\Delta'_2-\Delta'_3 \ ,$$

and

$$z = \frac{z_{12}z_{34}}{z_{14}z_{32}} \ , \qquad\qquad y = \frac{y_{12}y_{34}}{y_{14}y_{32}} \ . \tag{A.24}$$

The function $\mathcal{F}_{SU(2)}(z,\bar{z},y,\bar{y})$ is given in terms of the Dotsenko-Fateev integral

$$\mathcal{F}_{SU(2)}(z,\bar{z},y,\bar{y}) = \mathcal{N}(j_1,j_2,j_3,j_4)\ |z|^{\frac{4j_1j_2}{k'+2}}\ |1-z|^{\frac{4j_1j_3}{k'+2}} \times \tag{A.25}$$

$$\times \int \prod_{i=1}^{2j_1} \frac{dt'_i d\bar{t}'_i}{(2\pi i)}\ |t'_i - z|^{-\frac{2\beta'_1}{k'+2}}\ |t'_i|^{-\frac{2\beta'_2}{k'+2}}\ |t'_i - 1|^{-\frac{2\beta'_3}{k'+2}}\ |y - t'_i|^2\ |D(t')|^{\frac{4}{k'+2}} \ ,$$

where

$$D(t') = \prod_{i<j} (t'_i - t'_j) \ , \tag{A.26}$$

and

$$\begin{aligned}
\beta'_1 &= j_1 + j_2 + j_3 + j_4 + 1 \,, \\
\beta'_2 &= j_1 + j_2 - j_3 - j_4 + 1 + k' \,, \\
\beta'_3 &= j_1 + j_3 - j_2 - j_4 + 1 + k' \ .
\end{aligned} \tag{A.27}$$

The normalization is

$$\begin{aligned}
\mathcal{N}(j_1,j_2,j_3,j_4) &= \left[\gamma\left(\tfrac{1}{k'+2}\right)\right]^{2j_1+1} \frac{P(j_1+j_2+j_3+j_4+1)}{\gamma\left(\tfrac{2j_1+1}{k'+2}\right)^{1/2} P(2j_1)} \\
&\times \prod_{i=2}^{4} \frac{P(j_2+j_3+j_4-j_1-2j_i)}{\gamma\left(\tfrac{2j_i+1}{k'+2}\right)^{1/2} P(2j_i)} \ ,
\end{aligned} \tag{A.28}$$

with

$$P(n) = \prod_{m=1}^{n} \gamma\left(\tfrac{m}{k'+2}\right) , \qquad P(0) = 1 \ . \tag{A.29}$$

B Some correlators

In this appendix we give some more details on the computation of some correlators used in the main text.

For the computation of these correlators we will need the following OPEs (the dependence of the fields on z is suppressed):

$$
\begin{aligned}
j(x_k)\Phi_{h_i}(x_i) &= (-j^+ + 2x_k j^3 - x_k^2 j^-)\Phi_{h_i}(x_i) \\
&\sim \frac{1}{z_{ik}}\left(-D^+_{x_i} + 2x_k D^3_{x_i} - x_k^2 D^-_{x_i}\right)\Phi_{h_i}(x_i) \\
&= \frac{1}{z_{ik}}\left(-x_i^2\partial_{x_i} - 2h_i x_i + 2x_k(x_i\partial_{x_i} + h_i) - x_k^2\partial_{x_i}\right)\Phi_{h_i}(x_i) \\
&= \mathcal{D}^{(h_i)}_{ki}\Phi_{h_i}(x_i)\,, \qquad \text{(B.1)} \\
j(x_1)j(x_2) &\sim (k+2)\frac{x_{12}^2}{z_{12}^2} + \mathcal{D}^{(-1)}_{12} j(x_2)\,, \qquad \text{(B.2)} \\
\hat{\jmath}(x_1)\hat{\jmath}(x_2) &\sim -2\frac{x_{12}^2}{z_{12}^2} + \mathcal{D}^{(-1)}_{12}\hat{\jmath}(x_2)\,, \qquad \text{(B.3)} \\
\hat{\jmath}(x_1)\psi(x_2) &\sim \mathcal{D}^{(-1)}_{12}\psi(x_2)\,, \qquad \text{(B.4)}
\end{aligned}
$$

where we defined the operator $\mathcal{D}^{(h)}_{ki}$ as

$$\mathcal{D}^{(h)}_{ki} \equiv \frac{1}{z_{ki}}\left(x_{ki}^2\,\partial_{x_i} - 2h\,x_{ki}\right)\,. \qquad \text{(B.5)}$$

Recall that $j(x)$ generates a bosonic $SL(2)$ affine algebra at level $k_b = k+2$ (k is the supersymmetric level), while $\hat{\jmath}(x)$ forms a supersymmetric $SL(2)$ model at level -2.

We first show that an n-point correlator involving $j(x_k)$ ($k \in \{1, ..., n\}$) and n $SL(2)$ primaries $\Phi_{h_i}(x_i)$ ($i = 1, ..., n$) satisfies

$$d_k^{(n)} = \langle j(x_k)\prod_{i=1}^n \Phi_{h_i}(x_i)\rangle = \sum_{\substack{i=1\\ i\neq k}}^n \mathcal{D}^{(h_i)}_{ki}\langle\prod_{i=1}^n \Phi_{h_i}(x_i)\rangle\,. \qquad \text{(B.6)}$$

This follows directly from (B.1).

Acting with $j(x_m)$ ($m \in \{1, ..., n\}$) on (B.6), we find the n-point correlator

$$d_{k,m}^{(n)} = \langle j(x_k)j(x_m)\prod_{i=1}^n \Phi_{h_i}(x_i)\rangle = \big(\mathcal{D}^{(-1)}_{km} + \sum_{\substack{i=1\\ i\neq k}}^n \mathcal{D}^{(h_i)}_{ki}\big)d_m^{(n)}\,. \qquad \text{(B.7)}$$

Similarly, we may compute the fermionic correlators using[13]

$$
\begin{aligned}
\langle\psi(x_1)\psi(x_2)\rangle &= k\frac{(x_{12})^2}{z_{12}}\,, \\
\langle\hat{\jmath}(x_3)\psi(x_1)\psi(x_2)\rangle &= \sum_{i=1}^2 \mathcal{D}^{(-1)}_{3i}\langle\psi(x_1)\psi(x_2)\rangle\,, \\
\langle\hat{\jmath}(x_4)\hat{\jmath}(x_3)\psi(x_1)\psi(x_2)\rangle &= \sum_{j=1}^3 \mathcal{D}^{(-1)}_{4j}\langle\hat{\jmath}(x_3)\psi(x_1)\psi(x_2)\rangle\,. \qquad \text{(B.8)}
\end{aligned}
$$

[13]In the third equation we ignore a term of the type $\langle\hat{\jmath}(x_4)\hat{\jmath}(x_3)\rangle\langle\psi(x_1)\psi(x_2)\rangle$. It turns out to be subleading at small u.

C Comments on $SU(2)$ four-point function

In this appendix we derive the factorization (4.12) of the $SU(2)$ four-point function.

We start from the $SU(2)$ four-point function in y-space,

$$\left\langle \Phi'_{j_1}(0)\Phi'_{j_2}(y_2,\bar{y}_2)\Phi'_{j_3}(y_3,\bar{y}_3)\Phi'_{j_4}(\infty)\right\rangle \,, \tag{C.1}$$

in which we fixed $y_1 = 0$, $y_4 = \infty$. This corresponds to choosing states with $m_1 = j_1$ and $m_4 = -j_4$. This will now be expanded by means of the general OPE [27]

$$\Phi'_{j_2}(y_2,\bar{y}_2;z_2,\bar{z}_2)\Phi'_{j_1}(0) = \sum_j \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2))}}{|y_2|^{2(j-j_1-j_2)}} C'^{j}_{j_1,j_2}[\Phi'_j](y_2,\bar{y}_2;z_2,\bar{z}_2)\,, \tag{C.2}$$

where $C'^{j}_{j_1,j_2}$ are the $SU(2)$ structure constants given by (A.20) and the square brackets $[\Phi'_j]$ denote the contributions to the OPE from the primary field Φ'_j and all its descendants. This quantity can be presented in the form

$$[\Phi'_j](y_2,\bar{y}_2;z_2,\bar{z}_2) = R^{j}_{j_1,j_2}(y_2,z_2)\bar{R}^{j}_{j_1,j_2}(\bar{y}_2,\bar{z}_2)\Phi'_j(y_2,\bar{y}_2;z_2,\bar{z}_2)\,, \tag{C.3}$$

where the operator R is given by

$$R^{j}_{j_1,j_2}(y_2,z_2) = \sum_{n'=0}^{\infty}\frac{z_2^{n'}}{y_2^{n'}}\prod_{\alpha_i=1}^{3}\sum_{\{n^i p_i\}=n'} R_{n'}(n_i,p_i,j)(J^{\alpha_i}_{-n_i}(y_2,z_2))^{p_i} \tag{C.4}$$

where $i=1=+$, $i=2=-$, $i=3=3$ and $\{n^i p_i\} = n'$ means all combinations of $n_i p_i$ (partitions of n') such that $n_+p_+ + n_-p_- + n_3p_3 = n'$. In order to determine the coefficient $R_{n'}(n_i,p_i,j)$, let us take without loss of generality, a single combination of n_ip_i for each given n' (*i.e.* let us look at the contribution to the OPE from a single descendant for each level n'). In that case

$$\Phi'_{j_2}(y_2,\bar{y}_2;z_2,\bar{z}_2)\Phi'_{j_1}(0) = \\ \sum_{j,n',\bar{n'}} \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2))}z_2^{n'}\bar{z}_2^{\bar{n'}}}{|y_2|^{2(j-j_1-j_2)}y_2^{n'}\bar{y}_2^{\bar{n'}}} C'^{j}_{j_1,j_2}R_{n'}(n_i,p_i,j)\bar{R}_{\bar{n'}}(\bar{n}_i,\bar{p}_i,j)\Phi'^{jn'\bar{n'}}_{J,\bar{J}}(y_2,\bar{y}_2;z_2,\bar{z}_2)\,, \tag{C.5}$$

where $\Phi'^{jn'\bar{n'}}_{J,\bar{J}}$ defined by

$$\Phi'^{jn'\bar{n'}}_{J,\bar{J}} = \left[\prod_{\alpha_i=1}^{3}\sum_{\{n^ip_i\}=n'}(J^{\alpha_i}_{-n_i}(y_2,z_2))^{p_i}\sum_{\{\bar{n}^i\bar{p}_i\}=\bar{n'}}(\bar{J}^{\alpha_i}_{-\bar{n}_i}(\bar{y}_2,\bar{z}_2))^{\bar{p}_i}\right]\Phi'_j \tag{C.6}$$

is the descendant of Φ'_j at level $(n',\bar{n'})$. Let us now consider a three-point function with a descendant inside. Such a three-point function has the general form

$$\langle\Phi'_{j_1}(y_1,z_1)\Phi'_{j_2}(y_2,z_2)\Phi'^{j_3n'_3}_{J_3}(y_3,z_3)\rangle = C'_{j_1,j_2,j_3}\mathcal{D}(j_1,j_2,J_3)\prod_{i<j}\frac{|y_{ij}|^{2J_{ij}}}{|z_{ij}|^{2\tilde{\Delta}_{ij}}}\,, \tag{C.7}$$

with $J_{12} = j_1 + j_2 - J_3$, $\tilde{\Delta}_{12} = \Delta_{12} - n'$, *etc.*, where we have taken $n' = \bar{n}'$ for the sake of simplicity. Using the OPE (C.5) on the left hand side of (C.7) and putting $y_1 = z_1 = 0\,, y_2 = z_2 = 1\,, y_3 = z_3 = \infty$, we find

$$\mathcal{D}(j_1, j_2, J_3) = R_{n'_3}(n_i, p_i, j_3)\bar{R}_{n'_3}(n_i, p_i, j_3)\,. \tag{C.8}$$

This allows us to write

$$\begin{aligned}&\Phi'_{j_2}(y_2, \bar{y}_2; z_2, \bar{z}_2)\Phi'_{j_1}(0)\\ &\quad = \sum_{j,n'} \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|y_2|^{2(j-j_1-j_2+n')}}\, C'^{j}_{j_2,j_3}\mathcal{D}(j_1, j_2, J)\, \Phi'^{jn'}_{J}(y_2, \bar{y}_2; z_2, \bar{z}_2)\,.\end{aligned} \tag{C.9}$$

Inserting this into the $SU(2)$ four-point function, we get

$$\begin{aligned}&\left\langle \Phi'_{j_1}(0)\Phi'_{j_2}(y_2, \bar{y}_2)\Phi'_{j_3}(y_3, \bar{y}_3)\Phi'_{j_4}(\infty)\right\rangle\\ &= \sum_{j,n'} \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|y_2|^{2(j-j_1-j_2+n')}}\, C'^{j}_{j_1,j_2}\mathcal{D}(j_1, j_2, J)\, \langle \Phi'_{j_4}(\infty)\Phi'_{j_3}(y_3, \bar{y}_3)\Phi'^{jn'}_{J}(y_2, \bar{y}_2)\rangle\\ &= \sum_{j,n'} \mathcal{C}'(j)\frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}}\frac{|y_{23}|^{2(j+n'+j_3-j_4)}}{|y_2|^{2(j+n'-j_1-j_2)}}\mathcal{D}(j_1, j_2, J)\mathcal{D}(j_3, j_4, J)\,,\end{aligned} \tag{C.10}$$

with $\mathcal{C}'(j) = C'^{j}_{j_1,j_2}C'_{j,j_3,j_4}$.

We now convert the $SU(2)$ four-point function to the $m-$basis. This will be accomplished by the field transformation [27]

$$\Phi'_{j,m,\bar{m}} = \frac{1}{2\pi i}\oint d^2y|y|^{2(m-j-1)}c^{j+m}_{2j}\Phi'_j(y, \bar{y})\,, \tag{C.11}$$

where c are the inverse of the binomial coefficients,

$$c^{j+m}_{2j} = \frac{\Gamma(j+m+1)\Gamma(j-m+1)}{\Gamma(2j+1)}\,. \tag{C.12}$$

We have restricted the quantum numbers to $m = \bar{m}$. We then get

$$\begin{aligned}\langle \Phi'_{j_1,j_1}\Phi'_{j_2,m_2}\Phi'_{j_3,m_3}\Phi'_{j_4,-j_4}\rangle = \frac{1}{(2\pi i)^2}\sum_{j,n'}\Bigg[\mathcal{C}'(j)\mathcal{D}(j_1, j_2, J)\mathcal{D}(j_3, j_4, J)c^{j_2+m_2}_{2j_2}c^{j_3+m_3}_{2j_3}\\ \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}}\oint d^2y_2d^2y_3|y_2|^{2(j_1+m_2-j-n'-1)}|y_3|^{2(m_3-j_3-1)}|y_2-y_3|^{2(j+n'+j_3-j_4)}\Bigg]\end{aligned}$$

or, after changing variables from y_2 to $y = y_2/y_3$,

$$\begin{aligned}\frac{1}{(2\pi i)^2}\sum_{j,n'}\Bigg[\mathcal{C}'(j)\mathcal{D}(j_1, j_2, J)\mathcal{D}(j_3, j_4, J)c^{j_2+m_2}_{2j_2}c^{j_3+m_3}_{2j_3}\frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}}\\ \times\oint d^2y|y|^{2(j_1+m_2-j-n'-1)}|1-y|^{2(j+n'+j_3-j_4)}\oint d^2y_3|y_3|^{2(j_1+m_2+m_3-j_4-1)}\Bigg]\,.\end{aligned} \tag{C.13}$$

Both integrals can be carried out using the formula

$$\frac{1}{2\pi i}\oint \frac{dy}{y^n}\frac{1}{(1-y)^m} = \frac{\Gamma(n+m-1)}{\Gamma(n)\Gamma(m)} \tag{C.14}$$

such that the $SU(2)$ four-point function in $m-$basis becomes

$$\langle \Phi'_{j_1,j_1}\Phi'_{j_2,m_2}\Phi'_{j_3,m_3}\Phi'_{j_4,-j_4}\rangle = \sum_{j,n'}\Big[\, \mathcal{C}'(j)\mathcal{D}(j_1,j_2,J)\mathcal{D}(j_3,j_4,J)c_{2j_2}^{j_2+m_2}c_{2j_3}^{j_3+m_3} \tag{C.15}$$

$$\times \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}}\frac{\Gamma(j_4-j_1-m_2-j_3)^2}{\Gamma(j+n'-j_1-m_2+1)^2\Gamma(j_4-j-n'-j_3)^2}\delta^2_{j_1+m_2+m_3-j_4,0}\Big]\,.$$

We may eventually take $m_2 = j_2 - d$, $m_3 = j_3$ with $d \geq 0$ and set $z_{1,2,3,4} = 0, z, 1, \infty$. Then, $c_{2j_3}^{j_3+m_3} = 1$ (since $m_3 = j_3$) and (C.15) reduces to (4.12). Note that in the small z limit, the factor $|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}$ with $z_{23} = z - 1$ is just one.

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