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Local causal structures, Hadamard states and the principle of local covariance in quantum field theory.

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Abstract. In the framework of the algebraic formulation, we discuss and analyse some new features of the local structure of a real scalar quantum field theory in a strongly causal spacetime. In particular we use the properties of the exponential map to set up a local version of a bulk-to-boundary correspondence. The bulk is a suitable subset of a geodesic neighbourhood of any but fixed point p of the underlying background, while the boundary is a part of the future light cone having p as its own tip. In this regime, we provide a novel notion for the extended $*$-algebra of Wick polynomials on the said cone and, on the one hand, we prove that it contains the information of the bulk counterpart via an injective $*$-homomorphism while, on the other hand, we associate to it a distinguished state whose pull-back in the bulk is of Hadamard form. The main advantage of this point of view arises if one uses the universal properties of the exponential map and of the light cone in order to show that, for any two given backgrounds M and M' and for any two subsets of geodesic neighbourhoods of two arbitrary points, it is possible to engineer the above procedure such that the boundary extended algebras are related via a restriction homomorphism. This allows for the pull-back of boundary states in both spacetimes and, thus, to set up a machinery which permits the comparison of expectation values of local field observables in M and M'.

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Contents

1 Introduction

In the framework of quantum field theory over curved backgrounds, we witnessed a considerable series of leaps forward due to a novel use of advanced mathematical techniques combined with new physical insights leading to an improved understanding of the underlying foundations of the theory. It is far from our intention to give a recollection of all of them, but we would like to draw attention at least to some of them. On the one hand, in [9], the principle of general local covariance was formulated leading to the realisation of a quantum field theory as a covariant functor between the category of globally hyperbolic (four-dimensional) Lorentzian manifolds with isometric embeddings as morphisms and the category of C^*-algebras with unit-preserving monomorphisms as morphisms and also to the new interpretation of local fields as natural transformations from compactly supported smooth function to suitable operators. On the other hand, the presence of a nontrivial background comes with the grievous problem of the *a priori* absence of a sufficiently large symmetry group to identify a natural ground state as in Minkowski spacetime where Poincaré invariance enables this.

Nonetheless, it is still possible to identify a class of physically relevant states as those fulfilling the so-called Hadamard condition. This guarantees that the ultraviolet behaviour of the chosen state mimics that of the Minkowski vacuum at short distances as well as that the quantum fluctuations of observables such as the smeared components of the stress-energy tensor are bounded. From a practical point of view, the original characterisation of the Hadamard form was realised by means of the local structure of the integral kernel of the two-point function of the selected quasi-free state in a suitably small neighbourhood of a background point. Unfortunately, such a criterion is rather difficult to check in a concrete example and a real step forward has been achieved in [38, 39] in which the connection between the Hadamard condition and the microlocal properties of the two-point function is proven and fully characterised.

This result has prompted a series of interesting developments in the analysis of physically relevant states in a curved background, but we focus mainly on a few recent advances (*cf.* [13, 14, 15]) where it has been shown that, either in asymptotically flat or in cosmological spacetimes, it is possible to exploit the conformal structure of the manifold to identify a preferred null submanifold of codimension one, the conformal boundary. On the latter it is possible to coherently encode the information of the bulk algebra of observables and to identify a state fulfilling suitable uniqueness properties whose pull-back in the bulk satisfies the Hadamard condition, being at the same time invariant under all spacetime isometries.

The main problem in the above construction is the need to find a rigid and global geometric structure which acts as an auxiliary background out of which the bulk state is constructed. Hence the local applicability of a similar scheme seems rather limited; yet one of the main goals of the present paper is to show that such a procedure can indeed be set up at a local level and for all spacetimes of physical interest. In particular, this statement is established on the basis of a careful use of some rather well-known geometrical objects.

To be more precise, the point of view taken is the following: if one considers an arbitrary but fixed point p in a strongly causal four-dimensional spacetime, it is always possible to single out a geodesic neighbourhood where the exponential map is a local diffeomorphism. Within this set we can also always select a second point q such that the double cone $\mathscr{D} \equiv \mathscr{D}(p,q) \doteq I^+(p) \cap I^-(q)$ is a globally hyperbolic spacetime. This line of reasoning has a twofold advantage: on the one hand, one can single out a local natural null submanifold of codimension one, $\mathscr{C}_p^+$, as the portion of $J^+(p)$ contained in the closure of $\mathscr{D}$, while, on the other hand, we are free to repeat the very same construction for a second point p' with associated double cone $\mathscr{D}'$ in another spacetime M'. Since the exponential map is invertible and the tangent spaces $T_p(M)$ and $T_{p'}(M')$ are isomorphic, it turns out that it is possible to engineer all the geometric data in such a way that the two boundaries $\mathscr{C}_p^+$ and $\mathscr{C}_{p'}^+$ can be related by a suitable restriction map, the only freedom being the choice of a frame at p and at p'.

These two advantages can be used to draw some important conclusions on the structure of local quantum field theories. More precisely, we shall focus on a real scalar field theory in $\mathscr{D}$, with generic mass m and with generic curvature coupling ξ. The associated quantum observables are described by the Borchers-Uhlmann algebra or rather by the extended algebra of fields. In particular, we shall show that it is possible to construct a scalar field theory also on $\mathscr{C}_p$ and, as a novel result, that also, in the boundary, there exists a natural notion of extended algebra which is made precise here. Apart from the check of mathematical consistency of our definition, we reinforce our proposal by showing that there exists an injective *-homomorphism Π between the bulk and the boundary counterparts. The relevance of this result is emphasised by the identification of a natural state on the boundary, whose pull-back in $\mathscr{D}$ via Π turns out still to be invariant under a change of the frame (hence, physically speaking, it is the same for all inertial observers at p) and to be of Hadamard form. This result provides a potential candidate for a local vacuum in the large class of backgrounds to be considered here.

Yet we still have not made profitable use of the second advantage outlined before. As a matter of fact, we can now consider two arbitrary strongly causal spacetimes M and M' as well as two points therein so that the relevant portions of the two boundaries, $\mathscr{C}_p$ and $\mathscr{C}_{p'}$, say, associated with the double cones, can be related by a suitable restriction map. The construction of the boundary field theory shows that such a map becomes an injective homomorphism between the boundary extended algebras, hence allowing for the construction of a local Hadamard state in two different backgrounds starting from the same building block on the boundary.

The results presented in the present work have some antecedents in the concept of relative Cauchy evolution developed in [9]. Knowing a theory (and its corresponding Hadamard function) in the neighbourhood of a Cauchy surface, such a method permits one to reconstruct the theory in the neighbourhood of any other Cauchy surface of the same spacetime. The deformation arguments, see [18], play a crucial role in obtaining the Hadamard property for the deformed state in particular. Another related key result is also the one presented in [45] about the local quasi-equivalence of quasi-free Hadamard states. In the present paper, using the null cones as hypersurfaces on which to encode the quantum information, we succeed in giving an extended algebra of observables without knowing the state in a neighbourhood of such a surface. Hence, based on the new method presented here, it is possible to determine quantum states out of their form on null surfaces alone and thus in a spacetime-independent way.

We are now in a position to have a reference state with respect to which we can compare the expectation values of the same field observables in two different spacetimes. In particular, if one of these is (a portion of) Minkowski spacetime, it is obvious that the result of the comparison will be related to the geometric data of the second background which can now be assessed with a crystal clear procedure. Furthermore, we shall show that this method admits an interpretation within the language of category theory, so that it becomes manifest that our proposal is not in contrast with the principle of general local covariance, but can actually be seen as a generalisation. As a matter of fact, it reduces to the latter whenever isometric embeddings are involved, in which case the fields recover their interpretation as natural transformations as in [9], *i.e.*, they transform in a covariant manner under local isometries.

To reinforce the above procedure we also provide an explicit example of this "comparison" strategy considering a massless real scalar field minimally coupled to scalar curvature both in Minkowski and in a Friedman-Robertson-Walker spacetime with flat spatial sections. We demonstrate how the difference of the expectation values of the regularised squared scalar fields in these two spacetimes can be expanded into a power series of a suitable local coordinate system (null-advanced) yielding, at first order, a contribution dependent on the structure of the so-called scale factor of the curved background.

Since we have already extensively discussed the plan of action, we only briefly sketch the synopsis of the paper. In Section 2 we shall analyse all the geometric structure needed. Although most of the material, devoted to the construction of frames and of the exponential map, is rather well-known in the literature, we try nonetheless to recollect it here to provide guidance through the construction of the main geometric objects required, the boundary in particular. In Section 3 we shall tackle the problem of constructing a quantum scalar field theory on a null cone; in particular, in Subsections 3.1 and 3.2 we discuss the structure of the bulk and boundary algebras therein while, in Subsection 3.3, we identify the distinguished boundary state. The novel construction of the extended algebra on the boundary is presented in Subsection 3.4 and all these results are connected to the bulk counterpart in Subsection 3.5. Eventually, in Section 4, we discuss, by means of the language of categories, the scheme which leads to the possibility to compare field theories on different spacetimes. The concrete example mentioned above is in Subsection 4.2. Section 5 summarises the paper and sets out a few conclusions as well as possible future investigations.

2 Frames and Cones

As outlined in the introduction, the keyword of this paper is "comparison," *i.e.* our ultimate goal will be to correlate quantum field theories in different backgrounds both at the level of algebras and of states and, moreover, to try in the process also to extract information on the local geometry. To this avail one needs a crystal clear control both of the underlying background and of its properties. Therefore, we cannot consider arbitrary manifolds, but need to focus only on those which are of physical relevance insofar as they can carry a full-fledged quantum field theory.

If we keep in mind this perspective, we shall henceforth call *spacetime* a four-dimensional, Hausdorff, connected smooth manifold M endowed with a Lorentzian metric whose signature is $(-,+,+,+)$. Then, consequently, M is also second countable and paracompact [19, 20]. Customarily one also requires that M be globally hyperbolic (see for example [1] or [9]) in order to have a well-defined Cauchy problem for the equations of motion ruling the dynamics of standard free field theories.

The next natural step is the identification of further local geometric structures which could serve as a useful tool in the comparison of two different field theories on two different spacetimes, M and M'. It is known that, for a real scalar field theory, it suffices to require that the two spacetimes are either isometrically embedded into each other or conformally related [9, 34]. The drawback of this approach is that only few pairs M and M' fulfil such criteria and potentially interesting cases, such as when M coincides with Minkowski spacetime and M' with de Sitter, are excluded.

A natural alternative would be to consider pairs of spacetimes M and M' related by a global diffeomorphism, but, unfortunately, these maps do no preserve the geometric structures at the heart of the quantum or even of the classical field theory. A typical example of such a problem arises in connection with the equations of motion of a dynamical system whenever these are constructed out of the spacetime metric. The action of a generic diffeomorphism preserves their form only in special cases, *viz.* when they are related to isometries. Hence we would return to the original scenario.

Apart from these remarks we should also keep in mind the idea, briefly sketched in the introduction, to exploit a bulk-to-boundary reconstruction procedure along the lines of [13, 14]. At a global level, this requires the existence of a conformal boundary structure, a feature shared only by a certain class of manifolds. Since we want to consider a scenario as general as possible, a viable alternative is to focus only on the *local* structures of the underlying spacetimes. In the remainder of this section, we show how to substantiate this heuristic idea if one carefully uses certain properties of the exponential map.

2.1 Frames and the Exponential Map

The aim of this subsection is to introduce the basic geometric tools to be used. Most of the concepts are certainly well-known in the literature and the reader might refer either to [26] for a full-fledged analysis of those related to bundles and their properties or to [28, 33] for a discussion focused on the differential geometric aspects. Nonetheless, it is worthwhile to recapitulate part of them since they will play a pivotal role in this paper and we can, at the same time, fix the notation.

Consider an arbitrary four-dimensional differentiable manifold M. To any point $p \in M$, we can associate

- a *linear frame* F_p of the tangent space, *i.e.*, a non-singular linear mapping $e : \mathbb{R}^4 \to T_p(M)$, or, equivalently, an assignment of an ordered basis $e_1, \ldots, e_4$ of $T_p(M)$.

It is straightforward to infer that the set of all such linear frames FM at an arbitrary but fixed $p \in M$ naturally comes with a right and free action of the group $GL(4, \mathbb{R})$ which is tantamount to the possible changes of basis in $\mathbb{R}^4$, *i.e.*, $(A, e) \mapsto eA$ where eA denotes the ordered basis $A^i_j e_i$ for all $A \in GL(4, \mathbb{R})$. Thus FM can be endowed with the following additional structure:

- Given a four-dimensional differentiable manifold M, a *frame bundle* is the principal bundle $\widetilde{FM} = F[GL(4, \mathbb{R}), \pi', M]$ built from the disjoint union $\bigsqcup_p \widetilde{F_pM}$, where $\widetilde{F_pM}$ is identified with the typical fibre $GL(4, \mathbb{R})$ and $\pi' : \widetilde{FM} \to M$ is the projection map. Furthermore, the tangent bundle TM can be constructed as the associated bundle $TM = \widetilde{FM} \times_{GL(4,\mathbb{R})} \mathbb{R}^4$.

We emphasise the well-known fact that the structure introduced last guarantees that the typical fibre of the tangent bundle at any point p is $\mathbb{R}^4$ regardless of the chosen manifold, a fact we shall use in the forthcoming discussion. Following [28, 33], recall that

- for any $p \in M$, if D_p is the set of all vectors v in $T_p(M)$ such that the geodesic $\gamma_v : [0, 1] \to M$ admits v as tangent vector in 0, then the exponential map at p is $\exp_p : D_p \to M$ with $\exp_p(v) = \gamma_v(1)$;

- for any point $p \in M$ there always exists a neighbourhood $\widetilde{\mathcal{O}}$ of the 0-vector in $T_p(M)$ such that the exponential map is a diffeomorphism onto an open subset $\mathcal{O} \subset M$. Furthermore, whenever $\widetilde{\mathcal{O}}$ is star-shaped, $\mathcal{O}$ is called a *normal neighbourhood*, and the inverse map therein will be denoted $\exp_p^{-1} : \mathcal{O} \to \widetilde{\mathcal{O}}$.

Although the existence of open sets where the exponential map is a diffeomorphism suggests a way to compare local quantum field theories on different manifolds, we also need to single out a preferred structure of codimension 1, since we wish to implement a bulk-to-boundary procedure. To this avail all the manifolds are henceforth endowed with a smooth Lorentzian metric, which entails the following additional features:

- Since a linear frame at a point $p \in M$ can be seen as the assignment of an ordered basis of $\mathbb{R}^4$, one can endow this latter vector space with the standard Minkowski metric η, which, by construction, is invariant under the Lorentz group $SO(3,1)$. In this case the frame bundle becomes $FM = F[SO(3,1), \pi', M]$ which is also referred to as the *bundle of orthonormal frames over M*. Furthermore, if the spacetime is oriented and time-oriented, we can further reduce the group to $SO_0(3,1)$, the component of $SO(3,1)$ connected to the identity.

- Every point in a Lorentzian manifold admits a normal neighbourhood (see Proposition 7 and also Definition 5 in Chapter 5 of [33]).

- There is always a choice of coordinates, called *normal coordinates*, such that, in these coordinates, the pull-back of the metric g under the inverse of the exponential map equals η (the Minkowski metric in standard coordinates) on the inverse image of the point p.

- Since we shall ultimately need to single out a sort of preferred codimension 1 structure, it is rather important that, in a Lorentzian manifold, the so-called Gauss lemma holds true (Lemma 1 in Chapter 5 of [33]). In particular, this entails that, given any $p \in M$, if we consider the null cone $\widetilde{C} \subset T_p(M)$ having p as its own tip, then the subset $\widetilde{C} \cap \widetilde{\mathcal{O}}$ is mapped into a local null cone in $\mathcal{O} \subset M$ which consists of initial segments of all null geodesics starting at p.

We are now in a position to outline the building blocks of our geometric construction. Let us consider two spacetimes (M, g) and (M', g') and two generic points $p \in M$ and $p' \in M'$, together with their normal neighbourhoods $\mathcal{O}_p$ and $\mathcal{O}_{p'}$. If we equip each tangent space with an orthonormal basis via a frame, $e : \mathbb{R}^4 \to T_p(M)$ and $e' : \mathbb{R}^4 \to T_{p'}(M')$, we are also free to introduce a map $i_{e,e'} : T_p(M) \to T_{p'}(M')$ which is constructed simply by identifying the elements of the two ordered bases.

The strategy is now to exploit the fact that the exponential map is a diffeomorphism (hence invertible) in a geodesic neighbourhood to introduce a map $\imath_{e,e'} : \mathcal{O}_p \to \mathcal{O}_{p'}$ such that

$$\imath_{e,e'} \doteq \exp_{p'} \circ\, i_{e,e'} \circ \exp_p^{-1} . \tag{1}$$

It is important to stress a few further aspects of this last definition:

- The map $\imath_{e,e'}$ is well defined only when $\exp_{p'}^{-1}$ can be inverted on the image of $i_{e,e'} \circ \exp_p^{-1}$, that is when $i_{e,e'} \circ \exp_p^{-1}(\mathcal{O}_p) \subset \widetilde{\mathcal{O}}_{p'}$. Therefore, for the sake of notational simplicity, when we write $\imath_{e,e'}$ it is always assumed that such a requirement is satisfied. Furthermore, for every point p we can always consider a sufficiently smaller subset of $\mathcal{O}_p$, retaining all its properties, where the above inclusion holds true.

- The map $\imath_{e,e'}$, which maps a sufficiently small $\mathcal{O}$ to $\mathcal{O}'$, is not unique, in the sense that it depends on the chosen orthonormal frames e and e'. We have always the freedom to act with an element of the structure group of the fibre (be it $SO(3,1)$ or $SO_0(3,1)$ depending on the scenario considered) which maps an orthonormal basis into a second one, and this either on $T_p(M)$ or $T_{p'}(M')$. Such arbitrariness cannot be lifted and, for this reason, we have explicitly indicated the two frames in the mapping $\imath_{e,e'}$.
- Despite the freedom mentioned above, the map $\imath_{e,e'}$ is invariant under the action of a single element of the structure group of the fibre on both e and e', *i.e.*, there exists an equivalence relation: We say that
$$\imath_{e,e'} \sim \imath_{\tilde{e},\tilde{e}'} \tag{2}$$
if and only if there exists an element $\Lambda \in SO_0(3,1)$ such that $\tilde{e} = \Lambda e$ and $\tilde{e}' = \Lambda e'$. This equivalence relation shall actually play a relevant role in the discussion of Section 4.

As a related point, notice that, if the spacetime M is isometrically embedded into M', a scenario close to the hypotheses in [9], each isometry $\phi : M \to M'$ induces an isomorphism between the orthonormal frame bundles FM and FM' since the metric structure is preserved. In this case every local character of the manifold M is preserved under ϕ (see for example Chapter 3 of [33]) and, hence, one can consider a sufficiently small subset of the normal neighbourhood of any $p \in M$ as well as of $\phi(p) \in M'$ so that our construction yields the following commutative diagram,

$$\begin{array}{ccc} \mathcal{O}_p & \xrightarrow{\exp_p^{-1}} & T_p(M) \\ \phi\big\downarrow & & \big\downarrow \imath_{e,(\phi_*\circ e)} \\ \mathcal{O}_{\phi(p)} & \xleftarrow[\exp_{\phi(p)}]{} & T_{\phi(p)}(M') \end{array} .$$

Notice that the presence of $\phi_* \circ e$ in place of a generic e' can be justified as follows: If we call $(\ ,\ )_p$ the inner product between vectors in $T_p(M)$, then for any $v,\ w \in T_p(M)$, one has $(v,w)_p = (\phi_*(v), \phi_*(w))_{\phi(p)}$, which, upon introduction of a local frame $e : \mathbb{R}^4 \to T_p(M)$, yields $(v,w)_p = (e(v_i), e(w_i))_p = (\phi_* \circ e(v_i), \phi_* \circ e(w_i))_{\phi(p)}$ where $v_i,\ w_i \in \mathbb{R}^4$. Moreover, if a generic e' is used in place of $\phi_* \circ e$ there is no guarantee that the previous diagram commutes. A counterexample can actually be constructed considering two isometrically related spacetimes, which are not rotationally invariant and taking for e', $\phi_* \circ e$ rotated by some generic angle.

2.2 Double Cones and Their Past Boundary

The analysis of the previous subsection is a first step towards the setup of a full-fledged procedure which allows for the local comparison of quantum field theories on different spacetimes. We shall now single out a preferred submanifold of codimension 1 on which to apply a bulk-to-boundary reconstruction.

To this avail we have to ensure in the first place that one can consistently assign to the background M a well-defined quantum field theory. Since we are only interested in local quantities, the usual hypothesis of global hyperbolicity of the spacetime can be moderately relaxed and, henceforth, we shall assume M to be *strongly causal* [2], *i.e.*, for every point $p \in M$, there exists an arbitrarily small convex, causally convex neighbourhood $\mathcal{O}'_p$, which means that no non-spacelike curve intersects $\mathcal{O}'_p$ in a disconnected set. In other words, $\mathcal{O}'_p$ itself is globally hyperbolic.

From a physical point of view, this simply forces us to require that, ultimately, the theory coincides with the usual quantisation procedure on each of these subsets, while, from a geometrical perspective, the discussion of the preceding section still holds true since we are entitled to select $\mathcal{O}'_p \subseteq \mathcal{O}_p$, the normal neighbourhood of p, in such a way that the exponential map is a local diffeomorphism also on $\mathcal{O}'_p$. Furthermore, a rather useful class of sets is constructed out of the so-called *double cones*,

$$\mathscr{D}(p', q) = I^+(p') \cap I^-(q) \subset M,$$

where $I^{\pm}$ stand, respectively, for the chronological future and past while $q \in \mathcal{O}'_{p'}$. Notice that both p' and q can be arbitrary but, for our construction, we shall always suppose that at least one of them coincides with p, henceforth $p' \equiv p$. It is also interesting that $\mathscr{D}(p, q)$ is an open and still globally hyperbolic subset of $\mathcal{O}'_p$. In the forthcoming discussion the boundary of this region will also be relevant and we point out that the closure $\overline{\mathscr{D}(p, q)}$ is a compact set (see for example Chapter 8 in [46]) which coincides with $J^+(p) \cap J^-(q)$. Furthermore, it is also important to recall both that the set of (the closures of) double cones can be used as a base of the topology of $\mathcal{O}'_p$ and that, under the previous assumptions, we can also freely consider the image of $\overline{\mathscr{D}(p, q)}$ under the inverse exponential map $\exp_p^{-1}$, denoted by $U(p, q)$. The reader should bear in mind that $U(p, q)$ is not necessarily the closure of a double cone in $T_p(M) \sim \mathbb{R}^4$ with respect to the flat metric since only (portions of) cones in $T_p(M)$, having p as their tip, are mapped in (portions of) those in $\mathcal{O}_p$ and vice versa.

Nonetheless, this construction allows for the identification of the main geometrical structure needed, since the very existence of $\mathscr{D}(p, q)$ and the properties of this set as well as of $J^+(p)$ under the exponential map suggest to consider $\mathscr{C}_p^+ \doteq \partial J^+(p) \cap \overline{\mathscr{D}(p, q)}$ as the natural boundary on which to encode data from a field theory in the bulk. The bulk here means $\mathscr{D}(p, q)$ which is a genuine globally hyperbolic submanifold of M on which a full-fledged quantum field theory can indeed be defined.

From a geometrical point of view, a few interesting intrinsic properties of $\mathscr{C}_p^+$ can readily be inferred, namely, to start with, $\mathscr{C}_p^+$ is generated by future directed null geodesics in particular originating from p. Notice that the latter are not complete since the set we are interested in is constrained to $\overline{\mathscr{D}(p, q)} \subset \mathcal{O}'_p$ and, therefore, its image under $\exp_p^{-1}$ in $T_p(M)$ identifies a portion of a future directed null cone C^+ constructed with respect to the flat metric η, where this portion is topologically equivalent to $I \times \mathbb{S}^2$, $I \subseteq \mathbb{R}$. Yet all these properties are universal, thus they do not depend on the choice of a specific frame e at p. This is not the case for the form of the image of $\mathscr{C}_p^+$ in C^+ under $\exp_p^{-1}$ or the pull-back of the metric in normal coordinates under $\exp_p^*$. These clearly depend upon the coordinate system considered (individuated by e) and, hence, the possible choices of e and of coordinates on $\mathscr{C}_p^+$ deserve a more detailed discussion.

If one starts from the observation that the double cones of interest all lie in a normal neighbourhood, a first natural guess is to select the standard normal coordinates constructed out of the frame e. In this setting the metric can be expanded as

$$g_{\mu\nu}(q) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}(p)\sigma^{\alpha}(q,p)\sigma^{\beta}(q,p) + O(3),$$

where $\sigma(q,p)$ is the so-called Synge's world function, *i.e.* half of the square of the geodesic distance between p and q (see Section 2.1 of [35]). Here σ^{α} and σ^{β} denote the covariant derivatives of σ performed at p, whereas $O(3)$ is a shortcut to stress that the metric is approximated up to cubic quantities in the normal coordinates.

Unfortunately, both the coordinate system and the expansion are not well suited to be used in the analysis of the geometry of the null cone $\mathscr{C}_p^+$, since one would like to have a local chart where it is manifest that $\mathscr{C}_p^+$ is a null hypersurface. Furthermore, for later purposes, we also need to discuss some properties of the metric in the vicinity of $\mathscr{C}_p^+$ as a whole and not only in a neighbourhood of p. To this end, it is useful to employ the so called *retarded coordinates* as introduced in [36, 37]. We also refer to the review [35], which has the advantage to clearly discuss the explicit relation between these new coordinates and the normal ones (or, also, the Fermi-Walker ones).

Let us briefly recall the construction of these retarded coordinates. Consider a timelike geodesic $\tilde{\gamma}$ through p with unit tangent vector u. In this setting one can define a coordinate r as the field

$$r(q) \doteq -\sigma_{\alpha}(q,p')\, u^{\alpha}(p'),$$

where $q \in \mathcal{O}$ and $p' \in \tilde{\gamma}$ are connected by a light-like geodesic originating from p' and pointing towards the future. With $\sigma_{\alpha}(q,p')$ we mean the covariant derivative at p of the geodesic distance. The net advantage of r is that, on $\mathscr{C}_p^+$, it can be read as an affine parameter of the null geodesics emanating from p. In other words, once an orthonormal frame e is chosen in such a way that $e^0(p') = u$, the scalar field r on $C_{p'}^+$ is unambiguously fixed.

We can now define the full retarded coordinates as (u, r, x^A), where u labels the family of forward null cones with tips lying on $\tilde{\gamma}$ (see equation (154) and the preceding discussion in [35]), and

$$\mathscr{C}_p^+ = \left\{ p' \in \mathscr{D}(p,q) \mid u(p') = 0 \right\},$$

while x^A are local coordinates on $\mathbb{S}^2$. Notice that one could alternatively switch to the more common local chart (θ, φ) of $\mathbb{S}^2$ at p' and we shall do so whenever needed.

Moreover, in this coordinate system, the most generic form of the metric reads [12]

$$ds^2 = -\alpha\, du^2 + 2\upsilon_A du\, dx^A - 2e^{2\beta} du\, dr + g'_{AB} dx^A dx^B, \tag{3}$$

where α, υ_A, β and g'_{AB} are smooth functions depending on the coordinates. Notice that here $r \in (0,\infty)$ while u ranges over an open set $I \subseteq \mathbb{R}$ which contains 0. Moreover, the x-coordinates on the sphere give rise to a volume element with respect to (3) of the form

$$\sqrt{|g'_{AB}|}\, dx^A \wedge dx^B = \sqrt{|g_{AB}|}\ |\sin\theta|\ d\theta \wedge d\varphi, \tag{4}$$

where the symbol $|\cdot|$ under the square root is kept to recall that we are actually referring to the determinant of the matrices involved. Notice also that, depending on the chosen coordinates x^A on $\mathbb{S}^2$, the switch to (θ,φ) yields a harmless additional contribution to the metric coefficients; this justifies the two symbols g'_{AB} and g_{AB}, although, henceforth, we shall mostly stick to the last one.

It is also remarkable that, whenever $R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu = 0$, γ a generator of $\mathscr{C}_p^+$, one can prove that, on $\mathscr{C}_p^+$, (3) simplifies (see formula (2.36) in [12]) to

$$ds^2 = -\alpha\, du^2 - 2\, du\, dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{5}$$

where the standard coordinates (θ,φ) on the 2-sphere are used in place of x^A. Apart from being much simpler, this form is more closely related to the standard Bondi one which is canonically used in the implementation of bulk-to-boundary techniques as devised in [13, 14] for a large class of asymptotically flat and of cosmological spacetimes. Unfortunately, contrary to these papers, here the scenario is much more complicated and, furthermore, the cone does not seem to display any particular symmetry group to be exploited, such as for example the BMS in [13]. Yet, the situation is not as desperate as one might think, since, ultimately, for our purposes it will only be relevant that the metric at p becomes the Minkowski one, our coordinates being constructed out of an orthonormal frame at p. In particular, this means that, at p, $\sqrt{|g_{AB}|}$ will become proportional to r, which, in this special scenario, can be seen both as the affine null parameter introduced above, or, equivalently, as the standard radial coordinate in Minkowski spacetime constructed out of the orthonormal frame in $T_p(M)\sim\mathbb{R}^4$.

Before concluding this section, we briefly compare (5) with the corresponding expression in Minkowski spacetime, where the flat metric can be written as

$$ds^2 = -dU^2 + 2\, dU\, dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{6}$$

$U \doteq t + r$ denoting the light coordinate constructed out of the time and spherical coordinates. The cone with tip at 0 is characterised by $U = 0$ and, also in this case, r is an affine parameter along the null geodesics emanating from 0. It is important to stress that the pull-back of (3) under $\exp_p^*$ tends to (6) when approaching the point $\exp_p(0) = p$.

Finally, we comment on the behaviour of $\mathscr{D}(p,q)$ under (1). As mentioned before, $\exp_p^{-1}$ does not map a double cone in M into one in $T_p(M)$, but, nonetheless, we can still adapt the choice of q in such a way that $\imath_{e,e'}(\mathscr{D}(p,q))$ is properly contained in a sufficiently large double cone $\mathscr{D}(p,q')\subset\mathcal{O}'_p$.

3 Algebras of Observables on the Bulk and on the Boundary

In the previous section, we focused on the introduction and analysis of the main geometric tools needed. In particular, we recall once more that the main geometrical objects are the double cones $\mathscr{D}(p,q)$ which are globally hyperbolic spacetimes in their own right. Since, in the forthcoming discussion, neither p nor q will play a distinguished role, we shall omit them, hence using $\mathscr{D}$

in place of $\mathscr{D}(p,q)$. More importantly, we are now entitled to introduce a well-defined classical field theory and, for the sake of simplicity, we shall henceforth only deal with a free real scalar field with generic mass m and generic curvature coupling ξ.

Let us recollect some standard properties of such a physical system along the lines, *e.g.*, of [47]. Consider $\varphi : \mathscr{D} \to \mathbb{R}$ which fulfils the following equation of motion,

$$P\varphi \doteq \left(\Box_g + \xi R + m^2\right)\varphi = 0, \quad m^2 > 0 \text{ and } \xi \in \mathbb{R}, \tag{7}$$

where $\Box_g = -\nabla^\mu\nabla_\mu$ is the d'Alembert wave operator constructed out of the metric g while R is the scalar curvature. Since this is a second-order hyperbolic partial differential equation, each solution with smooth and compactly supported initial data on a Cauchy surface can be constructed as the image of the following map,

$$\Delta : C_0^\infty(\mathscr{D}) \to C^\infty(\mathscr{D}), \tag{8}$$

where Δ is the *causal propagator* defined as the difference of the advanced and the retarded fundamental solutions. Furthermore, each $\varphi_f \doteq \Delta(f)$ satisfies the following support property,

$$\text{supp}(\varphi_f) \subseteq J^+(\text{supp}(f)) \cup J^-(\text{supp}(f)),$$

and, if $\mathfrak{S}(\mathscr{D})$ denotes the set of solutions of (8) with smooth compactly supported initial data on any Cauchy surface Σ of $\mathscr{D}$, then this turns out to be a symplectic space when endowed with the weakly non-degenerate symplectic form,

$$\sigma(\varphi_f,\varphi_h) = \int_\Sigma d\mu(\Sigma)\left(\varphi_f\nabla_n\varphi_h - \varphi_h\nabla_n\varphi_f\right) = \int_{\mathscr{D}} d\mu(\mathscr{D})(f\,\Delta h), \quad \forall f,h \in C_0^\infty(\mathscr{D}). \tag{9}$$

Here the integral is independent of the choice of Cauchy surface Σ, as can be noticed from the last equation, while $d\mu(\Sigma)$, $d\mu(\mathscr{D})$ and n are, respectively, the metric-induced measures and the vector normal to Σ.

As a last ingredient, these properties can be exploited in combination with the fact that, by construction, $\mathscr{D}$ is contained in a larger globally hyperbolic open set ($\mathcal{O}'$ in the notation of the previous section), in order to conclude that φ_f can be unambiguously extended to a solution of the very same equation throughout $\mathcal{O}'$. This can be proved by recalling that both $\mathscr{D}$ and $\mathcal{O}'$ are globally hyperbolic and by invoking the uniqueness of the causal propagator. As a consequence we are entitled to consider the restriction of φ_f on $\mathscr{C}_p^+$ which yields

$$\varphi_f\big|_{\mathscr{C}_p^+} \in C^\infty(\mathscr{C}_p^+). \tag{10}$$

3.1 Quantum Algebras on $\mathscr{D}$

After the setup of a classical field theory, we consider a suitable quantisation scheme to be described as a two-fold process: in a first step we shall select a suitable algebra of fields which fulfils the necessary commutation relations and, second, we choose a quantum state as a functional on this algebra in order to compute the expectation values of the relevant observables.

Thus let us proceed in logical sequence starting from $\mathscr{D}$, the bulk spacetime, where we can introduce $\mathscr{F}_b(\mathscr{D})$ as the subset of sequences with a finite number of elements lying in

$$\bigoplus_{n\geq 0} \otimes_s^n C_0^\infty(\mathscr{D}),$$

where $n = 0$ yields $\mathbb{C}$ by definition while $\otimes_s^n$ denotes the n-fold symmetric tensor product. According to this definition it is customary to denote a generic $F \in \mathscr{F}_b(\mathscr{D})$ as a finite sequence $\{F_n\}_n$ where each $F_n \in \otimes_s^n C_0^\infty(\mathscr{D})$. We can now promote $\mathscr{F}_b(\mathscr{D})$ to a topological *-algebra equipping it with

- a tensor product $\cdot_S$ such that

$$(F \cdot_S G)_n = \sum_{p+q=n} \mathcal{S}(F_p \otimes G_q),$$

  where $\mathcal{S}$ is the operator which realises total symmetrisation;

- a *-operation via complex conjugation, *i.e.*, $\{F_n\}_n^* = \{\overline{F}_n\}_n$ for all $F \in \mathscr{F}_b(\mathscr{D})$;

- the topology induced by the natural one of $\otimes_s^n C_0^\infty(\mathscr{D})$.

The above more traditional realisation of $\mathscr{F}_b(\mathscr{D})$ can be replaced by a novel point of view, thoroughly developed in [7, 5]. To be specific, consider $\mathscr{F}_b(\mathscr{D})$ as a suitable subset of the functionals over $C^\infty(\mathscr{D})$, the smooth field configurations. Explicitly, $F \in \mathscr{F}_b(\mathscr{D})$ yields a functional $F : C^\infty(\mathscr{D}) \to \mathbb{R}$ out of the standard pairing between $\otimes^n C^\infty(\mathscr{D})$ and $\otimes^n C_0^\infty(\mathscr{D})$, denoted by $\langle\ ,\ \rangle$, via

$$F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \langle F_n, \varphi^n \rangle. \tag{11}$$

In order to grasp the connection between the two perspectives, it is useful to introduce a Gâteaux derivative,

$$F^{(n)}(\varphi) h^{\otimes n} = \frac{d^n}{d\lambda^n} F(\varphi + \lambda h)\bigg|_{\lambda=0}, \quad \forall h \in C^\infty(\mathscr{D}),$$

so that $F_n \equiv F^{(n)}(0)$. We shall use alternatively both pictures in the forthcoming analysis.

The key point in the subsequent quantisation scheme consists in the modification of the algebraic product $\cdot_S$ to yield a new one, $\star$, which is constructed out of the causal propagator Δ, unambiguously defined according to (8),

$$(F \star G)(\varphi) = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \big\langle F^{(n)}(\varphi), \Delta^{\otimes n} G^{(n)}(\varphi) \big\rangle, \qquad \forall F, G \in \mathscr{F}_b(\mathscr{D}). \tag{12}$$

By direct inspection one realises that $F \star G$ still lies in $\mathscr{F}_b(\mathscr{D})$ and, more importantly, that $(\mathscr{F}_b(\mathscr{D}), \star)$ gets the structure of a *-algebra under the operation of complex conjugation.

It is important to notice that, up to now, we have not used the existence of the equation of motion (7) and, therefore, we can refer to $\mathscr{F}_b(\mathscr{D})$ as an *off-shell* algebra. On the other hand, if one wants to encompass also the dynamics of the classical system, one needs only to divide $\mathscr{F}_b(\mathscr{D})$ by the ideal $\mathscr{I}$ which is the set of elements in $\mathscr{F}_b(\mathscr{D})$ generated by those of the form $P_j F_n(x_1, \ldots, x_n)$, where P_j is the operator in (7) applied to the j-th variable in $F_n \in \otimes_s^n C_0^\infty(\mathscr{D})$. The outcome is the *on-shell* algebra $\mathscr{F}_{bo}(\mathscr{D}) \doteq \frac{\mathscr{F}_b(\mathscr{D})}{\mathscr{I}}$ which inherits the *-operation from $\mathscr{F}_b(\mathscr{D})$ and is nothing but the more commonly used field algebra.

At this stage, it is important to remark that neither $\mathscr{F}_b(\mathscr{D})$ nor its on-shell version $\mathscr{F}_{bo}(\mathscr{D})$ contain all the elements needed to fully analyse the underlying quantum field theory. As a matter of fact, objects such as the components of the stress-energy tensor involve products of fields evaluated at the same spacetime point, an operation which is *a priori* not well-defined due to the distributional nature of the fields themselves. To circumvent this obstruction, a standard procedure calls for the regularisation of these ill-defined objects by means of a suitable scheme which goes under the name of Hadamard regularisation. We shall not dwell here on the technical details but only highlight some aspects in the appendix. The interested reader is referred to [23, 24] for a full account.

In the functional language used before, the problem mentioned above translates into the impossibility to include in $\mathscr{F}_b(\mathscr{D})$ objects of the form

$$F(\varphi) = \int_{\mathscr{D}} d\mu(g)\, f(x)\varphi^2(x),$$

where $d\mu(g)$ is the metric-induced volume form, while f is a test function in $C_0^\infty(\mathscr{D})$ and $\varphi \in C^\infty(\mathscr{D})$. Actually, the star product (12) applied to a couple of such fields involves the ill-defined pointwise product of Δ with itself.

To solve this problem, we shall follow the line of reasoning of [7]. Namely, we introduce a new class of functionals, $\mathscr{F}_e(\mathscr{D})$, which have a finite number of non-vanishing derivatives, the n-th of which has to be a symmetric element of the space of compactly supported distributions $\mathcal{E}'(\mathscr{D}^n)$, whose wave front sets, moreover, satisfy the following restriction

$$\mathrm{WF}(F_n) \cap \left\{ \left(\mathscr{D} \times \overline{V}^+\right)^n \cup \left(\mathscr{D} \times \overline{V}^-\right)^n \right\} = \emptyset, \tag{13}$$

where $\overline{V}^\pm$ corresponds to the forward and to the backward causal cone in the tangent space, respectively. The closure symbol indicates that we also include the tip of the cone in the set of future or past directed causal vectors.

We can make $\mathscr{F}_e(\mathscr{D})$ a *-algebra if we extend naturally the *-operation of $\mathscr{F}_b(\mathscr{D})$ and endow it with a new product, $\star_H$, whose well-posedness was first proved in [8, 6, 23, 24]. The explicit form is realised as

$$(F \star_H G)(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(n)}(\varphi), H^{\otimes n} G^{(n)}(\varphi) \right\rangle, \tag{14}$$

where $H \in \mathcal{D}'(\mathscr{D}^2)$ is the so-called Hadamard bi-distribution. We shall briefly introduce and discuss it in the appendix, but, for our purposes, it is important to recall that, on the one hand, it satisfies the microlocal spectrum condition, hence yielding a natural substitute for the notion of positivity of energy out of its wave front set, while on the other hand it suffers from an ambiguity. At the level of the integral kernel, only the antisymmetric part of the Hadamard bi-distribution is fixed to $i/2$ times the causal propagator Δ. Also the singular structure is unambiguously determined by the choice of the background. Otherwise there always exists the freedom to add a smooth symmetric function which, in our scenario, means that, if H, H' are two Hadamard distributions, then the integral kernel of $H - H'$ is a symmetric element of $C^\infty(\mathscr{D}^2)$. Yet, as far as the algebra is concerned, this freedom boils down to an algebraic isomorphism $\mathfrak{i}_{H',H} : (\mathscr{F}_e(\mathscr{D}), \star_H) \to (\mathscr{F}_e(\mathscr{D}), \star_{H'})$, namely (*cf.* [23, 5])

$$\begin{aligned} \mathfrak{i}_{H',H} &= \alpha_{H'} \circ \alpha_H^{-1}, \\ \alpha_H(F) &\doteq \sum_{n=0}^{\infty} \frac{1}{n!} \big\langle H^{\otimes n}, F^{(2n)} \big\rangle. \end{aligned} \tag{15}$$

As for the algebra generated by compactly supported smooth functions, also the extended one, $\mathscr{F}_e$, has its on-shell counterpart, $\mathscr{F}_{eo}$, constructed from the quotient with the ideal generated by the equation of motion applied to the elements of $\mathscr{F}_e$.

One of the advantages of the formalism employed is the possibility to easily transcribe the overall construction in terms of categories, hence yielding a crystal-clear mathematical picture of the relevant structures and their relations. This was first advocated and utilised in the seminal paper [9], where the principle of general local covariance was formulated in this language to which we shall also stick. In particular, we shall now recast the above discussion in this different perspective, while the actual relation with [9] will only later be outlined in Section 4. Hence, we shall use the following categories:

- DoCo: The objects are defined as follows: For every spacetime M, as in the previous section, we consider the oriented and time-oriented double cones $\mathscr{D}(p,q)$ with the property that there exists a normal neighbourhood $\mathcal{O}_p \subset M$ centred in p that contains $\mathscr{D}(p,q)$. Hence, an object is a triple $\mathscr{D} \equiv [\mathscr{D}(p,q), \mathcal{O}_p, e]$. Recall that since both $\mathcal{O}_p$ and the double cones are globally hyperbolic, the choice of time and space orientation is always possible. The morphisms are the maps $\imath_{e,e'} : \mathcal{O}_p \to \mathcal{O}_{p'}$ introduced in (1) such that $\imath_{e,e'}\big|_{\mathscr{D}}\big(\mathscr{D}(p,q)\big) \subset \mathscr{D}'(p',q')$. Although the subscript which refers to e and e' is a small abuse of notation, we feel that its presence might help the reader to focus on the ingredients here at play. The composition rule for the morphisms is defined in terms of the composition of the maps $\imath_{e,e'}$ and hence the associativity derives from the associativity of the composition. Notice that, as per (1), an arrow between two objects exists only if the source double cone is sufficiently small so that its image under $\exp_p^{-1}$ is contained in the domain of the definition of $\exp_{p'}$. This caveat does not spoil the associativity property of the composition of arrows.

- DoCo_{iso}: This is the subcategory of DoCo obtained by keeping the same objects but restricting the possible morphisms of DoCo to those which are isometric embeddings.

- Alg_i: The objects are unital, topological, *-algebras $\mathscr{F}_i(\mathscr{D})$ with $i = b$, bo, constructed as above, while for $i = e$, eo, we further restrict the objects to the equivalence classes generated by identifying extended algebras which are isomorphic under the map (15). The morphisms are equivalence classes of injective, unit-preserving, *-homomorphisms.

Since the key ingredients to construct both $\mathscr{F}_b(\mathscr{D})$ and $\mathscr{F}_{bo}(\mathscr{D})$ are just the causal propagator Δ from (8) and the operator (7) realising the equations of motion, their uniqueness in any $\mathscr{D}$ suggests that the association of a suitable algebra,

$$\mathscr{F}_i : \mathscr{D} \to \mathscr{F}_i(\mathscr{D}), \qquad i = b,\, bo, \tag{16}$$

with each double cone $\mathscr{D}$ can be promoted to a functor between DoCo_{iso} and Alg_i. To this end, in order to define the action of $\mathscr{F}_i$ on morphisms of DoCo_{iso}, notice that $\mathscr{F}_i(\mathscr{D})$ is an algebra generated by smooth and compactly supported functions on $\mathscr{D}$, hence $\mathscr{F}_i(\imath_{e,e'})$ is just the injective *-homomorphism which associates with every element in $\mathscr{F}_i(\mathscr{D})$ its image under $\imath_{e,e'}$. Furthermore, $\mathscr{F}_i$ defined in that way enjoys the covariance property and maps $\mathrm{id}_{\mathscr{D}}$, the identity of $\mathscr{D}$ in DoCo_{iso}, to $\mathrm{id}_{\mathscr{F}_i(\mathscr{D})}$, the identity of $\mathscr{F}_i(\mathscr{D})$ in Alg_i, *i.e.*,

$$\mathscr{F}_i(\imath_{e,e'}) \circ \mathscr{F}_i(\tilde{\imath}_{e',\tilde{e}'}) = \mathscr{F}_i(\imath_{e,e'} \circ \tilde{\imath}_{e',\tilde{e}'}) \qquad \text{and} \qquad \mathscr{F}_i(\mathrm{id}_{\mathscr{D}}) = \mathrm{id}_{\mathscr{F}_i(\mathscr{D})}\,.$$

Notice that, in the case of the on-shell algebra, the equation of motion is left unchanged by any morphism in DoCo_{iso}.

It is also important to note that singling out extended algebras which are related via (15) ensures that the ambiguity in the choice of the Hadamard bi-distribution does not spoil the well-posedness of (16) when $i = e$. All these assertions can be proved noticing that, due to the discussion presented after (1), the category DoCo_{iso} is just a subcategory of the category of local manifolds introduced in [9] where similar results are discussed.

It would be desirable to extend the functor (16) to the category DoCo that has a larger group of morphisms. Unfortunately, this is not straightforward and, actually, not even possible. If we consider two generic globally hyperbolic regions $\mathscr{D}$ and $\mathscr{D}'$ in DoCo, related by $\imath_{e,e'}$ as in (1), we can draw the following diagram,

$$\begin{array}{ccc} \mathscr{D} & \xrightarrow{\ \mathscr{F}\ } & \mathscr{F}_i(\mathscr{D}) \\ {\scriptstyle \imath_{e,e'}}\Big\downarrow & & \\ \mathscr{D}' & \xrightarrow[\ \mathscr{F}\ ]{} & \mathscr{F}_i(\mathscr{D}') \end{array} \qquad . \tag{17}$$

To have a functor between DoCo and some Alg_i requires existence of a well-defined morphism $\mathscr{F}(\imath_{e,e'})$ between $\mathscr{F}_i(\mathscr{D})$ and $\mathscr{F}_i(\mathscr{D}')$. But this is not possible since, in general, $\imath_{e,e'}$ is not an isometry and, thus, it does neither map solutions of (7) in $\mathscr{D}$ into those of the same equation (but out of a different metric) in $\mathscr{D}'$, nor does it preserve the causal propagator. Hence it spoils the $\star$-operation and does not preserve the singular structure of the Hadamard bi-distribution which only depends upon the underlying geometry.

As an aside, a positive answer to the present question will only be possible considering the off-shell classical *-algebras $(\mathscr{F}_b, \cdot_S)$, but, as soon as quantum algebras are employed, the situation looks grim. Yet it is possible to circumvent this obstruction making profitable use of the geometric properties of the portion of the future directed light cone in any $\mathscr{D}$ in order both to set up a bulk-to-boundary correspondence and to later compare the outcome on the boundary of different spacetimes.

3.2 Quantum Field Theory on the Boundary

In order to fulfil our goals, it is mandatory as a first step to understand how to construct a full-fledged quantum field theory on the light cone and this will be the main aim of this subsection. Our procedure derives from the experience gathered in similar scenarios where a field theory on a null surface was constructed such as, *e.g.*, in [13, 14, 15, 31] (see also [21, 43, 11] for further analyses in similar contexts).

Therefore, following the same philosophy as in these papers, we shall first show that it is possible to assign to the boundary a natural field algebra and that there is a "natural" choice for the relevant quantum state. To this avail, in this subsection, we shall consider the cone as an abstract manifold on its own, not regarding it as a particular portion of the boundary of a specific globally hyperbolic double cone $\mathscr{D}$, since the connection with the bulk theory will be presented only later. Nevertheless, we have to keep in mind that the algebra to be constructed has to be large enough in order to contain the images of some suitable projections of all the elements of the algebra in the bulk $\mathscr{D}$. This will be the most delicate point of the whole construction because it is not sufficient to consider an algebra generated by compactly supported data on the cone; we have to extend this set to more generic elements.

Let us start with the introduction of three distinct sets in $\mathbb{R}\times\mathbb{S}^2\subset\mathbb{R}^4$ relevant for the following construction, namely, employing the standard coordinates, we define

$$\mathscr{C}_p^+=\big\{(V,\theta,\varphi)\in\mathbb{R}\times\mathbb{S}^2\,|\,V\in I\subset\mathbb{R},\,(\theta,\varphi)\in\mathbb{S}^2\big\},\tag{18}$$

where I is the open interval $\big(0,V_0(\theta,\varphi)\big)$ with a positive, bounded smooth function $V_0(\theta,\varphi)$ on the sphere. The other two regions will be denoted $\mathscr{C}_p$ and $\mathscr{C}$, respectively, where the coordinate V is allowed to extend over $(0,\infty)$ or the full real line, so that $\mathscr{C}_p^+\subset\mathscr{C}_p\subset\mathscr{C}$. We stress that, with a slight abuse of notation, we employ the symbol $\mathscr{C}_p^+$ as in the previous sections although we are not referring to an actual cone since, ultimately, (18) will indeed coincide with $J_p^+\cap\partial\mathscr{D}$, if we employ the same conventions and nomenclatures as in the preceding analysis. Furthermore notice that $\mathscr{C}$ is not completely independent of p which still is required to be a point in this set. Yet, to avoid worsening an already cumbersome notation, we leave such a dependence implicit.

As a natural next step, we need to identify a suitable space of functions on the boundary and, to this avail, viewing $\mathscr{C}_p$ immersed in $\mathbb{R}^4$, we define

$$\mathscr{S}(\mathscr{C}_p)\doteq\Big\{\psi\in C^\infty(\mathscr{C}_p)\,|\,\psi=hf\big|_{\mathscr{C}_p},\,f\in C_0^\infty(\mathbb{R}^4)\text{ and }h\in C^\infty(\mathscr{C}_p)\Big\},\tag{19}$$

where h vanishes uniformly on $\mathbb{S}^2$, as $V\to 0$, while each derivative along V tends to a constant uniformly on $\mathbb{S}^2$. As far as this subsection is concerned we can safely choose h to be equal to

V. Furthermore, $\mathscr{S}(\mathscr{C}_p)$ turns out to be a symplectic space when endowed with the following strongly non-degenerate symplectic form,

$$\sigma_{\mathscr{C}}(\psi,\psi') \doteq \int_{\mathscr{C}_p}\left[\psi\frac{d\psi'}{dV}-\frac{d\psi}{dV}\psi'\right] dV\wedge d\mathbb{S}^2, \quad \forall\psi,\psi'\in\mathscr{S}(\mathscr{C}_p), \tag{20}$$

where $d\mathbb{S}^2$ is the standard measure on the unit 2-sphere.

The reason for such an apparently strange choice of $\mathscr{S}(\mathscr{C}_p)$ is the later need to relate the theory on these sets to the one in the bulk of a double cone. Hence the most natural choice of compactly supported smooth functions on $\mathscr{C}_p$ would not fit into the overall picture since a general solution of the Klein-Gordon equation (7) with smooth compactly supported initial data on some $\mathscr{D}$ would propagate on the light cone $\mathscr{C}_p$ to a function which is also supported on the tip. This point corresponds to $V=0$ in the above picture and, thus, it does not lie in $\mathscr{C}_p$. On the contrary, we shall show in the next subsection that (19) is indeed the natural counterpart on the boundary which arises from the set of solutions of (7).

In order to introduce the relevant algebra of observables, we follow the same philosophy as in Subsection 3.1, introducing $\mathscr{A}_b(\mathscr{C}_p)$, whose generic element F' is a sequence $\{F'_n\}_n$ with a finite number of elements in

$$\bigoplus_{n\geq 0}\otimes^n_s\mathscr{S}(\mathscr{C}_p), \tag{21}$$

where $\otimes^n_s$ again denotes the symmetrised n-fold tensor product and the first term in the sum is $\mathbb{C}$. Notice that the $'$-superscript is introduced in this subsection in order to avoid a potential confusion with the similar symbols used for the counterpart in the bulk. In order to promote (21) to a full topological *-algebra, we have to endow it with

- a *-operation which is the complex conjugation, *i.e.*, $\{F'_n\}^*_n = \{\overline{F'_n}\}_n$ for all $F'\in\mathscr{A}_b(\mathscr{C}_p)$;
- multiplication of elements such that for any F', $G'\in\mathscr{A}_b(\mathscr{C}_p)$,

$$(F'\cdot_S G')_n = \sum_{p+q=n}\mathcal{S}(F'_p\otimes G'_q);$$

- the topology induced by the natural topology of $\mathscr{S}(\mathscr{C}_p)$, namely the topology of smooth functions on $\mathscr{C}_p$.

Although well-defined, this algebra is not suited to be put in relation to data in the bulk and, thus, the above product has to be deformed once more. To this avail, we employ the functional point of view as in (11), *i.e.*, if $X\doteq\left\{\Phi\in C^\infty(\mathscr{C}_p)\mid V^{-1}\Phi\in C^\infty(\mathscr{C})\right\}$, then $F'\in\mathscr{A}_b(\mathscr{C}_p)$ yields a functional $F': X\to\mathbb{R}$ out of $\langle\ ,\ \rangle$, the pairing between $\otimes^n X$ and $\otimes^n\mathscr{S}(\mathscr{C}_p)$,

$$F'(\Phi)=\sum_{n=0}^{\infty}\frac{1}{n!}\langle F'_n,\Phi^n\rangle, \quad \forall\Phi\in X.$$

By direct inspection of the definition of $\mathscr{S}(\mathscr{C}_p)$ in (19), one notices that $\langle\ ,\ \rangle$ is nothing but the standard inner product on $(0,\infty)\times\mathbb{S}^2$ between compactly supported functions and smooth ones, taken with respect to the measure $dV\wedge d\mathbb{S}^2$.

Although the theory on $\mathscr{C}_p$ has no equation of motion built in and, hence, no causal propagator such as (8), we can nonetheless introduce a new $\star_B$-product on $\mathscr{A}_b(\mathscr{C}_p)$, namely

$$(F'\star_B G')(\Phi)\doteq\sum_{n=0}^{\infty}\frac{i^n}{2^n n!}\big\langle F'_n(\Phi),\Delta^n_{\sigma_{\mathscr{C}}}G'_n(\Phi)\big\rangle,\quad\forall\Phi\in X, \tag{22}$$

where $\Delta_{\sigma_{\mathscr{C}}}$ is the integral kernel of (20), *i.e.*,

$$\Delta_{\sigma_{\mathscr{C}}}\big((V,\theta,\varphi),(V',\theta',\varphi')\big)=-\frac{\partial^2}{\partial V\partial V'}\,\mathrm{sign}(V-V')\,\delta(\theta,\theta'), \tag{23}$$

where $\delta(\theta,\theta')$ stands for $\delta(\theta-\theta')\delta(\varphi-\varphi')$ and the derivatives have to be taken in the weak sense. Notice that $\Delta_{\sigma_{\mathscr{C}}}$ is defined as a distribution on $C^\infty_0(\mathscr{C}^2_p)$ which, by direct inspection, can be extended also to $\mathscr{S}^2(\mathscr{C}^2_p)$. Finally, $\star_B$ is well-defined because only a finite number of elements appear in the sum on the right-hand side of (23) and, thus, convergence is not an issue here. We can hence conclude this subsection with a proposition whose proof follows from the preceding discussion.

Proposition 3.1. *The pair $\big(\mathscr{A}_b(\mathscr{C}_p),\star_B\big)$ equipped with the *-operation introduced above is a well defined *-algebra.*

In the next section we shall discuss the form of a certain class of quantum states on this algebra to eventually use them to extend the boundary algebra of observables analogous to the procedure on the bulk.

3.3 Natural Boundary States

The next step in our construction is the introduction of an extended algebra on the boundary, but, in the present scenario, there is no standard definition of an Hadamard state or of a bi-distribution; and this lack of a class of *a priori* physically relevant states hinders an imitative repetition of the use of the function H as in (14). Therefore, a natural bi-distribution on $\mathscr{C}^+_p$ is needed and, for this purpose, our choice is the following weak limit,

$$\omega\big((V,\theta,\varphi),(V',\theta',\varphi')\big)\doteq-\frac{1}{\pi}\lim_{\epsilon\to0^+}\frac{1}{(V-V'-i\epsilon)^2}\,\delta(\theta,\theta'), \tag{24}$$

which has the advantage of being at the same time a well-defined element of $\mathcal{D}'(\mathscr{C}^2)$, where $\mathscr{C}\sim\mathbb{R}\times\mathbb{S}^2$ and $\mathcal{D}'$ denotes the space of distributions over the test functions in $C^\infty_0(\mathscr{C})$.

Such an expression already appeared in different, albeit related scenarios where a scalar quantum field theory was studied [13, 15, 16, 31, 27, 43, 44]. It is important to remark that in the first two of these papers, (24) was actually used as the building block to construct a quasi-free pure state for a scalar field theory built on a three-dimensional null cone which represented the conformal boundary of a suitable class of spacetimes. In all these cases, the particular geometric structure as well as the presence of a particular symmetry group entails that (24) fulfils suitable uniqueness properties and, furthermore, gives rise to a full-fledged Hadamard state in the bulk. Therefore we shall employ the above expression as the natural candidate bi-distribution on the boundary, proving in the next sections that, when we realise $\mathscr{C}_p^+$ as part of the boundary of a double cone, we can also construct a physically meaningful state on the bulk $\mathscr{D}$ out of (24).

The above bi-distribution can be read as a functional $\omega : \mathscr{S}(\mathscr{C}_p) \times \mathscr{S}(\mathscr{C}_p) \to \mathbb{R}$, but it is actually much more convenient to recall that $\mathscr{C}_p \subset \mathscr{C}$. Within this perspective, since $\mathscr{C}$ is topologically $\mathbb{R} \times \mathbb{S}^2$ and each element in $\mathscr{S}(\mathscr{C}_p)$, together with the V-derivative, also lies in $L^2(\mathscr{C}, dV \wedge d\mathbb{S}^2)$, the following expression for the 2-point function is meaningful,

$$\omega(\psi, \psi') = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^2 \times \mathbb{S}^2} dV dV' d\mathbb{S}^2(\theta, \varphi) \, \frac{\psi(V, \theta, \varphi)\psi'(V', \theta, \varphi)}{(V - V' - i\epsilon)^2}, \tag{25}$$

where the delta function over the angular coordinates is already integrated out. The distribution (25) satisfies a suitable continuity condition, as for example shown in [30],

$$|\omega(\psi, \psi')| \leqslant C\big(\|\psi\|_{L^2} \|\partial_V \psi\|_{L^2}\big)\big(\|\psi'\|_{L^2} + \|\partial_V \psi'\|_{L^2}\big) < \infty, \tag{26}$$

which allows for the extension of ω to the space of square-integrable functions whose derivative along the V-coordinate is also an element of L^2. Furthermore, this also entails the possibility to perform a Fourier-Plancherel transform along V (*cf.* Appendix C in [30]) resulting in a much more manageable form for (25),

$$\omega(\psi, \psi') = \int_{\mathbb{R} \times \mathbb{S}^2} dk \, d\mathbb{S}^2(\theta, \varphi) \; 2k \, \Theta(k) \, \overline{\widehat{\psi}(k, \theta, \varphi)} \, \widehat{\psi}'(k, \theta, \varphi) \,, \tag{27}$$

where $\Theta(k)$ is the step function equal to 1 if $k \geqslant 0$ and 0 otherwise. It should be stressed that the presence of $\Theta(k)$ corresponds to the physical intuition of taking only positive frequencies, also because, on the cone, the only causal directions are the lines at constant angular variables. Under special circumstances this idea has a clear connection with the geometrical bulk data as well as with the Hadamard property of a bulk state constructed out of (24) (*cf.* for example [29, 30, 15]).

As a last step in this subsection, we underline that the above analysis entails two relevant remarks. The first one concerns the wave front set of the bi-distribution ω on $\mathscr{C}^2 \sim (\mathbb{R} \times \mathbb{S}^2)^2$. This was already studied in Lemma 4.4. of [30] yielding

$$\mathrm{WF}(\omega) \subseteq A \cup B, \tag{28}$$

where

$$\begin{aligned} A = \big\{ \big((V, \theta, \varphi, \zeta_V, \zeta_\theta, \zeta_\varphi), (V', \theta', \varphi', \zeta_{V'}, \zeta_{\theta'}, \zeta_{\varphi'})\big) \in (T^*\mathscr{C})^2 \setminus \{0\} \mid \\ V = V', \, \theta = \theta', \, \varphi = \varphi', \, 0 < \zeta_V = -\zeta_{V'}, \, \zeta_\theta = -\zeta_{\theta'}, \, \zeta_\varphi = -\zeta_{\varphi'} \big\} \end{aligned} \tag{29}$$

and

$$B = \big\{ \big((V, \theta, \varphi, \zeta_V, \zeta_\theta, \zeta_\varphi), (V', \theta', \varphi', \zeta_{V'}, \zeta_{\theta'}, \zeta_{\varphi'}) \big) \in (T^*\mathscr{C})^2 \setminus \{0\} \mid \\ \theta = \theta',\, \varphi = \varphi',\, \zeta_V = \zeta_{V'} = 0,\, \zeta_\theta = -\zeta_{\theta'},\, \zeta_\varphi = -\zeta_{\varphi'} \big\}. \quad (30)$$

Although, at this stage, this result is only an aside, it will play a pivotal role in the discussion of Subsections 3.4 and 3.5. In particular, if we recall that $\mathscr{C}_p \subset \mathscr{C}$, it turns out that the wave front set of ω on $C_0^\infty(\mathscr{C}_p^2)$ can only be smaller or equal to $A \cup B$, and actually it corresponds to $A \cup B$ restricted to $(T^*\mathscr{C}_p)^2$.

As a second remark, note that it is possible to construct a new algebra, say $\mathscr{A}_b(\mathscr{C})$ on the full $\mathscr{C}$ starting from (19) and considering the set $L^2(\mathscr{C}, dV \wedge d\mathbb{S}^2)$ in place of $\mathscr{S}(\mathscr{C}_p)$, while keeping the same *-operation and composition rule. On the one hand, it is straightforward, that $\mathscr{A}_b(\mathscr{C}_p)$ is a *-subalgebra of $\mathscr{A}_b(\mathscr{C})$ while, on the other hand, we can see that the two-point function ω as in (24) can be used as a building block of a quasi-free state for $\mathscr{A}_b(\mathscr{C})$. Hence the same conclusion can be drawn for $\mathscr{A}_b(\mathscr{C}_p)$ since, by construction, the antisymmetric part of ω is equal to $\frac{i}{2}\Delta_\sigma$. The only possible issue is positivity, but this is solved by direct inspection of (27) whose right-hand side is manifestly greater than 0 once $\psi = \psi'$. It is important to point out, for the sake of completeness, that an almost identical analysis appears in [13, 15], though performed at the level of Weyl algebras. In summary, we get

Proposition 3.2. *The Gaussian (quasi-free) state constructed out of the distribution ω enjoys the following properties:*

1. *It is a well-defined algebraic state on $\mathscr{A}_b(\mathscr{C}_p)$ and on $\mathscr{A}_b(\mathscr{C})$.*
2. *It is a vacuum with respect to ∂_V, in the sense given in [40].*
3. *It is invariant under the change of the local frame, hence invariant under the action of $SO_0(1,3)$.*

Proof. The first point can be analysed by checking linearity, positivity and normalisability of the state on $\mathscr{A}_b(\mathscr{C}_p)$. Since the state is quasi-free, it is enough to examine these properties for the functional ω on $\mathscr{S}(\mathscr{C}_p) \times \mathscr{S}(\mathscr{C}_p)$ where they follow from the previous discussion.

The proof of the second point derives from the observation that ω, as a state on $\mathscr{A}_b(\mathscr{C})$, is invariant under translations and from the fact that the Fourier-Plancherel transform of the integral kernel of ω along the V-direction only contains positive frequencies, as is clear from (27).

The third point can be proved recalling a result in [13], namely, ω on $\mathscr{A}_b(\mathscr{C})$ is invariant under the action of an infinite-dimensional group, the so-called Bondi-Metzner-Sachs group (BMS). In short, if one switches from the coordinates (V, θ, φ) to $(V, z, \bar{z})$ obtained out of a stereographic projection, the BMS maps

$$\begin{cases} z \mapsto z' = \Lambda(z) \doteq \frac{az+b}{cz+d}, \quad ad - bc = 1, \quad a,\, b,\, c,\, d \in \mathbb{C}, \\ V \mapsto V' = K_\Lambda(z, \bar{z})(V + \alpha(z, \bar{z})), \end{cases}$$

where $\bar{z}$ transforms as the complex conjugate of z, $\alpha(z,\bar{z}) \in C^\infty(\mathbb{S}^2)$ and

$$K_\Lambda(z,\bar{z}) \doteq \frac{1+|z|^2}{|az+d|^2 - |bz+c|^2}.$$

Hence, by direct inspection of the above formulae, the BMS group is seen to be the regular semidirect product $SL(2,\mathbb{C}) \ltimes C^\infty(\mathbb{S}^2)$. Most notably one observes that there exists a proper subgroup which is homomorphism to $SO(3,1)$ and thus the state ω turns out to be invariant under the group sought-for. □ □

3.4 Extended Algebra on the Boundary

In the previous subsection, we introduced the boundary algebra together with a suitable notion of $\star$-product, but this is still not sufficient to intertwine the boundary data with those on the bulk because we lack a counterpart for the extended algebra of observables on $\mathscr{C}_p$. Yet, due to the results of the last subsection, we have all the ingredients to construct it.

As a starting point, we define the building block of the extended algebra as follows.

Definition 3.1. *We call $\mathcal{A}^n$ the set of elements $F'_n \in \mathcal{D}'(\mathscr{C}_p^n)$ that fulfil the following properties:*

1. ***Compactness**: The F'_n are compact towards the future, i.e., the support of F'_n is contained in a compact subset of $\mathscr{C}^n \sim (\mathbb{R}\times\mathbb{S}^2)^n$.*

2. ***Causal non-monotonic singular directions**: The wave front set of F'_n contains only causal non-monotonic directions which means that*

$$WF(F'_n) \subseteq W_n \doteq \left\{(x,\zeta) \in (T^*\mathscr{C}_p)^n \setminus \{0\} \mid (x,\zeta) \notin \overline{V}_n^+ \cup \overline{V}_n^-,\ (x,\zeta) \notin S_n\right\}, \tag{31}$$

   where $(x,\zeta) \equiv (x_1,\dots,x_n,\zeta_1,\dots,\zeta_n) \in \overline{V}_n^+$ if, employing the standard coordinates on $\mathscr{C}_p$, for all $i = 1, \dots, n$, $(\zeta_i)_V > 0$ or ζ_i vanishes. The subscript V here refers to the component along the V-direction on $\mathscr{C}_p$. Analogously, we say $(x,\zeta) \in \overline{V}_n^-$ if every $(\zeta_i)_V < 0$ or ζ_i vanishes. Furthermore, $(x,\zeta) \in S_n$ if there exists an index i such that, simultaneously, $\zeta_i \neq 0$ and $(\zeta_i)_V = 0$.

3. ***Smoothness Condition**: The distribution F'_n can be factorised into the tensor product of a smooth function and an element of $\mathcal{A}^{n-1}$ when localised in a neighbourhood of $V = 0$, i.e., there exists a compact set $\mathcal{O} \subset \mathscr{C}_p$ such that, if $\Theta \in C_0^\infty(\mathscr{C}_p)$ so that it is equal to 1 on $\mathcal{O}$ and $\Theta' \doteq 1 - \Theta$, then for every multi-index P in $\{1,\dots,n\}$ and for every $i \leqslant n$,*

$$f \doteq \tilde{F}'_n(u_{x_{P_{i+1}},\dots,x_{P_n}})\,\Theta'_{x_{P_1}} \cdots \Theta'_{x_{P_i}} \in C^\infty(\mathscr{C}_P^i), \tag{32}$$

   where $\tilde{F}'_n : C_0^\infty(\mathscr{C}_p^{n-i-1}) \to \mathcal{D}'(\mathscr{C}_p^i)$ is the unique map from $C_0^\infty(\mathscr{C}_p^{n-i-1})$ to $\mathcal{D}'(\mathscr{C}_p^i)$ determined by F'_n using the Schwartz kernel theorem. Furthermore, $u_{x_{P_{i+1}},\dots,x_{P_n}} \in C_0^\infty(\mathscr{C}_p^{n-i})$, and we have specified the integrated variables $x_{P_{i+1}}, \dots, x_{P_n}$. For every $j \leqslant i$, $\partial_{V_1}\cdots\partial_{V_j} f$ lies in $C^\infty(\mathscr{C}_p^i) \cap L^2(\mathscr{C}_p^i, dV_{P_1} \wedge d\mathbb{S}^2_{P_1} \cdots dV_{P_i} \wedge d\mathbb{S}^2_{P_i}) \cap L^\infty(\mathscr{C}_p^i)$, while the limit of f as V_j tends uniformly to 0 vanishes uniformly in the other coordinates.

Remark:

a) In property 2 of Definition 3.1 we required that $\mathrm{WF}(F'_n) \cap S_n = \emptyset$, *viz.*, no spatial directions are present in the wave front set of F'_n. Even if such an extra condition is not stipulated in the definition of the elements on $\mathscr{F}_e$, here we have to add it because, later on, we want to multiply elements of $\mathcal{A}$ with ω and in the wave front set of ω there are spatial directions, to be specific, the intersection $\mathrm{WF}(\omega) \cap S_2$ is not empty.

b) Thanks to the smoothness condition, the last requirement in Definition 3.1, the distributions in $\mathcal{A}^n$ can be extended over $\otimes^n \mathscr{S}(\mathscr{C}_p)$ and such an extension is unique.

Notice that $\mathscr{S}(\mathscr{C}_p)$ is strictly contained in $\mathcal{A}$ and that the candidate to play the role of the extended algebra on $\mathscr{C}_p$ thus is

$$\mathscr{A}_e(\mathscr{C}_p) = \bigoplus_{n \geqslant 0} \mathcal{A}^n_s,$$

where $\mathcal{A}^n_s$ is the subset of totally symmetric elements in $\mathcal{A}^n$ defined in Definition 3.1. Moreover, the first space in the previous direct sum is $\mathbb{C}$ and only sequences with a finite number of elements are considered. We can now endow this set with the structure of *-algebra by introducing the *-operation $\{F'_n\}^* \doteq \{\overline{F'_n}\}$ for all $F' \in \mathscr{A}_e$. The composition law arises from a modification of $\star_B$ by means of the state constructed in the sequel of (24). It is *a priori* clear that such a procedure intrinsically depends on the particular ω considered. Nonetheless, our choice will later be justified both through its connection with the bulk data and by well-posedness of the new structure. If we stick to the functional representation, we can thus introduce

$$\begin{aligned}
&\star_\omega : \mathscr{A}_e(\mathscr{C}_p) \times \mathscr{A}_e(\mathscr{C}_p) \to \mathscr{A}_e(\mathscr{C}_p), \\
&(F' \star_\omega G')(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \big\langle F'^{(n)}(\Phi), \omega^n G'^{(n)}(\Phi) \big\rangle,
\end{aligned} \tag{33}$$

for all F', $G' \in \mathscr{A}_e(\mathscr{C}_p)$ and for all $\Phi \in C^\infty(\mathscr{C}_p)$.

Proposition 3.3. *The operation* (33) *is a well-defined product in* $\mathscr{A}_e$.

Proof. As a starting point, notice that (33) is bilinear by construction and that, by definition of $\mathscr{A}_e(\mathscr{C}_p)$, there are only a finite number of non-vanishing elements $F'^{(n)}$ and $G'^{(n)}$. Accordingly, (33) consists of finite linear combinations of terms that formally look like

$$\mathbb{S} \int_{\mathscr{C}_p^{2k}} F'_j(x_1, \ldots, x_j)\, \omega(x_1, y_1) \cdots \omega(x_k, y_k)\, G'_l(y_1, \ldots, y_l) \prod_{i=1}^{k} d\mu(x_i)\, d\mu(y_i) \tag{34}$$

with $k \leqslant j$ and $k \leqslant l$, while $\mathbb{S}$ realises symmetrisation in the non-integrated variables and $d\mu$ is the measure $dV \wedge d\mathbb{S}^2(\theta, \varphi)$ on $\mathscr{C}_p$, written here in the usual coordinates. Therefore, the proof amounts to showing that it is possible to give a rigorous meaning to expressions like (34) and that the result of such an operation is an element of $\mathcal{A}^{j+l-2k}$.

First, we follow the proof of [23], and, to this avail, examine if (34) can be seen as the target of $1 \in C^\infty(\mathscr{C}_p^{2k})$ under the linear map determined, with the help of the Schwartz kernel theorem, by the distribution resulting from multiplication of the two distributions $\omega^{\otimes k} \otimes I^{\otimes(j+l-2k)} \in \mathcal{D}'(\mathscr{C}^{j+l})$ and $F'_j \otimes G'_l \in \mathcal{A}^{j+l}$, where I denotes the identity operator on $\mathcal{A}^1$. Let us start by discussing the well-posedness of the multiplication of distributions presented above. This can be checked by examining the structure of the wave front set of the single objects to verify that their composition never contains the zero section. The key ingredients for this can be readily inferred using Theorem 8.2.9 in [25],

$$\mathrm{WF}(F'_j \otimes G'_l) \subset \big(W_j \cup \{0\}\big) \times \big(W_l \cup \{0\}\big) \setminus \{0\}, \tag{35}$$

and

$$\mathrm{WF}(\omega^{\otimes k} \otimes I^{j+l-2k}) \subset \big(A \cup B \cup \{0\}\big)^k \times \{0\} \setminus \{0\}, \tag{36}$$

where, as usual, we have not specified the dimension of the zero section in the cotangent space. Furthermore, A, B and W_j are defined in (29), (30) and (31), respectively. It is now possible to apply Theorem 8.2.10 in [25] since the above wave front sets never sum up to the zero section. This is tantamount to realising that for every n and m, since $A^n \subset \overline{V}_n^+ \times \overline{V}_n^-$ and $\overline{V}_n^\pm \cap W_n = \emptyset$, $B^n \times \{\mathbb{R}^3\}^m \cap W_{2n+m} = \emptyset$ and $A^n \cap (W_n \times W_n) = \emptyset$. The outcome is that, in (34), the pointwise product of $F'_j \otimes G'_l \in \mathcal{D}'(\mathscr{C}_p^{j+l})$ with $\omega^{\otimes k} \otimes I^{j+l-2k} \in \mathcal{D}'(\mathscr{C}_p^{j+l})$ is still a well-defined element of $\mathcal{D}'(\mathscr{C}_p^{j+l})$, whose wave front set satisfies the inclusion

$$\mathrm{WF}\big(F'_j \otimes G'_l \cdot \omega^{\otimes k} \otimes I^{j+l-2k}\big) \subset \mathrm{WF}(F_j \otimes G_l) \cup \{0\} + \mathrm{WF}\big(\omega^{\otimes k} \otimes I^{j+l-2k}\big) \cup \{0\}, \tag{37}$$

where, as usual, the sum of two wave front sets is defined as the sum on the fibres of the cotangent spaces. Unfortunately, this does not suffice to show the well-posedness of (34). Since F'_j and G'_l are not compactly supported on $\mathscr{C}_p^j$ and $\mathscr{C}_p^l$, respectively, their product does not lie in $\mathcal{E}'\big(\mathscr{C}_p^{j+l}\big)$. Hence we cannot directly test the linear map stemming from $(F'_j \otimes G'_l) \cdot \big(\omega^{\otimes k} \otimes I^{j+l-2k}\big)$ on the unit constant function on $\mathscr{C}_p^{2k}$ in order to infer that (34) is an element in $\mathcal{A}^{j+l-2k}$.

Let us hence proceed by showing that in the case $k = 1$ we can test the linear map arising from $(F'_j \otimes G'_l) \cdot \big(\omega \otimes I^{j+l-2}\big)$ on 1 and that the result of this operation is an element of $\mathcal{A}^{j+l-2}$. The case for a generic k arises from of a recursive application of the very same procedure and, eventually, the application of an operator realising the total symmetrisation. Thus we are interested in

$$\int_{\mathscr{C}_p^2} (F'_j \otimes G'_l) \cdot \big(\omega \otimes I^{j+l-2}\big)\, d\mu(x_1)\, d\mu(y_1),$$

where $F'_j \in \mathcal{A}^j$ and $G'_l \in \mathcal{A}^l$. We exploit property 3 of Definition 3.1 and notice that, if the smoothness condition holds for a compact set $\mathcal{O}$, it also holds for every larger compact set $\mathcal{O}_1$ containing $\mathcal{O} \in \mathscr{C}_p$. We can thus find a common set $\mathcal{O}_1$ for which the smoothness condition property is true at the same time for $F'_j = (\Theta + \Theta')\, F'_j$ and $G'_l = (\Theta + \Theta')\, G'_l$ with respect to a common compactly supported function Θ equal to 1 on $\mathcal{O}_1$. Effectively, the above integral is divided into the sum of four different ones, which we now analyse separately.

Part I) The first term is

$$\int_{\mathscr{C}_p^2} (\Theta(x_1)F_j' \otimes \Theta(y_1)G_l') \cdot \big(\omega \otimes I^{j+l-2}\big)\, d\mu(x_1)\, d\mu(y_1). \tag{38}$$

In this case the integral can be considered as the smearing of a distribution in $\mathcal{D}'\big(\mathscr{C}_p^{j+l}\big)$ with a test function in $C_0^\infty(\mathscr{C}_p^2)$. Hence, Theorem 8.2.12 of [25] ensures that, using the notation introduced there, the result of (38) is a distribution whose wave front set is contained in $W_{j+l-2}\cup (W_j \times W_l) \circ (A \times \{0\}) \subset W_{j+l-2}$ as given in (31). Notice that, in the proof of the last inclusion, we have used (35) and (36). Hence property 2 in Definition 3.1 holds. That said, property 1 is automatically satisfied since, by hypothesis, $F_j' \in \mathcal{A}^j$ and $G_l' \in \mathcal{A}^l$, while property 3 holds true for the resulting distribution as (32) is valid *a priori* in all variables and, thus, left untouched for those which have not been integrated out in (38).

Part II) The second term is

$$\int_{\mathscr{C}_p^2} (\Theta'(x_1)F_j' \otimes \Theta'(y_1)G_l') \cdot \big(\omega \otimes I^{j+l-2}\big)\, d\mu(x_1)\, d\mu(y_1). \tag{39}$$

The analysis is rather simple if we make profitable use of (32) in the integrated variables and interpret the previous integral in the weak sense. Namely, let $f \doteq \Theta'(x_1)F_j'(u)$ for some $u \in C_0^\infty\big(\mathscr{C}_p^{j-1}\big)$ and $f' \doteq \Theta'(x_{j+1})G_l'(u')$ for some $u' \in C_0^\infty\big(\mathscr{C}_p^{l-1}\big)$, then we have that the operation $\omega(f, f')$ is well defined due to the continuity property (26) satisfied by ω. Hence, properties 1, 2 and 3 in Definition 3.1 hold true because they are satisfied by $\Theta'(x_1)F_j' \otimes \Theta'(y_1)G_l'$.

Parts III & IV) The remaining two terms are substantially identical and we treat only one of them. Hence consider

$$\int_{\mathscr{C}_p^2} (\Theta F_j' \otimes \Theta' G_l') \cdot \big(\omega \otimes I^{j+l-2}\big)\, d\mu(x_1)\, d\mu(y_1). \tag{40}$$

In order to cope with this integral we introduce a new larger factorisation $\eta + \eta' = 1$ with $\eta \in C_0^\infty(\mathscr{C}_p)$ such that $\eta = 1$ on a large compact set properly containing the closure of $\mathrm{supp}(\Theta)$ so that both $\mathrm{supp}(\eta'\Theta') \cap \mathrm{supp}(\Theta) = \emptyset$ and $\eta'\Theta' = \eta'$. If now G_l' is substituted with $(\eta + \eta')G_l'$ we obtain another splitting. On the one hand, since $\Theta'\eta \in C_0^\infty(\mathscr{C}_p)$, the analysis of $\Theta F_j' \otimes \Theta'\eta G_l'$ boils down to that of case I, while, on the other hand, $\frac{\Theta(V)[\Theta'\eta'](V')}{(V-V')^2}$ turns out to be smooth on $\mathscr{C}_p$, since by construction $(V - V')^2 > 0$ for V on the support of Θ and V' on that of η'. Hence, if we write the smoothness condition (32) by means of η as $\eta'(x_{j+1})G_l'(t_{l-1}) = f(x_{j+1})$, where $t_{l-1} \in C_0^\infty\big(\mathscr{C}_p^{l-1}\big)$, we obtain that $u \doteq \Theta\omega(f)$ is a compactly supported smooth function on $\mathscr{C}$, thus yielding that $F_j'(u)$ is a well-posed operation as u is compactly supported. Furthermore, in order to conclude the analysis of the present case, due to Theorem 8.2.12 in [25], we notice that the wave front set of (40) is contained in W_{j+l-2} given in (31) and that property 3 in Definition 3.1 holds true just by applying (32) before smearing it. □ □

The result of this subsection is that $\big(\mathscr{A}_e(\mathscr{C}_p), \star_\omega\big)$ is a full-fledged topological *-algebra. Furthermore, due to the compactness property stated in Definition 3.1, the subalgebra $\big(\mathscr{A}_e\big(\mathscr{C}_p^+\big), \star_\omega\big)$

defined by restriction of the domain of the test distributions to $\mathscr{C}_p^+$ is a well-defined topological *-algebra and, thus, we are ready to discuss the intertwining relations between bulk and boundary data.

3.5 Interplay Between the Algebras and the States on $\mathscr{D}$ and on $\mathscr{C}_p^+$

We are now in the position to discuss a connection between the field theories described above, hence setting up a bulk-to-boundary correspondence and identifying an Hadamard state in the bulk. The whole subsection is devoted to this issue, but, as a starting point, we need to recapitulate the geometric structure in order to clearly relate Subsections 2.2 and 3.2.

Recall that we consider the globally hyperbolic subset $\mathcal{O}'$ contained in a geodesic neighbourhood of an arbitrary but fixed point p in a strongly causal spacetime M. In $\mathcal{O}'$ we single out a double cone $\mathscr{D} \equiv \mathscr{D}(p,q)$, which plays the role of the bulk spacetime, while the set $\partial J^+(p) \cap \overline{\mathscr{D}}$ is our selected boundary. Up to the choice of an orthonormal frame in p, the latter can be seen as the locus $u = 0$ in the natural coordinate system $\left(u, r, x^A\right)$ introduced in Subsection 2.2 which is furthermore endowed with the metric (3). In terms of the structure of Subsection 3.2, we can identify the boundary as $\mathscr{C}_p^+$ with a small caveat with respect to the coordinates used. While it is always possible to switch from x^A ($A = 1,\ 2$) to the standard (θ, φ), the role of V as a coordinate is played by r, the affine parameter on the null geodesics of the cone. As a last point, the role of the function h in (19) is taken in general by $\sqrt[4]{|g_{AB}|}$, where g_{AB} are the metric components appearing in (3) evaluated at $u = 0$ and $|\cdot|$ is kept to indicate the determinant. It is interesting to notice, that, whenever the conditions for the reduction of (3) to (5) are fulfilled, h can be set to $V \equiv r$, (see also the relation between the volume elements of the sphere in different coordinates (4)). Furthermore, in the retarded coordinates used, the exponential map becomes an identity. Hence, if not strictly necessary, we shall not indicate it anymore.

Now we proceed in two steps. The first one consists in proving the possibility of introducing a well-defined map from the extended algebra in $\mathscr{D}$ to the one on $\mathscr{C}_p^+ \subset \mathscr{C}_p$, while, in the second, we prove that this map is also well-behaved with respect to the algebra structures.

Theorem 3.1. *Let $\mathscr{D}$ be a double cone and regard the portion $\mathscr{C}_p^+$ of the boundary as part of a cone $\mathscr{C}_p$. Let us introduce the linear map $\Pi : \mathscr{F}_e(\mathscr{D}) \to \mathscr{A}_e(\mathscr{C}_p^+)$ by setting*

$$\Pi_n(F_n) \doteq \sqrt[4]{|g_{AB}|}_1 \cdots \sqrt[4]{|g_{AB}|}_n\, \Delta^{\otimes n}(F_n)\big|_{(\mathscr{C}_p^+)^n}, \tag{41}$$

where Δ is the causal propagator (8), $|_{\mathscr{C}_p^+}$ denotes the restriction on $\mathscr{C}_p^+$ and the subscripts 1, …,n entail dependence of the root on the coordinates of the i-th cone. Then, the following properties hold true:

1) $\hat{\Pi}_n$, the integral kernel of Π_n, is equal to $\otimes^n \hat{\Pi}_1$ and is an element of $\mathcal{D}'\big((\mathscr{C}_p^+ \times \mathscr{D})^n\big)$. The wave front set of $\hat{\Pi}_n$ satisfies

$$WF(\hat{\Pi}_n) \subset (\,WF(\hat{\Pi}_1) \cup \{0\})^n \setminus \{0\}. \tag{42}$$

Furthermore, if $(x, \zeta_x; y, \zeta_y) \in WF(\hat{\Pi}_1)$, then:

(a) *neither* ζ_x *nor* ζ_y *vanish;*

(b) $(\zeta_x)_r \neq 0$

(c) $(\zeta_x)_r \geqslant 0$ *if and only if* $-\zeta_y$ *is future directed.*

2) *The image of* $\mathscr{F}_e(\mathscr{D})$ *under* Π *lies in* $\mathscr{A}_e(\mathscr{C}_p)$.

Proof. We prove the above properties in two separate steps.

I) Construction of $\left(\sqrt[4]{|g_{AB}|}\,\Delta_{\mathscr{C}}\right)^{\otimes n}$ *and the wave front set of its integral kernel*

In the normal neighborhood $\mathcal{O}_p \subset M$ which contains $\mathscr{D}$ we can select a subset $\mathcal{O}' \subset (J^-(\mathscr{D}) \setminus J^-(p)) \subset \mathcal{O}_p$ which is a globally hyperbolic open set that extends $\mathscr{D}$ over $\mathscr{C}_p^+$, but neither over p nor over the future of $\mathscr{D}$. The existence of a similar set results from the global hyperbolicity of M which contains $\mathcal{O}_p$ and thus also $\mathscr{D}$. Let us indicate by $\hat{\Delta} \in \mathcal{D}'(\mathcal{O}' \times \mathscr{D})$ the integral kernel of $\Delta : C_0^\infty(\mathscr{D}) \to C^\infty(\mathcal{O}')$ defined by restricting the map in (8). It holds true that

$$\mathrm{WF}(\hat{\Delta}) = \left\{(x_1, \zeta_1; x_2, \zeta_2) \in T^*\mathcal{O}' \times T^*\mathscr{D} \setminus \{0\} \,\middle|\, (x_1, \zeta_1) \sim (x_2, -\zeta_2)\right\}, \tag{43}$$

where the equivalence relation $(x_1, \zeta_1) \sim (x_2, \zeta_2)$ means that there exists a null geodesic γ with respect to the metric g in $\mathscr{D}$ which contains both x and y. Furthermore, $g^{\mu\nu}(\zeta_1)_\nu$ and $g^{\mu\nu}(\zeta_2)_\nu$ are the tangent vectors of the affinely parametrised geodesic γ in x and y, respectively [38, 39].

We proceed by restricting one entry of the causal propagator[1] on $\mathscr{C}_p^+$, while leaving the other localised in $\mathscr{D}$. To this end, let us define the embedding χ of $\mathscr{C}_p^+ \times \mathscr{D}$ to $\mathcal{O}' \times \mathscr{D}$, whose action, in retarded coordinates on $\mathcal{O}'$, is defined as $\chi(r, \theta, \varphi; x_2) = (0, r, \theta, \varphi; x_2)$. According to Theorem 8.2.4 of [25], the restriction of the first entry of $\hat{\Delta}$ on $\mathscr{C}_p^+$ by means of the pullback under χ is well defined provided that $M_\chi \cap \mathrm{WF}(\hat{\Delta}) = \emptyset$, where M_χ is the set of normals of χ. In order to verify this statement about the empty intersection, we notice that, using the definition employed in such a theorem,

$$M_\chi \subset N_\chi = \left\{(x_1, \zeta_1; x_2, \zeta_2) \in T^*\mathcal{O}' \times T^*\mathscr{D} \,\middle|\, x_1 \in \mathscr{C}_p^+ \subset \mathcal{O}',\, \zeta_1 = (\zeta_1)_u du,\, (\zeta_1)_u \in \mathbb{R}\right\}.$$

We shall now prove that $N_\chi \cap \mathrm{WF}(\hat{\Delta}) = \emptyset$, consider $(x_1, \zeta_1; x_2, \zeta_2) \in N_\chi$ and the null geodesic γ' originating from x_1 whose tangent vector in x_1 is equal to $g^{-1}(\zeta_1)$. Notice that, in the retarded coordinates, the only non-vanishing component of $g^{-1}(\zeta_1)$ is $(g^{-1}(\zeta_1))^r$ which implies that γ' is contained in $\mathscr{C}_p^+$ and, in particular, does not enter $\mathscr{D}$. For this reason the intersection of N_χ with $\mathrm{WF}(\hat{\Delta})$ is the empty set and hence the hypotheses of Theorem 8.2.4 of [25] are fulfilled. Thus $\hat{\Delta}_{\mathscr{C}}$, the pullback of $\hat{\Delta}$ under χ, can be defined in one and only one way and $\mathrm{WF}(\hat{\Delta}_{\mathscr{C}}) \subset \chi^*\mathrm{WF}(\hat{\Delta})$. In particular, this entails that, if $(x, \zeta_x; y, \zeta_y) \in \mathrm{WF}(\hat{\Delta}_{\mathscr{C}})$ with $x \in \mathscr{C}_p^+$ and $y \in \mathscr{D}$, it enjoys properties (a), (b) and (c) stated above.

In order to verify (b), suppose this were not true and consider $(x, \zeta_x; y, \zeta_y) \in \mathrm{WF}(\hat{\Delta}_{\mathscr{C}})$, where $(\zeta_x)_r = 0$. Thus there should exist an element $(x, \zeta'_x; y, \zeta_y)$ such that $\chi^*(x, \zeta'_x; y, \zeta_y) = (x, \zeta_x; y, \zeta_y)$, where ζ'_x is a null covector whose components are $(\zeta'_x)_r = 0$, $(\zeta'_x)_\theta = (\zeta_x)_\theta$ and $(\zeta'_x)_\varphi = (\zeta_x)_\varphi$ while $(\zeta'_x)_u$ is a fixed number in $\mathbb{R}$. Since $g^{-1}(\zeta'_x)$ has to be null and since

[1] The same procedure was employed in the proof of Proposition 4.3 in [30] or in the work [22].

$(g^{-1}(\zeta'_x))^u = 0$, the only possibility is $(\zeta'_x)_\theta = (\zeta'_x)_\varphi = 0$. Hence the only non-vanishing component of $g^{-1}(\zeta'_x)$ is the r-component, which implies that $\zeta'_x = (\zeta'_x)_u du$, thus $(x, \zeta'_x; y, \zeta_y) \in N_\chi$. At this point we have reached a contradiction because $\mathrm{WF}(\hat{\Delta}) \cap N_\chi = \emptyset$ so that $(\zeta_x)_r$ has to be different from zero. Notice that (a) and (c) result from the constraint imposed by the equivalence relation $\sim$ in the wave front set of $\hat{\Delta}$ in (43) and from the observation that the projection of ζ'_x on $\mathscr{C}_p^+$ under χ^* does not change the causal direction (past/future). In order to accomplish this part of the proof, we only need to multiply every $\hat{\Delta}_{\mathscr{C}}$ with $\sqrt[4]{|g_{AB}|} \otimes 1$ which is smooth because it is the 4-th order root of a smooth positive function. Hence the wave front set of the resulting distribution is left unchanged. Thus we define $\hat{\Pi}_n$ as the tensor product of distributions $\hat{\Pi}_n \doteq (h\,\hat{\Delta}_{\mathscr{C}})^{\otimes n} \in \mathcal{D}'((\mathscr{C}_p^+ \times \mathscr{D})^n)$, where $h = \sqrt[4]{|g_{AB}|} \otimes 1$, and, due to Theorem 8.2.9 in [25], $\mathrm{WF}(\hat{\Pi}_n)$ enjoys the inclusion (42). By the Schwartz kernel theorem we obtain the linear map $\Pi_n = (\sqrt[4]{|g_{AB}|}\,\Delta_{\mathscr{C}})^{\otimes n}$ whose integral kernel is $\hat{\Pi}_n$.

II) On the image of Π

Notice that, since every $F \in \mathscr{F}_e(\mathscr{D})$ is composed of a finite number of F_n, it is sufficient to prove that the generic F_n is mapped to an element of $\mathcal{A}^n_s$ by Π or, rather, by Π_n. Moreover, the pointwise product of $\hat{\Pi}_n$, the integral kernel of Π_n, and $I^n \otimes F_n$, I the unit constant function in $\mathcal{D}'(\mathscr{C}_p)$, is well-defined because their wave front sets do not sum up to the zero section, as one can infer from (13), from Theorem 8.2.9 in [25] and from (42) along with the discussion presented above. More precisely, we have that $\mathrm{WF}(I^n \otimes F_n) \subset \{0\} \times \mathrm{WF}(F_n)$ where $\{0\}$ is the zero section in $T^*\mathscr{C}_p^{+n}$ while every element in $\mathrm{WF}(\hat{\Pi}_n)$ has non-vanishing components on that cotangent space, thus $\mathrm{WF}(I^n \otimes F_n) + \mathrm{WF}(\hat{\Pi}_n)$ cannot contain the zero section. Hence the Hörmander criterion for the multiplication of distributions, Theorem 8.2.10 in [25], is fulfilled and, moreover, the resulting distribution $(\hat{\Pi}_n) \cdot (I^n \otimes F_n)$ can be tested on any compactly supported smooth characteristic function η on the support of F_n yielding the $\Pi_n(F_n)$ sought-after. Theorem 8.2.12 in [25] guarantees that $\Pi_n(F_n) \in \mathcal{D}'\big(\mathscr{C}_p^{+n}\big)$ and that $\mathrm{WF}(\Pi_n(F_n)) \subset \big\{(x_1, \zeta_{x_1}; \dots; x_n, \zeta_{x_n}; y_1, 0; \dots; y_n, 0) \in \mathrm{WF}\big((\hat{\Pi}_n) \cdot (I^n \otimes F_n)\big)\big\} \subset W_n$ as in (31). In order to verify the last inclusion, we notice that $\mathrm{WF}(\hat{\Pi}_1) \cap (S_1 \times T^*\mathscr{D}) = \emptyset$ and that, making once more use of Theorem 8.2.10 in [25], $WF(\hat{\Pi}_n) \cdot (I^n \otimes F_n) \cap (\overline{V}_n^{\pm} \times \{0\}) = \emptyset$, where $\{0\}$ is the zero section in $T^*\mathscr{D}^n$. Thus the wave front set of $\Pi_n(F_n)$ is contained in W_n, and this is tantamount to the second condition in Definition 3.1.

As for the first one, this can be shown to hold true since, by construction, $\mathrm{supp}(F_n) \subset K^n \subset \mathscr{D}^n$, K a compact set, and hence, due to the support property of Δ, there exists another compact set K' constructed as the closure of $J^-(K) \cap \mathscr{C}_p^+$ in $\mathscr{C}$ such that K'^n contains the support of $\Pi_n(F_n)$.

The third and last requirement can also be established by recalling that the singular support of the causal propagator (8) is contained in the set of the null geodesics. Furthermore, those emanating from the support of any F_n (recall that $\mathrm{supp}(F_n) \subset K$) intersect $\mathscr{C}_p$ on a compact set that is disjoint from p in particular. Hence the causal propagator is a smooth function whenever one entry is smoothly localised[2] on the support of F_n and the other one on a neighbourhood of

[2]The localisation is realised by pointwise multiplication with smooth functions of suitable support.

p. Furthermore, even after multiplication both by the function Θ' as in Definition 3.1 and by $\sqrt[4]{|g_{AB}|}$, such smooth function is square-integrable and bounded, together with its V-derivative, in a suitable open set of $\mathscr{C}_p \sim \mathbb{R}^+ \times \mathbb{S}^2$ such that $V \in (0, V_0)$, because it is a restriction to $\mathscr{C}_p^+$ of a smooth function defined in a neighbourhood of p multiplied by $\sqrt[4]{|g_{AB}|}$. For the same reason, the limit $V \to 0$ of $\sqrt[4]{|g_{AB}|}\,\hat{\Delta}_{\mathscr{C}}$ tends to zero whenever one entry of the causal propagator is localised on some compact set in $\mathscr{D}$. Finally, notice that $\Pi_n(F_n)$ is totally symmetric whenever F_n has this property, and this completes the proof that $\Pi_n(F_n) \in \mathcal{A}_s^n$. □

As an intermediate step, we proceed by discussing the effect of the map Π on the symplectic form.

Proposition 3.4. *The projection $\Pi : \mathscr{F}_b(\mathscr{D}) \to \mathscr{A}_e(\mathscr{C}_p^+)$ is a symplectomorphism, i.e., for every $f,\ h \in C_0^\infty(\mathscr{D})$,*

$$\sigma(\varphi_f, \varphi_h) = \sigma_{\mathscr{C}}(\Pi_1 f, \Pi_1 h), \tag{44}$$

with σ taken as in (9).

Proof. Let $\varphi_f = \Delta f$ and $\varphi_h = \Delta h$, where Δ is as in (8), and consider both a Cauchy surface Σ of $\mathscr{D}$ and the portion of $\mathcal{O}_1 \doteq \mathscr{D} \cap I^-(\Sigma)$ whose boundary is formed by the null surface $\mathscr{C}_p^+$ and by Σ. Then the current

$$J_\mu \doteq \varphi_f \partial_\mu \varphi_h - \varphi_h \partial_\mu \varphi_f$$

satisfies $\int_\Sigma d\mu(\Sigma)\, n^\mu J_\mu = \sigma(\varphi_f, \varphi_h)$ with n^μ the unit future directed vector normal to Σ. Hence we can apply the divergence theorem to J_μ in $\mathcal{O}_1$ considered as a subregion of a larger globally hyperbolic spacetime, $\mathcal{O}'$, that contains $\mathscr{D}$. The result is that, since $\nabla^\mu J_\mu = 0$ in $\mathcal{O}_1$ in particular, the following identity holds,

$$\sigma(\varphi_f, \varphi_h) = \int_{\mathscr{C}_p^+} d\mu(\mathscr{C}_p^+)\, n^\mu J_\mu.$$

Furthermore, the right-hand side of the preceding equation can be rewritten in terms of the retarded coordinates on $\mathscr{C}_p^+$ and, if one uses the relation between the volume elements on the sphere (4) in spherical and local coordinates, it becomes

$$\int_{\mathbb{R}^+ \times \mathbb{S}^2} \sqrt{|g_{AB}|} \left[\varphi_f \frac{\partial}{\partial r} \varphi_h - \varphi_h \frac{\partial}{\partial r} \varphi_f \right] dr \wedge d\mathbb{S}^2, \tag{45}$$

where both φ_f and φ_h are evaluated on $\mathscr{C}_p^+$, a legitimate operation as explained at the beginning of this section, and, furthermore, they vanish on the complement of $\mathscr{C}_p^+$ in $\mathscr{C}_p$. Finally, due to the antisymmetry of the preceding expression, we can consider $\sqrt[4]{|g_{AB}|}\,\varphi_f = \Pi_1 f$ as well as $\sqrt[4]{|g_{AB}|}\,\varphi_h = \Pi_1 h$, and a direct inspection shows that (45) equals $\sigma_{\mathscr{C}}(\Pi_1 f, \Pi_1 g)$ as given in (20) setting $r = V$. □ □

Remark: Notice that the ideal $\mathscr{I}$ generated by the equations of motion (7) is mapped by Π to $0 \in \mathscr{A}_e(\mathscr{C})$, because Δ is a weak solution of (7). Hence the image of both $\mathscr{F}_b(\mathscr{D})$ and $\mathscr{F}_{bo}(\mathscr{D})$ under Π lie in $\mathscr{A}_e(\mathscr{C})$; actually they coincide.

On the basis of this remark, we stress another important property of Π.

Proposition 3.5. *The map* Π *is injective when acting on the on-shell extended algebra* $\mathscr{F}_{eo}(\mathscr{D})$.

Proof. Let us recall that the action of Π on $F = \{F_n\}_n \in \mathscr{F}_e(\mathscr{D})$ is determined by the actions of $\Pi_n = \Pi_1^{\otimes n}$ on its components F_n. We shall hence analyse the kernel of Π_1 seen as a map from $\mathcal{T}'(\mathscr{D})$ to some functionals on $\mathscr{S}(\mathscr{C}_p^+)$, where the elements of $\mathcal{T}'(\mathscr{D})$ are the compactly supported symmetric distributions whose wave front sets enjoy (13). We shall prove that $\ker(\Pi_1) = \mathscr{K} \doteq \{Pu \,|\, u \in \mathcal{T}'(\mathscr{D})\}$. Given $u \in \mathcal{E}'(\mathscr{D})$ there exists a sequence of $u_j \in C_0^\infty(\mathscr{D})$ whose support is contained in K, a proper compact subset of $\mathscr{D}$, such that $u_j \to u$ weakly for $j \to \infty$. Consider then the following chain of equalities

$$-\int_M \Delta(f)\, u_j = \int_M \Delta(f)\, P\Delta_A(u_j) = \sigma(\Delta(f), \Delta(u_j)) = \sigma_{\mathscr{C}}(\Pi_1(f), \Pi_1(u_j)),$$

where P is the operator realising the equation of motion and Δ_A is its advanced fundamental solution. Furthermore, Σ does intersect neither K nor $J^+(K)$, and hence, on Σ, $\Delta_A(u_j)$ is equal to $\Delta_R(u_j)$. Notice that in order to obtain the second equality, we have to choose two elements of a family of Cauchy hypersurfaces, Σ and Σ', which do not intersect K and such that $J^+(K) \cap \Sigma = \emptyset$ and $J^-(K) \cap \Sigma' = \emptyset$, and integrate by parts twice. The resulting integral on Σ' vanishes due to the support property of Δ_A, while the integral on M is zero since $P(\Delta(f)) = 0$. Thus the remaining integral is precisely the symplectic form computed on Σ. Furthermore, the last equality derives from Proposition 3.4. We proceed by writing

$$\int_M \Delta(f)\, u_j = -2\int_{\mathscr{C}_p} \sqrt[4]{|g_{AB}|}\, \Delta(u_j)\frac{\partial}{\partial V}\Big(\sqrt[4]{|g_{AB}|}\, \Delta(f)\Big) dV d\mathbb{S}^2.$$

Passing now to the weak limit yields $\Pi_1(u)(-2\partial_V \Pi_1 f) = u(\Delta(f))$. From the previous discussion we obtain that, letting $S \doteq -2\partial_V \Pi_1(C_0^\infty(\mathscr{D}))$, the condition $\Pi_1(u)(S) = 0$ is equivalent to $u(\Delta(\mathcal{D}(\mathscr{D}))) = 0$, and the latter implies $u = Pu'$ for some $u' \in \mathcal{E}'(\mathscr{D})$. Since, as a functional, $\Pi_1(u)$ are defined on a set larger than S we have that $\ker(\Pi_1) \subset \mathscr{K}$. In order to obtain the opposite inclusion, let us define by R the operator that realises the restriction on $\mathscr{C}_p^+$ and the subsequent multiplication by $\sqrt[4]{|g_{AB}|}$. Notice that Π_1 is defined as the composition $R \circ \Delta$, where now Δ is the map from $\mathcal{T}'(\mathscr{D})$ to $\mathcal{D}'(\mathcal{O}')$, where $\mathcal{O}'$ is the normal neighbourhood containing $\mathscr{D}$. Hence $\ker(\Pi_1) \supset \ker(\Delta) = \mathscr{K}$. Note that $\mathscr{K}$ is contained in the ideal $\mathscr{I}$ divided out of $\mathscr{F}_e(\mathscr{D})$ in order to obtain $\mathscr{F}_{eo}(\mathscr{D})$. The proof can then be concluded by applying a similar procedure to Π_n to verify that also $\ker(\Pi_n)$ is contained in the ideal $\mathscr{I}$. □ □

Before continuing the analysis of the map Π acting on the extended algebra $\mathscr{F}_e(\mathscr{D})$, we show that the pull-backs both of the symplectic form $\sigma_{\mathscr{C}}$ and of the boundary state ω have a nice interplay with the symplectic form in the bulk and with the Hadamard states in general. The next proposition deals with the singular structure of the state ω when pulled back in the bulk.

Proposition 3.6. *Under the assumptions of Theorem 3.1*

$$H_\omega \doteq \Pi^*\omega \tag{46}$$

is an Hadamard bi-distribution constructed as the pull-back of ω *as in* (24) *under* Π *as in* (41).

Proof. The proof of this proposition can be performed by restricting attention to the compactly supported smooth functions on $\mathscr{D}$. Let us start by showing that H_ω is a good distribution on $C_0^\infty(\mathscr{D}^2)$, hence continuous in the topology of compactly supported smooth functions. To this end, notice that $H_\omega(f,g) = \omega(\Pi_1(f), \Pi_1(g))$, moreover, $\omega(\Pi_1(f), \Pi_1(g))$ enjoys the continuity stated in (26). Furthermore, the L^2 norms present in (26) are controlled by the supremum norms of $\Pi_1(f)$, $\Pi_1(g)$ and their r-derivatives on some compact set in $\mathscr{C}$ (here taken as the entire cone). The proof of the continuity of H_ω can be concluded employing the continuity of $\Delta : C_0^\infty(M) \to C^\infty(M)$, which is spoilt neither by the restriction on $\mathscr{C}_p^+$ nor by the multiplication by $\sqrt[4]{|g_{AB}|}$.

Notice that the antisymmetric part of H_ω equals the symplectic form (9) which is preserved by the action of Π (Proposition 3.4), and H_ω satisfies the equation weakly because so does Δ which is used in the definition of Π. Furthermore, $H_\omega(f,f)$ is positive for every f, because ω is a state for the boundary algebra. We thus conclude that H_ω is the two-point function of a quasi-free state for $\mathscr{F}_b(\mathscr{D})$. Hence, in order to prove the Hadamard property, due the the work of Radzikowksi [38], it is only necessary to check that the wave front set of H_ω satisfies the microlocal spectrum condition. This can be verified following the procedure envisaged in [30, 22]. For completeness, we shall summarise here the main steps of such a proof.

1.) It suffices to show that the microlocal spectrum condition holds locally in $\mathscr{D}$, namely, when H_ω is restricted on a generic compact set K^2 with $K \subset \mathscr{D}$. We hence have to show that

$$\mathrm{WF}(H_\omega) = \big\{(x_1,\zeta_1;x_2,-\zeta_2) \in T^*K^2 \setminus \{0\} \big| (x_1,\zeta_1) \sim (x_2,\zeta_2),\ \zeta_1 \triangleright 0\big\}, \tag{47}$$

where $\sim$ is the equivalence relation of (43), while $\zeta_1 \triangleright 0$ indicates that ζ_1 is a future directed vector. According to the preceding discussion, to show the inclusion $\supset$, we make use of Theorem 5.8 of [41] which can be applied once $\subset$ holds and yields the other relation as thesis.

2.) In order to show that $\subset$ holds in (47), notice that the past directed null geodesics originating from K in $\mathcal{O}_p$ intersect $\mathscr{C}_p^+$ on a region contained in a compact set $N \subset \mathscr{C}_p^+$. We stress that $p \notin \mathscr{C}_p^+$, and hence $p \notin N$. Thus, if we smoothly localise $\hat{\Pi}_1$ (the integral kernel of Π_1) on $N' \times K$, where N' is the complement of N in $\mathscr{C}_p^+$, the resulting object is described by a smooth function which is square-integrable on $\mathscr{C}_p^+$ and so is its V-derivative, also when an entry of $\hat{\Pi}_1$ is kept fixed in K.

3.) We shall hence introduce a partition of unity on $\mathscr{C}_p^+$, $\Theta_N + \Theta'_N = 1$, such that $\Theta_N \in C_0^\infty(\mathscr{C}_p^+)$ is equal to 1 on the compact set N. Hence it vanishes on the intersection of a sufficiently small neighbourhood of p with $\mathscr{C}_p^+$. Inserting two such partitions of unity in $\Pi^*\omega$ and employing multilinearity, H_ω becomes the sum of four terms, $\omega\big((\Theta_N + \Theta'_N)\hat{\Pi}_1 \otimes (\Theta_N + \Theta'_N)\hat{\Pi}_1\big)$.

4.) The only one which contributes to $\mathrm{WF}(H_\omega)$ is $\omega(\Theta_N\hat{\Pi}_1 \otimes \Theta_N\hat{\Pi}_1)$. In this case, we notice that, due to the form of $\mathrm{WF}(\omega)$ given in (28) and to the constraint enjoyed by $\mathrm{WF}(\hat{\Pi}_1)$ as discussed in 1) of Theorem 3.1, we can apply Theorem 8.2.13 of [25] in order to obtain that the inclusion sought holds for the wave front set of this term.

5.) All the other three terms have vanishing wave front sets. Let us briefly discuss them separately. In particular, due to the regularity shown by $\Theta'_N\hat{\Pi}_1$ when restricted to K, the composition of $(\Theta'_N \hat{\Pi}_1 \otimes \Theta'_N \hat{\Pi}_1)$ with ω on $\mathscr{C}_p^{+2}$ can be computed and yields a smooth function on K^2. The remaining two terms can be addressed similarly. To this end, let us concentrate on $\omega(\Theta'_N \hat{\Pi}_1 \otimes \Theta_N \hat{\Pi}_1)$. At this point, notice that the supports of Θ'_N and of Θ_N have non-vanishing intersection; however, such an intersection is contained in a further compact set R. We can thus insert another partition of unity, $\Theta_R + \Theta'_R$, in order to divide such a term in two parts $\omega\big(\Theta'_N(\Theta_R + \Theta'_R)\hat{\Pi}_1 \otimes \Theta_N \hat{\Pi}_1\big)$. Hence, $\omega\big((\Theta'_N \Theta'_R)\hat{\Pi}_1 \otimes \Theta_N \hat{\Pi}_1\big)$ has a vanishing wave front set because the supports of $\Theta'_N \Theta'_R$ and Θ_N are disjoint and ω is represented by a smooth function on their Cartesian product. Furthermore, since both $\Theta'_N \Theta_R$ and Θ_N are in $C_0^\infty(\mathscr{C}_p^+)$, we can estimate the wave front set of $\omega\big(\Theta'_N \Theta_R \hat{\Pi}_1 \otimes \Theta_N \hat{\Pi}_1\big)$ employing once more Theorem 8.2.13 of [25] to obtain that it is equal to the empty set. □

We can now prove a second theorem which focuses on the effect of the map Π on the algebraic structures and on the boundary state. In particular, we shall individuate an Hadamard state in the bulk.

Theorem 3.2. *Under the assumptions of Theorem 3.1, one has:*

1) Π *induces a unit-preserving* **-homomorphism between the algebras* $\big(\mathscr{F}_e(\mathscr{D}), \star_{H_\omega}\big)$ *and* $\big(\mathscr{A}_e(\mathscr{C}_p^+), \star_\omega\big)$.

2) Π *is an injective* **-homomorphism when acting "on shell" on* $\big(\mathscr{F}_{eo}(\mathscr{D}), \star_{H_\omega}\big)$.

Proof. We only prove the first statement. The second arises by direct inspection from this one and from Proposition 3.5. First notice that Π automatically preserves the *-operation because $\overline{\Pi}_n = \Pi_n$, hence

$$\Pi_n(F_n)^* = \Pi_n(F_n^*).$$

Thus we only need to verify the statement on the $\star$-products. In particular, we look for $\star_{H_\omega} : \mathscr{F}_e(\mathscr{D}) \times \mathscr{F}_e(\mathscr{D}) \to \mathscr{F}_e(\mathscr{D})$ such that,

$$\Pi(F \star_{H_\omega} G) = (\Pi F) \star_\omega (\Pi G), \quad \forall F,\, G \in \mathscr{F}_e(\mathscr{D}), \tag{48}$$

and, at the same time $\big(\mathscr{F}_e, \star_{H_\omega}\big)$ is isomorphic to $\big(\mathscr{F}_e, \star_H\big)$.

The natural candidate arises from the analysis performed in Proposition 3.6 and, in particular, from the distribution H_ω introduced in (46) to be plugged in (14) in place of H. This is a well-defined procedure since H_ω is of Hadamard form as proved in Proposition 3.6, and, hence, $(\mathscr{F}_e, \star_H)$ turns out to be isomorphic to $(\mathscr{F}_e, \star_{H_\omega})$, the isomorphism being realised as in (15). We are thus left with the task to verify (48) for every F and G in $\mathscr{F}_e(\mathscr{D})$. If we exploit both the definitions and the bilinearity of all the $\star$-products involved, this reduces to the requirement to show that

$$\Pi_{l+m-2k}\big((F_l \otimes G_m)({H_\omega}^{\otimes k})\big) = (\Pi_l F_l \otimes \Pi_m G_m)(\omega^{\otimes k})$$

for $l + m - 2k \geqslant 0$. The last relation directly results from bilinearity, (46) and from the fact that $\Pi_k = \Pi_{k_1} \otimes \Pi_{k_2}$ for all $k_1,\, k_2 > 0$ such that $k_1 + k_2 = k$. □ □

Remark: Notice that it is possible to turn the injective homomorphism into a bijection if we restrict attention to the local von Neumann algebras defined as the double commutant in the GNS representation of a quasifree Hadamard state of the C^*-algebra generated by the local Weyl operators constructed out of the symplectic forms (9) and (20) in $\mathscr{D}$ and on the boundary $\mathscr{C}_p^+$, respectively. This last claim is based on the invertibility of Π_2 on the weak solutions of the Klein-Gordon equation, where we recall that the Goursat problem with compact initial data on $\mathscr{C}_p$, in general, yields a solution of (7) whose restriction on any Cauchy surface of $\mathscr{D}$ is not compact.

Alas, the von Neumann algebra mentioned does not contain relevant physical observables such as the components of the regularised stress-energy tensor or the regularised squared fields, objects we would like to use in order to extract information about the local geometric data such as the scalar curvature.

4 Interplay with General Covariance and Comparison between Spacetimes

We are now in the position to collect all our results in a comprehensive framework which will exhibit both a nice interplay with the principle of local covariance, as devised in [9], and the possibility to compare quantum field theories on different backgrounds both at the level of algebras and of states.

To this avail, the construction in Subsection 3.5 will play a pivotal role, and the natural language we adopt is that of categories, which was already introduced in Subsection 2.1. In particular, notice that the construction of an extended algebra of observables on a double cone $\mathscr{D}$ can be realised as a suitable functor between the categories DoCo_{iso} and Alg, although it is not possible to extend such a functor to DoCo.

Notwithstanding this obstruction, the additional structure which arises from both the boundary and the field theory defined thereon allows us to circumvent the above problem in a way that also puts us in a position to compare field theories in different spacetimes. This requires the introduction of a further category, namely,

BAlg: The objects of this category are the extended boundary (topological *-)algebras presented in Subsection 3.4, constructed on all possible $\mathscr{C}_p^+$, while the morphisms are the unit preserving *-homomorphisms among them.

The key point consists in making a profitable use of the *-homomorphisms Π introduced in Theorem 3.1, in order to establish that $\Pi\circ\mathscr{F}$ indeed defines a functor between the two categories DoCo and BAlg, which admits the following pictorial, but inspiring diagrammatic representation,

$$\begin{array}{ccccc} \mathscr{D} & \xrightarrow{\mathscr{F}} & \mathscr{F}_e(\mathscr{D}) & \xrightarrow{\Pi} & \mathscr{A}_e(\mathscr{C}_p^+) \\ {\scriptstyle \imath_{e,e'}}\downarrow & & & & \downarrow{\scriptstyle \alpha_{\imath_{e,e'}}} \\ \mathscr{D}' & \xrightarrow[\mathscr{F}]{} & \mathscr{F}_e(\mathscr{D}') & \xrightarrow[\Pi']{} & \mathscr{A}_e(\mathscr{C}_p^{+\prime}) \end{array} \quad . \tag{49}$$

The arrow $\alpha_{\imath_{e,e'}}$ traces back to the analysis in Subsection 3.2, where it was shown that the boundary theory can be constructed and analysed independently of the specific bulk. Hence, $\alpha_{\imath_{e,e'}}$ is the *-homomorphism $\alpha_{\imath_{e,e'}} : \mathscr{A}_e(\mathscr{C}_p^+) \to \mathscr{A}_e(\mathscr{C}_p^{+\prime})$ whose action on the $F' \in \mathscr{A}_e(\mathscr{C}_p^+)$ is defined as follows: By means of the push-forward, it is

$$\alpha_{\imath_{e,e'}}(F'_n) = {\imath_{e,e'}}_* F'_n,$$

on $\left(\imath_{e,e'}(\mathscr{C}_p^+)\right)^n \subset (\mathscr{C}_p^{+\prime})^n$, while $\alpha_{\imath_{e,e'}}(F'_n) = 0$ on the complement of $\left(\imath_{e,e'}(\mathscr{C}_p^+)\right)^n$ in $(\mathscr{C}_p^{+\prime})^n$. Such an operation is well-defined because F'_n has compact support towards the future and, when extended on the closure of $\mathscr{C}_p^+$, $\imath_{e,e'}$ maps p to p'. We hence have the following

Proposition 4.1. *Consider* $\mathscr{A}_e : \mathsf{DoCo} \to \mathsf{BAlg}$*, whose action on the objects and morphisms is seen as follows,*

- *the action of* $\mathscr{A}_e$ *on the objects of* DoCo *is such that* $\mathscr{A}_e(\mathscr{D}) = \Pi \circ \mathscr{F}_e(\mathscr{D}) = \mathscr{A}_e(\mathscr{C}_p^+)$*;*
- *the action of* $\mathscr{A}_e$ *on the morphisms* $\imath_{e,e'}$ *is such that* $\mathscr{A}_e(\imath_{e,e'}) = \alpha_{\imath_{e,e'}}$*.*

Then $\mathscr{A}_e$ *is a functor between the two categories.*

Proof. In order to show that $\mathscr{A}_e : \mathsf{DoCo} \to \mathsf{BAlg}$ is a functor, notice that the covariance property holds and the identity is preserved, *i.e.*,

$$\mathscr{A}_e(\imath_{e,e'}) \circ \mathscr{A}_e(\tilde{\imath}_{e',\tilde{e}'}) = \mathscr{A}_e(\imath_{e,e'} \circ \tilde{\imath}_{e',\tilde{e}'}), \qquad \mathscr{A}_e(id_{\mathscr{D}}) = id_{\mathscr{A}_e(\mathscr{C}_p^+)},$$

as can be seen by direct inspection of the definition of $\mathscr{A}_e(\imath_{e,e'})$. □ □

As for the connection with the principle of general local covariance, we recall that, in the most general case, it is not possible to find a direct relation between $\mathscr{F}(\mathscr{D})$ and $\mathscr{F}(\mathscr{D}')$, unless the embedding $\Pi : \mathscr{F}(\mathscr{D}') \to \mathscr{F}(\mathscr{C}')$ can be inverted on the image of Π composed with $\alpha_{\imath_{e,e'}}$. This is indeed what happens whenever, *e.g*, $\imath_{e,e'}$ is an isometry (or, at worst, a conformal isometry [34]) which preserves the base point p. Hence we are working in DoCo_{iso}.

Under this assumption, the discussion about causality, usually an integral part of the reasoning as in [9, 7], does not have to be performed directly, since its essence is already encoded in the analysis of the properties of the map Π. A similar statement holds true also for the time-slice axiom (see [10], in particular). Especially, since the theory on the boundary is, to a certain extent, non-dynamical, there is no such axiom in our boundary framework and the one in the bulk is automatically assured by Π and its properties.

4.1 Comparison of Expectation Values in Different Spacetimes

The aim of this section is to clarify in which sense one can compare two field theories on two different backgrounds. We shall first explain the procedure abstractly and then give a concrete example.

Bearing in mind (49), it is straightforward to realise that, whenever we assign a state ω on $\mathscr{A}_e(\mathscr{C}'_p)$, we can pull it back either on $\mathscr{F}_e(\mathscr{D}')$ via Π' or on $\mathscr{F}_e(\mathscr{D})$ via $\alpha_{\imath_{e,e'}} \circ \Pi$, and the information about the bulk geometry is indeed restored by Π and Π'. This is a rather general feature which holds true regardless of the global structure of the spacetimes in which $\mathscr{D}$ and $\mathscr{D}'$ are embedded. Yet, on practical grounds, it is natural to choose one of the two double cones embedded in the four-dimensional Minkowski spacetime, where our capability of performing explicit computations of physical quantities is enhanced due to the large symmetry of the background.

To be more specific on this point, consider a double cone $\mathscr{D}$ as a subset of $(\mathbb{R}^4, \eta)$ while $\mathscr{D}'$ lies in a generic strongly causal spacetime M', chosen in such a way that there exist two frames e and e' in $T_p(\mathbb{R}^4)$ and $T_{p'}(M')$, respectively, yielding a well-defined $\imath_{e,e'} : \mathscr{D} \to \mathscr{D}'$. At this point we can apply the construction discussed for the local fields and algebras, related on the boundary via the map $\alpha_{\imath_{e,e'}}$.

As a next step, following also the general philosophy of [9], we consider observables constructed out of the same local fields either on $\mathscr{D}$ or $\mathscr{D}'$. Here we suppose that there exists $\imath_{e,e'} : \mathscr{D} \to \mathscr{D}'$ and hence, from the same field, we form $\Phi(f) \in \mathscr{F}_e(\mathscr{D})$ and $\Phi(\imath_{e,e'*} f) \in \mathscr{F}_e(\mathscr{D}')$, where Φ is a local field in the sense of [9]. We compute their expectation values on the pull-back of a suitable boundary state yielding an Hadamard counterpart in the bulk. In general, the difference between the results depends on the geometric data of both $\mathscr{D}$ and $\mathscr{D}'$, and thus we are ultimately comparing quantum field theories on different backgrounds.

Nonetheless, from a computational point of view, this procedure is still too involved, since one has to cope with the singular structure of the chosen state(s). Even if we restrict attention to those fulfilling the Hadamard condition, one would still need to take care of the regularisation procedure of the observables, a hassle which can be avoided. The idea is to consider on each double cone two bulk Hadamard states constructed out of the pull-back of different boundary counterparts and to work with their difference. In this case the integral kernel of such a difference is known to be smooth, an advantage which strongly reduces the computational efforts. Although the price to pay is the introduction of two states on the light cone, a natural candidate as one of them is the distinguished reference state ω which arises from (46) in Proposition 3.6. Hence we are left with the need to assign only one extra datum.

Before we discuss an explicit example of this procedure, we stress the most important properties of both ω and of its pull-back in the bulk, say $\Pi^*\omega$. These can be inferred from both Proposition 3.2 and Theorem 3.2:

1. *Local Lorentz invariance*: According to the third item of Proposition 3.2, ω turns out to be invariant under a set of geometric transformations on the full cone $\mathscr{C}$ which contains the boundary. In particular, ω is always invariant under the natural action of the subgroup of the Lorentz group which corresponds to isometries of the neighbourhood where the state is defined. Since the map Π is constructed substantially out of the causal propagator (8) in a geodesic neighbourhood, its action on ω via pull-back does not spoil the above property, *i.e.*, $\Pi^*\omega$ is invariant under the above assumptions.

2. *Microlocal structure*: The wave front set of ω is contained in the union of (29) and (30) and, most notably, it does not contain directions to the past. In particular, this allows for

the proof of Proposition 3.6 according to which $\Pi^*\omega$ satisfies the Hadamard property in the bulk double cone.

3. *Behaviour as a "vacuum"*: The boundary state ω turns out to be invariant under rigid translations of the V-coordinate, or, in other words, it is a vacuum with respect to the transformation generated by the vector ∂_V. This statement can be proved exactly as in [29, 15] for the counterpart on the conformal boundary of an asymptotically flat or of a cosmological spacetime, *viz.*, from the explicit form of the two-point function (27). This also entails that the energy computed on the cone with respect to ∂_V is minimised. Unfortunately, this property has not a strong counterpart in the bulk, but, if the bulk can be realised as an open set in $(\mathbb{R}^4, \eta)$, then $\Pi^*\omega$ is seen to coincide with the Minkowski vacuum for massless fields.

4.2 An Application: Extracting the Curvature

In this subsection we shall present an explicit application which follows the guidelines given above. For simplicity consider on the one hand a double cone $\mathscr{D}$ realised as an open subset of Minkowski spacetime $(\mathbb{R}^4, \eta)$, where the metric η has the standard diagonal form with respect to the Cartesian coordinates (t, x, y, z) induced by the standard orthonormal frame e of $\mathbb{R}^4$. As $\mathscr{D}'$ we consider a double cone which can be embedded in a homogeneous and isotropic solution of Einstein's equation with flat spatial section. This is a Friedman-Robertson-Walker spacetime (M', g), where $g = a^2(t)\,\eta$ and $a(t) \in C^\infty(I, \mathbb{R}^+)$ with $I \subseteq \mathbb{R}$ and $a(0) = 1$. Here t refers to the so-called conformal time and thus we still consider the coordinates (t, x, y, z) induced by the standard frame of $\mathbb{R}^4$, indicated as e' to distinguish it from the previous one. Furthermore, notice that, in view of the special form of g, $e' = \frac{e}{a(t)}$.

Since the underlying spacetimes are conformally related, their causal structures and, in particular, the double cones coincide. Consider two points $p = (0, 0, 0, 0)$ and $q = (t', 0, 0, 0)$ and the corresponding double cones $\mathscr{D}(p, q) \subset \mathbb{R}^4$ and $\mathscr{D}'(p, q) \subset M'$. In this framework the map $\imath_{e,e'} : \mathscr{D}(x_0, x_1) \to \mathscr{D}'(x_0, x_1)$ turns out to be trivial. Next, we choose a minimally coupled real scalar field theory, *viz.*,

$$\phi : \mathscr{D} \to \mathbb{R}, \quad \Box\phi = 0,$$

where $\Box$ is the d'Alembert wave operator constructed out of the metric in $\mathscr{D}$. We stress once more that we consider the very same equation also in $\mathscr{D}'$. If we follow the guidelines of Subsection 3.1, we can construct $\mathscr{F}_e(\mathscr{D})$ and $\mathscr{F}_e(\mathscr{D}')$ and their counterparts on the boundaries $\mathscr{C}_p$ and $\mathscr{C}'_p$. As outlined in the previous subsection, we now consider two algebraic states on $\mathscr{A}_e(\mathscr{C}_p)$, one, ω, is the reference state, while the other can be arbitrary, provided that the pull-back to the bulk via Π still fulfils the Hadamard condition. This requirement is not too restrictive since, *e.g.*, any state which differs from ω by a smooth function on the boundary and vanishes in a neighbourhood of the tip is an admissible choice.

In particular, consider another Gaussian state $\omega' : \mathscr{A}_e(\mathscr{C}_p) \to \mathbb{C}$, whose two-point function has the following form,

$$\omega'(\psi_1,\psi_2) = \omega(\psi_1,\psi_2) + \frac{1}{4\pi}\int\limits_{\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{S}^2} dr\,dr' d\mathbb{S}^2(\theta,\varphi)\,\psi_1(r,\theta,\varphi)\psi_2(r',\theta,\varphi). \tag{50}$$

Notice that the integral on the right-hand side entails that the integral kernel of $\omega' - \omega$ is not smooth because it contains a δ-like singularity in the angular coordinates. Despite this fact ω' can be pulled back to every spacetime still yielding an Hadamard bi-distribution.

The last statement can be proved operating in the same way as in the proof of Proposition 3.6 with ω replaced by $\omega' - \omega$. The key new feature, yielding $\mathrm{WF}\big(H_{\omega'-\omega}\big) = \emptyset$, is the fact that, while $(\zeta_x)_r$ is never equal to zero for every $(x,\zeta_x;y,\zeta_y) \in \mathrm{WF}(\Pi_1)$, every $(x,\zeta_x;y,\zeta_y) \in \mathrm{WF}(\omega'-\omega)$ is such that $(\zeta_x)_r = (\zeta_y)_r = 0$.

We are now ready to consider the expectation values of suitable observables from the point of view of both $\mathscr{D}$ and $\mathscr{D}'$. The most natural one is the expectation value of $\phi^2(f)$, where $f \in C_0^\infty(\mathscr{D})$ (and hence also in $C_0^\infty(\mathscr{D}')$), with respect to the bulk states constructed out of ω' and ω. Notice that $\phi^2(f)$ is a shortcut for saying that we are actually considering $u_f = f(x)\delta(x-y) \in \mathcal{A}^2_s(\mathscr{D}) \subset \mathscr{F}_e(\mathscr{D})$. One of the advantages of this construction is that, in this case, we are allowed to keep a more general stance, namely, we can substitute $f(x)$ by a Dirac function peaked at the point $x_t = (t,0,0,0)$ with $t \in (0,t')$. This is tantamount to consider $:\phi^2:(x_t)$, where $u = \delta(x-x_t)\delta(x-y) \in \mathscr{A}^2_s(\mathscr{D})$.

Now (41) can be used to evaluate $\Pi_2 u$ in both $(\mathbb{R}^4,\eta)$ and (M',g) by means of the explicit form of the causal propagator. In Minkowski spacetime it looks like (see [17] or [35])

$$\Delta(x,x') \doteq -\frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{4\pi\,|\mathbf{x}-\mathbf{x}'|} + \frac{\delta(t-t'+|\mathbf{x}-\mathbf{x}'|)}{4\pi\,|\mathbf{x}-\mathbf{x}'|}, \tag{51}$$

where t is the time coordinate and $\mathbf{x}$ the three-dimensional spatial vector in Euclidean coordinates. The counterpart of Δ in $\mathscr{D}'$ can be directly evaluated exploiting the conformal transformation between (M',g) and $(\mathbb{R}^4,\eta)$. The d'Alembert wave equation in the first spacetime (M',g) corresponds in the flat one to

$$\Box\phi - \frac{a''}{a}\phi = 0, \tag{52}$$

where $'$ stands for time derivation and $\Box = -\nabla^\mu\nabla_\mu$. Furthermore, the causal propagator $\widetilde{\Delta}$ of this partial differential equation is related to the one in M' via

$$\Delta_{M'}(x,y) = \frac{1}{a(t_x)a(t_y)}\widetilde{\Delta}(x,y). \tag{53}$$

If we follow the procedure discussed in Chapter 4 of [35], we get

$$\widetilde{\Delta}(x,x') = \Delta(x,x') + \frac{V(x,x')}{4\pi}\big(\Theta(t-t'-\left|\mathbf{x}-\mathbf{x}'\right|) - \Theta(-t+t'-\left|\mathbf{x}-\mathbf{x}'\right|)\big),$$

where $V(x,x')$ is a smooth function whose explicit form is derived from the Hadamard recursive relations for (52). In particular, it also holds that $V(x,x) = -\frac{a''(x)}{2a(x)}$.

We are now ready to compare the expectation values in $(\omega' - \omega)$. The Minkowski side of this operation yields, by direct computation,

$$(\omega' - \omega)(\Pi_2^{\mathbb{R}^4} u) = \frac{1}{4},$$

while in the cosmological setting

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[(4\pi) \int_0^\infty \frac{\widetilde{\Delta}(x_t, r_*)}{a(t)a(r_*)} r_* a(r_*) a(r_*)^2 dr_* \right]^2,$$

where we have rewritten the integral in the r-variable in terms of r_*, the affine parameter of the null cone in Minkowski spacetime. The defining relation between the two variables is

$$dr = a^2(r_*) dr_*.$$

The above integral can be rewritten by means of (53) as

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[4\pi \int_0^\infty \Delta(x_t, r_*) \frac{a(r_*)^2}{a(t)} r_* dr_* + \int_0^{t/2} V(x_t, r_*) \frac{a(r_*)^2}{a(t)} r_* dr_* \right]^2,$$

in which the first integral yields via (51)

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[\int_0^\infty \delta(t - 2r_*) \frac{a(r_*)^2}{a(t)} dr^* + \int_0^{t/2} V(x_t, r_*) \frac{a(r_*)^2}{a(t)} r_* dr_* \right]^2.$$

Let us now expand $a(t)$ in a power series around the point x_0,

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[\frac{1}{2} \frac{a(t/2)^2}{a(t)} + a''(x_0) \frac{t^2}{4} + O(t^3) \right]^2,$$

where, in the derivative, we have exploited that, at first order in t,

$$V(x_t, r_*) = \frac{a''(x_0)}{a(x_0)} + O(t),$$

also due to the rotational symmetry of M'. If we now expand both $a(t)$ and $a(t/2)$ in a Taylor series , we obtain

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[\frac{a}{2} \left[1 + \left(-\frac{1}{2} \frac{a''}{a} + \frac{3}{2} \left(\frac{a'}{a} \right)^2 \right) t^2 \right] + a'' \frac{t^2}{4} + O(t^3) \right]^2 =$$
$$\left[\frac{a}{2} + \frac{3a}{4} \left(\frac{a'}{a} \right)^2 t^2 + O(t^3) \right]^2,$$

where all functions a together with their derivatives are evaluated at x_0. We can summarize the discussion, finally calculating the difference between $(\omega' - \omega)(\Pi_2^{\mathbb{R}^4} u)$ and $(\omega' - \omega)(\Pi_2^{M'} u)$,

$$(\omega' - \omega)(\Pi_2^{\mathbb{R}^4} u) - (\omega' - \omega)(\Pi_2^{M'} u) = \frac{3}{4}(a'(x_0))^2 t^2 + O(t^3),$$

with $a(x_0) = 1$. The interpretation of this result is that, exactly as expected, the above comparison yields a result which, at first order, allows us to extract via a measurement precise information on the a priori unknown geometric data, in this case, the derivative of the scale factor at the point x_0 in a Friedman-Robertson-Walker universe.

5 Summary and Outlook

In this paper we have achieved a twofold goal: on the one hand we propose a novel way to look at the properties of a local quantum field theory in a suitable curved background, while, on the other hand, the very same construction yields a mechanism which allows for the comparison of expectation values of field observables in different spacetimes.

More specifically, starting from a careful analysis of the underlying geometry, we realise that only moderate assumptions are needed to reach our goals, *viz.*, our general setting consists of an arbitrary strongly causal manifold M in which we identify an arbitrary but fixed double cone $\mathscr{D} \equiv \mathscr{D}(p, q) = I^+(p) \cap I^-(q)$ strictly contained in a normal neighbourhood of p. Since $\mathscr{D}$ is globally hyperbolic, we can consider therein a real scalar field theory along the lines of (7) and therefore follow the general quantisation scheme which particularly calls for the association of a Borchers-Uhlmann algebra of observables with the chosen system. This algebra can be extended, both enlarging the set of its elements and the defining product, in order to encompass also *a priori* more singular objects, such as the Wick polynomials, which constitute the so-called extended algebra. The very deep reason for choosing $\mathscr{D} \subset M$ lies in its boundary and, more properly, on the portion of $J^+(p)$ which it contains. This is a differentiable submanifold of codimension 1 on which it is possible to construct a genuine free scalar field theory, following exactly the same procedure successfully employed for the causal boundary of an asymptotically flat or cosmological spacetime in [13, 14]. The main novel result in this framework arises from the construction of an extended algebra also for the boundary theory—$\mathscr{A}_e(\mathscr{C}_p)$ in the main body—whose well-posedness is justified both by its mathematical properties and by its relation to the bulk counterpart. Hence, the latter is embedded in $\mathscr{A}_e(\mathscr{C}_p)$ by means of Π, an injective *-homomorphism.

The advantage of this picture is the possibility to make use of a long tradition, originating from [27], which allows us to exploit the geometrical properties of the boundary to identify for the algebra thereon a natural state which can be pulled-back to the bulk via Π, yielding a counterpart which satisfies the microlocal spectrum condition, hence is of Hadamard form. This guarantees that we can identify a local state in $\mathscr{D}$ which is physically well-behaved. In physical terms it means that this state is the same for all inertial observers at p. In other words, the bulk state as well as the one on the boundary are invariant under a natural action of $SO_0(3, 1)$.

Thus it can be identified as a sort of local vacuum on a curved spacetime independent of the frame.

The second goal of comparing expectation values on different backgrounds is based on the above construction. More precisely, we consider not just one but actually two regions as above in two *a priori* different spacetimes M and M'. The construction of the field theories proceeds as usual, but now we make use of the invertibility of the exponential map in geodesic neighbourhoods in order to engineer the double cones $\mathscr{D} \subset M$ and $\mathscr{D}' \subset M'$ so that we can map the boundary in $\mathscr{D}$ to that in $\mathscr{D}'$ via a local diffeomorphism. This procedure can be brought to the level of boundary extended algebras which thus can be related by means of a suitable restriction homomorphism. The advantage is the previously unknown possibility to carefully use the distinguished state identified on each $\mathscr{C}_p$ in order to compare the bulk expectation values of field observables constructed for the theories in $\mathscr{D}$ and $\mathscr{D}'$. The important point is that this new perspective is completely compatible with the standard principle of general local covariance when applicable as devised in [9], and, actually, it complements it corroborating its significance.

Furthermore, since nothing prevents us from choosing one of the spacetimes as the Minkowski one, one can concretely check how the proposed machinery allows for the comparison of the expectation values of the field observables, making manifest the role and the magnitude of the geometric quantities. We stress this point by means of a simple example involving a massless minimally coupled field in the flat and in a cosmological spacetime. It seems safe to claim that there are several possibilities to apply our procedure to many other cases of physical interest. These are certainly not the only roads left open, and actually even the identified bulk Hadamard state should be studied in more detail. As a matter of fact, it is interesting to understand whether it is connected in any way with the states of minimum energy that appear in Friedman-Robertson-Walker spacetimes [32]. We leave this as well as the myriad of other questions for future investigations.

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A Hadamard States

This appendix briefly recollects some properties of Hadamard states which are used throughout the main text. Since most of the material has already been proved in several different alternative ways in the literature, we limit ourselves to giving the main statements and the necessary references. Let us stress that, from a physical perspective, Hadamard states are the natural candidates for physical ground states of a quantum field theory on a curved background, since their ultraviolet behaviour mimics that of the Minkowski vacuum at short distances and, furthermore, they guarantee that the quantum fluctuations of the expectation values of observables, such as the smeared components of the stress-energy tensor, are finite.

In the subsequent discussion we always assume that we are dealing with a quasi-free state on a suitable field algebra constructed on a globally hyperbolic spacetime (M, g) from a field satisfying an equation of motion such as (7). We stick to this assumption because it is consistent with the main body of the paper, but the reader should keep in mind that such an hypothesis could be relaxed (see for example [42]). As a starting point we state a global criterion characterising Hadamard states [38, 39].

Definition A.1. *A state ω satisfies the **Hadamard condition** and is thus called an **Hadamard state** if and only if*

$$WF(\omega) = \{(x, k_x; y, -k_y) \in T^*M^2 \setminus \{0\} \big| (x, k_x) \sim (y, k_y),\ k_x \triangleright 0\},$$

where, in this expression, ω actually stands for the integral kernel of the two-point function associated with ω. The relation $(x, k_x) \sim (y, k_y)$ indicates that there exists a null geodesic γ connecting x to y such that k_x is coparallel and cotangent to γ at x and k_y is the parallel transport of k_x from x to y along γ. The requirement $k_x \triangleright 0$ means that the covector k_x is future directed.

The above condition on the wave front set is rather useful and often employed on practical grounds to check whether a given state really is Hadamard or not. Nonetheless, it is possible to provide another definition via the so-called *Hadamard form*, which has been rigorously introduced in [27].

Definition A.2. *A state ω is said to be of the (local) **Hadamard form** if and only if in any convex normal neighbourhood the integral kernel of the associated two-point function can be written as*

$$\omega(x, y) = H(x, y) + W(x, y),$$

where

$$H(x, y) = \lim_{\epsilon \to 0^+} \frac{U(x, y)}{\sigma_\epsilon(x, y)} + V(x, y) \ln \frac{\sigma_\epsilon(x, y)}{\lambda^2}, \tag{54}$$

and the limit is to be understood in the weak sense. Here, U, V, as well as W are smooth functions, while λ is a reference length; furthermore,

$$\sigma_\epsilon(x, y) \doteq \sigma(x, y) \pm 2i\epsilon\big(T(x) - T(y)\big) + \epsilon^2$$

with $\epsilon > 0$. In the above formula, T is a time function, such that ∇T is timelike and future directed on the full spacetime (M, g). In addition, if we apply (7) either to the x- or to the y-variable, the result has to be a smooth function.

The existence of a time function T is guaranteed on any globally hyperbolic manifold [3, 4] as these can be decomposed as $\Sigma \times \mathbb{R}$, where Σ is a smooth Cauchy surface and $\mathbb{R}$ is the range of the time function T.

A completely satisfactory definition of the Hadamard form requires some more work to rule out spacelike singularities, to circumvent convergence problems of the series V, which is only asymptotic, and, finally, to assure that the definition depends neither on a special choice of the temporal function T nor on the convex normal neighbourhood employed.

In strict terms, we have only defined the local Hadamard form here. A stronger and more satisfactory definition, the so-called global Hadamard form, has been introduced in [27]. It reinforces the local form extending it from the convex normal neighbourhoods to certain "causally-shaped" neighbourhoods of a Cauchy surface, thereby ruling out spacelike singularities. However, in [39], it has been shown that the local Hadamard form already implies the global Hadamard form.

Another important fact is that the singular structure (54) is completely determined by the geometry of the background and the equation of motion. This of course does not hold for W which encodes the full state dependence.

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