DESY 11-004, SFB/CPP-11-02, LPN11-03

Angular integrals in d dimensions

Gábor Somogyi
DESY, Platanenalle 6, D-15738 Zeuthen, Germany

(Dated: March 11, 2011)

We discuss the evaluation of certain d dimensional angular integrals which arise in perturbative field theory calculations. We find that the angular integral with n denominators can be computed in terms of a certain special function, the so-called H-function of several variables. We also present several illustrative examples of the general result and briefly consider some applications.

I. INTRODUCTION

When computing higher order corrections in perturbative field theory, the following d dimensional angular integrals are encountered in many situations

$$\int \mathrm{d}\Omega_{d-1}(q)\, \frac{1}{(p_1\cdot q)^{j_1}\dots(p_n\cdot q)^{j_n}}\,, \tag{1}$$

where $p_1^\mu,\dots,p_n^\mu$ are fixed vectors in $d = 4-2\epsilon$ dimensional Minkowski space and $\mathrm{d}\Omega_{d-1}(q)$ is the rotationally invariant angular measure in d dimensions for the massless vector q^μ. For a single denominator, i.e. $n=1$, the integral in Eq. (1) is easy to evaluate, as it reduces to a single (trivial) integration in a properly chosen Lorentz frame. The case of two denominators, $n=2$, is already quite a bit more cumbersome, and it seems that general (i.e. j_1 and j_2 are symbolic) analytic expressions valid to all orders in ϵ are only available in the literature for the massless case ($p_1^2 = p_2^2 = 0$), as first derived in Ref. [1]. When one or both of the momenta p_1^μ and p_2^μ are massive, Appendix C of Ref. [2] provides a very useful compilation of known results. (See also [3] and references therein.) However, these are first of all limited to specific values of j_1 and j_2, specifically $j_1, j_2 = -2,-1,\dots,2$. (Some results for different specific values of j's — in particular integers with $|j_1|,|j_2| \le 4$ — are also known [4].) Furthermore, they are given as expansions in ϵ, up to and including $\mathrm{O}(\epsilon)$ terms for the case of a single massive momentum (e.g. $p_1^2 \neq 0$ and $p_2^2 = 0$), while for the case when both momenta are massive ($p_1^2\neq 0$ and $p_2^2 \neq 0$), only the four dimensional result is given. (Clearly the integral in Eq. (1) is finite in four dimensions if all p_i^μ, $i=1,\dots,n$, are massive.) Work towards deriving the $\mathrm{O}(\epsilon)$ terms for the angular integral with two denominators and two massive momenta was presented recently in Ref. [5]. As explained in Ref. [3], the most difficult of these two denominator integrals were computed by relating them to the imaginary parts of certain box integrals, which could be evaluated by Feynman parameters.

However, in certain cases, results going beyond those found in Refs. [1–5] are needed. For example, when integrating [6] the so-called iterated singly-unresolved approximate cross section of the NNLO subtraction scheme of Refs. [7–11], one requires a general (i.e. symbolic j_1 and j_2), all-order (in ϵ) expression for the two denominator angular integral with one massive momentum. In the same computation, one also encounters angular integrals with three denominators and general exponents. To the best of our knowledge, there is no systematic discussion of such d dimensional angular integrals with more than two denominators in the published literature.

In this paper, we use the method of Mellin–Barnes representations (see [12] and references therein) to evaluate the integral in Eq. (1) for arbitrary n, and massless or massive momenta p_i^μ $(i = 1, \ldots, n)$. The exponents j_i $(i = 1, \ldots, n)$ are also kept symbolic, and we tacitly assume that they satisfy any constraints that are needed to make our manipulations meaningful. In particular, it will be seen that our final expression for the general angular integral (and indeed its derivation) cannot be applied naively for nonpositive integer exponents. Nevertheless, some of our specific results will be valid even for j_i being a nonpositive integer.

The analytical expression for the general angular integral with n denominators is computed in Sect. II, and is given in terms of the H-function of several variables. The H-function of several variables has been discussed in various forms by a number of authors in the literature, see e.g. Ref. [13] and references therein. (See also the recent book [14], which however deals mostly with the single variable case.) For convenience, we recall the definition of the H-function as used in the present paper in Appendix A. Then, in Sect. III, we illustrate the general case by several specific examples. In particular, we rederive and extend all known results for $n = 2$ as special cases of the general expression, including a general, all-order (in ϵ) formula for the case with a single massive momentum. We also discuss the three denominator angular integral arising in Ref. [6]. We draw our conclusions in Sect. IV.

II. ANGULAR INTEGRAL WITH n DENOMINATORS

A. General result

To begin, we note that the overall normalization of the p_i^μ and q^μ plays no essential role, since clearly

$$\int \mathrm{d}\Omega_{d-1}(q)\,\frac{1}{(\lambda_1 p_1\cdot\lambda q)^{j_1}\dots(\lambda_n p_n\cdot\lambda q)^{j_n}} = \frac{1}{\lambda_1^{j_1}\dots\lambda_n^{j_n}\lambda^{j_1+\dots+j_n}}\int \mathrm{d}\Omega_{d-1}(q)\,\frac{1}{(p_1\cdot q)^{j_1}\dots(p_n\cdot q)^{j_n}}\,. \tag{2}$$

Hence, it is no loss of generality to choose the normalization of all vectors in whatever way is most convenient. In particular, to write the integral in Eq. (1) explicitly, one may choose a Lorentz frame where

$$\begin{aligned}
p_1^\mu &= (1, \mathbf{0}_{d-2}, \beta_1)\,,\\
p_2^\mu &= (1, \mathbf{0}_{d-3}, \beta_2\sin\chi_2^{(1)}, \beta_2\cos\chi_2^{(1)})\,,\\
p_3^\mu &= (1, \mathbf{0}_{d-4}, \beta_3\sin\chi_3^{(2)}\sin\chi_3^{(1)}, \beta_3\cos\chi_3^{(2)}\sin\chi_3^{(1)}, \beta_3\cos\chi_3^{(1)})\,,\\
&\;\;\vdots\\
p_n^\mu &= (1, \mathbf{0}_{d-1-n}, \beta_n\prod_{k=1}^{n-1}\sin\chi_n^{(k)}, \beta_n\cos\chi_n^{(n-1)}\prod_{k=1}^{n-2}\sin\chi_n^{(k)}, \dots, \beta_n\cos\chi_n^{(2)}\sin\chi_n^{(1)}, \beta_n\cos\chi_n^{(1)})\,,
\end{aligned} \tag{3}$$

while q^μ reads

$$q^\mu = (1, ..\text{`angles'}.., \cos\vartheta_n\prod_{k=1}^{n-1}\sin\vartheta_k, \cos\vartheta_{n-1}\prod_{k=1}^{n-2}\sin\vartheta_k, \dots, \cos\vartheta_2\sin\vartheta_1, \cos\vartheta_1)\,, \tag{4}$$

and we have used the freedom to choose the normalization to fix each zeroth component to be one. In Eq. (4), the notation ..'angles'.. stands for the $d-1-n$ angular variables that may be trivially integrated in Eq. (1). The explicit expression for the measure $\mathrm{d}\Omega_{d-1}(q)$ reads

$$\mathrm{d}\Omega_{d-1}(q) = \prod_{k=1}^{n}\mathrm{d}(\cos\vartheta_k)\,(\sin\vartheta_k)^{-k+1-2\epsilon}\mathrm{d}\Omega_{d-1-n}(q)\,, \tag{5}$$

and hence Eq. (1) leads to the integral

$$\begin{aligned}
\Omega_{j_1,\dots,j_n} \equiv &\int \mathrm{d}\Omega_{d-1-n}(q)\int_{-1}^{1}\left[\prod_{k=1}^{n}\mathrm{d}(\cos\vartheta_k)\,(\sin\vartheta_k)^{-k+1-2\epsilon}\right]\\
&\times\prod_{k=1}^{n}\left\{1-\beta_k\sum_{l=1}^{k}\left[\left(\delta_{lk}+(1-\delta_{lk})\cos\chi_k^{(l)}\right)\cos\vartheta_l\prod_{m=1}^{l-1}\left(\sin\chi_k^{(m)}\sin\vartheta_m\right)\right]\right\}^{-j_k}\,,
\end{aligned} \tag{6}$$

which we take as the definition of $\Omega_{j_1,\dots,j_n}$. Notice that for $n=2$ the normalization of Ω_{j_1,j_2} conforms to that of Ref. [1], while Refs. [2, 3] use a different normalization (the factor of $\int \mathrm{d}\Omega_{d-1-n}(q)$ is missing). As defined above, $\Omega_{j_1,\ldots,j_n}$ is a function of the $\frac{n(n-1)}{2}$ independent angles $\chi_1^{(1)},\ldots,\chi_n^{(n-1)}$, and n velocities, $\beta_1,\ldots,\beta_n$. However, it will be more natural to adopt the dot products between the various p_i^μ in Eq. (3) as the independent variables. We will set the following notation

$$v_{kl} \equiv \begin{cases} \dfrac{p_k \cdot p_l}{2} & ;\ k \neq l \\ \dfrac{p_k^2}{4} & ;\ k = l \end{cases}, \tag{7}$$

where the choice of normalization will become clear later. If the p_i^μ are all light-like or time-like (i.e. $0 \le \beta_i \le 1$), then we have $v_{kl} \ge 0$. In the following, we will assume that all v_{kl} are nonnegative.

We can now state our main result: the function $\Omega_{j_1,\ldots,j_n}$ is given by the following expression:

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\boldsymbol{v};(\boldsymbol{\alpha},\boldsymbol{A});(\boldsymbol{\beta},\boldsymbol{B});\boldsymbol{L_s}] \tag{8}$$

where H is the H-function of $N = \frac{n(n+1)}{2}$ variables [13]. In Eq. (8) above, $\boldsymbol{v}$ denotes the vector of N variables, $\boldsymbol{v} = (v_{11}, v_{12}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{n-1n}, v_{nn})$ while $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the following vectors of parameters

$$\boldsymbol{\alpha} = (\boldsymbol{0}_N, j_1, \ldots, j_n, 1-j-\epsilon), \qquad \boldsymbol{\beta} = (j_1, \ldots, j_n, 2-j-2\epsilon). \tag{9}$$

In Eqs. (8) and (9), j is the sum of exponents,

$$j = \sum_{k=1}^{n} j_k . \tag{10}$$

Notice that the number of components of $\boldsymbol{\alpha}$ is $\frac{(n+1)(n+2)}{2}$, while that of $\boldsymbol{\beta}$ is $(n+1)$. Finally, $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\frac{(n+1)(n+2)}{2} \times N$ and $(n+1) \times N$ matrices of parameters, respectively. We have

$$\boldsymbol{A} = \left[\begin{array}{c} -\boldsymbol{1}_{N\times N} \\ \hline \boldsymbol{M}_{n\times N} \\ \hline -1\cdots -1 \end{array}\right], \qquad \boldsymbol{B} = [(0)_{(n+1)\times N}], \tag{11}$$

i.e. $\boldsymbol{B}$ is zero, while the $n \times N$ dimensional matrix $\boldsymbol{M}$ has the following block form:

$$\boldsymbol{M}_{n\times N} = \left[\ \boldsymbol{m}_{n\times n} \middle| \boldsymbol{m}_{n\times(n-1)} \middle| \cdots \middle| \boldsymbol{m}_{n\times 1}\ \right] \qquad \text{with} \qquad \boldsymbol{m}_{n\times p} = \left[\begin{array}{c|c} 0 & (0)_{(n-p)\times(p-1)} \\ \hline 2 & 1\cdots 1 \\ \hline 0 & \\ \vdots & \boldsymbol{1}_{(p-1)\times(p-1)} \\ 0 & \end{array}\right]. \tag{12}$$

In Eqs. (11) and (12), $\mathbf{1}_{a\times a}$ denotes the $a \times a$ dimensional unit matrix, while $(0)_{a\times b}$ denotes an $a \times b$ dimensional block of zeros. To give some examples, we spell out the $\boldsymbol{A}$ matrix explicitly for the cases $n = 1$, 2 and 3, when $\boldsymbol{A}$ is a (3×1), (6×3) and (10×6) dimensional matrix, respectively:

$$\boldsymbol{A}(n=1) = \left[\begin{array}{c} -1 \\ \hline 2 \\ \hline -1 \end{array}\right], \quad \boldsymbol{A}(n=2) = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ \hline 2 & 1 & 0 \\ 0 & 1 & 2 \\ \hline -1 & -1 & -1 \end{array}\right], \quad \boldsymbol{A}(n=3) = \left[\begin{array}{ccc|cc|c} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ \hline -1 & -1 & -1 & -1 & -1 & -1 \end{array}\right]. \tag{13}$$

For clarity, in Eq. (13), we have indicated the block structure of the various $\boldsymbol{A}$ matrices explicitly. Finally, in Eq. (8) we have $\boldsymbol{L_s} = L_{s_1} \times \ldots \times L_{s_N}$, where L_{s_k} is an infinite contour in the complex s_k-plane running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$, whose properties we discuss below Eq. (15). Here $\times$ indicates the Cartesian product of contours.

We note in passing that the H-function of several variables satisfies various contiguous relations [15–17], i.e. algebraic relations between functions $H[\boldsymbol{v}; (\boldsymbol{\alpha}, \boldsymbol{A}); (\boldsymbol{\beta}, \boldsymbol{B}); \boldsymbol{L_s}]$ with the vectors of parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ shifted by vectors of integers. These relations may be used to reduce H-functions to a set of basis functions with parameters differing form the original values by integer shifts via the method of differential reduction [18]. (See also Ref. [19] for a short but clear introduction to the main ideas.) The differential reduction of H-functions is beyond the scope of this paper. Nevertheless, since the parameter of dimensional regularization, ϵ, appears in $\boldsymbol{\alpha}$ (see Eq. (9)), we speculate that this reduction will naturally include dimensional shift identities [20] for angular integrals.

Finally, using the defining Mellin–Barnes representation of the H-function as recalled in Appendix A, we find

$$\begin{aligned}\Omega_{j_1,\ldots,j_n}(\{v_{kl}\};\epsilon) &= 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)} \\ &\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^{n}\prod_{l=k}^{n}\frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\,\Gamma(-z_{kl})\,(v_{kl})^{z_{kl}}\right]\left[\prod_{k=1}^{n}\Gamma(j_k+z_k)\right]\Gamma(1-j-\epsilon-z)\,.\end{aligned} \tag{14}$$

Note that the N Mellin–Barnes integration variables are denoted z_{kl}, with $k = 1, \ldots, n$ and

$l = k, \dots, n$, i.e. z_{11}, z_{12}, $\dots$, z_{1n}, z_{22}, z_{23}, $\dots$, z_{n-1n}, z_{nn} and in Eq. (14), we have furthermore introduced the following notation:

$$z = \sum_{k=1}^{n} \sum_{l=k}^{n} z_{kl} \,, \qquad \text{and} \qquad z_k = \sum_{l=1}^{k} z_{lk} + \sum_{l=k}^{n} z_{kl} \,. \tag{15}$$

In words, z is the sum of all N Mellin–Barnes variables, while z_k is the sum of all variables that involve k as one of their indices, such that z_{kk} itself is counted twice, i.e. $z_k = z_{1k} + \dots z_{k-1k} + 2z_{kk} + z_{kk+1} + \dots + z_{kn}$. In Eq. (14), j is the sum of all exponents, see Eq. (10). The contours of integration for the z_{kl} are chosen in the standard way: the poles with a $\Gamma(\dots + z_{kl})$ dependence are to the left of the contour and poles with a $\Gamma(\dots - z_{kl})$ dependence are to the right of it.

B. Computation

We establish Eq. (14) by direct computation, as follows. Consider Eq. (1). First, use Feynman parametrization to combine all n denominators:

$$\Omega_{j_1 \dots j_n} \equiv \int \mathrm{d}\Omega_{d-1}(q) \, \frac{\Gamma(j)}{\prod_{k=1}^{n} \Gamma(j_k)} \int_0^1 \left[\prod_{k=1}^{n} \mathrm{d}x_k \, (x_k)^{j_k - 1} \right] \delta \left(\sum_{k=1}^{n} x_k - 1 \right) \left[\left(\sum_{k=1}^{n} x_k p_k \right) \cdot q \right]^{-j} , \tag{16}$$

where again j is the sum of exponents as in Eq. (10), and the p_i^μ are given in Eq. (3). By rotational invariance, we can choose a frame such that

$$\sum_{k=1}^{n} x_k p_k^\mu = (1, \mathbf{0}_{d-2}, \beta) \qquad \text{and} \qquad q^\mu = (1, ..\text{‘angles’}.., \sin\vartheta, \cos\vartheta) \,. \tag{17}$$

Then we have

$$1 - \beta^2 = \sum_{k=1}^{n} \sum_{l=k+1}^{n} 2 x_k x_l (p_k \cdot p_l) + \sum_{k=1}^{n} x_k^2 p_k^2 = 4 \sum_{k=1}^{n} \sum_{l=k}^{n} x_k x_l v_{kl} \,. \tag{18}$$

In the frame of Eq. (17), the integral in Eq. (16) reduces to

$$\begin{aligned} \Omega_{j_1 \dots j_n} = & \frac{\Gamma(j)}{\prod_{k=1}^{n} \Gamma(j_k)} \int_0^1 \left[\prod_{k=1}^{n} \mathrm{d}x_k \, (x_k)^{j_k - 1} \right] \delta \left(\sum_{k=1}^{n} x_k - 1 \right) \\ & \times \int \mathrm{d}\Omega_{d-2} \int_{-1}^{1} \mathrm{d}(\cos\vartheta) \, (\sin\vartheta)^{-2\epsilon} [1 - \beta(\{x_k, v_{kl}\}) \cos\vartheta]^{-j} \,, \end{aligned} \tag{19}$$

where $\beta(\{x_k, v_{kl}\})$ is given (implicitly) in Eq. (18). Hence, we can exchange all but one angular integration for an integration over a Feynman parameter.

The angular integral which appears on the second line of Eq. (19) is just the one denominator massive integral Ω_j, to be discussed in more detail in Sect. III B. For now, we simply state that it has a Mellin–Barnes representation of the form

$$\Omega_j = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\Gamma(j)\Gamma(2-j-2\epsilon)} \times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z_0}{2\pi\mathrm{i}}\,\Gamma(-z_0)\Gamma(j+2z_0)\Gamma(1-j-\epsilon-z_0)\left(\frac{1-\beta^2(\{x_k,v_{kl}\})}{4}\right)^{z_0}, \tag{20}$$

however, we defer the derivation of this until Sect. III B. Using Eq. (20) in Eq. (19), we obtain

$$\Omega_{j_1\dots j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)}\int_0^1\left[\prod_{k=1}^{n}\mathrm{d}x_k\,(x_k)^{j_k-1}\right]\delta\left(\sum_{k=1}^{n}x_k-1\right) \times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z_0}{2\pi\mathrm{i}}\,\Gamma(-z_0)\Gamma(j+2z_0)\Gamma(1-j-\epsilon-z_0)\left(\frac{1-\beta^2(\{x_k,v_{kl}\})}{4}\right)^{z_0}. \tag{21}$$

Next, we perform the integral over the Feynman parameters. The only nontrivial x dependence appears in $\frac{1-\beta^2(\{x_k,v_{kl}\})}{4}$, and this is given by (see Eq. (18) above)

$$\frac{1-\beta^2(\{x_k,v_{kl}\})}{4} = \sum_{k=1}^{n}\sum_{l=k}^{n}x_kx_lv_{kl}\,, \tag{22}$$

which explains our choice of normalization in Eq. (7). We can factorize all x dependence in Eq. (21) by writing $\left(\frac{1-\beta^2(\{x_k,v_{kl}\})}{4}\right)^{z_0}$ as a multidimensional (in fact $(N-1)$ dimensional; recall that $N=\frac{n(n+1)}{2}$) Mellin–Barnes integral:

$$\left(\frac{1-\beta^2(\{x_k,v_{kl}\})}{4}\right)^{z_0} = \frac{1}{\Gamma(-z_0)}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^{n-1}\prod_{l=k}^{n}\frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})(x_kx_lv_{kl})^{z_{kl}}\right] \times\Gamma\left(-z_0+\sum_{k=1}^{n-1}\sum_{l=k}^{n}z_{kl}\right)(x_n^2v_{nn})^{z_0-\sum_{k=1}^{n-1}\sum_{l=k}^{n}z_{kl}}\,. \tag{23}$$

Substituting Eq. (23) into Eq. (21), we obtain

$$\Omega_{j_1\dots j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)}\int_0^1\left[\prod_{k=1}^{n}\mathrm{d}x_k\,(x_k)^{j_k-1}\right]\delta\left(\sum_{k=1}^{n}x_k-1\right) \times\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z_0}{2\pi\mathrm{i}}\,\Gamma(j+2z_0)\Gamma(1-j-\epsilon-z_0)\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^{n-1}\prod_{l=k}^{n}\frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})(x_kx_lv_{kl})^{z_{kl}}\right] \times\Gamma\left(-z_0+\sum_{k=1}^{n-1}\sum_{l=k}^{n}z_{kl}\right)(x_n^2v_{nn})^{z_0-\sum_{k=1}^{n-1}\sum_{l=k}^{n}z_{kl}}\,. \tag{24}$$

Setting $z_0-\sum_{k=1}^{n-1}\sum_{l=k}^{n}z_{kl}\equiv z_{nn}$ and changing the variable of integration $z_0\to z_{nn}$, we find

$$\Omega_{j_1\dots j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)}\int_0^1\left[\prod_{k=1}^{n}\mathrm{d}x_k\,(x_k)^{j_k-1}\right]\delta\left(\sum_{k=1}^{n}x_k-1\right) \times\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^{n}\prod_{l=k}^{n}\frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})(x_kx_lv_{kl})^{z_{kl}}\right]\Gamma(j+2z)\Gamma(1-j-\epsilon-z)\,, \tag{25}$$

where z is the sum of all N integration variables as in Eq. (15). Collecting all factors of the various x's in Eq. (25), we obtain

$$\begin{aligned}\Omega_{j_1\dots j_n} &= 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^n \Gamma(j_k)\Gamma(2-j-2\epsilon)}\int_0^1 \left[\prod_{k=1}^n \mathrm{d}x_k\,(x_k)^{j_k-1+z_k}\right]\delta\left(\sum_{k=1}^n x_k - 1\right) \\ &\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^n\prod_{l=k}^n \frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})(v_{kl})^{z_{kl}}\right]\Gamma(j+2z)\Gamma(1-j-\epsilon-z)\,,\end{aligned} \tag{26}$$

where z_k is defined in Eq. (15). We can now perform the Feynman parameter integrals via

$$\int_0^1 \prod_{k=1}^N \mathrm{d}x_k x_k^{p_k-1}\delta\left(\sum_{k=1}^N x_k - 1\right) = \frac{\prod_{k=1}^N \Gamma(p_k)}{\Gamma\left(\sum_{k=1}^N p_k\right)}\,, \tag{27}$$

to obtain Eq. (14), as claimed:

$$\begin{aligned}\Omega_{j_1\dots j_n} &= 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^n \Gamma(j_k)\Gamma(2-j-2\epsilon)} \\ &\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^n\prod_{l=k}^n \frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})(v_{kl})^{z_{kl}}\right]\left[\prod_{k=1}^n \Gamma(j_k+z_k)\right]\Gamma(1-j-\epsilon-z)\,.\end{aligned} \tag{28}$$

In writing Eq. (28), we used $\sum_{k=1}^n (j_k + z_k) = j + 2z$. This completes the calculation.

Before moving on, some comments are in order. First, note that the derivation of Eq. (28) implicitly assumes that the exponents, j_k ($k = 1, \dots, n$) are not zero or negative integers, and indeed Eq. (28) is clearly not applicable as it stands when any j_k is a nonpositive integer. In such cases, when e.g. $-j_{k'} \in \mathbb{N}$, we can attempt to analytically continue Eq. (28) to the required value of $j_{k'}$, say by setting $j_{k'} \to j_{k'} + \delta$ and performing the analytic continuation $\delta \to 0$. The analytic continuation of Mellin–Barnes integrals has been automated in the `Mathematica` package `MB.m` [21].

Second, we call attention to the fact that that Eq. (28) was obtained under the assumption that $v_{kl} > 0$ for all $k = 1, \dots, n$ and $l = k, \dots, n$. However, sometimes it may happen that some v_{kl} is identically zero, as e.g. when, say, momentum $p^\mu_{k'}$ in Eq. (3) is massless, implying $v_{k'k'} \equiv 0$. In such situations we clearly cannot use Eq. (28) as it stands. Nevertheless, the derivation is trivial to adapt to such cases, since when say $v_{k'l'}$ is identically zero, the only change is that this term is missing form the sum in Eq. (22). Then, the corresponding Mellin–Barnes integration over $z_{k'l'}$ is absent in Eq. (23), but the rest of the derivation goes through unchanged. The end result is that we must drop integrations corresponding to variables that are identically zero from the final expression, Eq. (28). Ultimately, this amounts to simply restricting all products (such as the one in the first bracket on the second line of Eq. (28)) and sums (as in the definitions of z_k and z, Eq. (15))

over z_{kl} to values of k and l such that $v_{kl} \neq 0$. Needless to say, the H-function representation of the integral must also be adapted to accommodate the fact that some integration variables are missing.

III. EXAMPLES

In this section we illustrate the use of the general result in Eqs. (8) and (14) with several examples.

A. One denominator, massless

We begin with the simplest example, the massless one denominator angular integral, i.e. $n = 1$ and $p_1^2 = 0$ (hence $\beta_1 = 1$). In this case, Eq. (6) reduces to

$$\Omega_j(0;\epsilon) = \int \mathrm{d}\Omega_{d-2} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_1)(\sin\vartheta_1)^{-2\epsilon}(1-\cos\vartheta_1)^{-j}\,. \tag{29}$$

Since p_1^μ is massless, v_{11} is identically zero, and the discussion at the end of Sect. II B applies. Thus, the Mellin–Barnes integral representation is zero dimensional and so clearly $z_1 = z = 0$. Then we find

$$\Omega_j(0;\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{\Gamma(1-j-\epsilon)}{\Gamma(2-j-2\epsilon)}\,. \tag{30}$$

The result in Eq. (30) is easy to verify by explicit computation: recalling that

$$\int \mathrm{d}\Omega_p = \frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} \tag{31}$$

and setting $\cos\vartheta_1 \to 2s-1$ in Eq. (29), we obtain Eq. (30) immediately.

B. One denominator, massive

The next simplest example is the massive one denominator angular integral, i.e. $n = 1$, but $p_1^2 \neq 0$. In this case, Eq. (6) gives

$$\Omega_j(v_{11};\epsilon) = \int \mathrm{d}\Omega_{d-2} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_1)(\sin\vartheta_1)^{-2\epsilon}(1-\beta_1\cos\vartheta_1)^{-j}\,. \tag{32}$$

Now v_{11} is nonzero, and Eq. (14) yields the following one dimensional Mellin–Barnes integral representation

$$\Omega_j(v_{11};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\Gamma(j)\Gamma(2-j-2\epsilon)}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z_{11}}{2\pi\mathrm{i}}\,\Gamma(-z_{11})\Gamma(j+2z_{11})\Gamma(1-j-\epsilon-z_{11})(v_{11})^{z_{11}}\,. \tag{33}$$

We used that Eq. (15) gives $z_1 = 2z_{11}$ and $z = z_{11}$. In terms of the H-function, we have

$$\Omega_j(v_{11};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[(v_{11}); (\boldsymbol{\alpha}, \boldsymbol{A}); (\boldsymbol{\beta}, \boldsymbol{B}); \boldsymbol{L_s}] \,, \tag{34}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are given in Eqs. (9)–(13), with $n = 1$. Explicitly

$$\boldsymbol{\alpha} = (0, j, 1-j-\epsilon) \,, \qquad \boldsymbol{\beta} = (j, 2-j-2\epsilon) \,, \tag{35}$$

and

$$\boldsymbol{A} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} , \qquad \boldsymbol{B} = [(0)_{2\times 1}] \,. \tag{36}$$

We may compute the integral in Eq. (33) by using the doubling relation for the gamma function,

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma\left(x + \tfrac{1}{2}\right) \,, \tag{37}$$

to write Eq. (33) in the following form

$$\begin{aligned} \Omega_j(v_{11};\epsilon) &= 2^{1-2\epsilon}\pi^{\frac{1}{2}-\epsilon} \frac{1}{\Gamma(j)\Gamma(2-j-2\epsilon)} \\ &\times \int_{-i\infty}^{+i\infty} \frac{dz_{11}}{2\pi i} \Gamma(-z_{11})\Gamma\left(\tfrac{j}{2} + z_{11}\right)\Gamma\left(\tfrac{j+1}{2} + z_{11}\right)\Gamma(1-j-\epsilon-z_{11})(4v_{11})^{z_{11}} \,. \end{aligned} \tag{38}$$

Then the Mellin–Barnes integral on the second line can be evaluated in terms of Γ functions and the ${}_2F_1$ hypergeometric function. Indeed, we have

$${}_2F_1(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \Gamma(a+z)\Gamma(b+z)\Gamma(c-a-b-z)\Gamma(-z)(1-x)^z \,, \tag{39}$$

(see e.g. appendix D of Ref. [12]), and hence we find

$$\begin{aligned} \Omega_j(v_{11};\epsilon) &= 2^{1-2\epsilon}\pi^{\frac{1}{2}-\epsilon} \frac{1}{\Gamma(j)\Gamma(2-j-2\epsilon)} \\ &\times \frac{\Gamma\left(\frac{j}{2}\right)\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{3-j}{2}-\epsilon\right)\Gamma\left(\frac{2-j}{2}-\epsilon\right)}{\Gamma\left(\frac{3}{2}-\epsilon\right)} {}_2F_1\left(\frac{j}{2}, \frac{j+1}{2}, \frac{3}{2}-\epsilon, 1-4v_{11}\right) \,. \end{aligned} \tag{40}$$

Using Eq. (37), we can clean up the prefactor and obtain the final expression

$$\Omega_j(v_{11};\epsilon) = 2^{2-2\epsilon}\pi^{1-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} {}_2F_1\left(\frac{j}{2}, \frac{j+1}{2}, \frac{3}{2}-\epsilon, 1-4v_{11}\right) \,. \tag{41}$$

This result is also simple to verify by explicit calculation, since Eq. (32) is straightforward to evaluate via the substitution $\cos\vartheta_1 \to 2s - 1$. We obtain

$$\Omega_j(v_{11};\epsilon) = 2^{2-2\epsilon}\pi^{1-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} (1+\beta_1)^{-j} {}_2F_1\left(j, 1-\epsilon, 2-2\epsilon, \frac{2\beta_1}{1+\beta_1}\right) \,, \tag{42}$$

and Eq. (41) is then reproduced using the quadratic hypergeometric identity (see e.g. [22])

$$
{}_2F_1(a,b,2b,z) = \left(1-\frac{z}{2}\right)^{-a} {}_2F_1\left[\frac{a}{2},\frac{a+1}{2},b+\frac{1}{2},\left(\frac{z}{2-z}\right)^2\right], \tag{43}
$$

and the relation $v_{11} = \frac{1-\beta_1^2}{4}$. The above considerations in fact establish Eq. (33) independently of Eq. (14), thus the gap left in the derivation of the main result is closed.

Before moving on, we note that although Eq. (33) was derived under the assumption that j is not zero or a negative integer, our final result, Eq. (41), is in fact valid for negative integer j as well.

In practical applications, one is often interested in the ϵ-expansion of the final result, Eq. (41). When j is an integer, it is straightforward to obtain such expansions starting from the equivalent form of the result, Eq. (42). Indeed, for j a negative integer, the power series representation of the hypergeometric function in Eq. (42) terminates, and the ϵ-expansion in this case is trivial. For positive integer j, the method of nested sums of Ref. [23] or the integration method of Ref. [24] may be employed. The nested sums method has been implemented in several publicly available packages, such as `nestedsums` [25], `XSummer` [26] and `HypExp` [24], with the last of these implementing the integration method as well. Algorithms have also been developed for the expansion of (generalized) hypergeometric functions around half-integer values of the parameters [27–32], and the `HypExp2` package [32] provides a public implementation of one particular method. Finally, we note that whenever j is not zero or a negative integer, the direct numerical integration of the Mellin–Barnes representation in Eq. (33) provides a fast and reliable way to obtain numerical results.

By way of illustration, and for purposes of comparing with existing literature [2, 3], we obtain the ϵ-expansion of Eq. (42) for the specific values of $j = -2, -1, 1$ and 2, up to and including $\mathrm{O}(\epsilon^2)$ terms. (Note that for $j = 0$, $\Omega_0(v;\epsilon)$ just reduces to the massless integral $\Omega_0(0;\epsilon)$, and we do not discuss this case further.) The results, obtained with the method of nested sums and `XSummer` [26], are presented in Appendix C 1.

C. Two denominators, massless

Our next example is the massless two denominator angular integral, i.e. $n = 2$, with $p_1^2 = p_2^2 = 0$ (hence $\beta_1 = \beta_2 = 1$). Eq. (6) reads in this case

$$
\begin{aligned}
\Omega_{j,k}(v_{12},0,0;\epsilon) = \int \mathrm{d}\Omega_{d-3} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_1)(\sin\vartheta_1)^{-2\epsilon} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_2)(\sin\vartheta_2)^{-1-2\epsilon} \\
\times (1-\cos\vartheta_1)^{-j}(1-\cos\chi_2^{(1)}\cos\vartheta_1 - \sin\chi_2^{(1)}\sin\vartheta_1\cos\vartheta_2)^{-k}\,.
\end{aligned} \tag{44}
$$

Since both p_1^μ and p_2^μ are massless, we have $v_{11} = v_{22} \equiv 0$. Then the discussion at the end of Sect. II B applies and we find that Eq. (14) leads to the following one dimensional Mellin–Barnes integral representation

$$\begin{aligned}\Omega_{j,k}(v_{12},0,0;\epsilon) &= 2^{2-j-k-2\epsilon}\pi^{1-\epsilon}\frac{1}{\Gamma(j)\Gamma(k)\Gamma(2-j-k-2\epsilon)} \\ &\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z_{12}}{2\pi\mathrm{i}}\Gamma(-z_{12})\Gamma(j+z_{12})\Gamma(k+z_{12})\Gamma(1-j-k-\epsilon-z_{12})\,(v_{12})^{z_{12}}\,.\end{aligned} \tag{45}$$

We used that Eq. (15) gives $z_1 = z_2 = z = z_{12}$. The H-function representation of Eq. (45) reads

$$\Omega_{j,k}(v_{12},0,0;\epsilon) = 2^{2-j-k-2\epsilon}\pi^{1-\epsilon}H[(v_{12});(\boldsymbol{\alpha},\boldsymbol{A});(\boldsymbol{\beta},\boldsymbol{B});\boldsymbol{L_s}]\,, \tag{46}$$

where

$$\boldsymbol{\alpha} = (0,j,k,1-j-k-\epsilon)\,,\qquad \boldsymbol{\beta} = (j,k,2-j-k-2\epsilon)\,, \tag{47}$$

and

$$\boldsymbol{A} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}\,,\qquad \boldsymbol{B} = [(0)_{3\times 1}]\,. \tag{48}$$

Notice that $\boldsymbol{A}$ above may be obtained from the general expression for $\boldsymbol{A}(n=2)$ in Eq. (13) by removing the first and third columns which would correspond to the variables v_{11} and v_{22} which are identically zero, and then removing the first and third rows of the matrix so obtained, which contain only zeros. Correspondingly, the first and third components of $\boldsymbol{\alpha}$ (both zeros) are also removed as compared to the general formula for $n = 2$ in Eq. (9).

The Mellin–Barnes integral in Eq. (45) straightforwardly evaluates in terms of Γ functions and a ${}_2F_1$ hypergeometric function, see Eq. (39), and we find

$$\Omega_{j,k}(v_{12},0,0;\epsilon) = 2^{2-j-k-2\epsilon}\pi^{1-\epsilon}\frac{\Gamma(1-j-\epsilon)\Gamma(1-k-\epsilon)}{\Gamma(1-\epsilon)\Gamma(2-j-k-2\epsilon)}{}_2F_1(j,k,1-\epsilon,1-v_{12})\,. \tag{49}$$

Upon noting that

$$v_{12} = \frac{(p_1\cdot p_2)}{2} = \frac{1-\cos\chi_2^{(1)}}{2}\qquad\Rightarrow\qquad 1-v_{12} = \frac{1+\cos\chi_2^{(1)}}{2} = \frac{\cos^2\chi_2^{(1)}}{2}\,, \tag{50}$$

the result in Eq. (49) is seen to coincide with Eq. (A.11) of Ref. [1].

We remind the reader that Eq. (45) was derived under the assumption that j and k are not zero or negative integers. Nevertheless, the final result in Eq. (49) applies in such cases as well.

Finally, we mention that the expansion of Eq. (49) in ϵ for integer or half-integer j and k is straightforward, as discussed at the end of Sect. III B. Here, by way of illustration, we present these expansions for $j, k = -2, -1, 1$ and 2 in Appendix C 2. (We do not consider cases where either exponent is zero, since these are not genuinely two denominator angular integrals.) Expansions of the appropriate hypergeometric functions were computed with the nested sums method and `XSummer` [26].

D. Two denominators, one mass

Now consider the generalization of the previous example to the single mass case, i.e. when, say, $p_1^2 \neq 0$ but $p_2^2 = 0$ (hence $\beta_1 \neq 1$, but $\beta_2 = 1$). Then Eq. (6) gives

$$\begin{aligned}\Omega_{j,k}(v_{12}, v_{11}, 0; \epsilon) = \int \mathrm{d}\Omega_{d-3} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_1)(\sin\vartheta_1)^{-2\epsilon} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_2)(\sin\vartheta_2)^{-1-2\epsilon} \\ \times (1-\beta_1\cos\vartheta_1)^{-j}(1-\cos\chi_2^{(1)}\cos\vartheta_1 - \sin\chi_2^{(1)}\sin\vartheta_1\cos\vartheta_2)^{-k}\,.\end{aligned} \tag{51}$$

Since v_{22} still vanishes identically, the Mellin–Barnes representation of Eq. (14) is only two dimensional

$$\begin{aligned}\Omega_{j,k}(v_{12}, v_{11}, 0; \epsilon) = 2^{2-j-k-2\epsilon}\pi^{1-\epsilon}\frac{1}{\Gamma(j)\Gamma(k)\Gamma(2-j-k-2\epsilon)}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z_{11}\,\mathrm{d}z_{12}}{(2\pi\mathrm{i})^2}\Gamma(-z_{11})\Gamma(-z_{12}) \\ \times\Gamma(j+2z_{11}+z_{12})\Gamma(k+z_{12})\Gamma(1-j-k-\epsilon-z_{11}-z_{12})\,(v_{11})^{z_{11}}\,(v_{12})^{z_{12}}\,.\end{aligned} \tag{52}$$

We used that in this case, Eq. (15) evaluates as $z_1 = 2z_{11} + z_{12}$, $z_2 = z_{12}$ and $z = z_{11} + z_{12}$. Written in terms of the H-function, Eq. (52) is

$$\Omega_{j,k}(v_{12}, v_{11}, 0; \epsilon) = 2^{2-j-k-2\epsilon}\pi^{1-\epsilon}H[(v_{11}, v_{12}); (\boldsymbol{\alpha}, \boldsymbol{A}); (\boldsymbol{\beta}, \boldsymbol{B}); \boldsymbol{L_S}]\,, \tag{53}$$

where

$$\boldsymbol{\alpha} = (0, 0, j, k, 1-j-k-\epsilon)\,, \qquad \boldsymbol{\beta} = (j, k, 2-j-k-2\epsilon)\,, \tag{54}$$

and

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 1 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \boldsymbol{B} = [(0)_{3\times 2}]\,. \tag{55}$$

The $\boldsymbol{A}$ matrix above is obtained from the general expression for $\boldsymbol{A}(n=2)$ in Eq. (13) by removing the third column corresponding to the variable v_{22} which is identically zero, and then removing the third row of the matrix obtained, which contains only zeros. Accordingly, the third component of $\boldsymbol{\alpha}$ (again zero) is also dropped as compared to the general formula for $n=2$ in Eq. (9).

The two dimensional Mellin–Barnes integral in Eq. (52) can be evaluated in terms of the Appell function of the first kind. We show this in Appendix B, and only quote the final result here. We find

$$\Omega_{j,k}(v_{12}, v_{11}, 0; \epsilon) = 2^{2-j-k-2\epsilon}\pi^{1-\epsilon}\frac{\Gamma(1-k-\epsilon)}{\Gamma(2-k-2\epsilon)}\, v_{12}^{-j} \\ \times F_1\left(j, 1-k-\epsilon, 1-k-\epsilon, 2-k-2\epsilon, \frac{2v_{12}-1-\sqrt{1-4v_{11}}}{2v_{12}}, \frac{2v_{12}-1+\sqrt{1-4v_{11}}}{2v_{12}}\right) . \tag{56}$$

Let us make several comments. First, as in the previous two examples, the final expression in Eq. (56) is valid even for j or k zero or a negative integer, even though the Mellin–Barnes representation in Eq. (52), as it stands, does not apply to these cases.

Second, the Appell function of the first kind is precisely the type of generalized hypergeometric function whose expansion around integer or half-integer parameters can be solved with the methods of Refs. [23, 28–32]. In Appendix C 3, we present the expansion of Eq. (56) in ϵ up to and including finite terms, for $j, k = -2, -1, 1$ and 2. (Again, we only deal with cases which genuinely involve two denominator integrals.) We used the method of nested sums and `XSummer` [26] to compute the expansions of the appropriate Appell F_1 functions.

Finally, let us briefly discuss the application of the result in Eq. (56) to the computation of certain integrated counterterms in the NNLO subtraction scheme of Refs. [7–9]. As explained in [6], when computing the so-called integrated iterated singly-unresolved approximate cross section, several integrals must be evaluated which involve the the function $\mathcal{J}^{(1\mathrm{m})}(Y, \beta; \epsilon, y_0, d'_0)$ in their integrands, where (see (C.25) of Ref. [6])

$$\mathcal{J}^{(1\mathrm{m})}(Y, \beta; \epsilon, y_0, d'_0) \equiv -4Y\frac{\Gamma^2(1-\epsilon)}{2\pi\Gamma(1-2\epsilon)}\Omega_{11}(\cos\chi(Y,\beta), \beta, 1)\int_0^{y_0} \mathrm{d}y\, y^{-1-2\epsilon}(1-y)^{d'_0} , \tag{57}$$

and Y as well as β depend on further integration variables. Because of this, we require an all-order (in ϵ) result for $\mathcal{J}^{(1\mathrm{m})}(Y, \beta; \epsilon, y_0, d'_0)$. In Eq. (57), $\Omega_{11}(\cos\chi(Y,\beta), \beta, 1)$ is a special case of the general function $\Omega_{jk}(\cos\chi, \beta_1, \beta_2)$, defined in (C.19) of Ref. [6] as follows

$$\Omega_{j,k}(\cos\chi, \beta_1, \beta_2) \equiv \int_{-1}^{1} \mathrm{d}(\cos\vartheta)(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1} \mathrm{d}(\cos\varphi)(\sin\varphi)^{-1-2\epsilon} \\ \times (1-\beta_1\cos\vartheta)^{-j}[1-\beta_2(\sin\chi\sin\vartheta\cos\varphi+\cos\chi\cos\vartheta)]^{-k} . \tag{58}$$

For $\beta_2 = 1$, which is the case relevant in Eq. (57), this is clearly just proportional to the one mass, two denominator angular integral. Hence the results of this section can be used to evaluate $\mathcal{J}^{(1\mathrm{m})}(Y, \beta; \epsilon, y_0, d_0')$ analytically.

E. Two denominators, two masses

Next, consider the general massive two denominator angular integral, when both $p_1^2 \neq 0$ and $p_2^2 \neq 0$ (hence $\beta_1 \neq 1$ and $\beta_2 \neq 1$). In this case, Eq. (6) gives explicitly

$$\begin{aligned}\Omega_{j,k}(v_{12}, v_{11}, v_{22}; \epsilon) = \int \mathrm{d}\Omega_{d-3} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_1)(\sin\vartheta_1)^{-2\epsilon} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_2)(\sin\vartheta_2)^{-1-2\epsilon} \\ \times (1 - \beta_1 \cos\vartheta_1)^{-j} (1 - \beta_2 \cos\chi_2^{(1)} \cos\vartheta_1 - \beta_2 \sin\chi_2^{(1)} \sin\vartheta_1 \cos\vartheta_2)^{-k} .\end{aligned} \tag{59}$$

Now Eq. (14) leads to the following three dimensional Mellin–Barnes integral representation

$$\begin{aligned}\Omega_{j,k}(v_{12}, v_{11}, v_{22}; \epsilon) = 2^{2-j-k-2\epsilon} \pi^{1-\epsilon} &\frac{1}{\Gamma(j)\Gamma(k)\Gamma(2-j-k-2\epsilon)} \\ &\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_{11}\,\mathrm{d}z_{12}\,\mathrm{d}z_{22}}{(2\pi\mathrm{i})^3} \Gamma(-z_{11})\Gamma(-z_{12})\Gamma(-z_{22}) \\ &\times \Gamma(j + 2z_{11} + z_{12})\Gamma(k + z_{12} + 2z_{22}) \\ &\times \Gamma(1 - j - k - \epsilon - z_{11} - z_{12} - z_{22}) (v_{11})^{z_{11}} (v_{12})^{z_{12}} (v_{22})^{z_{22}} .\end{aligned} \tag{60}$$

We used that Eq. (15) gives $z_1 = 2z_{11} + z_{12}$, $z_2 = z_{12} + 2z_{22}$ and $z = z_{11} + z_{12} + z_{22}$. Since all variables ($v_{11}$, v_{12} and v_{22}) are now different from zero, the H-function representation of Eq. (60),

$$\Omega_{j,k}(v_{12}, v_{11}, v_{22}; \epsilon) = 2^{2-j-k-2\epsilon} \pi^{1-\epsilon} H[(v_{11}, v_{12}, v_{22}); (\boldsymbol{\alpha}, \boldsymbol{A}); (\boldsymbol{\beta}, \boldsymbol{B}); \boldsymbol{L_s}] , \tag{61}$$

is simply the general expression in Eqs. (9)–(13) for $n = 2$, i.e. we have

$$\boldsymbol{\alpha} = (0, 0, 0, j, k, 1 - j - k - \epsilon) , \qquad \boldsymbol{\beta} = (j, k, 2 - j - 2\epsilon) , \tag{62}$$

and

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} , \qquad \boldsymbol{B} = [(0)_{3\times 3}] . \tag{63}$$

In this case, we are no longer able to evaluate the Mellin–Barnes integrals in Eq. (60) in terms of functions more familiar than the H-function of several variables.

Nevertheless, the Mellin–Barnes representation of Eq. (60) is still a very useful starting point for computing the ϵ expansion of $\Omega_{j,k}(v_{12}, v_{11}, v_{22}; \epsilon)$. Let us briefly review the main steps involved.

1. In general, the contours of integration in Eq. (60) are not necessarily straight lines, and their standard definition is such that the poles with a $\Gamma(\ldots + z_{kl})$ dependence are to the left of the contour for the z_{kl} integration, while poles with a $\Gamma(\ldots - z_{kl})$ dependence are to the right of it. However, as a key observation, Ref. [33] realized straight line contours parallel to the imaginary axis in an algorithmic way. The basic idea is that starting with a curved contour that fulfills the condition on the poles, one may deform it into a straight line, taking into account the residua of the crossed poles according to Cauchy's theorem. This procedure lends itself to implementation in computer codes for the evaluation and manipulation of Mellin–Barnes integrals, such as the `MB.m` package of Ref. [21].

2. Upon deformation of the curved contours, all potential singularities in ϵ are extracted so that it is safe to expand in ϵ around zero before performing the complex integrations. In this way, the Mellin–Barnes representations of the required coefficients of the Laurent expansion of the original integral are obtained.

3. In the next step, we convert the complex contour integrations into sums over residua using Cauchy's theorem.

4. Finally, we evaluate the sums.

For the specific case of $\Omega_{j,k}(v_{12}, v_{11}, v_{22}; \epsilon)$ with integer j and k (we considered the cases $j, k = -2, -1, 1$ and 2 as before), this procedure leads to a representation of the $\mathrm{O}(\epsilon^0)$ coefficient which involves at most single sums. These are all straightforward to compute and we present the results in Appendix C 4.

However, starting from the linear term in ϵ, we are lead to representations of the coefficients involving up to triple sums, which are difficult to compute, and we made no severe effort to calculate them. In fact, this corroborates the findings of Ref. [5] nicely, where a completely different method leads to a one dimensional real integral representation of the linear term in the ϵ expansion of $\Omega_{j,k}(v_{12}, v_{11}, v_{22}; \epsilon)$ which "…involves three square roots so it is difficult to evaluate the integration analytically…" and hence Ref. [5] does "…not have an analytical answer." Nevertheless, we remind the reader that the direct numerical integration of the Mellin–Barnes representation provides a convenient and efficient way of obtaining numerical results for the higher order coefficients.

F. Three denominators, massless

As a final example, we present explicitly the massless angular integral with three denominators, i.e. $n = 3$ and $p_i^2 = 0$ (hence $\beta_i = 1$) $i = 1, 2, 3$. Eq. (6) reads

$$\begin{aligned}\Omega_{j,k,l}(v_{12}, v_{13}, v_{23}; \epsilon) &= \int \mathrm{d}\Omega_{d-4} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_1)(\sin\vartheta_1)^{-2\epsilon} \int_{-1}^{1} \mathrm{d}(\cos\vartheta_2)(\sin\vartheta_2)^{-1-2\epsilon} \\ &\times \int_{-1}^{1} \mathrm{d}(\cos\vartheta_3)(\sin\vartheta_3)^{-2-2\epsilon}\,(1-\cos\vartheta_1)^{-j}(1-\cos\chi_2^{(1)}\cos\vartheta_1 - \sin\chi_2^{(1)}\sin\vartheta_1\cos\vartheta_2)^{-k} \\ &\times (1-\cos\chi_3^{(1)}\cos\vartheta_1 - \cos\chi_3^{(2)}\sin\chi_3^{(1)}\sin\vartheta_1\cos\vartheta_2 - \sin\chi_3^{(2)}\sin\chi_3^{(1)}\sin\vartheta_1\sin\vartheta_2\cos\vartheta_3)^{-l}\,.\end{aligned} \tag{64}$$

Since $v_{ii} \equiv 0$ all for $i = 1, 2, 3$, the Mellin–Barnes representation of Eq. (14) collapses to only three integrations

$$\begin{aligned}\Omega_{j,k,l}(v_{12}, v_{13}, v_{23}; \epsilon) &= 2^{2-j-k-l-2\epsilon}\pi^{1-\epsilon}\frac{1}{\Gamma(j)\Gamma(k)\Gamma(l)\Gamma(2-j-k-l-2\epsilon)} \\ &\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_{12}\,\mathrm{d}z_{13}\,\mathrm{d}z_{23}}{(2\pi\mathrm{i})^3}\Gamma(-z_{12})\Gamma(-z_{13})\Gamma(-z_{23}) \\ &\times \Gamma(j+z_{12}+z_{13})\Gamma(k+z_{12}+z_{23})\Gamma(l+z_{13}+z_{23}) \\ &\times \Gamma(1-j-k-l-\epsilon-z_{12}-z_{13}-z_{23})\,(v_{12})^{z_{12}}\,(v_{13})^{z_{13}}\,(v_{23})^{z_{23}}\,.\end{aligned} \tag{65}$$

Eq. (15) gives $z_1 = z_{12} + z_{13}$, $z_2 = z_{12} + z_{23}$, $z_3 = z_{13} + z_{23}$ and $z = z_{12} + z_{13} + z_{23}$ in this case, which we used when writing Eq. (65). In terms of the H-function, Eq. (65) has the following representation

$$\Omega_{j,k,l}(v_{12}, v_{13}, v_{23}; \epsilon) = 2^{2-j-k-l-2\epsilon}\pi^{1-\epsilon} H[(v_{12}, v_{13}, v_{23}); (\boldsymbol{\alpha}, \boldsymbol{A}); (\boldsymbol{\beta}, \boldsymbol{B}); \boldsymbol{L_s}]\,, \tag{66}$$

where

$$\boldsymbol{\alpha} = (0, 0, 0, j, k, l, 1-j-k-l-\epsilon)\,, \qquad \boldsymbol{\beta} = (j, k, l, 2-j-k-l-2\epsilon)\,, \tag{67}$$

and

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \qquad \boldsymbol{B} = [(0)_{4\times 3}] \,. \tag{68}$$

We may obtain the $\boldsymbol{A}$ matrix above from the general expression for $\boldsymbol{A}(n = 3)$ in Eq. (13) by removing the first, fourth and sixth columns corresponding to the variables v_{11}, v_{22} and v_{33} which are identically zero, and then removing all rows of the resulting matrix, which contain only zeros (i.e. rows one, four and six). The corresponding components of $\boldsymbol{\alpha}$ (i.e. the first, fourth and sixth, all zeros) are also dropped as compared to the general formula for $n = 3$ in Eq. (9).

As in the previous example, we are again unable to evaluate the Mellin–Barnes integrals in Eq. (65) in terms of functions other than the H-function of several variables.

However, as discussed in Sect. III E, Eq. (65) provides a useful starting point for obtaining the ϵ expansion of $\Omega_{j,k,l}(v_{12}, v_{13}, v_{23}; \epsilon)$. In this particular case, for j, k and l integers (in fact we consider only $j, k, l = 1$ and 2), the procedure outlined below Eq. (60) leads to zero dimensional Mellin–Barnes integral representations for both the $\mathrm{O}(\epsilon^{-1})$ and $\mathrm{O}(\epsilon^0)$ coefficients in Step 2. Hence, there are no integrals or sums to compute at all. The expansions obtained, up to and including $\mathrm{O}(\epsilon^0)$ terms, are presented in Appendix C 5.

The situation with higher order expansion coefficients is very similar to the previous example of Sect. III E. Again, starting from the linear term in ϵ, we find representations involving triple sums which are hard to compute. In passing, we note that one may also attempt to evaluate the Mellin–Barnes representations of given expansion coefficients by means other than converting them into sums e.g. with methods along the lines of Ref. [34], where the authors compute a difficult three dimensional Mellin–Barnes integral in terms of Goncharov polylogarithms. If we are satisfied with numerical results for higher order expansion coefficients, then the direct numerical integration the Mellin–Barnes representation proves convenient.

We finish by noting that the knowledge of Eq. (65) was necessary to compute certain iterated singly-unresolved integrals in Ref. [6]. Specifically, when computing the so-called integrated soft-double soft counterterm, we encounter the massless angular integral with three denominators in intermediate stages of the calculation. In particular, in order to be able to write the Mellin–Barnes representation of the $\mathcal{I}^{(11)}_{\mathcal{S};ik,jk}$ master integral of Eq. (E.52) of Ref. [6], we required a Mellin–Barnes representation for the angular integral

$$\begin{aligned}&\int_{-1}^{1}\mathrm{d}(\cos\vartheta)(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1}\mathrm{d}(\cos\varphi)(\sin\varphi)^{-1-2\epsilon}\,(1-\cos\vartheta)^{-1}\\&\quad\times(1-\cos\chi_2^{(1)}\cos\vartheta-\sin\chi_2^{(1)}\sin\vartheta\cos\varphi)^{-z_1}(1-\cos\chi_3^{(1)}\cos\vartheta+\sin\chi_3^{(1)}\sin\vartheta\cos\varphi)^{-z_2}\,,\end{aligned} \tag{69}$$

where z_1 and z_2 are integration variables of further Mellin–Barnes integrals. The integral in Eq. (69) is clearly proportional to $\Omega_{1,z_1,z_2}(v_{12},v_{13},v_{23};\epsilon)$, in the special case where $\sin\chi_3^{(2)}=0$ and $\cos\chi_3^{(2)}=-1$ in Eq. (3). (This constraint means that only two variables out of v_{12}, v_{13} and v_{23} are independent.) The results of this section thus provide the necessary Mellin–Barnes representation of Eq. (69), and hence are needed to compute the $\mathcal{I}^{(11)}_{\mathcal{S};ik,jk}$ master integral.

IV. CONCLUSIONS

In this paper, we have evaluated some d dimensional angular integrals which arise in perturbative field theory calculations. We used the method of Mellin–Barnes representations to compute the general angular integral with n denominators, massive or massless momenta and (essentially) arbitrary powers of the denominators in terms of a certain special function, the so-called H-function of several variables. We pointed out that the existence of various contiguous relations for the H-function provides the opportunity to apply the method of differential reduction to angular integrals in d dimensions. It would be very interesting to expand the present results in this direction.

We illustrated the use of our general result with several examples of angular integrals with up to three denominators. We showed that some of these integrals can be computed in terms of (generalized) hypergeometric functions. In particular, the single denominator massless integral is fully expressed by Γ functions, while the massive integral involves the ${}_2F_1$ hypergeometric function. For the massless two denominator integral we recover the known result of Ref. [1] which again involves a ${}_2F_1$ hypergeometric function. However, our derivation is much more straightforward than the original computation. When precisely one of the momenta is massive, we obtain a new all-order (in ϵ) analytical expression for the two denominator angular integral which involves the Appell function of the first kind, F_1.

In some applications, one is interested in the expansion of the angular integrals in the parameter of dimensional regularization, ϵ. We discussed briefly how such expansions can be obtained starting from the corresponding Mellin-Barnes representations. By way of illustration, we have explicitly presented such expansions for all examples discussed, for a few specific values of exponents.

The results of this paper have already found applications in computing certain phase space integrals which appear when integrating NNLO subtraction terms. In fact, all specific angular integrals that are encountered in the integration of the so-called singly-unresolved and iterated singly-unresolved subtraction terms of Refs. [8, 9] were discussed in this paper explicitly. We expect that our present results will also prove valuable when computing the so-called integrated doubly-unresolved subtraction terms.

Acknowledgments

It is a pleasure to thank M. Y. Kalmykov, S.-O. Moch and Z. Trócsányi for useful discussions and comments on the manuscript and E. Laenen for providing copies of Ref. [5] and some unpublished notes of W. L. van Neerven. This work was supported in part by the Deutsche Forschungsgemeinschaft in SFB/TR 9 and.

Appendix A: The H-function of several variables

In this Appendix, we recall the particular definition of the H-function of several variables which we use in Sect. II A. This function has been discussed in various forms by several authors in the literature, here we adopt (essentially) the definition of Ref. [13]. In the most general case, the H-function of N variables is defined as follows:

$$H[\boldsymbol{x}, (\boldsymbol{\alpha}, \boldsymbol{A}), (\boldsymbol{\beta}, \boldsymbol{B}); \boldsymbol{L_s}] \equiv (2\pi\mathrm{i})^{-N} \int_{\boldsymbol{L_s}} \Theta(\boldsymbol{s})\, \boldsymbol{x}^{\boldsymbol{s}}\, \mathbf{d}\boldsymbol{s}\,, \tag{A1}$$

where

$$\Theta(\boldsymbol{s}) = \frac{\prod_{j=1}^{m} \Gamma\left(\alpha_j + \sum_{k=1}^{N} a_{j,k} s_k\right)}{\prod_{j=1}^{n} \Gamma\left(\beta_j + \sum_{k=1}^{N} b_{j,k} s_k\right)}\,. \tag{A2}$$

Here $\boldsymbol{s} = (s_1, \ldots, s_N)$, $\boldsymbol{x} = (x_1, \ldots, x_N)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ denote vectors of complex numbers; while

$$\boldsymbol{A} = (a_{j,k})_{m\times N} \qquad \text{and} \qquad \boldsymbol{B} = (b_{j,k})_{n\times N} \tag{A3}$$

are matrices of real numbers. Also

$$\boldsymbol{x}^{\boldsymbol{s}} = \prod_{k=1}^{N} (x_k)^{s_k}\,; \qquad \mathbf{d}\boldsymbol{s} = \prod_{k=1}^{N} \mathrm{d}s_k\,; \qquad \boldsymbol{L_s} = L_{s_1} \times \ldots \times L_{s_N}\,, \tag{A4}$$

where L_{s_k} is an infinite contour in the complex s_k-plane running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ such that $\Theta(\boldsymbol{s})$ has no singularities for $\boldsymbol{s} \in \boldsymbol{L_s}$.

The H-function of Eq. (A1) generalizes nearly all known special functions of N variables, e.g. Lauricella functions $F_A^{(N)}$, $F_B^{(N)}$, $F_C^{(N)}$ and $F_D^{(N)}$; the G-function of N variables; the special H-function of N variables, etc. For the specific cases of $N = 1$ and 2, it essentially reduces to the known Fox's H-function of one variable and the H-function of two variables defined by various authors scattered in the literature. The definition given in Eq. (A1) is different form the H-function considered by Ref. [13] only in the replacement of $\boldsymbol{x}^{-\boldsymbol{s}}$ by $\boldsymbol{x}^{\boldsymbol{s}}$. We have made this replacement for convenience in our applications.

Appendix B: A Mellin–Barnes Integral

In this Appendix, we evaluate the following two dimensional Mellin–Barnes integral analytically

$$I = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(2\pi\mathrm{i})^2} \Gamma(-z_1)\Gamma(-z_2)\Gamma(a+2z_1+z_2)\Gamma(b+z_2)\Gamma(c-z_1-z_2)\, x^{z_1}\, y^{z_2}\,, \tag{B1}$$

which we encounter in Sect. III D. Throughout this Appendix, we assume tacitly that all parameters and integration variables lie in a strip of the complex plane such that each integral we write converges.

We begin by writing the product of the second and fourth gamma functions above as a one dimensional real integral:

$$I = \Gamma(b) \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(2\pi\mathrm{i})^2} \int_0^1 \mathrm{d}s\, s^{-1-z_2}(1-s)^{b-1+z_2}\, \Gamma(-z_1)\Gamma(a+2z_1+z_2)\Gamma(c-z_1-z_2)\, x^{z_1}\, y^{z_2}\,. \tag{B2}$$

Now, it is somewhat easier to follow the manipulations below if we make the change of variables $z_2 \to -a - 2z_1 - z_2$:

$$\begin{aligned} I = \Gamma(b) \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(2\pi\mathrm{i})^2} \int_0^1 \mathrm{d}s\, s^{a-1+2z_1+z_2}(1-s)^{b-a-1-2z_1-z_2} \\ \times\, \Gamma(-z_1)\Gamma(-z_2)\Gamma(a+c+z_1+z_2)\, x^{z_1} \left(\frac{1}{y}\right)^{a+2z_1+z_2} \quad . \end{aligned} \tag{B3}$$

Next, we rearrange some factors and write I in the following form

$$\begin{aligned} I = y^{-a}\Gamma(b) \int_0^1 \mathrm{d}s\, s^{a-1}(1-s)^{a+b+2c-1} \\ \times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(2\pi\mathrm{i})^2} \Gamma(-z_1)\Gamma(-z_2)\Gamma(a+c+z_1+z_2) \left(\frac{xs^2}{y^2}\right)^{z_1} \left[\frac{s(1-s)}{y}\right]^{z_2} \left[(1-s)^2\right]^{-a-c-z_1-z_2} . \end{aligned} \tag{B4}$$

The Mellin–Barnes integrals are now easy to perform and we find

$$I = y^{-a}\Gamma(b)\Gamma(a+c)\int_0^1 \mathrm{d}s\, s^{a-1}(1-s)^{a+b+2c-1}\left[(1-s)^2 + \frac{s(1-s)}{y} + \frac{xs^2}{y^2}\right]^{-a-c} . \tag{B5}$$

Factoring the quadratic expression in the square brackets,

$$(1-s)^2 + \frac{s(1-s)}{y} + \frac{xs^2}{y^2} = \left(1 - \frac{2y-1-\sqrt{1-4x}}{2y}s\right)\left(1 - \frac{2y-1+\sqrt{1-4x}}{2y}s\right) , \tag{B6}$$

we obtain

$$\begin{aligned} I = {} & y^{-a}\Gamma(b)\Gamma(a+c)\int_0^1 \mathrm{d}s\, s^{a-1}(1-s)^{a+b+2c-1} \\ & \times \left(1 - \frac{2y-1-\sqrt{1-4x}}{2y}s\right)^{-a-c}\left(1 - \frac{2y-1+\sqrt{1-4x}}{2y}s\right)^{-a-c} . \end{aligned} \tag{B7}$$

The final integral can be performed in terms of the Appell function of the first kind (see e.g. [22]) and we find

$$\begin{aligned} I = {} & y^{-a}\frac{\Gamma(a)\Gamma(b)\Gamma(a+c)\Gamma(a+b+2c)}{\Gamma(2a+b+2c)} \\ & \times F_1\left(a, a+c, a+c, 2a+b+2c, \frac{2y-1-\sqrt{1-4x}}{2y}, \frac{2y-1+\sqrt{1-4x}}{2y}\right) , \end{aligned} \tag{B8}$$

which is our final result.

Appendix C: Expansions

1. One denominator, one mass

In this section, we present the ϵ-expansion of $\Omega_j(v;\epsilon)$ for the specific values of $j = -2, -1, 1$ and 2. In order not to clutter the following expressions with irrelevant constants like $\ln(4\pi)$ and γ_E, we extract a factor of $\int \mathrm{d}\Omega_{d-3}$ and present the ϵ-expansion of the function I_j, where

$$I_j(v;\epsilon) \equiv 2^{-1+2\epsilon}\pi^{\epsilon}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\Omega_j(v;\epsilon) , \tag{C1}$$

which has the further advantage that it may be directly compared with the appropriate expressions of Refs. [2, 3]. We find

$$I_{-2}(v;\epsilon) = 2\pi\left[\frac{4}{3} - \frac{4}{3}v + \left(\frac{26}{9} - \frac{32}{9}v\right)\epsilon + \left(\frac{160}{27} - \frac{208}{27}v\right)\epsilon^2 + \mathrm{O}(\epsilon^3)\right], \tag{C2}$$

$$I_{-1}(v;\epsilon) = 2\pi\left[1 + 2\epsilon + 4\epsilon^2 + \mathrm{O}(\epsilon^3)\right], \tag{C3}$$

$$\begin{aligned} I_1(v;\epsilon) = \frac{\pi}{\sqrt{1-4v}}\Bigg\{ & \ln\left(\frac{1+\sqrt{1-4v}}{1-\sqrt{1-4v}}\right) + \frac{1}{2}\left[\ln^2\left(\frac{1+\sqrt{1-4v}}{1-\sqrt{1-4v}}\right) + 4\,\mathrm{Li}_2\left(\frac{2\sqrt{1-4v}}{1+\sqrt{1-4v}}\right)\right]\epsilon \\ & + \frac{1}{6}\left[\ln^3\left(\frac{1+\sqrt{1-4v}}{1-\sqrt{1-4v}}\right) - 6\ln\left(\frac{2\sqrt{1-4v}}{1+\sqrt{1-4v}}\right)\ln^2\left(\frac{1+\sqrt{1-4v}}{1-\sqrt{1-4v}}\right)\right. \\ & + 12\ln\left(\frac{1+\sqrt{1-4v}}{1-\sqrt{1-4v}}\right)\mathrm{Li}_2\left(\frac{2\sqrt{1-4v}}{1+\sqrt{1-4v}}\right) + 12\ln\left(\frac{1+\sqrt{1-4v}}{1-\sqrt{1-4v}}\right)\mathrm{Li}_2\left(\frac{1-\sqrt{1-4v}}{1+\sqrt{1-4v}}\right) \\ & \left. + 24\,\mathrm{Li}_3\left(\frac{2\sqrt{1-4v}}{1+\sqrt{1-4v}}\right) + 12\,\mathrm{Li}_3\left(\frac{1-\sqrt{1-4v}}{1+\sqrt{1-4v}}\right) - 12\,\zeta_3\right]\epsilon^2 + \mathrm{O}(\epsilon^3)\Bigg\}, \end{aligned} \tag{C4}$$

$$\begin{aligned} I_2(v;\epsilon) = \frac{\pi}{4v\sqrt{1-4v}}\Bigg\{ & 2\sqrt{1-4v} + 2\ln\left(\frac{1+\sqrt{1-4v}}{1-\sqrt{1-4v}}\right)\epsilon + \left[\ln^2\left(\frac{1+\sqrt{1-4v}}{1-\sqrt{1-4v}}\right)\right. \\ & \left. + 4\,\mathrm{Li}_2\left(\frac{2\sqrt{1-4v}}{1+\sqrt{1-4v}}\right)\right]\epsilon^2 + \mathrm{O}(\epsilon^3)\Bigg\}. \end{aligned} \tag{C5}$$

2. Two denominators, massless

Here we present the ϵ-expansion of the massless angular integral with two denominators, $\Omega_{j,k}(v,0,0;\epsilon)$, for the specific values of $j,k = -2,-1,1$ and 2. More precisely, we extract a factor of $\int \mathrm{d}\Omega_{1-2\epsilon}$ and define

$$I_{j,k}(v;\epsilon) \equiv 2^{-1+2\epsilon}\pi^{\epsilon}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\Omega_{j,k}(v,0,0;\epsilon). \tag{C6}$$

This choice of normalization keeps the expanded expressions simpler and allows for a straightforward comparison with Refs. [2, 3]. Since the results are clearly symmetric in j and k, we can restrict to the cases where $j \geq k$. We find

$$I_{-2,-2}(v;\epsilon) = 2\pi\left[\frac{16}{5} - \frac{16}{5}v + \frac{8}{15}v^2 + \left(\frac{596}{75} - \frac{656}{75}v + \frac{368}{225}v^2\right)\epsilon + \mathrm{O}(\epsilon^2)\right], \tag{C7}$$

$$I_{-1,-2}(v;\epsilon) = 2\pi\left[2 - \frac{4}{3}v + \left(\frac{14}{3} - \frac{32}{9}v\right)\epsilon + \mathrm{O}(\epsilon^2)\right], \tag{C8}$$

$$I_{1,-2}(v;\epsilon) = 2\pi\left[-\frac{2v^2}{\epsilon} + 1 + 2v - 6v^2 + 2\Big(1 + 3v - 7v^2\Big)\epsilon + \mathrm{O}(\epsilon^2)\right], \tag{C9}$$

$$I_{2,-2}(v;\epsilon) = 2\pi\left[-\frac{2v(2-3v)}{\epsilon} + 1 - 8v + 6v^2 + 2\Big(1 - 8v + 9v^2\Big)\epsilon + \mathrm{O}(\epsilon^2)\right], \tag{C10}$$

$$I_{-1,-1}(v;\epsilon) = 2\pi\left[\frac{4}{3} - \frac{2}{3}v + \left(\frac{26}{9} - \frac{16}{9}v\right)\epsilon + \mathrm{O}(\epsilon^2)\right], \tag{C11}$$

$$I_{1,-1}(v;\epsilon) = 2\pi\left[-\frac{v}{\epsilon} + 1 - 2v + 2\Big(1 - 2v\Big)\epsilon + \mathrm{O}(\epsilon^2)\right], \tag{C12}$$

$$I_{2,-1}(v;\epsilon) = 2\pi\left[-\frac{1-2v}{2\epsilon} - v + v\epsilon + \mathrm{O}(\epsilon^2)\right], \tag{C13}$$

$$I_{1,1}(v;\epsilon) = \frac{\pi}{v}\left\{-\frac{1}{\epsilon} + \ln v - \frac{1}{2}\Big[\ln^2 v + 2\,\mathrm{Li}_2(1-v)\Big]\epsilon + \mathrm{O}(\epsilon^2)\right\}, \tag{C14}$$

$$I_{2,1}(v;\epsilon) = \frac{\pi}{2v^2}\left\{-\frac{1}{\epsilon} - 2 + v + \ln v + \frac{1}{2}\Big[2v + 4\ln v - \ln^2 v - 2\,\mathrm{Li}_2(1-v)\Big]\epsilon + \mathrm{O}(\epsilon^2)\right\}, \tag{C15}$$

$$\begin{aligned} I_{2,2}(v;\epsilon) = \frac{\pi}{2v^3}\bigg\{ & -\frac{2-v}{\epsilon} - 5 + 4v + (2-v)\ln v \\ & - \frac{1}{2}\Big[4 - 6v - (10-4v)\ln v + (2-v)\ln^2 v + 2(2-v)\,\mathrm{Li}_2(1-v)\Big]\epsilon + \mathrm{O}(\epsilon^2)\bigg\}. \end{aligned} \tag{C16}$$

3. Two denominators, one mass

Next, we present the ϵ-expansion of the angular integral with two denominators and one mass, $\Omega_{j,k}(v_{12}, v_{11}, 0; \epsilon)$, for the specific values $j, k = -2, -1, 1$ and 2. As before, we extract a factor of $\int \mathrm{d}\Omega_{1-2\epsilon}$ and set

$$I^{(1)}_{j,k}(v_{12}, v_{11}; \epsilon) \equiv 2^{-1+2\epsilon}\pi^{\epsilon}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\Omega_{j,k}(v_{12}, v_{11}, 0; \epsilon)\,. \tag{C17}$$

When $k < 1$, the integral is finite in ϵ and we have

$$I^{(1)}_{-2,-2}(v_{12}, v_{11}; \epsilon) = 2\pi \left[\frac{16}{5} - \frac{8}{5} v_{11} - \frac{16}{5} v_{12} + \frac{8}{15} v_{12}^2 + \mathrm{O}(\epsilon) \right], \tag{C18}$$

$$I^{(1)}_{-1,-2}(v_{12}, v_{11}; \epsilon) = 2\pi \left[2 - \frac{4}{3} v_{12} + \mathrm{O}(\epsilon) \right], \tag{C19}$$

$$\begin{aligned} I^{(1)}_{1,-2}(v_{12}, v_{11}; \epsilon) = \frac{2\pi}{(1-4v_{11})^{5/2}} \Bigg[& \sqrt{1-4v_{11}} \Big(1 - 10v_{11} + 2v_{12} + 16 v_{11} v_{12} - 6 v_{12}^2 \Big) \\ & - 2 \Big(6v_{11}^2 - 6 v_{11} v_{12} + v_{12}^2 + 2 v_{11} v_{12}^2 \Big) \ln \left(\frac{1 - \sqrt{1-4v_{11}}}{1 + \sqrt{1-4v_{11}}} \right) + \mathrm{O}(\epsilon) \Bigg], \end{aligned} \tag{C20}$$

$$\begin{aligned} I^{(1)}_{2,-2}(v_{12}, v_{11}; \epsilon) = \frac{2\pi}{v_{11}(1-4v_{11})^{5/2}} \Bigg[& \sqrt{1-4v_{11}} \Big(v_{11} + 8v_{11}^2 - 12 v_{11} v_{12} + v_{12}^2 + 8 v_{11} v_{12}^2 \Big) \\ & + 2v_{11} \Big(3v_{11} - 2v_{12} - 4 v_{11} v_{12} + 3 v_{12}^2 \Big) \ln \left(\frac{1 - \sqrt{1-4v_{11}}}{1 + \sqrt{1-4v_{11}}} \right) + \mathrm{O}(\epsilon) \Bigg], \end{aligned} \tag{C21}$$

$$I^{(1)}_{-2,-1}(v_{12}, v_{11}; \epsilon) = 2\pi \left[2 - \frac{4}{3} \Big(v_{11} + v_{12} \Big) + \mathrm{O}(\epsilon) \right], \tag{C22}$$

$$I^{(1)}_{-1,-1}(v_{12}, v_{11}; \epsilon) = 2\pi \left[\frac{4}{3} - \frac{2}{3} v_{12} + \mathrm{O}(\epsilon) \right], \tag{C23}$$

$$\begin{aligned} I^{(1)}_{1,-1}(v_{12}, v_{11}; \epsilon) = & \frac{2\pi}{(1-4v_{11})^{3/2}} \\ & \times \left[\sqrt{1-4v_{11}} \Big(1 - 2v_{12} \Big) + \Big(2v_{11} - v_{12} \Big) \ln \left(\frac{1 - \sqrt{1-4v_{11}}}{1 + \sqrt{1-4v_{11}}} \right) + \mathrm{O}(\epsilon) \right], \end{aligned} \tag{C24}$$

$$\begin{aligned} I^{(1)}_{2,-1}(v_{12}, v_{11}; \epsilon) = & -\frac{\pi}{v_{11}(1-4v_{11})^{3/2}} \\ & \times \left[\sqrt{1-4v_{11}} \Big(2v_{11} - v_{12} \Big) + v_{11} \Big(1 - 2v_{12} \Big) \ln \left(\frac{1 - \sqrt{1-4v_{11}}}{1 + \sqrt{1-4v_{11}}} \right) + \mathrm{O}(\epsilon) \right], \end{aligned} \tag{C25}$$

On the other hand, for $k \geq 1$, the integral has a pole in ϵ. We find

$$I^{(1)}_{-2,1}(v_{12}, v_{11}; \epsilon) = 2\pi\left[-\frac{2v_{12}^2}{\epsilon} + 1 - 2v_{11} + 2v_{12} - 6v_{12}^2 + \mathrm{O}(\epsilon)\right], \tag{C26}$$

$$I^{(1)}_{-1,1}(v_{12}, v_{11}; \epsilon) = 2\pi\left[-\frac{v_{12}}{\epsilon} + 1 - 2v_{12} + \mathrm{O}(\epsilon)\right], \tag{C27}$$

$$I^{(1)}_{1,1}(v_{12}, v_{11}; \epsilon) = \frac{\pi}{2v_{12}}\left[-\frac{1}{\epsilon} - \ln\left(\frac{v_{11}}{v_{12}^2}\right) + \mathrm{O}(\epsilon)\right], \tag{C28}$$

$$I^{(1)}_{2,1}(v_{12}, v_{11}; \epsilon) = \frac{\pi}{4v_{11}v_{12}^2}\left[-\frac{v_{11}}{\epsilon} - 2v_{11} + v_{12} - v_{11}\ln\left(\frac{v_{11}}{v_{12}^2}\right) + \mathrm{O}(\epsilon)\right], \tag{C29}$$

$$I^{(1)}_{-2,2}(v_{12}, v_{11}; \epsilon) = 2\pi\left[\frac{2(v_{11} - 2v_{12} + 3v_{12}^2)}{\epsilon} + 1 + 4v_{11} - 8v_{12} + 6v_{12}^2 + \mathrm{O}(\epsilon)\right], \tag{C30}$$

$$I^{(1)}_{-1,2}(v_{12}, v_{11}; \epsilon) = 2\pi\left[-\frac{1 - 2v_{12}}{2\epsilon} - v_{12} + \mathrm{O}(\epsilon)\right], \tag{C31}$$

$$I^{(1)}_{1,2}(v_{12}, v_{11}; \epsilon) = \frac{\pi}{4v_{12}^3}\left[\frac{2v_{11} - v_{12}}{\epsilon} + 2\Big(2v_{11} - 2v_{12} + v_{12}^2\Big) + \Big(2v_{11} - v_{12}\Big)\ln\left(\frac{v_{11}}{v_{12}^2}\right) + \mathrm{O}(\epsilon)\right], \tag{C32}$$

$$\begin{aligned} I^{(1)}_{2,2}(v_{12}, v_{11}; \epsilon) = \frac{\pi}{8v_{11}v_{12}^4}&\left[\frac{2v_{11}(3v_{11} - 2v_{12} + v_{12}^2)}{\epsilon} + 16v_{11}^2 - 16v_{11}v_{12} + v_{12}^2 + 10v_{11}v_{12}^2\right.\\ &\left. + 2v_{11}(3v_{11} - 2v_{12} + v_{12}^2)\ln\left(\frac{v_{11}}{v_{12}^2}\right) + \mathrm{O}(\epsilon)\right], \end{aligned} \tag{C33}$$

4. Two denominators, two masses

Here, we present the ϵ-expansion of the angular integral with two denominators and two masses, $\Omega_{j,k}(v_{12}, v_{11}, v_{22}; \epsilon)$, for the specific values $j, k = -2, -1, 1$ and 2. As before, we extract a factor of $\int \mathrm{d}\Omega_{1-2\epsilon}$ and define

$$I^{(2)}_{j,k}(v_{12}, v_{11}, v_{22}; \epsilon) \equiv 2^{-1+2\epsilon}\pi^{\epsilon}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\Omega_{j,k}(v_{12}, v_{11}, v_{22}; \epsilon)\,. \tag{C34}$$

Clearly this expression is symmetric under the simultaneous exchange of $j \leftrightarrow k$ and $v_{11} \leftrightarrow v_{22}$, thus we can restrict to the cases where $j \geq k$.

We find

$$I^{(2)}_{-2,-2}(v_{12},v_{11},v_{22};\epsilon) = 2\pi\left[\frac{16}{5} - \frac{8}{5}v_{11} - \frac{16}{5}v_{12} + \frac{8}{15}v_{12}^2 - \frac{8}{5}v_{22} + \frac{16}{15}v_{11}v_{22} + \mathrm{O}(\epsilon)\right], \tag{C35}$$

$$I^{(2)}_{-1,-2}(v_{12},v_{11},v_{22};\epsilon) = 2\pi\left[2 - \frac{4}{3}v_{12} - \frac{4}{3}v_{22} + \mathrm{O}(\epsilon)\right], \tag{C36}$$

$$\begin{aligned} I^{(2)}_{1,-2}(v_{12},v_{11},v_{22};\epsilon) = {} & \frac{2\pi}{(1-4v_{11})^{5/2}}\Bigg[\sqrt{1-4v_{11}}\Big(1 - 10v_{11} + 2v_{12} + 16v_{11}v_{12} - 6v_{12}^2 - 2v_{22} \\ & + 8v_{11}v_{22}\Big) - 2\Big(6v_{11}^2 - 6v_{11}v_{12} + v_{12}^2 + 2v_{11}v_{12}^2 + 2v_{11}v_{22} - 8v_{11}^2v_{22}\Big) \\ & \times \ln\left(\frac{1-\sqrt{1-4v_{11}}}{1+\sqrt{1-4v_{11}}}\right) + \mathrm{O}(\epsilon)\Bigg], \end{aligned} \tag{C37}$$

$$\begin{aligned} I^{(2)}_{2,-2}(v_{12},v_{11},v_{22};\epsilon) = {} & \frac{2\pi}{v_{11}(1-4v_{11})^{5/2}}\Bigg[\sqrt{1-4v_{11}}\Big(v_{11} + 8v_{11}^2 - 12v_{11}v_{12} + v_{12}^2 + 8v_{11}v_{12}^2 \\ & + 4v_{11}v_{22} - 16v_{11}^2v_{22}\Big) + 2v_{11}\Big(3v_{11} - 2v_{12} - 4v_{11}v_{12} + 3v_{12}^2 + v_{22} \\ & - 4v_{11}v_{22}\Big) \ln\left(\frac{1-\sqrt{1-4v_{11}}}{1+\sqrt{1-4v_{11}}}\right) + \mathrm{O}(\epsilon)\Bigg], \end{aligned} \tag{C38}$$

$$I^{(2)}_{-1,-1}(v_{12},v_{11},v_{22};\epsilon) = 2\pi\left[\frac{4}{3} - \frac{2}{3}v_{12} + \mathrm{O}(\epsilon)\right], \tag{C39}$$

$$\begin{aligned} I^{(2)}_{1,-1}(v_{12},v_{11},v_{22};\epsilon) = {} & \frac{2\pi}{(1-4v_{11})^{3/2}} \\ & \times\left[\sqrt{1-4v_{11}}\Big(1-2v_{12}\Big) + \Big(2v_{11}-v_{12}\Big)\ln\left(\frac{1-\sqrt{1-4v_{11}}}{1+\sqrt{1-4v_{11}}}\right) + \mathrm{O}(\epsilon)\right], \end{aligned} \tag{C40}$$

$$\begin{aligned} I^{(2)}_{2,-1}(v_{12},v_{11},v_{22};\epsilon) = {} & -\frac{\pi}{v_{11}(1-4v_{11})^{3/2}} \\ & \times\left[\sqrt{1-4v_{11}}\Big(2v_{11}-v_{12}\Big) + v_{11}\Big(1-2v_{12}\Big)\ln\left(\frac{1-\sqrt{1-4v_{11}}}{1+\sqrt{1-4v_{11}}}\right) + \mathrm{O}(\epsilon)\right], \end{aligned} \tag{C41}$$

$$I^{(2)}_{1,1}(v_{12},v_{11},v_{22};\epsilon) = \frac{\pi}{2\sqrt{v_{12}^2-4v_{11}v_{22}}}\left[\ln\left(\frac{v_{12}+\sqrt{v_{12}^2-4v_{11}v_{22}}}{v_{12}-\sqrt{v_{12}^2-4v_{11}v_{22}}}\right) + \mathrm{O}(\epsilon)\right], \tag{C42}$$

$$\begin{aligned} I^{(2)}_{2,1}(v_{12},v_{11},v_{22};\epsilon) = {} & -\frac{\pi}{4v_{11}(v_{12}^2-4v_{11}v_{22})^{3/2}}\Bigg[\sqrt{v_{12}^2-4v_{11}v_{22}}\Big(2v_{11}-v_{12}\Big) \\ & - v_{11}\Big(v_{12}-2v_{22}\Big)\ln\left(\frac{v_{12}+\sqrt{v_{12}^2-4v_{11}v_{22}}}{v_{12}-\sqrt{v_{12}^2-4v_{11}v_{22}}}\right) + \mathrm{O}(\epsilon)\Bigg], \end{aligned} \tag{C43}$$

$$I^{(2)}_{2,2}(v_{12}, v_{11}, v_{22}; \epsilon) = \frac{\pi}{8v_{11}v_{22}(v_{12}^2 - 4v_{11}v_{22})^{5/2}} \Bigg[\sqrt{v_{12}^2 - 4v_{11}v_{22}}\Big(v_{11}v_{12}^2 + 8v_{11}^2 v_{22}$$

$$- 12v_{11}v_{12}v_{22} + v_{12}^2 v_{22} + 4v_{11}v_{12}^2 v_{22} + 8v_{11}v_{22}^2 - 16v_{11}^2 v_{22}^2\Big)$$

$$- 2v_{11}v_{22}\Big(3v_{11}v_{12} - 2v_{12}^2 + v_{12}^3 - 4v_{11}v_{22} + 3v_{12}v_{22} - 4v_{11}v_{12}v_{22}\Big)$$

$$\times \ln\left(\frac{v_{12} + \sqrt{v_{12}^2 - 4v_{11}v_{22}}}{v_{12} - \sqrt{v_{12}^2 - 4v_{11}v_{22}}}\right) + \mathrm{O}(\epsilon)\Bigg]. \qquad \text{(C44)}$$

5. Three denominators, massless

Finally, we present the ϵ-expansion of the massless angular integral with three denominators, $\Omega_{j,k,l}(v_{12}, v_{13}, v_{23}; \epsilon)$, for the specific values $j, k = 1$ and 2. As in the previous examples, we extract a factor of $\int \mathrm{d}\Omega_{1-2\epsilon}$ and define

$$I_{j,k,l}(v_{12}, v_{13}, v_{23}; \epsilon) \equiv 2^{-1+2\epsilon}\pi^{\epsilon}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\Omega_{j,k,l}(v_{12}, v_{13}, v_{23}; \epsilon). \qquad \text{(C45)}$$

Clearly this expression is symmetric under the permutations of the indices j, k and l. Hence, we can restrict to the cases where $j \geq k \geq l$. We find

$$I_{1,1,1}(v_{12}, v_{13}, v_{23}; \epsilon) = -\frac{\pi}{4v_{12}v_{13}v_{23}}\Bigg[\frac{v_{12} + v_{13} + v_{23}}{\epsilon} + \Big(v_{12} - v_{13} - v_{23}\Big)\ln v_{12}$$

$$+ \Big(v_{13} - v_{12} - v_{23}\Big)\ln v_{13} + \Big(v_{23} - v_{12} - v_{13}\Big)\ln v_{23} + \mathrm{O}(\epsilon)\Bigg], \qquad \text{(C46)}$$

$$I_{2,1,1}(v_{12}, v_{13}, v_{23}; \epsilon) = -\frac{\pi}{8v_{12}^2 v_{13}^2 v_{23}}\Bigg[\frac{v_{12}^2 + v_{13}^2 + 2v_{12}v_{23} + 2v_{13}v_{23} - 2v_{12}v_{13}v_{23} - v_{23}^2}{\epsilon}$$

$$+ v_{12}^2 - 2v_{12}v_{13} + v_{13}^2 + 4v_{12}v_{23} + 4v_{13}v_{23} - 6v_{12}v_{13}v_{23} - v_{23}^2$$

$$+ \Big(v_{12}^2 - v_{13}^2 - 2v_{12}v_{23} - 2v_{13}v_{23} + 2v_{12}v_{13}v_{23} + v_{23}^2\Big)\ln v_{12}$$

$$+ \Big(v_{13}^2 - v_{12}^2 - 2v_{13}v_{23} - 2v_{12}v_{23} + 2v_{12}v_{13}v_{23} + v_{23}^2\Big)\ln v_{13}$$

$$- \Big(v_{12}^2 + v_{13}^2 - 2v_{12}v_{23} - 2v_{13}v_{23} + 2v_{12}v_{13}v_{23} + v_{23}^2\Big)\ln v_{23} + \mathrm{O}(\epsilon)\Bigg], \qquad \text{(C47)}$$

$$
\begin{aligned}
I_{2,2,1}(v_{12},v_{13},v_{23};\epsilon) = {} & -\frac{\pi}{16v_{12}^3v_{13}^2v_{23}^2}\Bigg[\Big(v_{12}^3+3v_{12}v_{13}^2-2v_{13}^3+6v_{13}^2v_{23}-6v_{12}v_{13}^2v_{23}+3v_{12}v_{23}^2 \\
& +6v_{13}v_{23}^2-6v_{12}v_{13}v_{23}^2-2v_{23}^3\Big)\frac{1}{\epsilon}+2v_{12}^3-3v_{12}^2v_{13}+8v_{12}v_{13}^2-3v_{13}^3-3v_{12}^2v_{23}-6v_{12}v_{13}v_{23} \\
& +6v_{12}^2v_{13}v_{23}+13v_{13}^2v_{23}-16v_{12}v_{13}^2v_{23}+8v_{12}v_{23}^2+13v_{13}v_{23}^2-16v_{12}v_{13}v_{23}^2-3v_{23}^3 \\
& +\Big(v_{12}^3-3v_{12}v_{13}^2+2v_{13}^3-6v_{13}^2v_{23}+6v_{12}v_{13}^2v_{23}-3v_{12}v_{23}^2-6v_{13}v_{23}^2+6v_{12}v_{13}v_{23}^2+2v_{23}^3\Big)\ln v_{12} \\
& -\Big(v_{12}^3-3v_{12}v_{13}^2+2v_{13}^3-6v_{13}^2v_{23}+6v_{12}v_{13}^2v_{23}+3v_{12}v_{23}^2+6v_{13}v_{23}^2-6v_{12}v_{13}v_{23}^2-2v_{23}^3\Big)\ln v_{13} \\
& -\Big(v_{12}^3+3v_{12}v_{13}^2-2v_{13}^3+6v_{13}^2v_{23}-6v_{12}v_{13}^2v_{23}-3v_{12}v_{23}^2-6v_{13}v_{23}^2+6v_{12}v_{13}v_{23}^2+2v_{23}^3\Big)\ln v_{23} \\
& +\mathrm{O}(\epsilon)\Bigg], \qquad \text{(C48)}
\end{aligned}
$$

$$
\begin{aligned}
I_{2,2,2}(v_{12},v_{13},v_{23};\epsilon) = {} & \frac{\pi}{16v_{12}^3v_{13}^3v_{23}^3}\Bigg[\Big(2v_{12}^4-4v_{12}^3v_{13}-4v_{12}v_{13}^3+2v_{13}^4-4v_{12}^3v_{23}+6v_{12}^3v_{13}v_{23} \\
& -4v_{13}^3v_{23}+6v_{12}v_{13}^3v_{23}-4v_{12}v_{23}^3-4v_{13}v_{23}^3+6v_{12}v_{13}v_{23}^3+2v_{23}^4\Big)\frac{1}{\epsilon}+4v_{12}^4-12v_{12}^3v_{13} \\
& +6v_{12}^2v_{13}^2-12v_{12}v_{13}^3+4v_{13}^4-12v_{12}^3v_{23}+6v_{12}^2v_{13}v_{23}+19v_{12}^3v_{13}v_{23}+6v_{12}v_{13}^2v_{23} \\
& -12v_{12}^2v_{13}^2v_{23}-12v_{13}^3v_{23}+19v_{12}v_{13}^3v_{23}+6v_{12}^2v_{23}^2+6v_{12}v_{13}v_{23}^2-12v_{12}^2v_{13}v_{23}^2 \\
& +6v_{13}^2v_{23}^2-12v_{12}v_{13}^2v_{23}^2+6v_{12}^2v_{13}^2v_{23}^2-12v_{12}v_{23}^3-12v_{13}v_{23}^3+19v_{12}v_{13}v_{23}^3+4v_{23}^4 \\
& +\Big(2v_{12}^4-4v_{12}^3v_{13}+4v_{12}v_{13}^3-2v_{13}^4-4v_{12}^3v_{23}+6v_{12}^3v_{13}v_{23}+4v_{13}^3v_{23} \\
& -6v_{12}v_{13}^3v_{23}+4v_{12}v_{23}^3+4v_{13}v_{23}^3-6v_{12}v_{13}v_{23}^3-2v_{23}^4\Big)\ln v_{12} \\
& -\Big(2v_{12}^4-4v_{12}^3v_{13}+4v_{12}v_{13}^3-2v_{13}^4-4v_{12}^3v_{23}+6v_{12}^3v_{13}v_{23}+4v_{13}^3v_{23} \\
& -6v_{12}v_{13}^3v_{23}-4v_{12}v_{23}^3-4v_{13}v_{23}^3+6v_{12}v_{13}v_{23}^3+2v_{23}^4\Big)\ln v_{13} \\
& -\Big(2v_{12}^4-4v_{12}^3v_{13}-4v_{12}v_{13}^3+2v_{13}^4-4v_{12}^3v_{23}+6v_{12}^3v_{13}v_{23}-4v_{13}^3v_{23} \\
& +6v_{12}v_{13}^3v_{23}+4v_{12}v_{23}^3+4v_{13}v_{23}^3-6v_{12}v_{13}v_{23}^3-2v_{23}^4\Big)\ln v_{23}+\mathrm{O}(\epsilon)\Bigg]. \qquad \text{(C49)}
\end{aligned}
$$

---

[1] W. L. van Neerven, Nucl. Phys. **B268**, 453 (1986).

[2] W. Beenakker, H. Kuijf, W. L. van Neerven, and J. Smith, Phys. Rev. **D40**, 54 (1989).

[3] J. Smith, D. Thomas, and W. L. van Neerven, Z. Phys. **C44**, 267 (1989).

[4] E. Laenen (2010), private communication, based on unpublished notes of W. L. van Neerven.

[5] J. Smith (2009), YITP-SB-09-13.

[6] P. Bolzoni, G. Somogyi, and Z. Trócsányi, JHEP **01**, 059 (2011), 1011.1909.

[7] G. Somogyi and Z. Trócsányi (2006), hep-ph/0609041.

[8] G. Somogyi, Z. Trócsányi, and V. Del Duca, JHEP **01**, 070 (2007), hep-ph/0609042.

[9] G. Somogyi and Z. Trócsányi, JHEP **01**, 052 (2007), hep-ph/0609043.

[10] U. Aglietti, V. Del Duca, C. Duhr, G. Somogyi, and Z. Trócsányi, JHEP **09**, 107 (2008), 0807.0514.

[11] P. Bolzoni, S.-O. Moch, G. Somogyi, and Z. Trócsányi, JHEP **08**, 079 (2009), 0905.4390.

[12] V. A. Smirnov, *Feynman integral calculus* (Springer, 2006), 283 p.

[13] N. T. Hai and H. M. Srivastava, Computers Math. Applic. **29**, 17 (1995).

[14] A. M. Mathai, R. K. Saxena, and H. H. J., *The H-function theory and applications* (Springer, 2010), 270 p.

[15] O. P. Tandon, Indian J. Pure Appl. Math. **11**, 321 (1980).

[16] C. M. Joshi and J. P. Arya, Indian J. Pure Appl. Math. **12**, 826 (1981).

[17] O. P. Tandon, Indian J. Math. **24**, 55 (1982).

[18] V. V. Bytev, M. Y. Kalmykov, and B. A. Kniehl, Nucl. Phys. **B836**, 129 (2010), 0904.0214.

[19] M. Y. Kalmykov, V. V. Bytev, B. A. Kniehl, B. F. L. Ward, and S. A. Yost, PoS **ACAT08**, 125 (2008), 0901.4716.

[20] O. V. Tarasov, Phys. Rev. **D54**, 6479 (1996), hep-th/9606018.

[21] M. Czakon, Comput. Phys. Commun. **175**, 559 (2006), hep-ph/0511200.

[22] I. S. Gradhsteyn and I. M. Ryzhik, *Table of integrals, series and products* (Elsevier, 2007), 7th ed., 1171 p.

[23] S. Moch, P. Uwer, and S. Weinzierl, J. Math. Phys. **43**, 3363 (2002), hep-ph/0110083.

[24] T. Huber and D. Maitre, Comput. Phys. Commun. **175**, 122 (2006), hep-ph/0507094.

[25] S. Weinzierl, Comput. Phys. Commun. **145**, 357 (2002), math-ph/0201011.

[26] S. Moch and P. Uwer, Comput. Phys. Commun. **174**, 759 (2006), math-ph/0508008.

[27] A. I. Davydychev and M. Y. Kalmykov, Nucl. Phys. **B699**, 3 (2004), hep-th/0303162.

[28] S. Weinzierl, J. Math. Phys. **45**, 2656 (2004), hep-ph/0402131.

[29] M. Y. Kalmykov, JHEP **04**, 056 (2006), hep-th/0602028.

[30] M. Y. Kalmykov, B. F. L. Ward, and S. Yost, JHEP **02**, 040 (2007), hep-th/0612240.

[31] M. Y. Kalmykov, B. F. L. Ward, and S. A. Yost, JHEP **10**, 048 (2007), 0707.3654.

[32] T. Huber and D. Maitre, Comput. Phys. Commun. **178**, 755 (2008), 0708.2443.

[33] J. B. Tausk, Phys. Lett. **B469**, 225 (1999), hep-ph/9909506.

[34] V. Del Duca, C. Duhr, and V. A. Smirnov, JHEP **05**, 084 (2010), 1003.1702.