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Cut-and-join operators and $\mathcal{N} = 4$ super Yang-Mills

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ABSTRACT

We show which multi-trace structures are compatible with the symmetrisation of local operators in $\mathcal{N} = 4$ super Yang-Mills when they are organised into representations of the global symmetry group. Cut-and-join operators give the non-planar expansion of correlation functions of these operators in the free theory. Using these techniques we find the $1/N$ corrections to the quarter-BPS operators which remain protected at weak coupling. We also present a new way of counting these chiral ring operators using the Weyl group S_N.

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1 Introduction and summary

Recent work on solving $\mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$ for finite N at weak coupling has focused on a basis of local operators which is well-defined regardless of the number of fields the operators contain [1]-[11]. In particular for classical conformal dimensions $\Delta \geq N$ the operators accommodate the Stringy Exclusion Principle [12] which limits the operators constructible from finite-size matrices. All the bases have been defined by generalising the half-BPS Schur polynomials [1] to the case where multiple fields of the theory are included and sorted into representations of the global symmetry group $PSU(2, 2|4)$ or its subgroups. Like the Schur polynomials these bases of operators are fully diagonal in the non-planar two-point function at tree level [4, 8], but unlike the Schur polynomials they mix at one loop [6] because they are no longer necessarily BPS.

These operators, which are traced with representations of the gauge group $U(N)$ like a Wilson loop, are linear combinations of operators with different trace structures. The goal of this paper is to return the non-planar analysis to operators which have a fixed trace structure for $\Delta < N^{\frac{1}{2}}$ but are still organised into representations of the global symmetry group. The latter is important because we expect the bulk spectrum to be organised into representations of $PSU(2, 2|4)$. The trace structure plays an important part in defining the operators with a given conformal dimension at weak coupling and hence identifying the map to string states in the bulk.

The complications of matching symmetrised representations to traces can be seen clearly with the $\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}$ representation of the subgroup $U(2) \subset SU(4)_R \subset PSU(2, 2|4)$ of the global symmetry group. The two columns of the Young diagram correspond to two commutators of X and Y: $[X, Y][X, Y]$. If we take a single trace of length 4 of these fields then we get the gauge-invariant operator $\mathrm{tr}([X, Y][X, Y]) = 2\,\mathrm{tr}(XYXY) - 2\,\mathrm{tr}(YXXY)$ (see Figure 1). However if we take two traces of length 2 then the operator $\mathrm{tr}([X, Y])\,\mathrm{tr}([X, Y])$ vanishes by cyclicity of the trace since $\mathrm{tr}([X, Y]) = \mathrm{tr}(XY) - \mathrm{tr}(YX) = 0$. To get a non-trivial operator with this trace structure you must 'twist' the fields with a permutation before tracing them. The result is $\mathrm{tr}(\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s)$ where each commutator $[X, Y] = \Phi^r\Phi_r = \epsilon_{rs}\Phi^r\Phi^s$ crosses between the traces (see Figure 1). Also, for a given trace structure and representation there are often multiple independent operators.

The crucial technical result in this paper is given in Section 3.2: for a given trace structure and representation of the global symmetry group we show how to write down the allowed operators

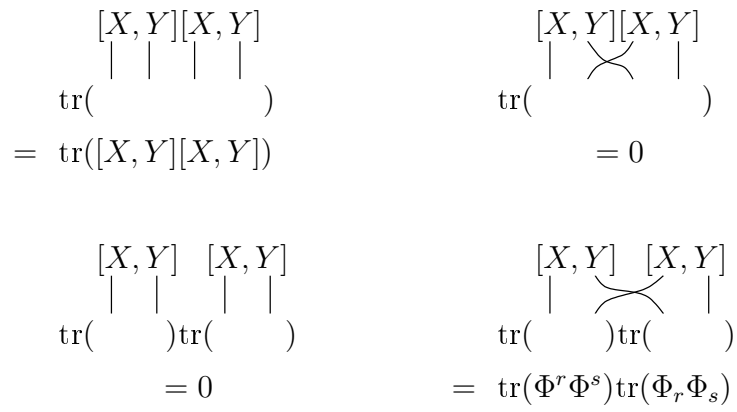

Figure 1: Different ways of traces the $U(2)$ representation $\boxplus$.

and count them correctly. These operators are diagonal in the planar two-point function. Once we have well-defined operators we can compute the full non-planar expansion of their free correlation functions order by order in $\frac{1}{N}$ using the string/spin bit formalism. For the half-BPS states this is reviewed in Section 2. Every time we cut a trace in two or join two traces together we gain a factor of $\frac{1}{N}$. Using cut-and-join operators in the class algebra of the symmetric group reduces the free theory combinatorics to multiplications within this algebra [13]. Mapping the factors of $\frac{1}{N}$ to string vertices makes the 't Hooft map from non-planar Feynman diagrams to two-dimensional Riemann surfaces transparent. Futhermore for large operators the higher-genus surfaces factorise into planar three-point functions corresponding to three-punctured spheres in the putative string dual to free super-Yang-Mills [14, 15, 13]. In Section 3.3 we show that this behaviour is not peculiar to the half-BPS sector but is universal for all operators in the free theory. This factorisation gives some evidence that the free theory becomes simpler when the operators are large.

In the AdS/CFT correspondence [16] it is not $\frac{1}{N}$ but rather $g_s = g_{\rm YM}^2 = \frac{\lambda}{N}$ which is identified with the physical string coupling in the bulk theory. However, in the free theory both $g_{\rm YM}$ and λ vanish while N remains finite. In the absence of a definite string theory dual to the free theory, we identify $\frac{1}{N}$ with the string coupling in this limit. It is with respect to this three-string vertex that the correlation functions factorise. In the free theory we also choose to identify single-trace operators with single-string states and correspondingly for the multi-trace operators, but this choice is by no means unique [17] and does not apply when we turn on the coupling.

At weak coupling the situation is subtle, but the techniques developed in Section 3 are still useful to identify the correct states. In Section 4 we consider the $U(2)$ subsector of operators built from two complex scalars X, Y where the first non-trivial complications arise. There is now a separation between those operators which gain an anomalous dimension when the coupling is turned on, and those which remain BPS (they are annihilated by a quarter of the supercharges). In this rearrangement of operators into eigenstates of the dilatation operator, different trace structures get mixed. The main result of this section is an application of the non-planar technology of Section 3 to find a new way of writing the $\frac{1}{N}$ corrections for the quarter-BPS operators based on the earlier work of [18] and [4]. Much of the free analysis remains, particularly in the fact that different trace structures come with powers of $\frac{1}{N}$ depending on how they are cut or joined.

The $U(2)$ operators which remain protected are in one-to-one correspondence with those operators which can be built assuming that the fields commute within each trace. These operators are part of the chiral ring of the theory. In Section 5 we give a new way to count the chiral ring by constructing operators which are functions of the eigenvalues of the fields X and Y.

2 Review of cut-and-join operators for half-BPS operators

In this section we synthesise analysis of the non-planar correlation functions of multi-trace half-BPS operators from across the literature [19, 1, 20, 14, 13, 21].

The main points of this review are:

- The Hilbert space of half-BPS operators with n fields maps to the conjugacy classes of the symmetric group S_n, where the cycles of the permutations correspond to traces.
- Different trace structures are orthogonal in the inner product given by the planar two-point function.
- The $\frac{1}{N}$ corrections to the extremal correlation functions of trace operators are captured by the algebra of the conjugacy classes of S_n.

In each half-BPS multiplet there is an operator built from n copies of the same complex scalar field X. The trace structure is captured by a permutation $\alpha \in S_n$

$$\mathrm{tr}(\alpha\, X) \equiv X^{i_1}_{i_{\alpha(1)}} X^{i_2}_{i_{\alpha(2)}} \cdots X^{i_n}_{i_{\alpha(n)}} \tag{1}$$

For example the single-trace operator can be written with a single cycle of length n

$$\mathrm{tr}(X^n) = \mathrm{tr}([n]\, X) = X^{i_1}_{i_2} X^{i_2}_{i_3} \cdots X^{i_n}_{i_1} \tag{2}$$

In this case α is an n-cycle $\alpha = [n] \equiv (123\cdots n)$. Similarly an operator with two traces of lengths n_1 and n_2 is written using $\alpha = [n_1, n_2] = (12\cdots n_1)(n_1+1\cdots n_1+n_2)$

$$\mathrm{tr}(X^{n_1})\,\mathrm{tr}(X^{n_2}) = \mathrm{tr}([n_1, n_2]\, X) = X^{i_1}_{i_2} X^{i_2}_{i_3} \cdots X^{i_{n_1}}_{i_1} X^{i_{n_1+1}}_{i_{n_1+2}} \cdots X^{i_{n_1+n_2}}_{i_{n_1+1}} \tag{3}$$

The final operator only depends on the conjugacy class of $\alpha \in S_n$ and is invariant under conjugation by any element $\sigma \in S_n$: $\alpha \mapsto \sigma^{-1}\alpha\sigma$. The number of traces of length k in an operator is just the number of cycles of length k in α. In this way the different operators with n fields map to the different conjugacy classes of S_n, which are in one-to-one correspondence with the partitions of n into integer parts.[1]

The scalar propagator

$$\left\langle (X^\dagger)^i_j(x)\; X^k_l(y) \right\rangle = \delta^i_l \delta^k_j \frac{1}{(x-y)^2} \tag{4}$$

can now be used to compute correlation functions of these operators. We choose the positions and coordinate frames of the operators in the two-point function to remove the spacetime dependence

[1] We shall be loose in our notation: $[n_1, n_2, \cdots n_k]$ refers both to the ordered partition of n that defines a conjugacy class of S_n and to the canonical permutation within that conjugacy class $(12\cdots n_1)(n_1+1\cdots n_1+n_2)\cdots(\cdots n)$.

so that we can focus on the combinatorics of the contraction of the gauge indices.[2] The simplest example is the two-point function of two single-trace operators[3]

$$\left\langle \mathrm{tr}(X^{\dagger n})\,\mathrm{tr}(X^n)\right\rangle = nN^n\left\{1+\left[\binom{n}{3}+\binom{n}{4}\right]\frac{1}{N^2}+\mathcal{O}\left(\frac{1}{N^4}\right)\right\} \tag{5}$$

We will now review how, once we map the operators to conjugacy classes of S_n, all the $\frac{1}{N}$ corrections are captured by class functions of S_n acting on these states. All the interactions can be encoded in the element Ω_n familiar from the $\frac{1}{N}$ expansion of 2d Yang-Mills [22, 23]

$$\begin{aligned}\Omega_n &= \sum_{\sigma\in S_n} N^{C(\sigma)-n}\ \sigma \\ &= 1+\frac{1}{N}\Sigma_{[2]}+\frac{1}{N^2}\left(\Sigma_{[3]}+\Sigma_{[2,2]}\right)+\frac{1}{N^3}\left(\Sigma_{[4]}+\Sigma_{[3,2]}+\Sigma_{[2,2,2]}\right)+\mathcal{O}\left(\frac{1}{N^4}\right)\end{aligned} \tag{6}$$

$C(\sigma)$ is the number of cycles in σ. The sum over transpositions $[2,1^{n-2}]=[2,\overbrace{1,1,\cdots 1}^{n-2}]$ (a total of $n-1$ cycles) is written $\Sigma_{[2]}$ and similarly $\Sigma_{[3,2]}$ sums over all permutations of the form $[3,2,1^{n-5}]$ (a total of $n-3$ cycles). Since each Σ_C is of the form $\Sigma_C\propto\sum_{\rho\in S_n}\rho\,\sigma\,\rho^{-1}$ for some σ in the conjugacy class C, Σ_C commutes with every element of S_n and is hence in the centre of S_n.

The non-planar two-point function of two single-trace operators can now be written

$$\left\langle \mathrm{tr}(X^{\dagger n})\,\mathrm{tr}(X^n)\right\rangle = N^n\,\langle n|\Omega_n|n\rangle \tag{7}$$

Furthermore for large n (but still $n<N^{\frac{1}{2}}$ so that mixing between operators with different trace structures is suppressed)[4] the dominant part of Ω_n exponentiates [13]

$$\Omega_n\to\exp\left(\frac{1}{N}\Sigma_{[2]}\right) \tag{8}$$

Geometrically this corresponds to the factorisation of higher genus two-dimensional surfaces with marked points into three-punctured spheres.

2.1 The Hilbert space and its inner product

The half-BPS Hilbert space is defined by conjugacy classes of S_n

$$|n_1,n_2,\cdots n_k\rangle\equiv\mathrm{tr}([n_1,n_2,\cdots n_k]\ X)=\mathrm{tr}(X^{n_1})\,\mathrm{tr}(X^{n_2})\cdots\mathrm{tr}(X^{n_k}) \tag{9}$$

Its conjugate is

$$\langle n_1,n_2,\cdots n_k|\equiv\mathrm{tr}([n_1,n_2,\cdots n_k]\ X^\dagger)=\mathrm{tr}(X^{\dagger n_1})\,\mathrm{tr}(X^{\dagger n_2})\cdots\mathrm{tr}(X^{\dagger n_k}) \tag{10}$$

[2] This choice is akin to the 2d Zamolodchikov metric: we put the operators at opposite poles of S^4 in coordinate frames centred about each pole respectively.

[3] Higher genus corrections are computed in Appendix Section A.

[4] The limit of n large with $\frac{n^2}{N}<1$ fixed is the same as the BMN limit [24] for the half-BPS sector. However, as we will show later, the exponentiation of Ω_n works in any sector, even when the number of fields n is not related to the R-charge J as it is for the half-BPS sector where $n=\Delta=J$.

The structure of the Hilbert space is exactly the same as that in [13].

Define an inner product by taking the planar part of the two-point function

$$\begin{aligned}\langle m_1, \dots m_l | n_1, \dots n_k \rangle &= \frac{1}{N^n} \left\langle \mathrm{tr}(X^{\dagger m_1}) \cdots \mathrm{tr}(X^{\dagger m_l}) \quad \mathrm{tr}(X^{n_1}) \cdots \mathrm{tr}(X^{n_k}) \right\rangle_{\text{planar}} \\ &= |\mathrm{Sym}([n_1, \dots n_k])| \;\; \delta_{[m_1, \dots m_l] = [n_1, \dots n_k]} \end{aligned} \tag{11}$$

The conjugacy class $[\vec{m}]$ must be the same as $[\vec{n}]$ for the inner product to be non-zero. This means that the trace structures must be the same. When it is non-zero the value is the size of the symmetry group of $[\vec{n}]$, i.e. the group of permutations that leave $[\vec{n}]$ invariant under conjugation. If $[\vec{n}]$ has i_k k-cycles then its size is $i_1! 1^{i_1} i_2! 2^{i_2} \cdots i_n! n^{i_n}$. The factors k^{i_k} corresponds to the cyclic symmetry $\mathbb{Z}_k$ of each cycle while $i_k!$ is the permutation factor for the i_k identical k-cycles.

By taking the planar part of the two-point function we have in mind the 2d Zamolodchikov metric where we get an inner product for a 2d theory by taking the two-point function on the sphere S^2 with operators at opposite poles.

In the simplest example for single traces

$$\langle n | n \rangle = \frac{1}{N^n} \left\langle \mathrm{tr}(X^{\dagger n}) \, \mathrm{tr}(X^n) \right\rangle_{\text{planar}} = n \tag{12}$$

where n is the size of the cyclic group $\mathbb{Z}_n$ that leaves the n-cycle $(12 \cdots n)$ invariant.

For two-particle states

$$\langle n_1, n_2 | n_1, n_2 \rangle = \begin{cases} n_1 n_2 & \text{for} \quad n_1 \neq n_2 \\ 2 n_1 n_2 & \text{if} \quad n_1 = n_2 \end{cases} \tag{13}$$

which is just the product of two single-trace inner products. For $n_1 = n_2$ we must be more careful: there is an extra factor of 2 because the symmetry group now includes the exchange of the two identical cycles.

2.2 Planar three-point function

The leading planar contribution to the extremal three-point function [25] of single-trace half-BPS operators is the same as that of the two-point function between a single-trace operator and a two-trace operator[5]

$$\left\langle \mathrm{tr}(X^{\dagger n}) \, \mathrm{tr}(X^{n_1}) \, \mathrm{tr}(X^{n_2}) \right\rangle = n n_1 n_2 N^{n-1} + \mathcal{O}(N^{n-3}) \tag{14}$$

for $n_1 + n_2 = n$. The inner product between these states vanishes $\langle n | n_1, n_2 \rangle = 0$ so we must introduce an interaction vertex of order $\frac{1}{N}$ to get this non-trivial result. Define an element of the algebra of the symmetric group $\mathbb{Q}S_n$ (elements of S_n with rational coefficients) by a sum over all of the $\binom{n}{2}$ two-cycles, i.e. all elements of S_n in the conjugacy class $[2, 1^{n-2}]$

$$\Sigma_{[2]} = \sum_{i<j} (ij) = \sum_{\sigma \in [2, 1^{n-2}]} \sigma \tag{15}$$

[5]The spacetime dependence of the three-point function is generically $(x-y)^{-2n_1}(x-z)^{-2n_2}$. However, because there is no interaction between $\mathrm{tr}(X^{n_1})$ and $\mathrm{tr}(X^{n_2})$ we can take these operators to be close together and treat the three-point function like the two-point function between $[\mathrm{tr}(X^{\dagger n})](x)$ and $[\mathrm{tr}(X^{n_1}) \, \mathrm{tr}(X^{n_2})](y)$.

It acts on $|n_1, \dots n_k\rangle$ by left action on the permutation defining the state

$$\Sigma_{[2]}|n_1, \dots n_k\rangle = \sum_{\sigma\in[2,1^{n-2}]} |\sigma[n_1, \dots n_k]\rangle \tag{16}$$

Applying it to $|n\rangle$, $\Sigma_{[2]}$ splits the single trace into all possible double traces[6]

$$\Sigma_{[2]}|n\rangle = \sum_{k=1}^{n-1} \frac{n}{2}|n-k, k\rangle \tag{17}$$

If $n_1 \neq n_2$ we get the resulting vertex for the three-point function

$$\langle n_1, n_2|\Sigma_{[2]}|n\rangle = \frac{n}{2}\langle n_1, n_2|n_1, n_2\rangle + \frac{n}{2}\langle n_1, n_2|n_2, n_1\rangle = nn_1n_2 \tag{18}$$

For $n_1 = n_2$ we arrive at the same result

$$\langle n_1, n_1|\Sigma_{[2]}|n\rangle = \frac{n}{2}\langle n_1, n_1|n_1, n_1\rangle = nn_1n_2 \tag{19}$$

To check the symmetry under conjugation, $\Sigma_{[2]}$ also joins traces when acting on the double trace

$$\Sigma_{[2]}|n_1, n_2\rangle = n_1n_2|n\rangle + \sum_{k=1}^{n_1-1} \frac{n_1}{2}|n_1-k, k, n_2\rangle + \sum_{k=1}^{n_2-1} \frac{n_2}{2}|n_1, n_2-k, k\rangle \tag{20}$$

so that the result is preserved

$$\langle n|\Sigma_{[2]}|n_1, n_2\rangle = nn_1n_2 \tag{21}$$

Including the N dependence the 3-point function vertex is

$$\frac{1}{N}\Sigma_{[2]} \tag{22}$$

2.3 Torus two-point function

We now use the same formalism to compute the first non-planar correction to the single-trace two-point function (5). The lowest genus Riemann surface on which the fat graphs of these Feynman diagrams can be drawn is the torus.

For a correction of $\frac{1}{N^2}$ the cut-and-join operators come from a sum over the elements of S_n with $n-2$ cycles. These elements are composed of two transpositions, i.e. either two 2-cycles or one 3-cycle. If we define $\Sigma_{[3]}$ to be the sum of all permutations with one 3-cycle (a sum of $2\binom{n}{3}$ terms) and $\Sigma_{[2,2]}$ to be the sum of all permutations with two 2-cycles (a sum of $3\binom{n}{4}$ terms) then we find

$$\begin{aligned}\Sigma_{[3]}|n\rangle &= \binom{n}{3}|n\rangle + \cdots \\ \Sigma_{[2,2]}|n\rangle &= \binom{n}{4}|n\rangle + \cdots\end{aligned} \tag{23}$$

[6]In more detail: consider the transposition (ij) acting on the canonical n-cycle $(12\cdots i\cdots j\cdots n)$. Just cycling around the elements, the n-cycle is identical to $(i\cdots j\cdots n12\cdots)$. Denote the first sequence abbreviated to $\cdots$ by $s_1 = i+1\cdots j-1$ and the second by $s_2 = j+1\cdots n12\cdots i-1$. The computation is now $(ij)(i\,s_1\,j\,s_2) = (i\,s_1)(j\,s_2)$.

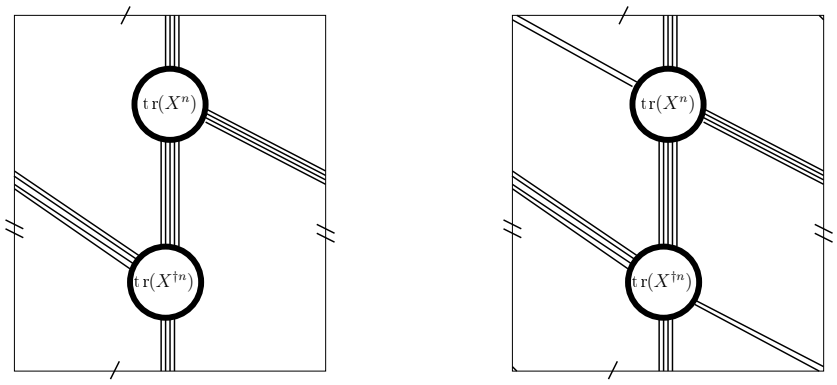

Figure 2: The two different bunchings of propagators that can be drawn on the torus with no crossing: three bunchings from $\Sigma_{[3]}$ on the left and four bunchings from $\Sigma_{[2,2]}$ on the right. This figure is copied from Figure 1 of [20] and Figure 2 of [14].

Roughly, each of these is akin to two actions of $\Sigma_{[2]}$ so that in the first term a single trace has split and rejoined. In the $\cdots$ are three-particle states from two splittings (see Appendix Section B for these terms and their use in the extremal four-point function).

The torus two-point function is then given by

$$\langle n| \left(\Sigma_{[3]} + \Sigma_{[2,2]}\right) |n\rangle = n \left[\binom{n}{3} + \binom{n}{4}\right] \tag{24}$$

This agrees with the calculation in (5). Including the N dependence the correct vertex is

$$\frac{1}{N^2} \left(\Sigma_{[3]} + \Sigma_{[2,2]}\right) \tag{25}$$

2.4 Relation to worldsheet models

What do the combinatorial numbers in (24) capture in the dual worldsheet theory? For two ordered circular chains of fields (the single traces in index space) the binomials count the number of ways of Wick contracting the fields on the two-dimensional torus so that none of the propagators cross [20, 14] (excluding those diagrams which can also be drawn on the sphere). Where propagators run parallel to each other they can be bunched together into homotopic groups. On the torus this can be done into either three groups, of which there are $\binom{n}{3}$ ways of bunching them, or four groups, of which there are $\binom{n}{4}$ ways. See Figure 2 copied from [20, 14].

The bunching of propagators into homotopic groups to form 'skeleton graphs' is an important feature of Gopakumar's mapping of the free theory Feynman diagrams to worldsheet correlators [26]. The vertex (25) is also the relevant one for the extremal four-point function. The two classes of bunchings for the four-point function, called the Y and lollipop diagrams in [27], also correspond to $\Sigma_{[3]}$ and $\Sigma_{[2,2]}$ respectively, see Appendix Section B. We are therefore lead to make the conjecture:

For each extremal k-point function on a genus g surface the different homotopic bunchings of propagators correspond to the different conjugacy classes that appear at order $\frac{1}{N^{k+2g-2}}$ in Ω_n.	(26)

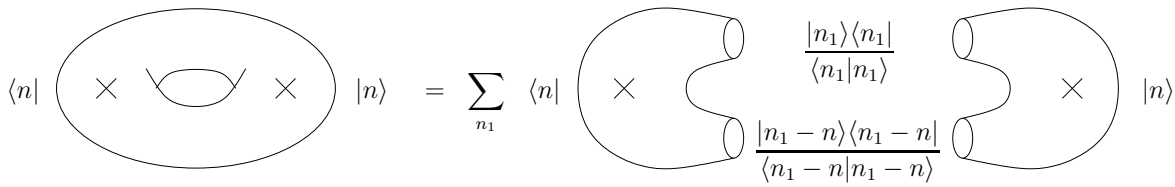

Figure 3: Torus factorisation.

The number of bunches for Σ_C is given by the sum of the non-trivial parts of C. For example, $\Sigma_{[2,2]}$ splits $|n\rangle$ into $2+2=4$ bunches because the typical element of C, say the permutation $(ij)(kl)$, reorders the n-cycle $(12\cdots n)$ into four separate pieces across one n-cycle or three smaller cycles.

Berkovits' model for the free field theory [28] also captures the non-crossing of propagators.

2.5 Factorisation of the torus two-point function into planar three-point functions

An interesting feature of the torus two-point function is that it factorises into the product of two planar three-point functions in the limit when n is large (so that the operators have many fields). This factorisation was first pointed out in investigations [14, 15] of the BMN limit [24][4], in particular using the string bit formalism [13] where these results appeared in this language. Such a factorisation takes the form of a sum over two sets of intermediate states (see Figure 3)

$$\langle n|\Sigma_{[2]}\Sigma_{[2]}|n\rangle = \sum_{n_1\leq\frac{n}{2}} \langle n|\Sigma_{[2]}\frac{|n_1,n-n_1\rangle\langle n_1,n-n_1|}{\langle n_1,n-n_1|n_1,n-n_1\rangle}\Sigma_{[2]}|n\rangle \tag{27}$$

Factorisations like this are familiar from two-dimensional topological field theories, but this association may be naïve given that the factorisation only takes place in index space: we have ignored the non-trivial spacetime dependencies of the correlation functions, particular the three-point function.[5]

To see the equality in (27) note that only two-particle states appear in the image of $\Sigma_{[2]}$ acting on $|n\rangle$ so that the projector

$$\sum_{n_1\leq\frac{n}{2}} \frac{|n_1,n-n_1\rangle\langle n_1,n-n_1|}{\langle n_1,n-n_1|n_1,n-n_1\rangle} \tag{28}$$

has a trivial action on $\Sigma_{[2]}|n\rangle$. Thus we can just insert it in the middle of $\langle n|\Sigma_{[2]}\Sigma_{[2]}|n\rangle$ to get (27). The half-BPS operators only mix with other half-BPS operators in the extremal three-point function, which is why the sum over intermediate states truncates to the half-BPS sector.

Next we want to show that (27) agrees with the correct result (24) in the large n limit.

$$\Sigma_{[2]}\Sigma_{[2]} = \binom{n}{2} + 3\Sigma_{[3]} + 2\Sigma_{[2,2]} \tag{29}$$

so that if we use the results for the actions on $|n\rangle$ in (23) we find

$$\langle n|\Sigma_{[2]}\Sigma_{[2]}|n\rangle = n\left[\binom{n}{2} + 3\binom{n}{3} + 2\binom{n}{4}\right] \tag{30}$$

This is not exactly the same as the correct field theory result (24). However in the large $n = \Delta = J$ limit the first terms in both (24) and (30) are suppressed and the leading behaviour $\binom{n}{4} \sim \frac{n^4}{4!}$ is the same for both up to a factor of 2. Essentially the term $\Sigma_{[2,2]}$ dominates over the other terms because its size is greatest, of order $\binom{n}{4}$. This result generalises as we see in the next section.

2.6 Factorisation for the full string cut-and-join operator

The full string cut-and-join operator is [13]

$$\begin{aligned}\Omega_n &= \sum_{\sigma\in S_n} \frac{1}{N^{n-C(\sigma)}}\ \sigma \\ &= 1 + \frac{1}{N}\Sigma_{[2]} + \frac{1}{N^2}\left(\Sigma_{[2,2]} + \Sigma_{[3]}\right) + \frac{1}{N^3}\left(\Sigma_{[2,2,2]} + \Sigma_{[3,2]} + \Sigma_{[4]}\right) + \mathcal{O}\left(\frac{1}{N^4}\right)\end{aligned} \tag{31}$$

$C(\sigma)$ is the number of cycles in σ. The negative power of N is also the minimum number $T(\sigma)$ of transpositions it takes to build each element σ, which is related to $C(\sigma)$ by

$$T(\sigma) = n - C(\sigma) \tag{32}$$

A check on Ω is that it reproduces the full non-planar two-point function. For operators built with two permutations $\alpha, \alpha' \in S_n$ the two-point function is a sum over all permutations $\tau \in S_n$ of Wick contractions of the fields that gives [1]

$$\begin{aligned}\left\langle \mathrm{tr}(\alpha'\, X^{\dagger n})\, \mathrm{tr}(\alpha\, X^n)\right\rangle &= \sum_{\tau\in S_n} N^{C(\alpha\tau\alpha'\tau^{-1})} \\ &= \sum_{\sigma\in S_n}\sum_{\tau\in S_n} N^{C(\sigma)}\delta(\sigma\alpha\tau\alpha'\tau^{-1} = 1)\end{aligned} \tag{33}$$

To see why the second equality holds, consider the coefficient of a particular power h of N. To contribute to this coefficient $\alpha\tau\alpha'\tau^{-1}$ must have h cycles. To test this condition, multiply $\alpha\tau\alpha'\tau^{-1}$ by all possible permutations σ with h cycles and see if $\sigma\alpha\tau\alpha'\tau^{-1} = 1$. This is equivalent to the condition that $\tau\alpha'\tau^{-1}$ is the inverse of $\sigma\alpha$, which because we sum over $\tau \in S_n$ happens $|\mathrm{Sym}(\alpha)|$ times (if $\sigma\alpha$ is in the same conjugacy class as α'). But this is exactly what we do when we insert Ω in the inner product: for N^h we multiply $|\alpha\rangle$ by all possible permutations with h cycles then $\langle\alpha'|\Omega|\alpha\rangle$ is only non-zero if $\sigma\alpha$ is in the same conjugacy class as α' (and the inner product gives us a factor of $|\mathrm{Sym}(\alpha)|$). Thus we get

$$\left\langle \mathrm{tr}(\alpha'\, X^{\dagger n})\, \mathrm{tr}(\alpha\, X^n)\right\rangle = N^n \langle\alpha'|\Omega_n|\alpha\rangle \tag{34}$$

Taking the large n limit at each order in $\frac{1}{N}$ we find that Ω exponentiates

$$\boxed{\Omega_n \to \sum_i \frac{1}{i!}\frac{1}{N^i}\left(\Sigma_{[2]}\right)^i = \exp\left(\frac{1}{N}\Sigma_{[2]}\right)} \tag{35}$$

This limit for the operator is meant in the sense that every matrix element $\langle\vec{m}|\Omega|\vec{n}\rangle$ satisfies this limit. The number of elements in $\Sigma_{[2]}$ is $\binom{n}{2} \sim n^2$ so that for Ω_n to remain non-vanishing requires the BMN ratio[4] $g_2 \equiv \frac{n^2}{N} < 1$ to be fixed in the large n, N limit. The exponentiation of Ω can in

fact be made exact for any n by adding extra terms, see equation (136) in Appendix C, from which it is clear that the corrections to (35) are subleading in g_2.

Equation (35) holds because the size of the conjugacy class $\Sigma_{[2^i]}$ grows like $\binom{n}{2i} \sim n^{2i}$ which is a factor of n larger than any other operator that appears at order N^{-i} (see Appendix Section A on the general form of higher genus operators). Similarly it dominates $\left(\Sigma_{[2]}\right)^i$ in the large n limit

$$\left(\Sigma_{[2]}\right)^i \to i!\, \Sigma_{[2^i]} \tag{36}$$

This equivalent behaviour gives equation (35).

The exponentiated term in equation (35) is the three-point function vertex studied in Section 2.2. Thus the large n limit exponentiation corresponds geometrically to the factorisation of higher genus correlation functions into the three-point functions which map to three-puncture spheres [13]. For small n the exponentiation of Ω still holds, see equation (136) in Appendix C, but there are higher order terms so we lose the geometrical interpretation.

2.7 Relation to Schur polynomials and $U(N)$ Casimirs

At finite N linear relations appear between trace operators with large dimension $\Delta \geq N$. This is known as the Stringy Exclusion Priniciple. A well-defined operator can be constructed by 'Fourier transforming' to representations R of S_n. Because it is used to trace the gauge indices, the Young diagram R also labels a representation of $U(N)$. This limits the number of rows to N, which neatly encodes the Stringy Exclusion Principle. Columns and rows correpond to giant gravitons in the bulk [1] and more general geometries via the LLM description [29].

The Schur polynomial is a linear combination of the trace operators where the coefficient is the S_n character $\chi_R(\sigma)$ of R

$$|R\rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) |\sigma\rangle \tag{37}$$

This basis is not only diagonal in the planar inner product

$$\langle S|R\rangle = \delta_{RS} \tag{38}$$

but also in the full non-planar two-point function. To see this compute the action of a generic cut-and-join operator Σ_C where C labels a conjugacy class of S_n

$$\Sigma_C |R\rangle = \frac{\chi_R(\Sigma_C)}{d_R} |R\rangle \tag{39}$$

It acts simply by multiplication. This is a result of Schur's lemma, because the Σ_C are in the centre of the group S_n. d_R is the symmetric group dimension. This means the non-planar two-point function is

$$\begin{aligned} \langle S|\Omega_n|R\rangle &= \frac{\chi_R(\Omega_n)}{d_R} \langle S|R\rangle \\ &= \frac{n! \dim R}{N^n d_R} \ \delta_{RS} \end{aligned} \tag{40}$$

We have used formula (133) for the $U(N)$ dimension $\dim R$.

The commuting Σ_C are linear combinations of the commuting higher Hamiltonians defined in [1, 9] which correspond to Casimirs of the gauge group $U(N)$. They distinguish the different Schur polynomials $|R\rangle$ by their eigenvalues. For example the second and third Casimirs in the representation R are

$$\begin{aligned} C_2(R) &= nN + 2\frac{\chi_R(\Sigma_{[2]})}{d_R} \\ C_3(R) &= n(n-1) + nN^2 + 4N\frac{\chi_R(\Sigma_{[2]})}{d_R} + 3\frac{\chi_R(\Sigma_{[3]})}{d_R} \end{aligned} \tag{41}$$

See Section 4.10.1 of the review of two-dimensional Yang-Mills [23] for more examples.

These charges have been identified in the dual gravitational description with asymptotic multipole moments of the spacetime [30].

3 Extension to general representations in the free theory

In the previous section we used cut-and-join operators to compute non-planar correlation functions of half-BPS operators. These operators were constructed from a single complex scalar X. In this section we allow more than one field from the theory. Non-trivial symmetrisations are now possible, such as the commutator of two complex fields $[X, Y]$. For non-planar calculations in the free theory only the symmetrisation of the operator is important, which is indexed by a Young diagram of n boxes (for n fields). This is equivalent to treating the global symmetry group as $U(\infty)$ where the fundamental multiplet is just the infinite number of single fields in the singleton representation of $PSU(2,2|4)$, including derivatives. These representations are the same as those for the higher spin theory, which are organised into YT-pletons in [31].

Only at one-loop do the specific $PSU(2,2|4)$ representations become important. Each $U(\infty)$ representation breaks down into an infinite number of $PSU(2,2|4)$ representations. In [8] it was shown how to do this using Schur-Weyl duality for $SO(6)$ and $SO(2,4)$. The general story is outlined in Section 3.5.

To start with we focus on the $U(2) \subset SU(4)_R \subset PSU(2,2|4)$ example but the language will be kept general enough that the extension to $U(\infty)$ is obvious. Some of the initial parts of this analysis were also carried out in [21, 32].

3.1 Not organising into representations of $U(2)$

In the $U(2)$ subsector of the global symmetry group there are two species of field X and Y. To compute the two-point function it is necessary to modify the inner product appropriately, but the non-planar corrections are still given by Ω_n. We borrow heavily from [4] where the non-planar correlation functions for the free theory were calculated.

To begin with we won't organise the operators into representations of $U(2)$. The state space is defined by a field content of μ_1 X's and μ_2 Y's and a permutation $\alpha \in S_n$

$$|\vec{\mu}, \alpha\rangle \equiv \mathrm{tr}(\alpha\, X^{\mu_1} Y^{\mu_2}) = X^{i_1}_{i_{\alpha(1)}} \cdots X^{i_{\mu_1}}_{i_{\alpha(\mu_1)}} Y^{i_{\mu_1+1}}_{i_{\alpha(\mu_1+1)}} \cdots Y^{i_{\mu_1+\mu_2}}_{i_{\alpha(\mu_1+\mu_2)}} \tag{42}$$

The states are not defined by conjugacy classes of α, but up to a new equivalence class $[\alpha]$ defined by the relation

$$\alpha \sim \rho^{-1}\alpha\rho \qquad \rho \in S_{\mu_1} \times S_{\mu_2} \tag{43}$$

Under this conjugation the operator $\mathrm{tr}(\alpha\, X^{\mu_1} Y^{\mu_2})$ remains invariant because the canonical choice of ordering of $X^{\otimes\mu_1} \otimes Y^{\otimes\mu_2}$ is unchanged by $\rho\, X^{\otimes\mu_1} \otimes Y^{\otimes\mu_2}\, \rho^{-1}$ for $\rho \in S_{\mu_1} \times S_{\mu_2}$ (see equation (51) for an example of this kind of S_n action).[7]

In the bra the order of the fields is reversed so that it is defined with the inverse α^{-1}

$$\langle \vec{\mu}, \alpha| \equiv \mathrm{tr}(\alpha^{-1}\, X^{\dagger\mu_1} Y^{\dagger\mu_2}) \tag{44}$$

The symmetry (43) is reflected in the planar inner product, where the non-zero value is given by the size of the intersection of $S_{\mu_1} \times S_{\mu_2}$ with α's symmetry group

$$\langle \vec{\mu}\,', \alpha' | \vec{\mu}, \alpha \rangle = |\mathrm{Sym}(\alpha) \cap (S_{\mu_1} \times S_{\mu_2})|\ \ \delta_{\vec{\mu}\vec{\mu}\,'} \delta_{\alpha\sim\alpha'} \tag{45}$$

The cut-and-join operators act by left-multiplication on α, just as before

$$\Sigma_{[2]} |\vec{\mu}, \alpha\rangle = \sum_{\sigma\in[2,1^{n-2}]} |\vec{\mu}, \sigma\alpha\rangle \tag{46}$$

For example take the single-trace state

$$|\vec{\mu} = (2,2), \alpha = (1324)\rangle = X^{i_1}_{i_3} X^{i_2}_{i_4} Y^{i_3}_{i_2} Y^{i_4}_{i_1} = \mathrm{tr}(XYXY) \tag{47}$$

The action of $\Sigma_{[2]}$ splits this state into three different kinds of double-trace operator

$$\begin{aligned} \Sigma_{[2]} |\vec{\mu}, \alpha = (1324)\rangle &= 2|\vec{\mu}, \alpha = (13)(24)\rangle + 2|\vec{\mu}, \alpha = (1)(324)\rangle + 2|\vec{\mu}, \alpha = (132)(4)\rangle \\ &= 2\,\mathrm{tr}(XY)\,\mathrm{tr}(XY) + 2\,\mathrm{tr}(X)\,\mathrm{tr}(YXY) + 2\,\mathrm{tr}(XYX)\,\mathrm{tr}(Y) \end{aligned} \tag{48}$$

Once again $\frac{1}{N}\Sigma_{[2]}$ defines the planar three-point function vertex.[8]

Using the inner product (45) the non-planar expansion of the correlation function is given by Ω_n by the same line of reasoning below (33)

$$\Big\langle \mathrm{tr}(\alpha'^{-1}\, X^{\dagger\mu_1} Y^{\dagger\mu_2})\, \mathrm{tr}(\alpha\, X^{\mu_1} Y^{\mu_2}) \Big\rangle = \sum_{\sigma\in S_{\mu_1}\times S_{\mu_2}} N^{C(\alpha\sigma\alpha'^{-1}\sigma^{-1})} = N^n\, \langle \vec{\mu}, \alpha' |\, \Omega_n\, | \vec{\mu}, \alpha \rangle \tag{49}$$

The exponentiation of the S-matrix $\Omega_n = \exp\left(\frac{1}{N}\Sigma_{[2]}\right)$ and hence the factorisation into planar three-point functions for large $n = \mu_1 + \mu_2$ follows exactly the same reasoning as in the half-BPS sector in Section 2.6.

3.2 Organising into representations

In this section we show how to organise operators built from Xs and Ys with a given trace structure α into representations Λ of $U(2)$. Λ is a Young diagram with n boxes and at most two rows, which we write $\Lambda \in P(n,2)$ to indicate that Λ is a partition of n into at most 2 parts. Λ is also a representation of S_n which shows how the operator is symmetrised. The cut-and-join technology follows through because these representations are just linear combinations of the operators constructed above.

[7] We can cycle permutations around the trace so that $\mathrm{tr}(\alpha\rho\, X^{\otimes\mu_1} \otimes Y^{\otimes\mu_2}\, \rho^{-1}) = \mathrm{tr}(\rho^{-1}\alpha\rho\, X^{\otimes\mu_1} \otimes Y^{\otimes\mu_2})$.

[8] Note that the only double-trace operator with $\vec{\mu} = (2,2)$ which doesn't appear here, $\mathrm{tr}(XX)\,\mathrm{tr}(YY)$, has no overlap with $\mathrm{tr}(XYXY)$ until N^{-3}.

To organise the fields into representations of $U(2)$ first strip the fields of their gauge indices so that they live in an abstract tensor space of $U(2)$. If $V_{\mathbf{2}}$ is the fundamental of $U(2)$ consisting of the two complex scalars X and Y then n copies of this rep $V_{\mathbf{2}}^{\otimes n}$ can be sorted into irreps of $U(2) \times S_n$ using Schur-Weyl duality

$$V_{\mathbf{2}}^{\otimes n} = \sum_{\Lambda \in P(n,2)} V_\Lambda^{U(2)} \otimes V_\Lambda^{S_n} \tag{50}$$

In each summand the Young diagram Λ for the $U(2)$ representation and the S_n representation is the same. This diagonal decomposition works because $U(2)$ and S_n have a commuting action on $V_{\mathbf{2}}^{\otimes n}$. To implement the decomposition concretely we define an action of $\sigma \in S_n$ on $V_{\mathbf{2}}^{\otimes n}$ by conjugation. For example on an element of $V_{\mathbf{2}}^{\otimes 4}$ the permutation $\sigma = (13) \in S_4$ has the action

$$\sigma \;\; X \otimes X \otimes Y \otimes Y \;\; \sigma^{-1} = Y \otimes X \otimes X \otimes Y \tag{51}$$

The decomposition (50) means that a complete basis is given by the linear combination of elements of $V_{\mathbf{2}}^{\otimes n}$ (to avoid confusion we will keep the standard braket notation $|\cdots\rangle$ for gauge-invariant states; for other states which live in general tensor spaces we will use $|\cdots\}$)

$$|\Lambda, M, a\} \equiv \frac{1}{n!} \sum_{\sigma \in S_n} B_{b\beta}^{\vec{\mu}} \, D_{ab}^{\Lambda}(\sigma) \quad \sigma \;\; \overbrace{X \otimes X \otimes \cdots X}^{\mu_1} \otimes \overbrace{Y \otimes Y \otimes \cdots Y}^{\mu_2} \;\; \sigma^{-1} \tag{52}$$

The operator is labelled by

- A Young diagram Λ with n boxes and at most 2 rows;
- The label $M = \{\vec{\mu}, \beta\}$ which tells us which $U(2)$ state from the representation $V_\Lambda^{U(2)}$ it is. $\vec{\mu}$ labels the field content and β the semi-standard tableau for $\vec{\mu}$ and Λ;
- a is the S_n state for $V_\Lambda^{S_n}$.

$D_{ab}^{\Lambda}(\sigma)$ is the orthogonal matrix for $\sigma \in S_n$ in the representation Λ; in (52) there is Einstein summation over the $V_\Lambda^{S_n}$ state b. The parameter $B_{b\beta}^{\vec{\mu}}$ is a branching coefficient for $S_n \to S_{\mu_1} \times S_{\mu_2}$ described in more detail below.

Reintroducing the gauge indices and tracing with α gives a gauge-invariant operator

$$|\Lambda, M, a; \alpha\rangle \equiv \frac{1}{n!} \sum_{\sigma \in S_n} B_{b\beta}^{\vec{\mu}} \, D_{ab}^{\Lambda}(\sigma) \;\; \mathrm{tr}(\alpha \sigma X^{\mu_1} Y^{\mu_2} \sigma^{-1}) \tag{53}$$

There are a large number of elements $\alpha \in S_n$ that give the same operator for a given $\{\Lambda, M, a\}$. We want to describe this degeneracy.

α is invariant under its symmetry group $\alpha = \rho^{-1}\alpha\rho$ for $\rho \in \mathrm{Sym}(\alpha)$. We can rotate σ^{-1} around the trace in (53) to get $\mathrm{tr}(\sigma^{-1}\rho^{-1}\alpha\rho\sigma X^{\mu_1} Y^{\mu_2})$ and then redefine the sum over σ using $\tau = \rho\sigma$. We get a state that is $D^{\Lambda}(\rho^{-1})$ times the original one. But we haven't actually altered the operator since $\alpha = \rho^{-1}\alpha\rho$ so there must be an equivalence class on both α and the S_n state a

$$(\alpha, a) \sim \left(\rho^{-1}\alpha\rho, \sum_a D_{ca}^{\Lambda}(\rho^{-1}) \;\; a \right) \quad \rho \in \mathrm{Sym}(\alpha) \tag{54}$$

To remove this redundancy we will back-track and see how the branching coefficient $B_{b\beta}^{\vec{\mu}}$ deals with a similar redundancy [4].

3.2.1 Revisiting the branching coefficient

The canonical choice of ordering of $X^{\otimes\mu_1} \otimes Y^{\otimes\mu_2}$ before we symmetrise in (52) remains invariant under $\rho\ X^{\otimes\mu_1} \otimes Y^{\otimes\mu_2}\ \rho^{-1}$ for $\rho \in S_{\mu_1} \times S_{\mu_2}$. For σ this is a symmetry on the right $\sigma \to \sigma\rho$ for $\rho \in S_{\mu_1} \times S_{\mu_2}$ that leaves the state invariant. To remove this redundancy we decompose the projector $P = \frac{1}{|S_{\mu_1} \times S_{\mu_2}|} \sum_{\rho \in S_{\mu_1} \times S_{\mu_2}} \rho$ into branching coefficients $B^{\vec{\mu}}_{b\beta}$

$$\frac{1}{|S_{\mu_1} \times S_{\mu_2}|} \sum_{\rho \in S_{\mu_1} \times S_{\mu_2}} D^{\Lambda}_{ab}(\rho) = \sum_{\beta} B^{\vec{\mu}}_{a\beta} B^{\vec{\mu}}_{b\beta} \tag{55}$$

This can be understood more intuitively as picking out the trivial representation $\mathbf{1}$ of $S_{\mu_1} \times S_{\mu_2}$ when we break Λ down into irreps of this subgroup. (55) then becomes

$$\{\Lambda, a|P|\Lambda, b\} = \sum_{\beta} \{\Lambda, a|\Lambda \to \mathbf{1} \text{ of } S_{\mu_1} \times S_{\mu_2}; \beta\}\{\Lambda \to \mathbf{1} \text{ of } S_{\mu_1} \times S_{\mu_2}; \beta|\Lambda, b\} \tag{56}$$

We choose (non-uniquely) the $B^{\vec{\mu}}_{b\beta}$ to be orthogonal in the sense

$$\sum_{a} B^{\vec{\mu}}_{a\beta} B^{\vec{\mu}}_{a\beta'} = \delta_{\beta\beta'} \tag{57}$$

β labels the degeneracy of the appearance of the trivial $\mathbf{1}$ representation when Λ is broken down into irreps of $S_{\mu_1} \times S_{\mu_2}$. The number of values for β is the Kostka number $K(\vec{\mu}, \Lambda)$ given by

$$K(\vec{\mu}, \Lambda) \equiv g([\mu_1], [\mu_2]; \Lambda) = \frac{1}{|S_{\mu_1} \times S_{\mu_2}|} \sum_{\rho \in S_{\mu_1} \times S_{\mu_2}} \chi_{\Lambda}(\rho) \tag{58}$$

Note that it is just the trace of the projecting matrix in (55). The Kostka number $K(\vec{\mu}, \Lambda)$ counts the number of ways the field content $\vec{\mu}$ can fit into a semi-standard tableau for Λ. It can be written using the Littlewood-Richardson coefficient $g([\mu_1], [\mu_2]; \Lambda)$ for the appearance of Λ in the $U(2)$ tensor product $[\mu_1] \circ [\mu_2]$. The different semi-standard tableaux label the states of the $U(2)$ rep Λ, so $\{\vec{\mu}, \beta\}$ completely label them too.

To check that the operator (52) has the required symmetry on the right, multiply σ by $\rho \in S_{\mu_1} \times S_{\mu_2}$ to get

$$B^{\vec{\mu}}_{b\beta}\ D^{\Lambda}_{ab}(\sigma\rho) = D^{\Lambda}_{bc}(\rho) B^{\vec{\mu}}_{b\beta}\ D^{\Lambda}_{ac}(\sigma) = B^{\vec{\mu}}_{c\beta}\ D^{\Lambda}_{ac}(\sigma) \tag{59}$$

This follows by using (57) to prove that $B^{\vec{\mu}}_{b\beta} = \sum_{\beta'} B^{\vec{\mu}}_{b\beta'}\ B^{\vec{\mu}}_{d\beta'}\ B^{\vec{\mu}}_{d\beta}$ and then using (55) to absorb $D^{\Lambda}_{bc}(\rho)$. The $B^{\vec{\mu}}_{b\beta}$ are an orthogonal basis for the eigenspace of the projector (55) with eigenvalue 1.

3.2.2 Removing trace description redundancy

α is a way of describing the trace structure of the operator. For each conjugacy class of S_n choose α to be the canonical permutation. The presence of α in the gauge-invariant operator (53) with $\sigma^{-1}\alpha\sigma$ induces a symmetry, this time on the left $\sigma \to \rho\sigma$ where ρ is in the symmetry group $\mathrm{Sym}(\alpha)$ of α so that $\alpha = \rho^{-1}\alpha\rho$. To remove this redundancy we proceed in exactly the same way using a coefficient $S^{\alpha}_{a\gamma}$ for $S_n \to \mathrm{Sym}(\alpha)$ that decomposes the projector

$$\frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\rho \in \mathrm{Sym}(\alpha)} D^{\Lambda}_{ab}(\rho) = \sum_{\gamma} S^{\alpha}_{a\gamma} S^{\alpha}_{b\gamma} \tag{60}$$

We choose the $S^{\alpha}_{a\gamma}$ to be orthogonal in the sense

$$\sum_a S^{\alpha}_{a\gamma} S^{\alpha}_{a\gamma'} = \delta_{\gamma\gamma'} \tag{61}$$

This orthogonal choice is not unique. Examples of the $S^{\alpha}_{a\gamma}$ for $\Lambda = [2,2]$ and for $\Lambda = [4,2]$ with trace structure $\alpha = [4,2]$ are computed in Appendix Section E. Contracting $S^{\alpha}_{a\gamma}$ with the degenerate operator (53) gives us our final non-degenerate operator

$$\boxed{|\Lambda, M; \alpha, \gamma\rangle = \frac{1}{n!} \sum_{\sigma \in S_n} S^{\alpha}_{a\gamma}\ B^{\vec{\mu}}_{b\beta}\ D^{\Lambda}_{ab}(\sigma)\ \mathrm{tr}(\alpha\sigma X^{\mu_1} Y^{\mu_2} \sigma^{-1})} \tag{62}$$

This is the crucial result of this paper. Note that we pick a single canonical α from each conjugacy class of S_n, so that α is just a partition of n. The γ label runs over the elements of each conjugacy class (i.e. trace structure) which are compatible with the symmetry imposed by Λ.

In Appendix Section D.1 it is checked that the basis defined by (62) is complete, in that any multi-trace operator of X's and Y's can be built out of a linear combination of this basis. We check below that these operators give the correct counting.

The number of values for γ is given by the trace of the projecting matrix

$$\boxed{S(\alpha, \Lambda) = \frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\rho \in \mathrm{Sym}(\alpha)} \chi_{\Lambda}(\rho)} \tag{63}$$

Given the symmetry imposed by Λ this number tells us how many operators there are with trace structure α. It gives an analytic expression for the examples computed in [31]. The partition function for global group $U(2)$ for $N \to \infty$ can thus be decomposed into $U(2)$ characters $\chi_{\Lambda}(x, y)$

$$Z(x,y) = \prod_{i=1}^{\infty} \frac{1}{1 - (x^i + y^i)} = \sum_n \sum_{\Lambda \in P(n,2)} \sum_{\alpha \in P(n)} S(\alpha, \Lambda)\ \chi_{\Lambda}(x, y) \tag{64}$$

This result is checked against known results [33, 4] in Appendix Section D.2.

Note that the $\sigma \in S_n$ sum in the final operator (62) is invariant both on the left by $\mathrm{Sym}(\alpha)$ and on the right by $S_{\mu_1} \times S_{\mu_2}$ so we could just sum over the double coset of S_n

$$\mathrm{Sym}(\alpha) \setminus S_n \,/\, S_{\mu_1} \times S_{\mu_2} \tag{65}$$

3.3 Inner product and two-point function

Before we give the inner product and the two-point function for these operators, we need to define the conjugate with care. The order of the fields in each trace are reversed so that the bra is defined with the inverse α^{-1} of the canonical permutation in each conjugacy class[9]

$$\langle \Lambda, M; \alpha, \gamma| = \frac{1}{n!} \sum_{\sigma \in S_n} S^{\alpha}_{a\gamma}\ B^{\vec{\mu}}_{b\beta}\ D^{\Lambda}_{ab}(\sigma)\ \mathrm{tr}(\alpha^{-1}\sigma X^{\dagger\mu_1} Y^{\dagger\mu_2} \sigma^{-1}) \tag{66}$$

[9]NB: The symmetry group of α and its inverse are the same $\mathrm{Sym}(\alpha) = \mathrm{Sym}(\alpha^{-1})$.

The inner product for the gauge-invariant operators defined by (62) then corresponds to the planar two-point function function, which is now diagonal

$$\boxed{\langle \Lambda', M'; \alpha', \gamma' | \Lambda, M; \alpha, \gamma \rangle = \delta^{\Lambda\Lambda'} \delta^{MM'} \delta^{\alpha\alpha'} \delta^{\gamma\gamma'} \ \frac{|\mathrm{Sym}(\alpha)||S_{\mu_1} \times S_{\mu_2}|}{n! d_\Lambda}} \tag{67}$$

The matrix elements for the cut-and-join operators are

$$\langle \Lambda', M'; \alpha', \gamma' | \Sigma_C | \Lambda, M; \alpha, \gamma \rangle = \delta^{\Lambda\Lambda'} \delta^{MM'} \frac{|S_{\vec{\mu}}|}{n! d_\Lambda} \sum_{\tau \in S_n} \delta(\Sigma_C \alpha = \tau^{-1} \alpha' \tau) \ S^{\alpha}_{a\gamma} \, S^{\alpha'}_{a'\gamma'} \ D^{\Lambda}_{a'a}(\tau) \tag{68}$$

The canonical planar three-point function between a single-trace state $\alpha = [n]$ and a double-trace state $\alpha = [n_1, n_2]$ can then be computed

$$\langle \Lambda', M'; \alpha' = [n_1, n_2], \gamma' | \, \Sigma_{[2]} \, | \Lambda, M; \alpha = [n], \gamma \rangle \tag{69}$$

This is easier to understand if we split the left state into two single-trace states

$$\boxed{\left(\langle \Lambda'_1, M'_1; [n_1], \gamma'_1 | \otimes \langle \Lambda'_2, M'_2; [n_2], \gamma'_2 | \right) \, \Sigma_{[2]} \, | \Lambda, M; \alpha = [n], \gamma \rangle = C^{\Lambda'_1 \circ \Lambda'_2 = \Lambda}_{M'_1 \ M'_2 \ M} \ f(\gamma'_1, \gamma'_2; \gamma)} \tag{70}$$

We get a Clebsch-Gordan coefficient coupling the $U(2)$ states and a function f of the γ's. See Appendix Section D.3 for more details.

The full non-planar free two-point function follows from the cut-and-join operators (68)

$$\begin{aligned} \left\langle \mathcal{O}^\dagger[\Lambda', M'; \alpha', \gamma'] \mathcal{O}[\Lambda, M; \alpha, \gamma] \right\rangle &= \delta^{\Lambda\Lambda'} \delta^{MM'} \frac{|S_{\vec{\mu}}|}{n! d_\Lambda} \sum_{\tau \in S_n} S^{\alpha}_{a\gamma} S^{\alpha'}_{a'\gamma'} D^{\Lambda}_{a'a}(\tau) N^{C(\tau^{-1}\alpha\tau\alpha'^{-1})} \\ &= \delta^{\Lambda\Lambda'} \delta^{MM'} \ N^n \, \langle \Lambda, M; \alpha', \gamma' | \, \Omega_n \, | \Lambda, M; \alpha, \gamma \rangle \end{aligned} \tag{71}$$

This is only diagonal in the global symmetry group $U(2)$ labels. As in all previous cases, the appearance of Ω_n means that the non-planar correlation functions factorise into planar three-point functions when n is large as in equation (35). Because the notation above is general the extension to all other unitary groups $U(K)$ is trivial. Thus the structure observed in the half-BPS sector, including the large n factorisation, is universal for the whole of $\mathcal{N} = 4$ SYM when the coupling is turned off.

3.4 Relation to finite N bases

To take account of the Stringy Exclusion Principle for operators with $\Delta \geq N$ the traces can be reorganised into representations R of $U(N)$ just as for the Schur polynomials of half-BPS operators in Section 2.7.

First consider the basic operator $|\vec{\mu}, \alpha\rangle$ from (42), before we've organised into $U(2)$ reps. Unlike the Schur polynomials, this is not a class function of α, so we cannot change basis using the class-invariant S_n character; we must use the more general matrix representation $D^R_{pq}(\alpha)$ of S_n

$$|\vec{\mu}; R, p, q\rangle = \frac{1}{n!} \sum_{\alpha} D^R_{pq}(\alpha) |\vec{\mu}; \alpha\rangle \tag{72}$$

The floating p, q state indices of $V_R^{S_n}$ will be dealt with later. To work out the action of the cut-and-join operators on these states, first note that as a consequence of Schur's lemma the central elements Σ_C act multiplicatively

$$D^R_{pq}(\Sigma_C \alpha) = \frac{\chi_R(\Sigma_C)}{d_R} D^R_{pq}(\alpha) \tag{73}$$

This means that the $U(N)$-organised trace is an eigenvector of the Σ_C

$$\Sigma_C |\vec{\mu}; R, p, q\rangle = \frac{\chi_R(\Sigma_C)}{d_R} |\vec{\mu}; R, p, q\rangle \tag{74}$$

One way to absorb the p, q indices is with a particular type of branching coefficient to build the 'restricted' Schur polynomials [34, 35, 5]. Another way [4] is to organise the $\mathbf{X}^{\vec{\mu}}$ into representations $\{\Lambda, M, a\}$ of $U(2) \times S_n$ as we did in Section 3.2 and contract the remaining S_n indices with an S_n $3j$ Clebsch-Gordan coefficient $S^{\hat{\tau}, \Lambda\ R\ R}_{\ \ a\ p\ q}$

$$|\Lambda, M; R, \hat{\tau}\rangle = \frac{1}{n!} \sum_{\alpha \in S_n} S^{\hat{\tau}, \Lambda\ R\ R}_{\ \ a\ p\ q}\ D^R_{pq}(\alpha)\ \ |\Lambda, M, a; \alpha\rangle \tag{75}$$

The relation to the $|\Lambda, M; \alpha, \gamma\rangle$ trace basis is given in Appendix Section D.4. Just like the Schur polynomials they are diagonal in the planar inner product

$$\langle \Lambda', M', R', \hat{\tau}' | \Lambda, M, R, \hat{\tau}\rangle = \delta_{\Lambda\Lambda'} \delta_{MM'} \delta_{RR'} \delta_{\hat{\tau}\hat{\tau}'} \frac{|H_\mu|}{n! d_R} \tag{76}$$

From equation (74) they are eigenstates of the cut-and-join operators

$$\Sigma_C |\Lambda, M, R, \hat{\tau}\rangle = \frac{\chi_R(\Sigma_C)}{d_R} |\Lambda, M, R, \hat{\tau}\rangle \tag{77}$$

The commuting Σ_C distinguish the $U(N)$ representations R by their different eigenvalues. They are linear combinations of the $U(N)$ Casimirs.

As a conseqence of equation (77) the full non-planar correlation function is also diagonal

$$N^n \langle \Lambda', M', R', \hat{\tau}' | \Omega_n | \Lambda, M, R, \hat{\tau}\rangle = \delta_{\Lambda\Lambda'} \delta_{MM'} \delta_{RR'} \delta_{\hat{\tau}\hat{\tau}'} \frac{|H_\mu|}{n! d_R^2} \mathrm{Dim} R \tag{78}$$

3.5 Trace operators for reps of general groups

In equation (50) n copies of the fundamental representation $V_{\mathbf{2}}$ of $U(2)$ were decomposed into general irreps of $U(2) \times S_n$. A precise formula for the states in these irreps was given in equation (52). Here we will give a more abstract description of these decompositions using Clebsch-Gordan coefficients. Label the fundamental fields of $U(2)$ by $W_1 = X$ and $W_2 = Y$. Define the Clebsch-Gordan coefficient for

$$V_{\mathbf{2}}^{\otimes n} \to V_\Lambda^{U(2)} \otimes V_\Lambda^{S_n} \tag{79}$$

by $C^{\vec{m}}_{\Lambda, M, a}$ so that the state in (52) is

$$|\Lambda, M, a\} = \sum_{\vec{m}} C^{\vec{m}}_{\Lambda, M, a}\, W_{m_1} \otimes W_{m_2} \otimes \cdots \otimes W_{m_n} \tag{80}$$

For a group G other than $U(K)$ the decomposition of n copies of one of its representations V_F into irreps $\Lambda \otimes \lambda$ of $G \times S_n$ is not necessarily multiplicity-free as it was in (50). An irrep $\Lambda \otimes \lambda$ generically appears with an integer multiplicity $\text{mult}(\Lambda, \lambda)$

$$V_F^{\otimes n} = \bigoplus_{\Lambda,\lambda} \text{mult}(\Lambda, \lambda) \quad V_\Lambda^G \otimes V_\lambda^{S_n} \tag{81}$$

In the states for this decomposition this multiplicity is labelled by τ

$$|\Lambda, M, \lambda, a, \tau\} = \sum_{\vec{m}} C^{\vec{m}}_{\Lambda,M,\lambda,a,\tau} \, W_{m_1} \otimes W_{m_2} \otimes \cdots \otimes W_{m_n} \tag{82}$$

M is still the state of the G irrep Λ; a is the S_n state of λ. The Clebsch-Gordan coefficient is defined to satisfy various orthogonality conditions; see [8] for more details.

This decomposition was outlined in [4] for $G = U(K_1|K_2)$ and in [8] for $G = SO(6)$, the symmetry for the six real scalars of $\mathcal{N} = 4$ SYM, and $G = SO(2,4)$, where V_F is now a single scalar with all possible onshell combinations of the four derivatives (for one derivative and $G = SL(2,\mathbb{R})$ this was done in detail). Extending these results to $PSU(2,2|4)$ where V_F is the singleton representation is tricky because of shortening conditions [36, 37], however it is done for $n = 2$ in [38] and for $n = 3$ in [31] (the higher spin YT-pletons in the latter are our λ). For the inner product use the spin bit metric from [39], see [8] for more details.

To get the trace operators with trace structure α (α is the canonical choice within its conjugacy class) we contract the S_n state a of $V_\lambda^{S_n}$ with the same coefficient $S^\alpha_{a\gamma}$ that we used before for $U(2)$ in equation (62) to get

$$\boxed{|\Lambda, M, \lambda, \tau; \alpha, \gamma\rangle = \sum_{\vec{m}} \; S^\alpha_{a\gamma} \; C^{\vec{m}}_{\Lambda,M,\lambda,a,\tau} \; \text{tr}(\alpha \, W_{m_1} \, W_{m_2} \cdots W_{m_n})} \tag{83}$$

The counting for a representation Λ and trace structure α must now take account of the multiplicity in the decomposition (81)

$$S(\alpha, \Lambda) = \sum_{\lambda \in P(n)} \text{mult}(\Lambda, \lambda) \; \frac{1}{|\text{Sym}(\alpha)|} \sum_{\rho \in \text{Sym}(\alpha)} \chi_\lambda(\rho) \tag{84}$$

The orthogonality in the planar inner product and the non-planar structure of the free theory through Ω_n follow exactly as for $U(K)$. Essentially the trace only sees how the operator is symmetrised through the S_n irrep λ.

4 Application to the chiral ring at weak coupling

In this section we present an application of the technology developed in Section 3. We construct the genuine quarter-BPS operators at weak coupling, including all their $\frac{1}{N}$ corrections. Crucial to this result is the understanding of how to enumerate the different trace structures for each $U(2)$ representation and then how to ensure orthogonality in the non-planar two-point function to the newly descendant anomalous operators at weak coupling. The extension to eighth-BPS states in $SU(3|2)$ representations should be possible using the Schur-Weyl technology for supergroups developed in [4].

4.1 Introduction

At one-loop the spectrum of anomalous dimensions and the mixing in the two-point function are connected via the one-loop dilatation operator [14, 40, 17, 41, 42]

$$\Delta = \mathrm{tr}([X,Y][\tilde{X},\tilde{Y}]) \tag{85}$$

where $\tilde{X}^i_j \equiv \frac{\partial}{\partial X^j_i}$. Δ can act on operators to give the matrix of anomalous dimensions or it can be inserted into the two-point function as an effective vertex for the one-loop correction.

If we diagonalise the action of Δ on the space of $U(2)$ multi-trace operators we find that the space splits into those operators with non-trivial anomalous dimensions and those operators with no correction to their dimension at one-loop. Because Δ always inserts a commutator $[X,Y]$ into the trace on which it acts, the non-trivial eigenstates always have commutators within traces. At one-loop these operators have in fact become descendants in long representations of other operators (i.e. they are no longer highest-weight states). For example the operator

$$\mathrm{tr}([X,Y][X,Y]) \tag{86}$$

becomes a descendant of the Konishi operator at one-loop. This discontinuous change in the spectrum from zero coupling to one-loop can also be seen from the action of the supercharge on the fermion which gains an additional term at weak coupling

$$Q\lambda \sim F + g[X,Y] \tag{87}$$

On the other hand the operators which remain BPS at weak coupling, with no correction to their dimension, are given at large N by those multitrace operators built assuming that the fields commute within each trace. In these symmetrised traces we sum over all orders of the fields within the trace. This process removes all commutators inside traces, but commutators can still cross between two different traces. These operators built from symmetrised traces are part of the *chiral ring* [43]. For $\Lambda = \begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}$ there are two such operators, $\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(\Phi^r)\,\mathrm{tr}(\Phi^s)$ and $\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(\Phi^r\Phi^s)$ where $\Phi^1 = X$, $\Phi^2 = Y$ and $\Phi_r\Phi^r = \epsilon_{rs}\Phi^r\Phi^s = [X,Y]$.

For N finite this does not completely describe the BPS operators. The BPS operators must be annihilated by the dilatation operator and be orthogonal in the full non-planar two-point function to the anomalous descendant operators (these two conditions are 'if and only if' since on general grounds operators with different dimensions are orthogonal in the two-point function of a CFT). This requires $\frac{1}{N}$ corrections to be added to the operators built from symmetrised traces [44, 18]. For example for the $\Lambda = \begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}$ case we must add the descendant operator from (86) to $\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(\Phi^r\Phi^s)$ to get the genuine BPS operator

$$\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(\Phi^r\Phi^s) + \frac{2}{N}\,\mathrm{tr}(\Phi_r\Phi^r\Phi_s\Phi^s) \tag{88}$$

The second term is suggestive of a joining accompanied by $\frac{1}{N}$.

The goal of this section is to capture these non-planar corrections in a precise analytic way, building on work in [18, 4], and connect the analysis to the cut-and-join operators constructed above. This is only possible with the technology from Section 3, because we need to know exactly how the $U(2)$ representation fits into each trace.

4.2 The dual basis

Take a representation and state $\{\Lambda, M\}$ of $U(2)$ and split the compatible trace structures $\{\alpha, \gamma\}$ into two groups (α is the canonical permutation for a particular conjugacy class of S_n and γ labels its multiplicity for this $U(2)$ representation)

- Identify $\{\alpha^d, \gamma^d\}$ exactly with those operators with commutators inside traces. These operators may be linear combinations of the original $\{\alpha, \gamma\}$ identified in the tree-level section. They coincide with the image of Δ and each of them is a descendant at one-loop. Because they all appear in the image of Δ, they have anomalous dimensions. However in this basis they may not be diagonal under Δ: the $\{\alpha^d, \gamma^d\}$ could be a linear combination of the non-trivial eigenstates of Δ.[10]

- Identify the remainder by $\{\alpha^c, \gamma^c\}$ such that they are orthogonal to all the $\{\alpha^d, \gamma^d\}$ in the *planar* inner product. These will give the leading terms for the genuine quarter-BPS operators, which are known on general grounds to be built from symmetrised traces. This is a much simpler job than insisting on orthogonality in the non-planar two-point function: if α^c has a different trace structure to each descendant in $\{\alpha^d, \gamma^d\}$ then we are done because different trace structures are automatically orthogonal in the planar inner product (67). If however for a given trace structure α some operators contain commutators within traces and some don't, then we must take care to guarantee orthogonality. An example of this type for $\Lambda = [4, 2]$ is computed in Appendix Section E.2.

Thus under the dilatation operator the $\{\alpha^d, \gamma^d\}$ furnish the image of Δ

$$\Delta|\Lambda, M; \alpha, \gamma\rangle = \sum_{\{\alpha^d, \gamma^d\}} D_{\{\alpha,\gamma\};\{\alpha^d,\gamma^d\}} |\Lambda, M; \alpha^d, \gamma^d\rangle \tag{89}$$

where D is a non-diagonal matrix of anomalous dimensions. The $\{\alpha^c, \gamma^c\}$ satisfy the condition

$$\langle \Lambda, M; \alpha^c, \gamma^c | \Lambda, M; \alpha^d, \gamma^d\rangle = 0 \qquad \forall \ c, d \tag{90}$$

We are now ready to define the genuine quarter-BPS operators with all their $\frac{1}{N}$ corrections:

$$\boxed{|\Lambda, M; \alpha^c, \gamma^c; \perp\rangle \equiv \Omega_n^{-1} \, |\Lambda, M; \alpha^c, \gamma^c\rangle} \tag{91}$$

Note that Ω_n^{-1} only exists if $N \geq n$; its first few terms are given in Appendix Section C. In previous work [46, 4] such operators have been called a *dual basis* because they are dual to the original trace basis in the two-point function. The connection to these papers is spelt out below.

We will now show that these operators satisfy two important properties:

- They are orthogonal to the descendant non-trivial eigenstates in the full non-planar two-point function at tree level and at one loop;

- They are annihilated by the dilatation operator.

[10] See Table 2 in Appendix Section E.2 for examples of eigenstates that mix trace-structures for $\Lambda = [4, 2]$.

That the second property follows from the first on general CFT grounds is the basis of the analysis in [18, 4]. The exact description of the non-planar corrections in Section 3 for each trace structure allows us to give precise analytic formulae here.

The orthogonality in the tree-level two-point function follows by definition

$$\begin{aligned}\left\langle \mathcal{O}^\dagger[\Lambda, M; \alpha^c, \gamma^c; \perp]\, \mathcal{O}[\Lambda, M; \alpha^d, \gamma^d]\right\rangle_{\text{tree}} &= \langle \Lambda, M; \alpha^c, \gamma^c; \perp |\Omega_n| \Lambda, M; \alpha^d, \gamma^d\rangle \\ &= \langle \Lambda, M; \alpha^c, \gamma^c | \Omega_n^{-1}\Omega_n | \Lambda, M; \alpha^d, \gamma^d\rangle \\ &= \langle \Lambda, M; \alpha^c, \gamma^c | \Lambda, M; \alpha^d, \gamma^d\rangle \\ &= 0 \end{aligned} \tag{92}$$

Orthogonality in the two-point function at one-loop follows automatically because if we insert Δ above only descendants appear in its image so the same logic follows.

The annihilation of the quarter-BPS operators by Δ follows in a similar way by considering the one-loop two-point function

$$\left\langle \mathcal{O}^\dagger[\Lambda, M; \alpha, \gamma]\, \mathcal{O}[\Lambda, M; \alpha^c, \gamma^c; \perp]\right\rangle_{\text{1-loop}} = \langle \Lambda, M; \alpha, \gamma | \Omega_n \Delta | \Lambda, M; \alpha^c, \gamma^c; \perp\rangle \tag{93}$$

Since Ω_n is non-singular and the bra $\langle \cdots |$ contains everything that could possibly be in the image of $\Omega_n\Delta$, if this object vanishes for all trace structures $\{\alpha, \gamma\}$ then it will follow that

$$\Delta | \Lambda, M; \alpha^c, \gamma^c, \perp\rangle = 0 \tag{94}$$

By the symmetry of the two-point function, (93) equals its conjugate

$$\begin{aligned}\langle \Lambda, M; \alpha^c, \gamma^c; \perp |\Omega_n \Delta | \Lambda, M; \alpha, \gamma\rangle &= \sum_{\{\alpha^d, \gamma^d\}} \langle \Lambda, M; \alpha^c, \gamma^c | \Omega_n^{-1}\, \Omega_n\, D_{\{\alpha,\gamma\};\{\alpha^d,\gamma^d\}}\, | \Lambda, M; \alpha^d, \gamma^d\rangle \\ &= \sum_{\{\alpha^d, \gamma^d\}} D_{\{\alpha,\gamma\};\{\alpha^d,\gamma^d\}}\, \langle \Lambda, M; \alpha^c, \gamma^c | \Lambda, M; \alpha^d, \gamma^d\rangle \\ &= 0 \end{aligned} \tag{95}$$

Thus the $|\Lambda, M; \alpha^c, \gamma^c, \perp\rangle$ are indeed annihilated by Δ and hence have a protected scaling dimension, at least at one-loop. It is conjectured [45] that these operators remain protected for higher loops too, since the higher order dilatation operators all contain the common element $[\tilde{X}, \tilde{Y}]$, which annihilates the operators constructed here.

4.3 Further analysis of dual basis using other 1-loop formulations

In this section we sketch a complementary analysis of why the the dual basis for the $\{\alpha^c, \gamma^c\}$ is annihilated by the dilatation operator $\Delta = \text{tr}([X, Y][\tilde{X}, \tilde{Y}])$ using two alternative formulations of the action of Δ.

Δ acts on two sites at a time, cycling through all possible pairs ($1 \leq i, j \leq n$) by the product rule. When it acts on the global symmetry group part it projects onto the antisymmetric $\begin{smallmatrix}\square\\\square\end{smallmatrix} \sim [X, Y]$ combination, splitting this off from the $U(2)$ representation $\{\Lambda, M\}$ (this separation is expressed clearly in [8]).

It also alters the trace structure. There are two equivalent ways of writing the action on the trace permutation α:

- The first way was first demonstrated in [21] using the spin bit formalism. The action of Δ is summarised by the left action on α by transpositions $(i\,\alpha(j))$

$$\Delta \sim \sum_{i\neq j} P_{ij} \quad (i\,\alpha(j))\,\alpha \tag{96}$$

  P_{ij} is the projector onto $[X,Y]$ acting on the fields. In some ways this is similar to the action of $\Sigma_{[2]}$, except that P_{ij} is only non-vanishing when a commutator is present and the transposition $(i\,\alpha(j))$ requires knowledge of α beforehand. When $\alpha(j)=i$ then $(i\,\alpha(j))\equiv N$; in this case the two sites of the commutator are adjacent in the trace.

- To avoid the dependence of Δ on α and derive explicit expressions for non-planar one-loop mixing matrices this action can also be written by introducing an extra $(n{+}1)$th permutation index and both pre- and post-multiplying α [6]

$$\Delta \sim \sum_{i\neq j} P_{ij} \quad (i\,n{+}1)\,\alpha\,(j\,n{+}1) \tag{97}$$

  For the finite N bases this gives the one-loop two-point function as an expansion of dimensions of $U(N)$ representations with $n{+}1$ boxes. Operators with n-box $U(N)$ representations only mix if they are related by moving a single box in the Young diagram.

In Appendix Section F we show the equivalence of these two formulations; in the remainder of this section we use the first.

We want to show that the dilatation operator of the form

$$\Delta = \sum_{\{i,j\}\sim[X,Y]} (i\,\alpha(j)) \tag{98}$$

annihilates the quarter-BPS operators defined in (91). The latter have a trace structure

$$\Omega_n^{-1}\alpha = \left[1 - \frac{1}{N}\Sigma_{[2]} + \mathcal{O}\left(\frac{1}{N^2}\right)\right]\,\alpha \tag{99}$$

where in α the sites of each commutator i and j are always in different cycles (i.e. the operator's leading term is the product of symmetrised traces). More terms of Ω_n^{-1} are given in Section C.

For the leading term, α, the dilatation operator Δ will join the two cycles which contain i and j so that they are in the same cycle (and hence describe a descendant operator).

When $\Sigma_{[2]}$ joins traces in α we get operators which are descendant (i and j in same cycle). Under Δ this gets a factor N, cancelling the $\frac{1}{N}$ and then cancelling against the descendant from the leading term $\Delta\,\alpha$.

If $\Sigma_{[2]}$ splits traces in α then we still get something that is in the chiral ring. Acting with Δ we get a descendant at order $\frac{1}{N}$ which must be cancelled by a descendant from $\frac{1}{N^2}\left(\Sigma_{[3]}+\Sigma_{[2,2]}\right)\alpha$ that gains a factor of N from Δ.

This cancellation continues at each order in $\frac{1}{N}$ until there is nothing left. Thus the quarter-BPS operator is indeed in the kernel of Δ.

4.4 Dual basis for $N < n$ and $SU(N)$

At finite N, in particular when N is less than the length of the operator $N < n$, we cannot define Ω_n^{-1}. Fortunately by expanding Ω_n^{-1} in terms of $U(N)$ dimensions using (134) we can define the dual basis by truncating the sum over R in (134) to those Young diagrams with N or fewer rows

$$|\Lambda, M; \alpha^c, \gamma^c; \perp\rangle = \frac{N^n}{(n!)^2} \sum_{\sigma \in S_n} \sum_{R \in P(n,N)} \frac{d_R^2}{\dim R} \chi_R(\sigma)\, \sigma\, |\Lambda, M; \alpha^c, \gamma^c\rangle$$

This continues to be dual to the trace basis in the non-planar two-point function for $N < n$. It is the generalisation to arbitrary representation Λ of the dual basis constructed in [46] for the half-BPS operators. If we remove the restriction on $R \in P(n, N)$ that there are no more than N rows in the Young diagram then this expression is identical to (91).

To extend the trace basis to $SU(N)$, where the traces of the single adjoint fields vanish, just drop the α with 1-cycles in them. It turns out that the dual basis as defined in (91) for such α is also dual to the trace basis in the $SU(N)$ two-point function [46], so these results for the quarter-BPS operators should follow through for the more physical $SU(N)$ gauge group.

5 Counting the chiral ring using the Weyl group S_N

In this section we characterise the chiral ring of 4d $\mathcal{N} = 4$ super Yang-Mills at finite N in terms of functions of the eigenvalues of the matrix fields and representations of the subgroup S_N of the gauge group $U(N)$. The number of these operators matches the finite N partition functions computed in Dolan [33] and furthermore provides a counting of chiral ring operators for each representation of the global symmetry group G. G is always a subgroup of $SU(3|2)$ for the chiral ring of $\mathcal{N} = 4$ SYM, corresponding to $\frac{1}{8}$th-BPS operators, but because these methods are applicable to any eigenvalue system we leave the group general.

5.1 Invariant functions of eigenvalues

In previous sections we considered gauge-invariant operators built out of generic matrices transforming in the adjoint of $U(N)$. Here we consider the chiral ring, a subset of operators built out of commuting matrices. These are functions only of the eigenvalues, since the matrices are simultaneously diagonalisable. These symmetric functions of eigenvalues are organised by irreps of the S_N which permutes the eigenvalues and the S_n which permutes tensor products of fundamental fields.

In Section 3.5 we organised tensor products of the fundamental fields $V_F^{\otimes n}$ for a global symmetry group G into representations $\Lambda \times \lambda$ of $G \times S_n$

$$|\Lambda, M, \lambda, a, \tau\} = \sum_{\vec{m}} C^{\vec{m}}_{\Lambda, M, \lambda, a, \tau}\, W_{m_1} \otimes W_{m_2} \otimes \cdots \otimes W_{m_n} \tag{100}$$

M is the state of the representation Λ of G, a is the state of λ of S_n and τ labels the multiplicity with which $\Lambda \times \lambda$ appear in $V_F^{\otimes n}$.

Now consider the eigenvalues of these fundamental fields w_m^e where $e \in \{1, 2, \ldots N\}$. The subgroup of the gauge group $U(N)$ which acts on these eigenvalues is S_N, the symmetric group

which permutes the eigenvalues. The eigenvalues are in the *natural representation* $V_{\rm nat}^{S_N}$ of S_N, the N-dimensional representation where S_N acts by just permuting the elements.[11] $V_{\rm nat}^{S_N}$ is reducible

$$V_{\rm nat}^{S_N} = V_{[N]}^{S_N} \oplus V_{[N-1,1]}^{S_N} \tag{101}$$

The trivial representation $V_{[N]}^{S_N}$ is the sum of the eigenvalues, which is invariant under S_N; $V_{[N-1,1]}^{S_N}$, referred to as the 'standard' representation in [47] and the 'hook' representation in [8], represents the $N-1$ differences of the eigenvalues.[12] To change the gauge group from $U(N)$ to $SU(N)$, for which S_N is the Weyl group, just substitute $V_{\rm nat}^{S_N}$ with $V_{[N-1,1]}^{S_N}$ in any of the following discussion.

We can use Schur-Weyl duality on the n-tensor product of the natural representation of S_N to decompose it into representations $K \times \kappa$ of $S_N \times S_n$

$$\left(V_{\rm nat}^{S_N}\right)^{\otimes n} = \bigoplus_{K\vdash N, \kappa \vdash n} {\rm mult}(K,\kappa)\ \ V_K^{S_N} \otimes V_\kappa^{S_n} \tag{102}$$

$K \vdash N$, which is equivalent to $K \in P(N)$ in our previous notation, means that K is a partition of N. The multiplicity ${\rm mult}(K,\kappa)$ for the appearance of $K \times \kappa$ is labelled with $\check{\tau}$ in the Clebsch-Gordan coefficent $C^{\vec{e}}_{K,M_K,\kappa,a_\kappa,\check{\tau}}$ for the decomposition (102). Projecting onto $K \times \kappa$ in $(V_{\rm nat}^{S_N})^{\otimes n}$ gives a formula for ${\rm mult}(K,\kappa)$, see equation (160) in Appendix Section G.[13]

We can thus map the space of eigenvalues $(V_F^G \otimes V_{\rm nat}^{S_N})^{\otimes n}$ to the linear combinations

$$C^{\vec{m}}_{\Lambda,M,\lambda,a,\tau}\ \ C^{\vec{e}}_{K,M_K,\kappa,a_\kappa,\check{\tau}}\ \ w_{m_1}^{e_1}\ w_{m_2}^{e_2}\ \cdots\ w_{m_n}^{e_n} \tag{103}$$

For the operators of the chiral ring, we know that they are invariant under the S_N that permutes the eigenvalues (this is the remnant of the $U(N)$ gauge invariance that survives for the eigenvalues). This means K is the trivial representation of S_N, $[N]$. Furthermore the final operators should be an overall S_n invariant too, because the eigenvalues are commuting bosons. This forces $\lambda = \kappa$ and requires us to sum over the S_n states $a = a_\kappa$. Thus we get the chiral ring as functions of eigenvalues

$$\boxed{|\Lambda, M, \lambda, \tau, \check{\tau}\rangle = \sum_a C^{\vec{m}}_{\Lambda,M,\lambda,a,\tau}\ \ C^{\vec{e}}_{[N],\lambda,a,\check{\tau}}\ \ w^{\vec{e}}_{\vec{m}}} \tag{104}$$

The physics of eighth-BPS states and their partition functions from both the field theory and the supergravity point of view (where they are product of the half-BPS supergravity multiplet) were studied in [50]. The quarter and eighth-BPS gauge invariant operators should be related to giant gravitons generalizing the analogous connection in the half-BPS case. It has been argued that the physics of the eighth-BPS giants [51] is given by the dynamics of N particles in a 3D simple harmonic oscillator [52, 53, 54, 55]. The states built from these commuting bosons can be counted combinatorially in terms of vector partitions [56]. Giant gravitons with strings attached were considered in [57, 35]. In the supergravity approximation generalisations of the LLM solution [29] were investigated in [58, 59, 60].

[11] This representation of S_N is also known as the permutation representation.

[12] This split between differences and the sum was used in [8] to distinguish highest weight states from descendants in the tensor products of $SL(2)$ spin $-\frac{1}{2}$ representations.

[13] The multiplicity-free Schur-Weyl dual of S_N is the maximal algebra in the space of endomorphisms of $(V_{\rm nat}^{S_N})^{\otimes n}$ that commutes with S_N: it is known as the partition algebra $P_n(N)$ [48, 49]. The symmetric group algebra is a subalgebra of $P_n(N)$ via the Brauer algebra $B_n(N)$ (which is the Schur-Weyl dual of $O(N)$), $\mathbb{C}S_n \subset B_n(N) \subset P_n(N)$, which mirrors the fact that $U(N) \supset O(N) \supset S_N$. As the group gets smaller, the commuting algebra grows.

5.2 Counting at finite N

Counting the multiplicities of the operators we have constructed (104) for a given G rep Λ gives

$$\#\text{ of operators for }\Lambda \;=\; \sum_{\lambda(S_n)} \text{mult}(V_F^{\otimes n} \to \Lambda\otimes\lambda)\;\;\text{mult}((V_{\text{nat}}^{S_N})^{\otimes n} \to [N]\otimes\lambda) \tag{105}$$

Summing over all representations with their characters $\chi_\Lambda(\mathbf{x})$ gives the partition function for the chiral ring

$$Z^{cr}_{U(N)}(\mathbf{x}) = \sum_n \sum_{\Lambda(G)} \sum_{\lambda(S_n)} \text{mult}(V_F^{\otimes n} \to \Lambda\otimes\lambda)\;\;\text{mult}((V_{\text{nat}}^{S_N})^{\otimes n} \to [N]\otimes\lambda)\;\;\chi_\Lambda(\mathbf{x}) \tag{106}$$

For global symmetry group $G = U(2)$ the Young diagram for the S_n representation λ is the same as that for the $U(2)$ representation Λ, so $\Lambda = \lambda$ and $\text{mult}(V_{\mathbf{2}}^{\otimes n} \to \Lambda\otimes\lambda) = 1$. This gives

$$Z^{cr}_{U(N)}(x,y) = \sum_n \sum_{\Lambda(U(2),S_n)} \text{mult}((V_{\text{nat}}^{S_N})^{\otimes n} \to [N]\otimes\Lambda)\;\;\chi_\Lambda(x,y) \tag{107}$$

In the remainder of this section we verify the counting in the partition function (107) by comparing it to known formulae for the $U(2)$ sector, i.e. the genuine quarter-BPS operators. The generating function for $Z^{cr}_{U(N)}(x,y)$ is given by [50][61][62]

$$Z^{cr}(\nu,x,y) = \prod_{n,m=0}^{\infty} \frac{1}{1-\nu x^n y^m} = \sum_{N=0}^{\infty} \nu^N Z^{cr}_{U(N)}(x,y) \tag{108}$$

In [33] Dolan showed that

$$Z^{cr}_{U(N)}(x,y) = \sum_{K\vdash N} s_K(1,x,x^2,\dots)s_K(1,y,y^2,\dots) \tag{109}$$

where the sum is over partitions K of N. $s_K(1,q,q^2,\dots)$ is the Schur polynomial defined for the partition K by

$$s_K(1,q,q^2,\dots) = \frac{1}{N!}\sum_{\sigma\in S_N} \chi_K(\sigma)\,\text{tr}\left(\sigma\, Q^{\otimes N}\right) \tag{110}$$

Q is the infinite matrix with $(1,q,q^2,q^3,\dots)$ on its diagonal.

Note that our formula (107) is a refinement of (109) in that it can count the chiral ring operators for each representation of $U(2)$ rather than as a total sum. A proof that (107) and (109) are the same can be found in Appendix Section G.

5.3 Check of counting for half-BPS operators

In the half-BPS case the global symmetry group representation is symmetrised $\Lambda = \lambda = [n]$. The counting of the chiral ring gives

$$\text{mult}((V_{\text{nat}}^{S_N})^{\otimes n} \to [N]\otimes[n]) = p(n,N) \tag{111}$$

$p(n,N)$ is the number of partitions of n into at most N parts. This counts the Schur polynomials $|R\rangle$ discussed in Section 2.7 where R, as a representation of $U(N)$, is a Young diagram with n boxes and at most N rows.

The combinatorics of this counting can be connected directly with the half-BPS supergravity geometries constructed by Lin, Lunin and Maldacena [29] (LLM) in the bulk. In [1, 63] it was shown that the half-BPS sector may be reduced to a complex matrix model. This in turn can be reduced to a system of the N eigenvalues in a harmonic ocillator. The eigenvalues become fermionic due to the change in the path integral measure; their excitation levels above the ground state then map to a partition into N parts, corresponding to the Young diagrams R for the Schur polynomials. The fermions can be represented as a Fermi droplet in phase space, where a filled circle is the ground state and disturbances of this are excitations. The Young diagram $[1^N]$ gives each eigenvalue one excitation, leaving a hole in the filled circle Fermi droplet (this is the giant graviton expanded in the S^5). $[N]$ gives only the top eigenvalue a large excitation, leaving a small blob separated from the filled Fermi droplet of the ground state (the giant graviton in the AdS_5).

Approaching from the supergravity side, Lin, Lunin and Maldacena [29] (LLM) searched for all the half-BPS geometries with $SO(4) \times SO(4) \times \mathbb{R}$ symmetry which are asymptotically $AdS_5 \times S^5$. They found smooth solutions determined by a bi-coloured plane, which correspond exactly to the Fermi droplets of the gauge theory matrix model. Geometries with extremely large R charge are similar to incipient black hole states and can be studied as such [64].

5.4 A combinatorial description of the quarter-BPS operators

Given the connection between the LLM supergravity solutions and the combinatorics of how half-BPS operators are counted for given global charges, it is worth pursuing potential descriptions of the quarter-BPS counting that might connect with quarter-BPS supergravity solutions. Investigations in the supergravity approximation for quarter- and eighth-BPS geometries have been carried out in [58, 59, 60]. Here we give a partial description of the combinatorics from the gauge theory point of view. The key result we show is that

The number of quarter-BPS operators for the $U(2)$ representation Λ is the number of times the trivial S_N representation $[N]$ appears when you decompose the representation of $U(N)$ with the same Young diagram Λ into irreps of its subgroup $S_N \subset U(N)$.

(112)

From (107), for a given representation Λ of $U(2)$ the number of quarter-BPS operators is

$$\mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes n} \to [N] \otimes \Lambda) \tag{113}$$

This multiplicity can also be expressed in another way. Starting from (102) we could have treated $V_{\mathrm{nat}}^{S_N}$ as the N-dimensional fundamental of $U(N)$, $V_{\mathbf{N}}^{U(N)}$, since they are both of the same size. If we had decomposed n copies of this representation using $U(N)$ and S_n instead of S_N and S_n we would have had a multiplicity-free decomposition

$$\left(V_{\mathbf{N}}^{U(N)}\right)^{\otimes n} = \bigoplus_{\kappa \vdash n} V_{\kappa}^{U(N)} \otimes V_{\kappa}^{S_n} \tag{114}$$

Now identifying this with (102) we conclude that $\mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes n} \to K \otimes \kappa)$ is the number of times $V_K^{S_N}$ appears when we decompose $V_{\kappa}^{U(N)}$ into irreps of the subgroup $S_N \subset U(N)$

$$V_{\kappa}^{U(N)} = \bigoplus_{K \vdash N} \mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes n} \to K \otimes \kappa) \;\; V_K^{S_N} \tag{115}$$

Thus the number of quarter-BPS operators for the $U(2)$ representation $\kappa = \Lambda$ is the number of times the S_N representation $[N]$ appears when you decompose the representation $\kappa = \Lambda$ of $U(N)$ into irreps of its subgroup S_N. In the remainder of this section we will try to describe this number better and we will only partially succeed.

The states of $V_\kappa^{U(N)}$ consist of semi-standard tableaux where we fill the n boxes of the Young diagram κ with numbers in $\{1, \dots N\}$ such that they are weakly increasing along the rows and strongly increasing down the columns. Label the 'field content' of each semi-standard tableau by $\vec{\mu}$ so that we have μ_1 1's, μ_2 2's, ... and μ_N N's. $\sum_{i=1}^N \mu_i = n$ so that $\vec{\mu}$ is an ordered partition of n into at most N parts.

Under permutations in S_N the N fields transform as the natural representation of S_N. The overall (unordered) partition type of each field content is invariant under S_N so that for example for $S_4 \subset U(4)$ the field content $\vec{\mu} = (2, 1, 0, 0)$ can be transformed into $(1, 0, 0, 2)$ but not into $(3, 0, 0, 0)$:

$$(142) \in S_4 : \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline 4 & 4 \\ \hline 1 \\ \cline{1-1}\end{array} = - \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} \tag{116}$$

This means that we can partially reduce $V_\kappa^{U(N)}$ into (reducible) representations $V_{\kappa,[\vec{\mu}]}^{S_N}$ of S_N corresponding to the equivalence classes of $\vec{\mu}$.

For a given $[\vec{\mu}]$ there are a certain number of fields which don't appear in $\vec{\mu}$. Label this number of 0's in $\vec{\mu}$ by m_0. Since the semi-standard tableaux are left invariant by permutations of these fields, $V_{[\vec{\mu}]}^{S_N}$ must take the form of an outer product[14] with the trivial representation $[m_0]$ of S_{m_0}

$$V_{\kappa,[\vec{\mu}]}^{S_N} = V_{\kappa,[\vec{\mu}]'}^{S_{N-m_0}} \circ V_{[m_0]}^{S_{m_0}} \tag{117}$$

where $[\vec{\mu}]'$ has the m_0 0's of $[\vec{\mu}]$ removed. To summarise the result so far

$$V_\kappa^{U(N)} = \sum_{[\vec{\mu}]} V_{\kappa,[\vec{\mu}]'}^{S_{N-m_0}} \circ V_{[m_0]}^{S_{m_0}} \tag{118}$$

Decomposing $V_{\kappa,[\vec{\mu}]'}^{S_{N-m_0}}$ is an open problem. When $N - m_0 = n$ then $[\vec{\mu}]' = [1^n]$ so that there are n different fields appearing only once each in the semi-standard tableau. Under these conditions the semi-standard tableau becomes standard (where the numbers along each row must strictly increase). This means that $V_{\kappa,[1^n]}^{S_n} = V_\kappa^{S_n}$ since the standard tableaux label the states of κ of S_n.

As an example consider the decomposition of $\kappa = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$ of $U(N)$ into S_N representations

$$\begin{aligned} V_{\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}}^{U(N)} &= V_{\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array},[\vec{\mu}]'=[1,1,1]}^{S_3} \circ V_{[N-3]}^{S_{N-3}} + V_{\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array},[\vec{\mu}]'=[2,1]}^{S_2} \circ V_{[N-2]}^{S_{N-2}} \\ &= \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} \circ [N-3] \; + \; \left(\begin{array}{|c|c|}\hline & \\ \hline \end{array} \; + \; \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \right) \circ [N-2] \\ &= [N-3,2,1] + 2[N-2,1,1] + 2[N-2,2] + 3[N-1,1] + [N] \end{aligned} \tag{119}$$

The single appearance of the S_N trivial representation $[N]$ corresponds to the chiral ring $U(2)$ operator $\mathrm{tr}(\Phi_r)\,\mathrm{tr}(\Phi^r X)$. This result degrades appropriately for $n > N$, so for example for $N = 2$

$$V_{\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}}^{U(2)} = \begin{array}{|c|c|}\hline & \\ \hline \end{array} \; + \; \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \tag{120}$$

[14] The symmetric group outer product denoted $\circ$ is the same as the tensor product for unitary group Young diagrams described by the Littlewood-Richardson coefficients.

More examples of this type are available [65].

In the simplest case when the $U(N)$ representation is totally symmetric $\kappa = [n]$ (corresponding as a $U(2)$ representation to the half-BPS operators) we know the full answer: $V^{S_{N-m_0}}_{[n],[\vec{\mu}]'} = [m_1] \circ [m_2] \circ \cdots \circ [m_n]$ where m_i is the number of i's in $\vec{\mu}$ ($\sum_{i=0}^{n} m_i = N$, $\sum_{i=0}^{n} i m_i = n$). Thus

$$V^{U(N)}_{[n]} = \sum_{[\vec{\mu}]} [m_0] \circ [m_1] \circ \cdots \circ [m_n] \tag{121}$$

The sum on the right is over symmetric group outer products that give representations of S_N. The trivial representation $[N]$ appears once in this outer product for each $[\vec{\mu}]$ so that it appears in total $p(n, N)$ times, as known for the half-BPS case.

Finding such a graphical description of the decomposition of more general representations of $U(N)$ into irreps of S_N is an outstanding problem that is also of interest to mathematicians [66]. Solving it would give us a good combinatorial description of the multiplicity of quarter-BPS operators for each $U(2)$ representation Λ.

6 Conclusion

The purpose of studying the combinatorics of the weakly-coupled gauge theory is to understand better the putative weak-weak dual worldsheet theory. It is hoped that such a theory for the small radius $AdS_5 \times S^5$ background geometry is related by a weak-strong duality to the standard large radius closed string theory. Such relationships are known for sigma models of compact symmetric superspaces [67, 68].

In Section 2 we reviewed the non-planar expansion of extremal correlation functions for half-BPS operators. The expansion can be captured by the class algebra of the symmetric group, which introduces a factor of $\frac{1}{N}$ each time a trace (i.e. a permutation cycle) is cut or joined. The combinatorics of the non-planar corrections can be modelled by a two-dimensional theory on higher genus surfaces with rings of spins connected by propagators that do not cross [20, 14]. As previously shown in [14, 15, 13], in the BMN limit of operators with many fields the correlation functions factorise into planar three-point functions.

The half-BPS sector is especially simple because the fields in each operator are symmetrised. In Section 3 we showed how representations of the global symmetry group are compatible with specific trace structures. This is important because we expect the spectrum of the dual string theory to be organised into representations of $PSU(2,2|4)$, while the trace structure should roughly correspond to the multi-string Hilbert space. We also showed that much of the behaviour of the half-BPS sector is universal for all operators: in the free theory the cut-and-join operators capture the non-planar expansion and correlation functions of general states factorise when the number of fields is large.

The weakly-coupled theory is discontinuously different to the zero-coupling theory. To find the correct state space our best guide is to look for operators with well-defined conformal dimensions, i.e. eigenstates of the dilatation operator. After the half-BPS operators, the next simplest sector is the $U(2)$ operators. Using technology from the free theory we showed in Section 4 how to construct the genuine quarter-BPS operators at weak coupling, including all their $\frac{1}{N}$ corrections, by using their orthogonality in the non-planar two-point function to the newly descendant anomalous operators. This fits the general expectation that the $\frac{1}{N}$ expansion is not physical at non-zero

coupling (the correct stringy expansion is in $\frac{\lambda}{N}$) but $\frac{1}{N}$ corrections are important in identifying the right mapping between SYM operators and states in the bulk. In addition to constructing the quarter-BPS operators we also have a novel combinatorial way to count them in Section 5, which may be useful in finding quarter-BPS geometries analogous to the half-BPS solutions constructed by LLM [29].

Understanding precisely how closed strings arise from $\mathcal{N} = 4$ super Yang-Mills is an on-going and challenging problem. The worldsheet theory must somehow capture the finite discrete combinatorics of the gauge theory. In the absence of a final model for the dual of the weakly-coupled gauge theory, we hope analysis of the field theory in this limit will shed some light on this problem.

6.1 Further directions

- Following the rudimentary comments in Section 2.4 a priority is to understand the dual discrete worldsheet theory better. The conjecture (26) suggests connections between how homotopic propagators bunch in Gopakumar's model and the symmetric group conjugacy classes. The universality of the factorisation suggests that the dual of the free theory is somewhat insensitive to the sector of the global symmetry group.

- Part of the purpose of refocusing the finite N technology on traces was to find the non-planar eigenstates of the dilatation operator. We have done this for the simplest case by finding the $\frac{1}{N}$ corrections to the quarter-BPS operators; further results for anomalous operators are desirable. The $\frac{1}{N}$ corrections to the anomalous eigenstates are not given by the conjugacy-class-invariant cut-and-join operators, see the discussion at the end of Appendix E.2. However the dilatation operator (98) is not a conjugacy class invariant of S_n either, so perhaps there is scope for a description of the anomalous eigenstates using cut-and-join operators that vary over each conjugacy class (while inheriting its N-dependence). Extending the three-string vertex (70) to one-loop eigenstates would be exciting.

- There are several important features of $\mathcal{N} = 4$ SYM that this formalism doesn't incorporate. As explained in Section 3.5 we can do the Schur-Weyl duality for global groups $SO(2,4)$, $SO(6)$ and $U(K_1|K_2)$ but the full $PSU(2,2|4)$ representations involve subtle shortening conditions. Higher loop corrections also change the lengths of operators in the $SU(3|2)$ sector [69], of which we haven't taken any account.

- In this paper we have focused on extremal correlation functions. The full non-planar expansion of non-extremal three-point functions in the free theory was computed in [70]. The structure was very similar so we expect many of the results here to follow through.

- We have only solved the chiral ring counting problem as formulated in (112) for special cases. Finding a general graphical way of reducing representations of $U(N)$ to representations of its subgroup S_N is an important mathematical problem in itself.

- In this approach the underlying fields are replaced by gauge-invariant trace operators and interactions are replaced by the cutting and joining of traces. Connections with the collective field theory approach to $\mathcal{N} = 4$ SYM [71] could be explored further.

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A Genus ≥ 2

A.1 Genus 2

The degree 4 cut-and-join operator is

$$\frac{1}{N^4}\left(\Sigma_{[5]}+\Sigma_{[4,2]}+\Sigma_{[3,3]}+\Sigma_{[3,2,2]}+\Sigma_{[2,2,2,2]}\right) \tag{122}$$

It gives the genus 2 contribution to the 2-point function (cf. equation (2.6) of [20])

$$\left\langle \mathrm{tr}(X^{\dagger n})\,\mathrm{tr}(X^n)\right\rangle_{g=2} = nN^{n-4}\left[8\binom{n}{5}+24\binom{n}{6}+12\binom{n}{6}+49\binom{n}{7}+21\binom{n}{8}\right] \tag{123}$$

corresponding to each splitting operator. The final factor 21 comes from (128). For n large

$$\left\langle \mathrm{tr}(X^{\dagger n})\,\mathrm{tr}(X^n)\right\rangle_{g=2} \to \frac{1}{4!}\langle n|\Sigma_{[2]}^4|n\rangle \tag{124}$$

To see how the factorisation works, insert schematic complete sets of states (see equation (20) for the exact form of the image of $\Sigma_{[2]}$ on $|n_1,n_2\rangle$)

$$\langle n|\Sigma_{[2]}\frac{|n_1,n_2\rangle\langle n_1,n_2|}{\langle n_1,n_2|n_1,n_2\rangle}\Sigma_{[2]}\left(\frac{|n\rangle\langle n|}{\langle n|n\rangle}+\frac{|n_1,n_2,n_3\rangle\langle n_1,n_2,n_3|}{\langle n_1,n_2,n_3|n_1,n_2,n_3\rangle}\right)\Sigma_{[2]}\frac{|n_1,n_2\rangle\langle n_1,n_2|}{\langle n_1,n_2|n_1,n_2\rangle}\Sigma_{[2]}|n\rangle \tag{125}$$

This encodes $1\to 2\to 1\to 2\to 1$ and $1\to 2\to 3\to 2\to 1$.

A.2 General genus

We generally expect the form for the degree k cut-and-join operator

$$\frac{1}{N^k}\left(\Sigma_{[k+1]}+\Sigma_{[k,2]}+\cdots+\Sigma_{[3,2^{k-2}]}+\Sigma_{[2^k]}\right) \tag{126}$$

and for the 2-point function for $k=2g$

$$nN^{n-k}\left[c_{[k+1]}\binom{n}{k+1}+c_{[k,2]}\binom{n}{k+2}+\cdots+c_{[3,2^{k-2}]}\binom{n}{2k-1}+c_{[2^k]}\binom{n}{2k}\right] \tag{127}$$

where $c_{[2^k]}$ is given by

$$c_{[2^k]}=\frac{(4g)!}{2^{2g}(2g+1)!}=\frac{1\cdot 3\cdot 5\cdots(4g-1)}{2g+1} \tag{128}$$

as described in [20, 14]. In the large n limit

$$\frac{(4g)!}{2^{2g}(2g+1)!}\binom{n}{2g}\sim\frac{(4g)!}{2^{2g}(2g+1)!}\frac{n^{4g}}{(4g)!}=\frac{n^{4g}}{2^{2g}(2g+1)!} \tag{129}$$

Summing over g with N^{-2g} we get $\frac{2\sinh(g_2/2)}{g_2}$ with the BMN non-planar coupling $g_2=\frac{n^2}{N}$ [20, 14].

B Extremal k-point functions

For the extremal four-point function the action of the vertex (25) on $|n\rangle$ in more detail is

$$\Sigma_{[3]}|n\rangle = \binom{n}{3}|n\rangle + \sum_{[n_1,n_2,n_3]} \frac{2nn_1n_2n_3}{|\mathrm{Sym}([n_1,n_2,n_3])|}|n_1,n_2,n_3\rangle$$
$$\Sigma_{[2,2]}|n\rangle = \binom{n}{4}|n\rangle + \sum_{[n_1,n_2,n_3]} \frac{(n^2-3n)n_1n_2n_3}{|\mathrm{Sym}([n_1,n_2,n_3])|}|n_1,n_2,n_3\rangle \tag{130}$$

Once again the $\Sigma_{[2,2]}$ term dominates over $\Sigma_{[3]}$ as a function of n, so the factorisation into two planar three-point functions for large n applies here too.

Applying the techniques used in Footnote 6 for $\Sigma_{[2]}$ it can be shown that when an element of $\Sigma_{[3]}$ splits a single trace into three the propagators are bunched into three groups like the Y diagram from [27]. However, when an element of $\Sigma_{[2,2]}$ is applied the trace is split into four and then two are rejoined out of order, just like the lollipop diagram of [27].

Adding the two terms together, the leading planar term of the extremal four-point function is

$$\frac{1}{N^2}\langle n_1,n_2,n_3|\left(\Sigma_{[3]}+\Sigma_{[2,2]}\right)|n\rangle = \frac{n(n-1)n_1n_2n_3}{N^2} \tag{131}$$

The general formula for the leading term of the extremal k-point function is

$$\frac{1}{N^n}\left\langle \mathrm{tr}(X^{\dagger n})\,\mathrm{tr}(X^{n_1})\cdots\mathrm{tr}(X^{n_k})\right\rangle_{\text{leading}} = \frac{n!}{(n-k+1)!N^{k-2}}\prod_{i=1}^{k} n_i \tag{132}$$

C Properties of Ω_n

Ω_n appeared in studies of 2d Yang-Mills [22, 23] when expanding unitary group dimensions in $\frac{1}{N}$. The relation to $U(N)$ dimensions is given by

$$(\dim R)^m = \left(\frac{N^n d_R}{n!}\right)^m \frac{\chi_R(\Omega_n^m)}{d_R} \tag{133}$$

This equation can be reversed to give the expansion of Ω_n^m in terms of dimensions

$$\Omega_n^m = \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{R\in P(n)} d_R\left(\frac{n!\dim R}{N^n d_R}\right)^m \chi_R(\sigma)\,\sigma \tag{134}$$

Ω_n^{-1} only exists if $N \geq n$. Its expansion is an infinite series in $\frac{1}{N}$

$$\Omega_n^{-1} = 1-\frac{1}{N}\Sigma_{[2]}+\frac{1}{N^2}\left[\binom{n}{2}+2\Sigma_{[3]}+\Sigma_{[2,2]}\right]-\frac{1}{N^3}\left[\frac{n^2+3n-8}{2}\Sigma_{[2]}+5\Sigma_{[4]}+2\Sigma_{[3,2]}+\Sigma_{[2,2,2]}\right]\cdots$$

To calculate the contribution for each Σ_C we can use (134) to get a finite series. For example

$$\Omega_2^{-1} = \frac{N^2}{N^2-1}\left(1-\frac{1}{N}\Sigma_{[2]}\right)$$
$$\Omega_4^{-1} = \frac{N^6}{(N^2-1)(N^2-4)(N^2-9)}\left(1-\frac{1}{N}\Sigma_{[2]}+\frac{1}{N^2}\left[-8+2\Sigma_{[3]}+\Sigma_{[2,2]}\right]+\frac{1}{N^3}\left[4\Sigma_{[2]}-5\Sigma_{[4]}\right]\right.$$
$$\left.+\frac{1}{N^4}\left[6-3\Sigma_{[3]}+6\Sigma_{[2,2]}\right]\right) \tag{135}$$

The exact exponential of Ω_n (cf. equation (2.5) of [72]) is

$$\Omega_n = \exp\left(\frac{1}{N}\Sigma_{[2]} - \frac{1}{2N^2}\left[\binom{n}{2} + \Sigma_{[3]}\right] + \frac{1}{3N^3}\left[(2n-3)\Sigma_{[2]} + \Sigma_{[4]}\right] + \mathcal{O}\left(\frac{1}{N^4}\right)\right) \tag{136}$$

Note that the additional terms that are exponentiated are all subleading in $\frac{n^2}{N}$ (which is the same as the BMN expansion parameter $g_2 = \frac{J^2}{N}$ in the half-BPS sector when $n = \Delta = J$).

D Basis details

Symmetric group identities used here are listed in Appendix B of [70].

D.1 Inverting

We fix the permutation α in its conjugacy class. If $\alpha = [n_1, n_2, \dots n_k]$ then $\alpha = (1 \cdots n_1)(n_1 + 1 \cdots n_1 + n_2) \cdots (\cdots n)$. Any trace operator with field content and trace structure α can be written

$$\mathrm{tr}(\alpha\rho\mathbf{X}^{\vec{\mu}}\rho^{-1}) \tag{137}$$

for some $\rho \in S_n$. To get this trace from the $|\Lambda, M; \alpha, \gamma\rangle$ basis

$$\begin{aligned}\sum_{\Lambda,\beta,\gamma} d_\Lambda D^\Lambda_{pq}(\rho) S^\alpha_{p\gamma} B^{\vec{\mu}}_{q\beta} |\Lambda, \vec{\mu}, \beta; \alpha, \gamma\rangle &= \sum_\Lambda \frac{1}{n!} \sum_{\sigma\in S_n} d_\Lambda \chi_\Lambda(\sigma\rho^{-1})\, \mathrm{tr}(\alpha\sigma\mathbf{X}^{\vec{\mu}}\sigma^{-1}) \\ &= \mathrm{tr}(\alpha\rho\mathbf{X}^{\vec{\mu}}\rho^{-1})\end{aligned} \tag{138}$$

So the basis $|\Lambda, \vec{\mu}, \beta; \alpha, \gamma\rangle$ does hit all the operators.

D.2 Counting

The counting formula (63) exactly matches known counting formulae. For example, counting single trace operators $\alpha = (123 \cdots n)$ where $\mathrm{Sym}(\alpha) = \mathbb{Z}_n$, we can read off the coefficients from equation (3.6) of [31] where they're counting the appearance of YT-pletons for $hs(2,2|4)$ in the single trace partition function.

All multi-trace operators with a given symmetrisation Λ are counted by

$$\begin{aligned}\sum_{\alpha\in P(n)} S(\alpha, \Lambda) &= \sum_{\alpha\in P(n)} \frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\rho\in\mathrm{Sym}(\alpha)} \chi_\Lambda(\rho) \\ &= \frac{1}{n!} \sum_{\alpha\in S_n} \sum_{\rho\in S_n} \delta(\rho\alpha\rho^{-1}\alpha^{-1} = 1)\ \chi_\Lambda(\rho)\end{aligned} \tag{139}$$

$\rho \in |\mathrm{Sym}(\alpha)|$ is enforced using a δ-function. Now expand the δ-function using a sum over S_n representations R

$$\begin{aligned}\sum_{\alpha\in P(n)} S(\alpha, \Lambda) &= \frac{1}{n!} \sum_{\alpha\in S_n} \sum_{\rho\in S_n} \frac{1}{n!} \sum_{R\in P(n)} d_R\ \chi_R(\rho\alpha\rho^{-1}\alpha^{-1})\ \chi_\Lambda(\rho) \\ &= \frac{1}{n!} \sum_{\alpha\in S_n} \sum_{\rho\in S_n} \frac{1}{n!} \sum_{R\in P(n)} \chi_R(\rho)\ \chi_R(\alpha\rho^{-1}\alpha^{-1})\ \chi_\Lambda(\rho) \\ &= \sum_{R\in P(n)} \frac{1}{n!} \sum_{\rho\in S_n} \chi_R(\rho)\ \chi_R(\rho)\ \chi_\Lambda(\rho)\end{aligned} \tag{140}$$

This matches the large N result from [33, 4].

D.3 Three-point vertex

We want to provide the details for equation (70). In particular we want to find the decomposition coefficients between the double-trace $[n_1, n_2]$ and the product of two single-traces

$$|\Lambda, M; \alpha = [n_1, n_2], \gamma\rangle = \sum_{\Lambda_i, M_i, \gamma_i} C^{\Lambda, M, \gamma}_{\{\Lambda_i, M_i, \gamma_i\}} \; |\Lambda_1, M_1; \alpha_1 = [n_1], \gamma_1\rangle \otimes |\Lambda_2, M_2; \alpha_1 = [n_2], \gamma_2\rangle$$

A vanilla double-trace operator defined by $\alpha = [n_1, n_2]$ and σ can be written as the product of two traces $\alpha_1 = [n_1]$ and $\alpha_2 = [n_2]$ with $\vec{\mu}_1$, $\vec{\mu_2}$, σ_1 and σ_2 depending on how the original double-trace laced the fields

$$\mathrm{tr}(\sigma^{-1}[n_1, n_2]\sigma \; \mathbf{X}^{\vec{\mu}}) = \mathrm{tr}(\sigma_1^{-1}[n_1]\sigma_1 \; \mathbf{X}^{\vec{\mu}_1}) \; \mathrm{tr}(\sigma_2^{-1}[n_2]\sigma_2 \; \mathbf{X}^{\vec{\mu}_2}) \tag{141}$$

To get the coefficients $C^{\Lambda, M, \gamma}_{\{\Lambda_i, M_i, \gamma_i\}}$ use the definition of $|\Lambda, M; \alpha = [n_1, n_2], \gamma\rangle$ in terms of $\mathrm{tr}(\sigma^{-1}\alpha\sigma \; \mathbf{X}^{\vec{\mu}})$ from (62) and then use (141) and the inversion procedure (138) on the $\mathrm{tr}(\sigma_i^{-1}[n_i]\sigma_i \; \mathbf{X}^{\vec{\mu}_i})$.

D.4 Change of basis to finite N basis

We want to describe the matrix of the change of basis to

$$\mathcal{O}[\Lambda, \mu, \beta; R, \tau] = \frac{1}{n!} \sum_{\rho} B^{\vec{\mu}}_{j\beta} \; S^{\tau, \Lambda}_{j} \; {}^{R}_{p} \; {}^{R}_{q} \;\; D^{R}_{pq}(\rho) \;\; \mathrm{tr}(\rho \; \mathbf{X}^{\mu}) \tag{142}$$

A first step to fix the canonical members of each conjugacy class and sum over their conjugations

$$\sum_{\rho \in S_n} f(\rho) = \sum_{\alpha \in P(n)} \frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\tau \in S_n} f(\tau^{-1}\alpha\tau) \tag{143}$$

We get

$$\mathcal{O}[\Lambda, \mu, \beta; R, \tau] = \sum_{\alpha \in P(n)} \frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\gamma} S^{\alpha}_{j\gamma} \; S^{\tau, \Lambda}_{j} \; {}^{R}_{p} \; {}^{R}_{q} \;\; D^{R}_{pq}(\alpha) \;\; \mathcal{O}[\Lambda, \mu, \beta; \alpha, \gamma] \tag{144}$$

The other way round

$$\mathcal{O}[\Lambda, \mu, \beta; \alpha, \gamma] = \sum_{R, \tau} \frac{d_R}{d_\Lambda} S^{\alpha}_{j\gamma} \; S^{\tau, \Lambda}_{j} \; {}^{R}_{p} \; {}^{R}_{q} \;\; D^{R}_{pq}(\alpha) \;\; \mathcal{O}[\Lambda, \mu, \beta; R, \tau] \tag{145}$$

These formulae are compatible with each other.

E Trace operator examples

Computer code used for these examples is available on the internet [65].

α	$S(\alpha, \Lambda = [2,2])$	operator(s)
$[1,1,1,1]$	0	
$[2,1,1]$	1	$\mathcal{O}_1 = \mathrm{tr}(\Phi^r \Phi^s)\,\mathrm{tr}(\Phi_r)\,\mathrm{tr}(\Phi_s)$
$[2,2]$	1	$\mathcal{O}_2 = \mathrm{tr}(\Phi^r \Phi^s)\,\mathrm{tr}(\Phi_r \Phi_s)$
$[3,1]$	0	
$[4]$	1	$\mathcal{O}_3 = \mathrm{tr}(\Phi^r \Phi_r \Phi^s \Phi_s)$

Table 1: Operators for $\Lambda = [2,2]$.

E.1 $\Lambda = [2,2]$ example

The operators for $\Lambda = [2,2]$ are listed in Table 1. This case is not so interesting because there is never any multiplicity for the α.

For $\alpha = (12)$ decompose the projector (60) to get the S_a^α

$$\frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\rho \in \mathrm{Sym}(\alpha)} D^\Lambda_{ab}(\rho) = \frac{1}{4} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1,0) = S_a^{[2,1,1]} S_b^{[2,1,1]} \tag{146}$$

For $\alpha = (12)(34)$

$$\frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\rho \in \mathrm{Sym}(\alpha)} D^\Lambda_{ab}(\rho) = \frac{1}{8} \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1,0) = S_a^{[2,2]} S_b^{[2,2]} \tag{147}$$

For $\alpha = (1234)$

$$\frac{1}{|\mathrm{Sym}(\alpha)|} \sum_{\rho \in \mathrm{Sym}(\alpha)} D^\Lambda_{ab}(\rho) = \frac{1}{4} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) = S_a^{[4]} S_b^{[4]} \tag{148}$$

We get

$$\begin{aligned} |\Lambda = [2,2], M = [2,2]; \alpha = [2,1,1]\rangle &= \frac{1}{6}\mathcal{O}_1 \\ |\Lambda = [2,2], M = [2,2]; \alpha = [2,2]\rangle &= \frac{1}{6}\mathcal{O}_2 \\ |\Lambda = [2,2], M = [2,2]; \alpha = [4]\rangle &= \frac{7}{24}\mathcal{O}_3 \end{aligned} \tag{149}$$

At weak coupling the eigenstates of the dilatation operator are

$$\begin{aligned} &\Delta\ \mathcal{O}_1 = 0 \\ &\Delta\ \left[\mathcal{O}_2 + \frac{2}{N}\mathcal{O}_3\right] = 0 \\ &\Delta\ \mathcal{O}_3 = -6N\ \mathcal{O}_3 \end{aligned} \tag{150}$$

The only descendant operator is $\alpha^d = [4]$ corresponding to $\mathcal{O}_3$. The remainder $\alpha_1^c = [2,1,1]$ and

$\alpha_2^c = [2,2]$ give the leading terms of the two genuine quarter-BPS operators following equation (91)

$$\begin{aligned}
|\Lambda = [2,2], M = [2,2]; \alpha_1^c = [2,1,1]; \perp\rangle &= \Omega_4^{-1}|\Lambda = [2,2], M = [2,2]; \alpha_1^c = [2,1,1]\rangle \\
&= \frac{N^4}{6(N^2-1)(N^2-4)}\left[\mathcal{O}_1 - \frac{1}{N}\mathcal{O}_2 - \frac{2}{N^2}\mathcal{O}_1 - \frac{2}{N^2}\mathcal{O}_3\right] \\
|\Lambda = [2,2], M = [2,2]; \alpha_2^c = [2,2]; \perp\rangle &= \Omega_4^{-1}|\Lambda = [2,2], M = [2,2]; \alpha_2^c = [2,2]\rangle \\
&= \frac{N^4}{6(N^2-1)(N^2-4)}\left[\mathcal{O}_2 + \frac{2}{N}\mathcal{O}_3 - \frac{2}{N}\mathcal{O}_1\right]
\end{aligned} \tag{151}$$

These are a linear combination of the protected operators identified in (150).

E.2 $\Lambda = [4,2], \alpha = [4,2]$ example

With trace structure $\alpha = [4,2] = (1234)(56)$ there are two independent operators

$$\begin{aligned}
\mathcal{O}_1 &= \mathrm{tr}([X,Y][X,Y])\,\mathrm{tr}(XX) = \mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_s)\,\mathrm{tr}(XX) \\
&= 2\,\mathrm{tr}(XYXY)\,\mathrm{tr}(XX) - 2\,\mathrm{tr}(XXYY)\,\mathrm{tr}(XX)
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}_2 &= \mathrm{tr}(\Phi^r\Phi^s XX)\,\mathrm{tr}(\Phi_r\Phi_s) \\
&= \mathrm{tr}(XXXX)\,\mathrm{tr}(YY) - 2\,\mathrm{tr}(XXXY)\,\mathrm{tr}(XY) + \mathrm{tr}(YYXX)\,\mathrm{tr}(XX)
\end{aligned} \tag{152}$$

Decompose the projector (60) to get the two orthogonal $S^\alpha_{a\gamma}$

$$\begin{aligned}
\frac{1}{|\mathrm{Sym}(\alpha)|}\sum_{\rho\in\mathrm{Sym}(\alpha)} D^\Lambda_{ab}(\rho) &= \frac{1}{8}\begin{pmatrix} 8 & \cdots & & \\ \vdots & \ddots & & \\ & & 2 & -2\sqrt{3} \\ & & -2\sqrt{3} & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}(1,0,\cdots) + \begin{pmatrix} 0 \\ \vdots \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}(0,\cdots,-\tfrac{1}{2},\tfrac{\sqrt{3}}{2}) \\
&= S^{[4,2]}_{a1}S^{[4,2]}_{b1} + S^{[4,2]}_{a2}S^{[4,2]}_{b2}
\end{aligned} \tag{153}$$

Note that the trace is 2 which is the correct number of operators for this trace structure.

We get

$$\begin{aligned}
|\Lambda = [4,2], M = [4,2]; \alpha = [4,2], \gamma = 1\rangle &= \frac{1}{15}\left(\mathcal{O}_2 + \frac{1}{6}\mathcal{O}_1\right) \\
|\Lambda = [4,2], M = [4,2]; \alpha = [4,2], \gamma = 2\rangle &= \frac{1}{9\sqrt{5}}\mathcal{O}_1
\end{aligned} \tag{154}$$

Fortunately we don't have to do any rearrangement to get the descendant operator $\mathcal{O}_1$ with commutators inside the trace: $\{\alpha^d, \gamma^d\} = \{[4,2],2\}$ is precisely $\mathcal{O}_1$. $\{\alpha^c, \gamma^c\} = \{[4,2],1\}$ is orthogonal to $\{[4,2],2\}$ in the planar inner product by construction so it is exactly the correct operator for the leading term of the quarter-BPS state. If a single symmetrised trace is written

$A^{r_1\cdots r_p} = \mathrm{tr}(\Phi^{(r_1}\cdots\Phi^{r_p)})$ then this operator is $\epsilon_{r_1s_1}\epsilon_{r_2s_2}A^{r_1r_2}11A^{s_1s_2}$ cf. equation (2.28) of [18].[15]

$$
\begin{aligned}
&|[4,2],[4,2];\alpha^c=[4,2],\gamma^c=1;\perp\rangle\\
&=\Omega_6^{-1}|[4,2],[4,2];\alpha^c=[4,2],\gamma^c=1\rangle\\
&=\frac{1}{15}\Bigg[\mathcal{O}_2+\frac{1}{6}\mathcal{O}_1+\frac{8}{3N}\,\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_sXX)-\frac{16}{3N}\,\mathrm{tr}(\Phi^r\Phi^s\Phi_r\Phi_sXX)\\
&-\frac{4}{3N}\,\mathrm{tr}(\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(XX)-\frac{1}{N}\,\mathrm{tr}(\Phi^r\Phi^sXX)\,\mathrm{tr}(\Phi_r)\,\mathrm{tr}(\Phi_s)-\frac{1}{6N}\,\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_s)\,\mathrm{tr}(X)\,\mathrm{tr}(X)\\
&-\frac{4}{N}\,\mathrm{tr}(\Phi^r\Phi^sX)\,\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(X)+\frac{2}{N}\,\mathrm{tr}(\Phi^r\Phi^sX)\,\mathrm{tr}(\Phi_rX)\,\mathrm{tr}(\Phi_s)+\mathcal{O}\left(\frac{1}{N^2}\right)\Bigg]
\end{aligned}
\tag{155}
$$

To show that this is correct to this order in $\frac{1}{N}$ we give the leading terms of the relevant eigenstates of Δ for $\Lambda=[4,2]$ (for which there are 15 independent trace operators in total) in Table 2. Unfortunately the subleading terms in $\frac{1}{N}$ of the anomalous eigenstates don't seem to be given in

leading terms of eigenstates	eigenvalue
$\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_sXX)+(-2\pm\sqrt{5})\,\mathrm{tr}(\Phi^r\Phi^s\Phi_r\Phi_sXX)+\mathcal{O}\left(\frac{1}{N}\right)$	$-(5\pm\sqrt{5})N+\mathcal{O}(N^{-1})$
$\mathcal{O}_1-\frac{14}{N}\,\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_sXX)-\frac{2}{N}\,\mathrm{tr}(\Phi^r\Phi^s\Phi_r\Phi_sXX)+\mathcal{O}\left(\frac{1}{N^2}\right)$	$-6N+\mathcal{O}(N^{-1})$
$\mathcal{O}_2+\frac{1}{6}\mathcal{O}_1+\frac{8}{3N}\,\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_sXX)-\frac{16}{3N}\,\mathrm{tr}(\Phi^r\Phi^s\Phi_r\Phi_sXX)+\mathcal{O}\left(\frac{1}{N^2}\right)$	0
$\mathrm{tr}(\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(XX)+\frac{2}{N}\mathcal{O}_1-\frac{8}{N}\mathcal{O}_2+\mathcal{O}\left(\frac{1}{N^2}\right)$	0
$\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_s)\,\mathrm{tr}(X)\,\mathrm{tr}(X)$	$-6N$
$\mathrm{tr}(\Phi^r\Phi^sXX)\,\mathrm{tr}(\Phi_r)\,\mathrm{tr}(\Phi_s)+\frac{1}{6}\,\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_s)\,\mathrm{tr}(X)\,\mathrm{tr}(X)$	0
$\mathrm{tr}(\Phi^r\Phi^sX)\,\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(X)+\mathcal{O}\left(\frac{1}{N}\right)$	0
$\mathrm{tr}(\Phi^r\Phi^sX)\,\mathrm{tr}(\Phi_rX)\,\mathrm{tr}(\Phi_s)+\mathcal{O}\left(\frac{1}{N}\right)$	0

Table 2: Leading terms of some of the eigenstates of Δ for $\Lambda=[4,2]$.

any obvious way by the cut-and-join operators. For example applying $\Sigma_{[2]}$ to $\mathcal{O}_1$ we get

$$
\begin{aligned}
\Sigma_{[2]}\mathcal{O}_1=&8\,\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_sXX)-4\,\mathrm{tr}(\Phi^r\Phi^s\Phi_r\Phi_sXX)\\
&-\mathrm{tr}(\Phi^r\Phi^s)\,\mathrm{tr}(\Phi_r\Phi_s)\,\mathrm{tr}(XX)+\mathrm{tr}(\Phi^r\Phi_r\Phi^s\Phi_s)\,\mathrm{tr}(X)\,\mathrm{tr}(X)
\end{aligned}
\tag{156}
$$

which doesn't match the correct $\frac{1}{N}$ term for the operator leading with $\mathcal{O}_1$ in Table 2.

F Equivalence of one-loop formulations

Here we demonstrate the equivalence of the two one-loop actions of [21] and [6] used in Section 4.3. Suppose P_{ij} projects onto two sites $1\le i,j\le n$. If i and j sit within a single trace $\alpha=(i\,s_1\,j\,s_2)$ where s_1,s_2 are some sequences of integers then

$$
(i\,n{+}1)\,\alpha\,(j\,n{+}1)=(i\,s_1\,j)(n{+}1\,s_2)=\begin{cases}(i\,s_1\,j)(s_2) & \text{if } s_2\neq 0\\ N(i\,s_1\,j) & \text{if } s_2=0\end{cases}
\tag{157}
$$

[15] One might wonder why the answer dropped out so easily in this case. We expect the leading terms of the eigenstates of Δ to be orthogonal in the planar tree-level two-point function. Thus it's possible that by diagonalising the planar two-point function with the $\{\alpha,\gamma\}$ we hit on the same operators that are eigenstates of Δ to planar order. However this is not guaranteed and is most likely a coincidence in this case.

If we split off the first integer of $s_2 = \alpha(j)\, s_2{}'$ then

$$(i\, n{+}1)\, \alpha\, (j\, n{+}1) = (i\, \alpha(j))\, \alpha = (i\, s_1\, j)(\alpha(j)\, s_2{}') = (i\, s_1\, j)(s_2) \tag{158}$$

which agrees if $s_2 \neq 0$. If s_2 vanishes then $\alpha(j) = i$ and $(i\, \alpha(j)) = N$, cf. [21] equation (2.15).

If i and j sit within two different traces $\alpha = (i\, s_1)(j\, s_2)$ then

$$(i\, n{+}1)\alpha(j\, n{+}1) = (j\, i\, s_1\, s_2) = (i\, \alpha(j))\, \alpha \tag{159}$$

G Proof of chiral ring counting formula

To show that our result (106) is the same as that in the literature (109), first note that for representations of $U(2)$ the Young diagram for the S_n representation λ is the same as that for the $U(2)$ representation Λ, so $\Lambda = \lambda$ and $\mathrm{mult}(V_{\mathbf{2}}^{\otimes n} \to \Lambda \otimes \lambda) = 1$.

In general the multiplicity $\mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes n} \to K \otimes \kappa)$ in (102) can be calculated using projectors for K and κ in $(V_{\mathrm{nat}}^{S_N})^{\otimes n}$

$$\mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes n} \to K \otimes \kappa) = \frac{1}{N!} \sum_{\sigma \in S_N} \chi_K(\sigma) \frac{1}{n!} \sum_{\tau \in S_n} \chi_\kappa(\tau) \prod_i (\chi_{\mathrm{nat}}(\sigma^i))^{c_i(\tau)} \tag{160}$$

where $c_i(\tau)$ is the number of cycles of length i in $\tau \in S_n$.

For the specialisation to $\kappa = [n]$ we will also use

$$\mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes n} \to K \otimes [n]) = \text{coefficient of } q^n \text{ in } s_K(1, q, q^2, \dots) \tag{161}$$

Alternatively this can be stated

$$s_K(1, q, q^2, \dots) = \sum_{n=0}^{\infty} \mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes n} \to K \otimes [n])\ \ q^n \tag{162}$$

Our expression for the $U(2)$ partition function (107) for $\frac{1}{4}$-BPS chiral ring states is

$$\begin{aligned} Z^{cr}_{U(N)}(x, y) &= \sum_{\Lambda} \mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes n} \to [N] \otimes \Lambda)\ \ \chi_\Lambda(x, y) \\ &= \sum_{\Lambda} \mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes n} \to [N] \otimes \Lambda)\ \ \sum_{\mu,\nu} g([\mu], [\nu]; \Lambda)\ x^\mu y^\nu \end{aligned} \tag{163}$$

where we've expanded out the Schur polynomial using the Littlewood-Richardson coefficient g. Next use (160) and the formula for the Littlewood-Richardson coefficient g from (58) to get

$$\begin{aligned} Z^{cr}_{U(N)}(x, y) &= \sum_{\Lambda} \frac{1}{N!} \sum_{\sigma \in S_N} \frac{1}{n!} \sum_{\tau \in S_n} \chi_\Lambda(\tau) \prod_i (\chi_{\mathrm{nat}}(\sigma^i))^{c_i(\tau)}\ \ \sum_{\mu,\nu} \frac{1}{\mu!\nu!} \sum_{\rho \in S_\mu \times S_\nu} \chi_\Lambda(\rho)\ x^\mu y^\nu \\ &= \sum_{\mu,\nu} \frac{1}{\mu!\nu!} \sum_{\rho \in S_\mu \times S_\nu} \frac{1}{N!} \sum_{\sigma \in S_N} \prod_i (\chi_{\mathrm{nat}}(\sigma^i))^{c_i(\rho)}\ x^\mu y^\nu \end{aligned} \tag{164}$$

Next, working from Dolan's formula we use (162) to get

$$\begin{aligned} Z^{cr}_{U(N)}(x, y) &= \sum_{K \vdash N} s_K(1, x, x^2, \dots) s_K(1, y, y^2, \dots) \\ &= \sum_{K \vdash N} \sum_{\mu,\nu} \mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes \mu} \to K \otimes [\mu])\ \ \mathrm{mult}((V_{\mathrm{nat}}^{S_N})^{\otimes \nu} \to K \otimes [\nu])\ \ x^\mu y^\nu \end{aligned} \tag{165}$$

Now use (160) to get

$$
\begin{aligned}
Z^{cr}_{U(N)}(x,y) =& \sum_{K\vdash N}\sum_{\mu,\nu}\frac{1}{N!}\sum_{\sigma_1\in S_N}\chi_K(\sigma_1)\frac{1}{\mu!}\sum_{\rho_1\in S_\mu}\prod_i(\chi_{\rm nat}(\sigma_1^i))^{c_i(\rho_1)} \\
& \frac{1}{N!}\sum_{\sigma_2\in S_N}\chi_K(\sigma_2)\frac{1}{\nu!}\sum_{\rho_2\in S_\nu}\prod_j(\chi_{\rm nat}(\sigma_2^j))^{c_j(\rho_2)}\ \ x^\mu y^\nu \\
=& \sum_{\mu,\nu}\frac{1}{N!}\sum_{\sigma\in S_N}\frac{1}{\mu!}\sum_{\rho_1\in S_\mu}\prod_i(\chi_{\rm nat}(\sigma^i))^{c_i(\rho_1)}\frac{1}{\nu!}\sum_{\rho_2\in S_\nu}\prod_j(\chi_{\rm nat}(\sigma^j))^{c_j(\rho_2)}\ \ x^\mu y^\nu \\
=& \sum_{\mu,\nu}\frac{1}{\mu!\nu!}\sum_{\rho\in S_\mu\times S_\nu}\frac{1}{N!}\sum_{\sigma\in S_N}\prod_i(\chi_{\rm nat}(\sigma^i))^{c_i(\rho)}\ x^\mu y^\nu
\end{aligned}
\tag{166}
$$

This is identical to (164) so we are done.

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