LOGARITHMIC GROMOV-WITTEN INVARIANTS

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CONTENTS

INTRODUCTION.

The purpose of this paper is the development of a general theory of Gromov-Witten invariants in logarithmically smooth situations. Concrete examples of such situations are nonsingular projective varieties with a normal crossing divisor, central fibres of semi-stable degenerations, or toroidal pairs. They occur naturally in Gromov-Witten theory in imposing constraints, in dimensional reduction and in degeneration situations. The easiest, and already quite useful special case is Gromov-Witten invariants relative to a smooth divisor. The first

treatments used symplectic techniques [LiRu],[IoPa]. Algebraically a direct approach for very ample divisors is possible [Ga], the much more complicated general case is due to Jun Li [Li1], [Li2]. Any of the general approaches use a geometrically beautiful, but technically unpleasant change of target space with the purpose of avoiding irreducible components mapping into the divisor. As an application these authors also developed decomposition formulas of absolute Gromov-Witten invariants into sums of relative Gromov-Witten invariants under a semistable degeneration of the target variety into two components intersecting along a smooth divisor (symplectically, a symplectic cut).

While Jun Li's theory was under development the second named author suggested that a far more general and potentially technically simpler theory should be possible using abstract log geometry [Si2]. This paper is the late realization of this proposal. While several problems could be solved back then, notably stable reduction and the construction of virtual fundamental classes, there remained a fundamental problem of selecting the natural ("basic") log structure on the base scheme of a family of stable log maps. Without a notion of basicness it is virtually impossible to prove algebraicity and quasi-compactness of the relevant moduli stack. With hindsight one can say that the suggestion of log Gromov-Witten theory was premature in 2001 because two major ingredients were only just developing, tropical geometry on the one hand and more powerful techniques for the treatment of log moduli problems on the other hand. In fact, the essential insight for the notion of basicness came in a discussion of the authors in August 2008 and was guided by our understanding of moduli of tropical curves and their relation to log geometry. As for log moduli problems Olsson's algebraic stack of fine log structures [Ol3] is crucial to prove algebraicity of our stack of basic stable log maps, and Olsson's many other works on problems and applications of log geometry served as a blueprint at many stages of this work.

The main concept is that of a *basic stable log map* (Definition 1.5 and Definition 1.19). A stable log map is just a stable map with all arrows defined in the category of log schemes and the condition on the domain to be pre-stable replaced by log smoothness. We show that stable log maps to a fixed log scheme $X = (\underline{X}, \mathcal{M}_X)$ form an algebraic stack $\tilde{\mathscr{M}}(X)$ (Theorem 2.4). The main ingredients in this proof are Olsson's algebraic stack $\mathcal{L}og$ of fine log schemes, the understanding of pre-stable curves from the log point of view [Kf], and a representability result for spaces of log morphisms in relative situations (Proposition 2.9). The stack $\tilde{\mathscr{M}}(X)$ is far too large because it allows arbitrary log structures on the base of a family of stable log maps. The notion of basicness (Definition 1.19) selects a universal choice that interacts well with geometry. Basicness only depends on the log morphism on the level of ghost sheaves. The relation to tropical geometry comes by pulling back to the standard log point (§1.3). In situations where tropical geometry applies, the moduli space of associated tropical curves is the dual of the basic monoid (the stalk of the ghost sheaf of the log structure on the base). Basicness selects an open substack $\mathscr{M}(X)$ of $\tilde{\mathscr{M}}(X)$ which now also has a separated diagonal. It also comes with a natural log structure. This is the first

main result (stated as Proposition 2.3 and Corollary 2.6 in the text). We work over a fixed fine saturated log scheme $S = (\underline{S}, \mathcal{M}_S)$.

Theorem 0.1. *Let $X = (\underline{X}, \mathcal{M}_X)$ be a fine saturated log scheme. Then the stack $\mathscr{M}(X/S)$ of basic stable log maps to X over S is an algebraic log stack locally of finite type over $\underline{S}$. The forgetful morphism $\mathscr{M}(X/S) \to \mathbf{M}(\underline{X}/\underline{S})$ to the stack of ordinary stable maps over $\underline{S}$ is representable.*

A basic problem, which we could not solve in complete generality, is to identify quasi-compact substacks of $\mathscr{M}(X/S)$. Natural conditions concern the classical conditions on genus, number of marked points and homology class of the underlying ordinary stable map, plus logarithmic conditions at the marked points similar to the orders of tangency with the divisor in the classical situation. Denoting $\mathscr{M}(X/S, \beta)$ the substack of $\mathscr{M}(X/S)$ fulfilling such a set of conditions, a stratawise approach reduces the question of boundednesss to a generally subtle question in convex geometry (Theorem 3.14), summarized in the concept of *combinatorial finiteness* of a class β (Definition 3.3). We have complete results provided the pull-back of $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\underline{\mathbb{Z}}} \mathbb{Q}$ to any stable log map is globally generated, that is, if any germ of a section is the restriction of a global section. Then any β is combinatorially finite (Theorem 3.9). This criterion holds in the important special cases of genus 0, of simple normal crossing divisors and in toric situations. We also have boundedness without further conditions as long as the dual intersection graph of the domain has genus at most one (Theorem 3.13) and in certain favourable stratawise situations (Theorem 3.12). In concrete situations one can check combinatorial finiteness by examining a finite list of ordinary stable maps.

Arguably the most difficult single result in this paper is stable reduction (Theorem 4.1), which requires a solid understanding of the interaction between the basicness condition and geometry. Taken together with boundedness we obtain the second main result (Corollary 4.2 in the text).

Theorem 0.2. *For $\underline{X}$ projective over $\underline{S}$ and β a combinatorially finite class of stable log maps, $\mathscr{M}(X/S, \beta)$ is proper over $\underline{S}$.*

The construction of a *virtual fundamental class* on $\mathscr{M}(X/S, \beta)$ now is completely straightforward via the approach of [BeFa] using Olsson's log cotangent complex [Ol3], as already worked out by Kim [Ki]. One can then define log Gromov-Witten invariants in the usual way by pairing with cohomology classes on $\underline{X}$ via evaluation at the marked points. From this construction we readily obtain our third main result.

Theorem 0.3. *In the situation of Theorem 0.2 there exists a virtual fundamental class $[\![\mathscr{M}(X/S, \beta)]\!]$ leading to log Gromov-Witten invariants with the expected properties.* $\quad\square$

Among the expected properties are compatibility with base change, equality with the ordinary virtual fundamental class for trivial log structures and with the usual fundamental class in unobstructed situations, and many more that follow by strict analogy with the ordinary case.

While this work was in progress we learnt of ongoing work of Abramovich, Chen and coworkers aiming at carrying out our program in the special case that there exists a surjection $\underline{\mathbb{N}}^r \to \overline{\mathcal{M}}_X$ [Ch],[ACGM].

Conventions. We work in the categories of schemes of finite type over a field $\Bbbk$ of characteristic 0 and of fine saturated log schemes over a fixed base log scheme S, itself fine and saturated. Our standard notation for log schemes is $X = (\underline{X}, \mathcal{M}_X)$, but we do not underline ordinary schemes that we do not want to endow with a log structure unless there is a chance of confusion. Similarly, for a morphism of log schemes the notation is $f = (\underline{f}, f^\flat)$. As usual the *ghost sheaf* (also called *characteristic*) of a log structure $\mathcal{M}$ is denoted $\overline{\mathcal{M}} := \mathcal{M}/\mathcal{O}^\times$. Throughout X is a log scheme over S such that $\mathcal{M}_S$, $\mathcal{M}_X$ and the structure morphism $(\underline{X}, \mathcal{M}_X) \to (\underline{S}, \mathcal{M}_S)$ are defined in the Zariski topology. To have a good theory of ordinary stable maps at our disposal we require $\underline{X}$ is quasi-projective over $\underline{S}$. If Y is a scheme (algebraic space, stack) then $|Y|$ denotes the set of geometric points of Y endowed with the Zariski topology, see [Kt], II.6 or [LaMB], Ch.5. A *toric monoid* is a fine, saturated, torsion-free monoid without non-trivial invertibles. For a monoid P we write $P^\vee := \operatorname{Hom}(P, \mathbb{N})$ for the dual in the category of monoids, P^{gp} for the associated abelian group and $P^* := \operatorname{Hom}(P, \mathbb{Z})$.

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1. Stable log maps

1.1. **Log smooth curves.** In this subsection we work in the absolute situation $S = \operatorname{Spec} \Bbbk$ with the trivial log structure. A logarithmic version of the theory of (pre-) stable curves of Deligne, Mumford and Knudsen [DeMu], [Kd] has been developed by F.Kato [Kf], see also [Ol1]. The starting point is the following result on the structure of log smooth curves.

Theorem 1.1. [Kf, p.222] *Let $\pi : C \to W$ be a smooth and integral morphism of fine saturated log schemes such that every geometric fibre is a reduced curve. Assume that $\underline{W} = \operatorname{Spec} A$ for $(A, \mathfrak{m})$ a strictly Henselian local ring. Let $0 \in \underline{W}$ be the closed point, $Q = \overline{\mathcal{M}}_{W,0}$ and $\sigma : Q \to A$ a chart for the log structure on W. Then étale locally C is isomorphic to one of the following log schemes V over W.*

(i) $\operatorname{Spec}(A[z, w]/(zw - t))$ *with $t \in \mathfrak{m}$ and with the log structure induced from the homomorphism*

$$Q \oplus_\mathbb{N} \mathbb{N}^2 \longrightarrow \mathcal{O}_V, \quad (q, (a, b)) \longmapsto \sigma(q)z^a w^b.$$

Here $\mathbb{N} \to \mathbb{N}^2$ is the diagonal embedding and $\mathbb{N} \to Q$, $1 \mapsto \rho_q$ is some homomorphism uniquely defined by $C \to W$. Moreover, $\rho_q \neq 0$.

(ii) $\operatorname{Spec}(A[z])$ *with the log structure induced from the homomorphism*

$$Q \oplus \mathbb{N} \longrightarrow \mathcal{O}_V, \quad (q, a) \longmapsto z^a \sigma(q).$$

(iii) $\mathrm{Spec}(A[z])$ *with the log structure induced from the homomorphism*

$$Q \longrightarrow \mathcal{O}_V, \quad q \longmapsto \sigma(q).$$

In this list, the morphism $C \to W$ is represented by the canonical maps of charts $Q \to Q \oplus_{\mathbb{N}} \mathbb{N}^2$, $Q \to Q \oplus \mathbb{N}$ and $Q \to Q$, respectively. $\square$

Case (i) covers the situation in a neighbourhood of the degeneracy locus of π, while (iii) deals with smooth points of the central fibre where the log structure comes entirely from the base. To interpret case (ii) observe that the preimage of $z = 0$ defines a section of π on an étale neighbourhood of x. This should be viewed as the section defining a marked point. Let $\Gamma \subset \underline{C}$ be the image of this section and $\iota : \underline{C} \setminus \Gamma \hookrightarrow \underline{C}$ the inclusion. Then the log structure of C near x is the sum of the log structure of the base and the log structure $\iota_*(\mathcal{O}_{\underline{C}\setminus\Gamma}^{\times}) \cap \mathcal{O}_{\underline{C}}$ associated to Γ.

Remark 1.2. (1) The monoid Q together with $\rho_q \in Q$ for the nodes $q \in \underline{C}$ determine the log structure on the closed fibre $\underline{C}_0$ and the morphism to the log point $(\mathrm{Spec}(A/\mathfrak{m}), Q)$ on the level of ghost sheaves. In fact, let $\bar{\eta}$ be the generic point of the branch of $\underline{C}_0$ defined by $w = 0$ in (i). Then at $\bar{\eta}$ the element z^a becomes invertible. Hence the compatibility of the charts in (i) and (iii) implies that the generization map $\overline{\mathcal{M}}_{C_0,\bar{q}} \longrightarrow \overline{\mathcal{M}}_{C_0,\bar{\eta}}$ equals

$$Q \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow Q, \quad (m,(a,b)) \longmapsto m + b \cdot \rho_q.$$

These generization maps together with one copy of $\mathbb{N}$ at each of the special points in (ii) define $\overline{\mathcal{M}}_C$ and the structure homomorphism $Q \to \Gamma(\underline{C}_0, \overline{\mathcal{M}}_{C_0})$ uniquely up to isomorphism.

Note that if $\underline{C}$ has non-normal irreducible components then $\mathcal{M}_C$ is only defined over the étale site. Still, $\overline{\mathcal{M}}_C$ can be described completely by generization maps, the only difference now being that at a node $q \in \underline{C}$ in the closure of only one generic point η there are two generization maps $\overline{\mathcal{M}}_{C,\bar{q}} \to \overline{\mathcal{M}}_{C,\bar{\eta}}$.

2) At a node $q \in \underline{C}_0$ the two generization maps $Q \oplus_{\mathbb{N}} \mathbb{N}^2 \to Q$ define the homomorphism

$$\iota : Q \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow Q \times Q, \quad (m,(a,b)) \longmapsto (m + a \cdot \rho_q, m + b \cdot \rho_q).$$

If $\iota\big((m,(a,b))\big) = 0$ then $m + a\rho_q = m + b\rho_q = 0$, and hence $a = b$ since $\rho_q \neq 0$. In view of the definition of $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ this implies

$$(m,(a,b)) = (m,(a,a)) = \big(m + a\rho_q,(0,0)\big) = 0.$$

Thus ι exhibits $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ as a submonoid of $Q \times Q$:

(1.1) $\qquad Q \oplus_{\mathbb{N}} \mathbb{N}^2 \simeq \big\{ (m_1, m_2) \in Q \times Q \,\big|\, m_1 - m_2 \in \mathbb{Z}\rho_q \text{ in } Q^{\mathrm{gp}} \big\} \subset Q \times Q.$

In fact, if $m_1 - m_2 = \alpha\rho_q$ then

$$(m_1, m_2) = \begin{cases} \iota\big(m_2,(\alpha,0)\big), & \alpha \geq 0 \\ \iota\big(m_1,(0,-\alpha)\big), & \alpha < 0. \end{cases}$$

$\square$

The theorem suggests the following generalization of the notion of marked (pre-) stable curves to log geometry.

Definition 1.3. A *pre-stable (marked) log curve* over W is a pair $(C/W, \mathbf{x})$ consisting of a proper log smooth and integral morphism $\pi : C \to W$ of fine saturated log schemes over S together with a tuple of sections $\mathbf{x} = (x_1, \dots, x_l)$ of $\underline{\pi}$, such that every geometric fibre of π is a reduced and connected curve, and if $U \subset \underline{C}$ is the non-critical locus of π then $\overline{\mathcal{M}}_C|_U \simeq \underline{\pi}^* \overline{\mathcal{M}}_W \oplus \bigoplus_i x_{i*} \mathbb{N}_W$.

A pre-stable log curve is *stable* if forgetting the log structure leads to an ordinary stable curve.   □

Remark 1.4. 1) The underlying morphism of a pre-stable log curve is flat and its geometric fibres have at most ordinary double points. The underlying morphism of schemes of a pre-stable log curve is hence an ordinary pre-stable curve.

2) The condition on $\overline{\mathcal{M}}_C$ says that the sections x_i label precisely the special non-nodal points occurring in Theorem 1.1,(ii). A straightforward generalization would only label a subset of the special non-nodal points.

3) Another way to state the stability condition is to say that the geometric fibres do not have infinitesimal (log-) automorphisms, that is, $H^0(\underline{C}_{\overline{w}}, \Theta_{C/W} \times_{\mathcal{O}_{\underline{C}}} \mathcal{O}_{\underline{C}_{\overline{w}}}) = 0$ for every geometric point $\overline{w}$ of $\underline{W}$.

Kato also introduces the notion of basic log structure of a marked *stable* curve ([Kf] p.227f). A simple way to think about it is as follows. Let $(\underline{\pi} : \underline{C} \to \underline{W}, \mathbf{x} = (x_1, \dots, x_k))$ be a stable marked curve. Locally with respect to the base, $(\underline{C}/\underline{W}, \mathbf{x})$ is the pull-back by a morphism $\underline{W} \to \underline{T}$ of a pre-stable marked curve $(\underline{q} : \underline{U} \to \underline{T}, \mathbf{y})$ that is formally versal at any point of $\underline{T}$. Versality implies that the image of the subspace of $\underline{U}$ defined by the first Fitting ideal of $\Omega^1_{\underline{U}/\underline{T}}$ is a normal crossings divisor $D \subset \underline{T}$. This divisor is the scheme theoretic version of the subset of $\underline{T}$ parametrizing singular curves. We endow $\underline{T}$ and $\underline{U}$ with the log structures associated to the divisors D in $\underline{T}$ and $q^{-1}(D)$ and the divisor of marked points in $\underline{U}$, respectively. The basic log structure on $C \to W$ is then obtained by pull-back via $\underline{W} \to \underline{T}$. Globally one obtains an étale descent datum for $\mathcal{M}_C$, $\mathcal{M}_W$ and for the morphism $\pi^* \mathcal{M}_W \to \mathcal{M}_C$. Note also that at a geometric point $\overline{w}$ of $\underline{W}$ there is an isomorphism $\overline{\mathcal{M}}_{W,\overline{w}} \simeq \mathbb{N}^r$ where r is the number of double points of $\underline{C}_{\overline{w}}$, which by versality equals the number of branches of D at the image of $\overline{w}$ in $\underline{T}$.

The existence of basic log structures on stable curves makes it possible to endow the stacks $\mathbf{M}_{g,k}$ of l-marked stable curves with a logarithmic structure [Kf, p.230f]. This means ([Ol2], Definition 5.1 and Corollary 5.8) that there exists a factorization

(1.2) $$\mathbf{M}_{g,k} \longrightarrow (\mathrm{Log}) \longrightarrow (\mathrm{Sch})$$

of the functor defining the stack $\mathbf{M}_{g,k}$. Here (Log) is the category of fine saturated log schemes with strict morphisms. The first arrow maps a family $(\underline{C}/\underline{W}, \mathbf{x})$ of k-marked stable curves of

genus g to its base scheme $\underline{W}$ endowed with the basic log structure. The factorization (1.2) endows $\mathbf{M}_{g,k}$ with a log structure, defining a log stack $\mathscr{M}_{g,k}$. Of course, not every stable log curve carries the basic log structure of the underlying stable marked curve. The log structure can rather always be obtained from the basic log structure by a unique base change inducing the identity on the underlying spaces ([Kf], Proposition 2.1). See also [Ol1], Ch.5, for an extended treatment.

Analogous statements hold for pre-stable curves, leading to the log algebraic stack $\mathscr{M}$, an Artin stack (see Appendix).

1.2. Stable log maps.

1.2. **Stable log maps.** We now turn to the main concept of this paper, a logarithmic version of the notion of stable map. Recall that X is a log scheme over S, with log structures defined on the Zariski sites.

Definition 1.5. A *log curve over* X *with* *base* W is a pre-stable marked log curve $(C/W, \mathbf{x})$ (Definition 1.3) together with a morphism $f : C \to X$ fitting into a commutative diagram of log schemes

(1.3)
$$\begin{array}{ccc} C & \xrightarrow{\ f\ } & X \\ \pi \downarrow & & \downarrow \\ W & \longrightarrow & S \end{array}$$

A log curve over X is a *stable log map* if for every geometric point $\overline{w} \to \underline{W}$ the restriction of $\underline{f}$ to the underlying pre-stable marked curve belonging to $\underline{C_{\overline{w}}} \to \overline{w}$ is an ordinary stable map. The notation is $(C/W, \mathbf{x}, f)$ with the morphisms $\pi : C \to W$ and $W \to S$ usually understood.

A morphism of stable log maps

$$\Phi : (C_1/W_1, \mathbf{x}_1, f_1) \longrightarrow (C_2/W_2, \mathbf{x}_2, f_2)$$

is a cartesian diagram of log curves $\Phi : C_1/W_1 \to C_2/W_2$ over S with $W_1 \to W_2$ strict and such that $f_1 = f_2 \circ \Phi$ and $\mathbf{x}_2 = \underline{\Phi} \circ \mathbf{x}_1$.

The category of stable log maps thus obtained is denoted $\tilde{\mathscr{M}}(X/S)$ or just $\tilde{\mathscr{M}}(X)$. $\square$

Let κ be a field. For a toric monoid Q with $Q^{\times} = \{0\}$ denote by $(\operatorname{Spec} \kappa, Q)$ the associated logarithmic point, that is, $\operatorname{Spec} \kappa$ with log structure

$$Q \times \kappa^{\times} \longrightarrow \kappa, \quad (q,a) \longmapsto \begin{cases} 0, & q \neq 0 \\ a, & q = 0. \end{cases}$$

Now let $\underline{C}$ be a pre-stable curve over κ, and assume given a fine saturated log structure $\alpha : \mathcal{M} \to \mathcal{O}_{\underline{C}}$ on the Zariski site of $\underline{C}$. Of course, we will have $\mathcal{M} = \underline{f}^* \mathcal{M}_X$ for $\underline{f} : \underline{C} \to \underline{X}$, but this is not relevant for the following discussion. In any case, this log structure $\mathcal{M}$ can be rather arbitrary and certainly does not need to be smooth for any log structure on $\operatorname{Spec} \kappa$.

We need to understand diagrams

(1.4)
$$(\underline{C}, \mathcal{M}_C) \longrightarrow (\underline{C}, \mathcal{M})$$
$$\downarrow$$
$$(\operatorname{Spec}\kappa, Q)$$

with left-hand vertical arrow smooth, for some toric monoid Q. In the simplest case of trivial $\mathcal{M}$ we are in the situation recalled in Section 1.1. Thus in this case there is a universal such diagram with $Q = \mathbb{N}^r$ and r the number of nodes of C, in the sense that any other diagram is obtained by unique pull-back. We will see that there is a similarly universal object in complete generality. The essential step is the characterization of a universal such diagram on the level of ghost sheaves.

To study this situation let $\overline{\mathcal{M}}$ be a fine saturated sheaf on a pre-stable curve $\underline{C}/\operatorname{Spec}\kappa$. We consider the following category.

Definition 1.6. Let $\underline{\operatorname{GS}}(\overline{\mathcal{M}})$ (for "ghost sheaves") be the category with objects

$$(Q, \overline{\mathcal{M}}_C, \psi : Q \to \Gamma(\underline{C}, \overline{\mathcal{M}}_C), \varphi : \overline{\mathcal{M}} \to \overline{\mathcal{M}}_C),$$

where Q is a toric monoid, $\overline{\mathcal{M}}_C$ is a fine sheaf on $\underline{C}$ and ψ and φ are sharp[1] homomorphisms of fine (sheaves of) monoids. We require that $\overline{\mathcal{M}}_C$ and ψ endow $\underline{C} \to \operatorname{Spec}\kappa$ with the structure of a pre-stable log curve over $(\operatorname{Spec}\kappa, Q)$ on the level of ghost sheaves, as described explicitly in Theorem 1.1. Thus the objects of $\underline{\operatorname{GS}}(\overline{\mathcal{M}})$ can alternatively be taken as diagrams

(1.5)
$$\overline{\mathcal{M}}_C \xleftarrow{\varphi} \overline{\mathcal{M}}$$
$$\psi \uparrow$$
$$Q_{\underline{C}}$$

of fine saturated sheaves on $\underline{C}$ with sharp homomorphisms. A morphism

$$(Q_1, \overline{\mathcal{M}}_{C,1}, \psi_1, \varphi_1) \longrightarrow (Q_2, \overline{\mathcal{M}}_{C,2}, \psi_2, \varphi_2)$$

in $\underline{\operatorname{GS}}(\overline{\mathcal{M}})$ is given by a pair of homomorphisms $Q_1 \to Q_2$ and $\overline{\mathcal{M}}_{C,1} \to \overline{\mathcal{M}}_{C,2}$ with the obvious compatibilities with ψ_i and φ_i, $i = 1, 2$.

Saying that the category $\underline{\operatorname{GS}}(\overline{\mathcal{M}})$ has a universal object essentially means that any two log enhancements of an ordinary stable map fit into one family, at least on the level of ghost sheaves. This is not true in general. One central insight in this paper is the characterization of the connected components of $\underline{\operatorname{GS}}(\overline{\mathcal{M}})$ by what we call the *type* of a stable log map. Even more usefully, for each type we construct a universal object of the corresponding connected component (see Proposition 1.18 below).

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[1] A homomorphism of monoids $\varphi : P \to Q$ is called *sharp* if $\varphi^{-1}(Q^\times) = P^\times$. Thus for the case of toric monoids this means $\varphi^{-1}(0) = 0$.

Discussion 1.7. To introduce the concept of type let us reformulate Diagram (1.5) on the level of stalks. By the structure of stable log curves over $(\operatorname{Spec}\kappa, Q)$ there are three types of points x on $\underline{C}$, depending on the stalks of $\overline{\mathcal{M}}_C$, as follows. Write $P_x := \overline{\mathcal{M}}_x$.[2]

(i) $x = \eta$ is a generic point or a general closed point. Then $\overline{\mathcal{M}}_{C,\bar\eta} = Q$ and φ defines a homomorphism

$$\varphi_{\bar\eta} : P_\eta \longrightarrow Q.$$

(ii) $x = p$ is a marked point. Then $\overline{\mathcal{M}}_{C,\bar p} = Q \oplus \mathbb{N}$ with $\psi_{\bar p}$ inducing the inclusion of Q as first factor. Then $\varphi_{\bar p}$ is determined by $\varphi_{\bar\eta}$ for $\bar\eta$ the generic point of the irreducible component containing p together with

(1.6) $$u_p := \operatorname{pr}_2 \circ \varphi_{\bar p} : P_p \longrightarrow \mathbb{N}.$$

(iii) $x = q$ is a node. Then $\overline{\mathcal{M}}_{C,\bar q} \simeq Q \oplus_{\mathbb{N}} \mathbb{N}^2$ with $\mathbb{N} \to Q$, $1 \mapsto \rho_q$ and $\mathbb{N} \to \mathbb{N}^2$, $1 \mapsto (1,1)$. Let $\bar\eta_1, \bar\eta_2$ be the geometric generic points of branches of $\underline{C}$ at $\bar q$. We have a commutative diagram

where the diagonal arrows are generization maps. Recall from (1.1) that ι is injective with $(m_1, m_2) \in Q \times Q$ in the image of $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ iff $m_1 - m_2 \in \mathbb{Z}\rho_q$, viewed as equation in Q^{gp}. Thus $\varphi_{\bar q}$ is determined uniquely by $\varphi_{\bar\eta_1}$, $\varphi_{\bar\eta_2}$, and there exists a homomorphism

(1.7) $$u_q : P_q \longrightarrow \mathbb{Z}$$

defined (since $\rho_q \neq 0$) by the equation

(1.8) $$\varphi_{\bar\eta_2}\big(\chi_2(m)\big) - \varphi_{\bar\eta_1}\big(\chi_1(m)\big) = u_q(m) \cdot \rho_q.$$

Note that the definition of u_q in (iii) depends on the choice of an ordering of the adjacent branches of $\underline{C}$. Such a choice can be implemented by orienting the *dual intersection graph* $\Gamma_{\underline{C}}$ of $\underline{C}$. The graph $\Gamma_{\underline{C}}$ has one vertex v_η for each generic point $\eta \in \underline{C}$ and an edge E_q for each node q joining v_{η_1} and v_{η_2} for η_1, η_2 the generic points of the two adjacent branches at q. Note that we can have $\eta_1 = \eta_2$ if the component is not normal. In addition there is an unbounded edge (a flag) E_p for each marked point p, with adjacent vertex v_η for the unique generic point η with $p \in \operatorname{cl}(\eta)$. $\qquad\square$

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[2]Recall that we assumed $\overline{\mathcal{M}}$ is a sheaf on the Zariski site, so $\overline{\mathcal{M}}_x = \overline{\mathcal{M}}_{\bar x}$.

Remark 1.8. In the above discussion, we only needed to use the stalks P_x of $\overline{\mathcal{M}}$ at points of $\underline{C}$ which were either generic, marked or double points. In fact, for any closed point $x \in \underline{C}$ which is neither a double point or marked point, we must have $P_x = P_\eta$, where η is the generic point of the component of C containing x. Indeed, we have a surjective generization map $\chi : P_x \to P_\eta$. If χ is not an isomorphism we can always find $0 \neq m \in P_x$ with $\chi(m) = 0$ (since generization is dual to an inclusion of faces). But $\overline{\mathcal{M}}_{C,\bar{x}} = \overline{\mathcal{M}}_{C,\bar{\eta}} = Q$, and since generization is compatible with the map φ on stalks, we have $\varphi_{\bar{\eta}} \circ \chi = \varphi_{\bar{x}}$, hence $\varphi_{\bar{x}}(m) = 0 \in Q$. This contradicts sharpness of $\varphi_{\bar{x}}$.

Definition 1.9. 1) The *type* of an object $(Q, \overline{\mathcal{M}}_C, \psi, \varphi)$ of the category $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ is the set $\mathbf{u} := \{u_p \in P_p^\vee, u_q \in P_q^*\}$ as defined in (1.6) and (1.7). Here p and q run over the marked and nodal points of $\underline{C}$, respectively. Given $\mathbf{u}$, the full subcategory of $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ with objects of type $\mathbf{u}$ is denoted $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u})$.

2) The *type* of a stable log map over a log point $(C/(\mathrm{Spec}\,\kappa, Q), \mathbf{x}, f)$ is the dual intersection graph $\Gamma_{\underline{C}}$ of $\underline{C}$, together with the type $\mathbf{u}$ of the corresponding object of $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$.

The type is compatible with generization:

Lemma 1.10. *Let* $\mathbf{u}$, $\mathbf{u}'$ *be the types of a stable log map* $(C/W, \mathbf{x}, f)$ *at two geometric points* $\overline{w} \to \underline{W}$, $\overline{w}' \to \underline{W}$ *with* $\overline{w} \in \mathrm{cl}(\overline{w}')$. *For* $x \in \underline{C}_{\overline{w}}$, $x' \in \underline{C}_{\overline{w}'}$ *and* $x \in \mathrm{cl}(x')$ *let* $\chi_{x'x} : P_x \to P_{x'}$ *be the generization map of the stalks of* $\underline{f}^* \overline{\mathcal{M}}_X$. *Then for marked or nodal points* x, x' *with* $x \in \mathrm{cl}(x')$ *it holds*

$$u_x = u_{x'} \circ \chi_{x'x}.$$

Proof. For marked points p, p' this follows readily from compatibility of (1.6) with generization. For nodal points q, q' comparing the generization of (1.7) for q with the equation for q' yields

$$u_q \cdot \kappa(\rho_q) = (u_{q'} \circ \chi_{q'q}) \cdot \rho_{q'}.$$

where $\kappa : \overline{\mathcal{M}}_{W,\overline{w}} \to \overline{\mathcal{M}}_{W,\overline{w}'}$. The claimed equation now follows from $\kappa(\rho_q) = \rho_{q'} \neq 0$. $\square$

1.3. **The standard log point and tropical curves.** An interesting special case is stable log maps over standard log points $(\mathrm{Spec}\,\kappa, \mathbb{N})$. This provides the connection to tropical geometry. Another motivation is that this case suffices to characterize universal stable log maps. To explain this we consider the situation of Diagram 1.4 of a pre-stable curve $\underline{C}/\mathrm{Spec}\,\kappa$ and a fine saturated log structure $\mathcal{M}$ on $\underline{C}$. We think of the case $\mathcal{M} = \underline{f}^* \mathcal{M}_X$ for an ordinary stable map $(\underline{C}/\mathrm{Spec}\,\kappa, \mathbf{x}, \underline{f})$. Now if Diagram 1.4 is universal (for a fixed type) then diagrams of the same form over the standard log point are given by morphisms

$$(\mathrm{Spec}\,\kappa, \mathbb{N}) \longrightarrow (\mathrm{Spec}\,\kappa, Q).$$

Moreover, two such morphisms lead to isomorphic log maps if and only if they differ by a homomorphism $Q \to \kappa^\times$. Now a morphism of log structures $Q \times \kappa^\times \longrightarrow \mathbb{N} \times \kappa^\times$ has the form

$$(m, a) \longmapsto (\varphi(m), h(m) \cdot a)$$

for some $\varphi \in \mathrm{Hom}(Q, \mathbb{N})$ with $\varphi^{-1}(0) = \{0\}$ and $h \in \mathrm{Hom}(Q, \kappa^\times)$. Composing with the automorphism

$$(m, a) \longmapsto (m, h(m) \cdot a)$$

of $(\mathrm{Spec}\,\kappa, Q)$ we may assume $h = 1$. Hence the set of isomorphism classes of stable log maps over $(\mathrm{Spec}\,\kappa, \mathbb{N})$ obtained from (1.4) by base change is in one-to-one correspondence with

$$\left\{ \varphi \in \mathrm{Hom}(Q, \mathbb{N}) \,\middle|\, \varphi^{-1}(0) = \{0\} \right\}.$$

The upshot of this discussion is that $\mathrm{Int}(Q^\vee)$ is equal to the set of isomorphism classes of Diagrams 1.4 with $Q = \mathbb{N}$. Note that while this is a discrete set, the set of isomorphism classes of Diagrams 1.4 relative to a *fixed* log point $(\mathrm{Spec}\,\kappa, Q)$ is fibred over this discrete set with fibres $\mathrm{Hom}(Q, \kappa^\times)$.

Now specializing Discussion 1.7 to $Q = \mathbb{N}$ the following data capture the situation on the level of ghost sheaves. For a node q the element $\rho_q \in Q \setminus \{0\}$ now is a number $e_q \in \mathbb{N} \setminus \{0\}$, and $Q \oplus_\mathbb{N} \mathbb{N}^2$ is isomorphic to the submonoid S_{e_q} of $\mathbb{N}^2$ generated by $(e_q, 0)$, $(0, e_q)$, $(1, 1)$.[3]

   (i) $x = \eta$ is a generic point. Then $\varphi_{\overline{\eta}} : P_\eta \longrightarrow \mathbb{N}$ defines an element $V_\eta \in P_\eta^\vee$.

   (ii) $x = p$ is a marked point. As in the general case, φ_p is determined by (i) and by $u_p \in P_p^\vee$, fixed by the type.

   (iii) $x = q$ is a node and $\bar{\eta}_1$, $\bar{\eta}_2$ are the geometric generic points of the adjacent branches of $\underline{C}$. Letting $i_{q,\eta_i} : P_{\eta_i}^\vee \to P_q^\vee$ be the inclusion induced by the generization maps Equation 1.8 now reads

(1.9) $$i_{q,\eta_1}(V_{\eta_2}) - i_{q,\eta_1}(V_{\eta_1}) = e_q u_q.$$

Thus apart from the type $\mathbf{u}$ a stable log map over a standard log point defines points $V_\eta \in P_\eta^\vee$ and $e_q \in \mathbb{N} \setminus \{0\}$. We call the tuple $\big((V_\eta)_\eta, (e_q)_q\big)$ *tropical data* of a stable log map over a standard log point. Similarly we can talk of tropical data for an object $(Q, \overline{\mathcal{M}}_C, \psi, \varphi)$ of $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ with $Q = \mathbb{N}$.

To discuss the relationship with tropical geometry we recall the basic definition of a tropical curve.

Definition 1.11. Let $\overline{\Gamma}$ be a connected graph and let Γ be the topological space obtained by removing from $\overline{\Gamma}$ a subset of the set of univalent vertices of $\overline{\Gamma}$, so that Γ has both compact and non-compact edges. We assume that Γ has at least one vertex. Let $N \simeq \mathbb{Z}^n$ be a lattice, $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. A *tropical curve* in $N_\mathbb{R}$ *with domain* Γ consists of the following data:

   (i) For each flag (v, E) of Γ, where v is a vertex of Γ and E an edge containing v, we are given a *weight vector* $u_{(v,E)} \in N$. If E has two vertices v_1 and v_2, then $u_{(v_1,E)} = -u_{(v_2,E)}$, and if E is a loop, then $u_{(v,E)} = 0$. The graph Γ along with the weight vectors is called the *type* of the tropical curve.

   (ii) A map $h : \Gamma \to N_\mathbb{R}$ with the following properties:

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[3]In terms of generators and relations we have $S_{e_q} = \langle a_1, a_2, a_3 \,|\, a_1 + a_2 = e_q \cdot a_3 \rangle$, where $a_1 = (e_q, 0)$, $a_2 = (0, e_q)$, $a_3 = (1, 1)$.

(a) For any edge E of Γ with vertex v, $h|_E$ is constant if $u_{(v,E)} = 0$, and otherwise $h|_E$ is proper and identifies E with an affine line segment or ray. Furthermore, $u_{(v,E)}$ is a tangent vector to $h(E)$ pointing away from $h(v)$.

(b) For each vertex v, we have the *balancing condition*

$$\sum_E u_{(v,E)} = 0,$$

where the sum is over all edges E with vertex v.

Discussion 1.12. The term "tropical data" is motivated by the case that $\overline{\mathcal{M}}^{\mathrm{gp}}$ is globally generated. In this case the tropical data gives rise to a generalized tropical curve with domain $|\Gamma_{\underline{C}}|$, the geometric realization of $\Gamma_{\underline{C}}$, and image in $N_{\mathbb{R}}$ for

$$N := \mathrm{Hom}(\Gamma(\underline{C}, \overline{\mathcal{M}}^{\mathrm{gp}}), \mathbb{Z}),$$

as follows. The restriction maps $\Gamma(\underline{C}, \overline{\mathcal{M}}^{\mathrm{gp}}) \to \overline{\mathcal{M}}^{\mathrm{gp}}_{\overline{x}}$ induce injections

$$P_x^\vee \longrightarrow N.$$

Denote by $\tilde{V}_\eta, \tilde{u}_p, \tilde{u}_q$ the images of V_η, u_p, u_q in N. Define a continuous map

$$h : |\Gamma_{\underline{C}}| \longrightarrow N_{\mathbb{R}}$$

by sending v_η to $\tilde{V}_\eta$, an edge E_q with adjacent vertices v_{η_1}, v_{η_2} to the line segment connecting $\tilde{V}_{\eta_1}$ and $\tilde{V}_{\eta_2}$, and an unbounded edge E_p with adjacent vertex v_η to the ray $\tilde{V}_\eta + \mathbb{R}_{\geq 0}\tilde{u}_p$. The weight vectors are given by $u_{(v_\eta, E_p)} := \tilde{u}_p$ for the unbounded edges and $u_{(v_{\eta_i}, E_q)} := \pm\tilde{u}_q$, with the sign chosen so that $u_{(v_{\eta_i}, E_q)}$ points away from $\tilde{V}_{\eta_i}$. Note that by (1.9) for an edge E_q with vertices v_{η_1}, v_{η_2} it holds $h(v_{\eta_2}) - h(v_{\eta_1}) = \pm e_q \tilde{u}_q$, so e_q is the integral length of the corresponding edge of the tropical curve, as a multiple of the weight vector, at least for $\tilde{u}_q \neq 0$.

As is, this does not in general fulfill the balancing condition. However, in Proposition 1.14 below we will identify the lack of balancing, which turns out to depend only on the given log structure $\mathcal{M}$ on $\underline{C}$, that is, on the underlying ordinary stable map. The balancing condition holds if for any irreducible components $D \subset \underline{C}$ and $m \in \Gamma(D, \overline{\mathcal{M}}|_D)$ the degree of the corresponding $\mathcal{O}_D^\times$-torsor $L_m \subset \mathcal{M}|_D$ vanishes. In general one can add one more unbounded edge at each vertex v_η with the weight vector τ_η^X derived from $\underline{f}$ via (1.10) to obtain an honest tropical curve in $N_{\mathbb{R}}$.

In the toric degenerations of toric varieties of [NiSi] already $\overline{\mathcal{M}}_X^{\mathrm{gp}}$ is globally generated. It is then appropriate to consider the composition with

$$N \longrightarrow N' := \mathrm{Hom}\left(\Gamma(\underline{X}, \overline{\mathcal{M}}_X^{\mathrm{gp}}), \mathbb{Z}\right).$$

Moreover, since in [NiSi] we work relative $(\mathrm{Spec}\,\Bbbk, \mathbb{N})$ the images of the generator of $\overline{\mathcal{M}}_{\mathrm{Spec}\,\Bbbk} = \mathbb{N}$ under $X \to (\mathrm{Spec}\,\Bbbk, \mathbb{N})$ and under $(\mathrm{Spec}\,\kappa, \mathbb{N}) \to (\mathrm{Spec}\,\Bbbk, \mathbb{N})$ define a global section ρ of $\overline{\mathcal{M}}_X$ and an element $b \in \mathbb{N} = \overline{\mathcal{M}}_{(\mathrm{Spec}\,\kappa, \mathbb{N})}$, respectively. Commutativity of (1.3) at a generic point η now implies $V_\eta(\rho_\eta) = b$. Thus h maps the vertices of Γ to the affine hyperplane $H \subset N_{\mathbb{R}}'$ defined by $\langle \rho, . \rangle = b$. Finally, from the structure of log smooth curves at a marked point, it

holds $\langle \rho, \tilde{u}_p \rangle = u_p(\rho_p) = 0$, and hence $\mathrm{im}(h) \subset H$. It is the map to H that traditionally and in [NiSi] is called a tropical curve. $\square$

As suggested by the tropical curve interpretation of Discussion 1.12 there should be a balancing condition at each vertex v_η of $\Gamma_{\underline{C}}$ imposing restrictions on u_p, u_q for the adjacent edges E_p, E_q. Denote $\mathcal{M} := \underline{f}^* \mathcal{M}_X$. For a generic point $\eta \in \underline{C}$ let $D := \mathrm{cl}(\eta)$ and $g : \tilde{D} \to \underline{C}$ the normalization of D. This gives rise to maps which are compositions

$$(1.10) \qquad \begin{array}{llll} \tau_\eta^X : & \Gamma(\tilde{D}, g^* \overline{\mathcal{M}}) & \longrightarrow & \mathrm{Pic}\,\tilde{D} \overset{\mathrm{deg}}{\longrightarrow} \mathbb{Z} \\ \tau_\eta^C : & \Gamma(\tilde{D}, g^* \overline{\mathcal{M}}_C) & \longrightarrow & \mathrm{Pic}\,\tilde{D} \overset{\mathrm{deg}}{\longrightarrow} \mathbb{Z}. \end{array}$$

The first map associates to a section of $g^* \overline{\mathcal{M}}$ or $g^* \overline{\mathcal{M}}_C$ the corresponding $\mathcal{O}_{\tilde{D}}^\times$-torsor, and the second map is the degree homomorphism.

The balancing condition is due to the basic fact that $f^\flat$ must induce isomorphisms of torsors, so that the pull-back $\varphi : g^* \overline{\mathcal{M}} \to g^* \overline{\mathcal{M}}_C$ of $\overline{f}^\flat$ to $\tilde{D}$ fits into the commutative diagram

$$(1.11) \qquad \begin{array}{ccc} \Gamma(\tilde{D}, g^* \overline{\mathcal{M}}) & \overset{\varphi}{\longrightarrow} & \Gamma(\tilde{D}, g^* \overline{\mathcal{M}}_C) \\ & {\scriptstyle \tau_\eta^X} \searrow & \downarrow {\scriptstyle \tau_\eta^C} \\ & & \mathbb{Z} \end{array}$$

The map τ_η^X is given by $\underline{f}$ and $\mathcal{M}$, so we have no control over it, except that if $\underline{f}$ contracts D, then $\tau_\eta^X = 0$. Otherwise there is nothing general we can say.

Similarly, τ_η^C is determined by $\mathcal{M}_C$. Explicitly, for $q \in D$ identify $S_{e_q} \subset \mathbb{N}^2$ with the submonoid generated by $(0, e_q), (e_q, 0), (1, 1)$ so that the generization map $\chi_q : \overline{\mathcal{M}}_{C, \bar{q}} = S_{e_q} \to \overline{\mathcal{M}}_{D, \bar{\eta}} = \mathbb{N}$ is the projection onto the second coordinate: $\chi_q(a, b) = b$. We have the identification

$$\Gamma(\tilde{D}, g^* \overline{\mathcal{M}}_C) = \big\{ (n_q)_{q \in \tilde{D}} \,\big|\, n_q \in S_{e_q} \text{ and } \chi_q(n_q) = \chi_{q'}(n_{q'}) \text{ for } q, q' \in \tilde{D} \big\} \oplus \bigoplus_{p \in \tilde{D}} \mathbb{N}.$$

In particular, if $\big((a_q, b_q), (n_p) \big)$ represents an element of $\Gamma(\tilde{D}, g^* \overline{\mathcal{M}}_C)$ then all second entries b_q agree.

Lemma 1.13. $\tau_\eta^C \big(((a_q, b)_{q \in \tilde{D}}, (n_p)_{p \in \tilde{D}}) \big) = -\sum_{p \in \tilde{D}} n_p + \sum_{q \in \tilde{D}} \dfrac{b - a_q}{e_q}.$

Proof. By log smoothness the element $\big((b, b)_q, (0)_p \big) \in \Gamma(\tilde{D}, g^* \overline{\mathcal{M}}_C)$ with all $a_q = b$, $n_p = 0$, maps to the trivial $\mathcal{O}_{\tilde{D}}^\times$-torsor. Thus it suffices to consider sections of $g^* \overline{\mathcal{M}}_C$ of the form $\big((a_q, 0), (n_p) \big)$ with $a_q / e_q \in \mathbb{Z}$. Let $L \subset g^* \mathcal{M}_C$ be the corresponding $\mathcal{O}_{\tilde{D}}^\times$-torsor. The structure map $L \to g^* \mathcal{M}_C \to \mathcal{O}_{\tilde{D}}$ identifies L with the sheaf of regular functions on $\tilde{D}$ with zeros of order a_q / e_q at q and of order n_p at p. In fact, if the log structure at q is induced from the toric model $\mathrm{Spec}\,\kappa[x, y, t]/(xy - t^{e_q})$ with $\tilde{D}$ corresponding to $V(y)$, then x defines an element

of $\mathcal{M}_{C,\bar{q}}$ mapping to $(e_q, 0) \in S_{e_q}$. Hence

$$\deg(L) = \deg \mathcal{O}_{\tilde{D}}\Big(-\sum_q \frac{a_q}{e_q} q - \sum_p n_p p \Big) = -\sum_q \frac{a_q}{e_q} - \sum_p n_p.$$

This is the claimed formula.   □

The equation $\tau_\eta^X = \tau_\eta^C \circ \varphi$ is a formula in $N_D := \Gamma(\tilde{D}, g^*\overline{\mathcal{M}}^{\mathrm{gp}})^*$, which is the inductive limit of abelian groups $(P_x^\vee)^{\mathrm{gp}} = (P_x^{\mathrm{gp}})^*$ with respect to the homomorphisms $\iota_{x,\eta} : (P_\eta^\vee)^{\mathrm{gp}} \to (P_x^\vee)^{\mathrm{gp}}$, $x \in \tilde{D}$. Here P_x for $x \in \tilde{D}$ means $P_{g(x)}$. More explicitly, if $\Sigma \subset \tilde{D}$ is the set of special points p, q, that is, mapping to a special point in D,

(1.12)
$$N_D = \varinjlim_{x \in \tilde{D}} (P_x^\vee)^{\mathrm{gp}} = \Big(\bigoplus_{x \in \Sigma} (P_x^\vee)^{\mathrm{gp}} \Big) \Big/ \sim,$$

where for any $a \in (P_\eta^\vee)^{\mathrm{gp}}$ and $x, x' \in \Sigma$,

(1.13)
$$(0, \ldots, 0, \iota_{x,\eta}(a), 0, \ldots, 0) \sim (0, \ldots, 0, \iota_{x',\eta}(a), 0, \ldots, 0).$$

We may thus represent an element of N_D as a tuple $(a_x)_{x \in \Sigma}$, but keep in mind the relations (1.13).

With this representation of N_D we are now in position to write down the balancing condition.

Proposition 1.14. *Consider a Diagram 1.4 with $Q = \mathbb{N}$ and $D \subset \underline{C}$ an irreducible component with generic point η and $\Sigma \subset \tilde{D}$ the preimage of the set of special points. If τ_η^X defined in (1.10) is represented by $(\tau_x)_{x \in \Sigma}$ then*

$$(u_x)_{x \in \Sigma} + (\tau_x)_{x \in \Sigma} = 0$$

in $N_D = \Gamma(\tilde{D}, \overline{\mathcal{M}}^{\mathrm{gp}})^*$.

Proof. Let $m \in \Gamma(\tilde{D}, g^*\overline{\mathcal{M}})$. Recall that φ denoted the pull-back of $\overline{f^\flat}$ by g. In view of Lemma 1.13 it holds

$$\begin{aligned}
\tau_\eta^C(\varphi(m)) &= \tau_\eta^C\Big(\big(\langle V_{\eta_q}, m \rangle, \langle V_\eta, m \rangle \big)_{q \in \tilde{D}}, \big(\langle u_p, m \rangle \big)_{p \in D} \Big) \\
&= \sum_{q \in \tilde{D}} \frac{1}{e_q} \Big(\langle V_\eta, m \rangle - \langle V_{\eta_q}, m \rangle \Big) - \sum_{p \in \tilde{D}} \langle u_p, m \rangle.
\end{aligned}$$

Thus since $u_q(m) = \frac{1}{e_q}\big(\langle V_{\eta_q}, m \rangle - \langle V_\eta, m \rangle \big)$,

$$\tau_\eta^C \circ \varphi = \big((-u_q)_{q \in \tilde{D}}, (-u_p)_{p \in \tilde{D}} \big),$$

and the claimed formula follows from $\tau_\eta^C \circ \varphi = \tau_\eta^X$, the commutativity of (1.11).   □

1.4. **The basicness condition.** For a fine saturated sheaf $\overline{\mathcal{M}}$ over a pre-stable curve over a field $\underline{C}/\operatorname{Spec}\kappa$, in Definition 1.6 we introduced the category $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$. Given a type $\mathbf{u}$ for objects of $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ (Definition 1.9) we are now in position to construct a universal object for the full subcategory $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u})$.

Construction 1.15. Let $\mathbf{u} = \{u_p, u_q\}$ be a type for $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ and assume $\underline{\mathrm{GS}}(\overline{\mathcal{M}}) \neq \emptyset$. For a node $q \in \underline{C}$ denote by $\chi_{\eta_i, q} : P_q \to P_{\eta_i}$ the two generization maps, ordered as in the definition of u_q in (1.8). Then if $m \in P_q$ let

$$a_q(m) := \big((\ldots, \chi_{\eta_1, q}(m), \ldots, -\chi_{\eta_2, q}(m), \ldots), (\ldots, u_q(m), \ldots) \big) \in \Big(\prod_\eta P_\eta \times \prod_q \mathbb{N} \Big)^{\mathrm{gp}}$$

be the element with all entries vanishing except the indicated ones at places η_1, η_2 and q. Let $R \subset \big(\prod_\eta P_\eta \times \prod_q \mathbb{N} \big)^{\mathrm{gp}}$ be the saturated subgroup generated by the $a_q(m)$ for all nodes $q \in \underline{C}$ and $m \in P_q$. Now define the *basic monoid* Q as the saturation of the quotient by R:

$$(1.14) \qquad Q := \left[\iota \Big(\prod_{\eta \in \underline{C}} P_\eta \times \prod_{q \in \underline{C}} \mathbb{N} \Big) \Big/ R \right]^{\mathrm{sat}}.$$

Here ι denotes the inclusion of $\prod_\eta P_\eta \times \prod_q \mathbb{N}$ into its groupification. By the very definition Q is fine and saturated. Taking the saturation of R amounts to dividing out any torsion of the groupification, so Q is also torsion-free. But note that at this point there is no reason to infer $Q^\times = \{0\}$, and in fact, this is not true in general.

The inclusion of the various factors into $\prod_\eta P_\eta \times \prod_q \mathbb{N}$ composed with the surjection to Q defines homomorphisms

$$\varphi_{\bar\eta} : P_\eta \longrightarrow \prod_\eta P_\eta \times \prod_q \mathbb{N} \longrightarrow Q,$$

$$\mathbb{N} \longrightarrow \prod_\eta P_\eta \times \prod_q \mathbb{N} \longrightarrow Q, \qquad 1 \longmapsto \rho_q.$$

Since the difference of the two sides of Equation (1.8) is nothing but $a_q(m)$, we have the following equality of maps $P_q \to Q$:

$$\varphi_{\bar\eta_2} \circ \chi_2 - \varphi_{\bar\eta_1} \circ \chi_1 = u_q \cdot \rho_q.$$

From (i)–(iii) after Definition 1.6 the data Q, ρ_q and $\varphi_{\bar\eta}$ thus define a distinguished object $(Q, \overline{\mathcal{M}}_C, \psi, \varphi)$ of $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u})$, except that we do not know at this point that $Q^\times = 0$ and that all morphisms are sharp. This property will be established in Proposition 1.18 provided $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u}) \neq \emptyset$. $\qquad \square$

Example 1.16. Let us illustrate some features of this definition with some simple examples, notably concerning sharpness and saturation issues.

(1) Consider a curve with two irreducible components mutually intersecting in two nodes q_1, q_2. Assume that the whole curve maps to a standard log point, and hence $P_x = \mathbb{N}$

for all x and all generization maps are isomorphisms. Choose $u_{q_1}(1) = 0$, $u_{q_2}(1) = 1$. Then in $\prod_\eta P_\eta^{\mathrm{gp}} \times \prod_q \mathbb{Z} = \mathbb{Z}^4$ we have

$$a_{q_1}(1) = (1, -1, 0, 0), \quad a_{q_2}(1) = (1, -1, 0, 1).$$

Thus $(0, 0, 0, 1) \in R$ and hence $\rho_{q_2} = 0$. In particular, $\overline{\mathcal{M}}_C$ is not the ghost sheaf of a pre-stable log curve. Geometrically this means that the u_q can not force the node at q_2 to smooth in a local deformation while keeping the node at q_1.

(2) Considering the same situation as in (1) but with $u_{q_1}(1) = 2$, $u_{q_2}(1) = 3$, leads to a non-saturated image of $\prod_\eta P_\eta \times \prod_q \mathbb{N}$. In fact, R is now generated by

$$a_{q_1}(1) = (1, -1, 2, 0), \quad a_{q_2}(1) = (1, -1, 0, 3),$$

and the map

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -6 & 0 & 3 & 2 \end{pmatrix} : \mathbb{Z}^4 \longrightarrow \mathbb{Z}^2$$

describes the quotient of $\prod_\eta P_\eta^{\mathrm{gp}} \times \prod_q \mathbb{Z} = \mathbb{Z}^4$ by R. The image of $\prod_\eta P_\eta \times \prod_q \mathbb{N} = \mathbb{N}^4$ is then generated by the columns of the matrix, and hence is not saturated for it contains $(0, 2)$ and $(0, 3)$ but not $(0, 1)$. Thus we need to saturate in the definition of Q to stay in the category of fine saturated log schemes.[4]

(3) Consider again a curve as in (1), but now mapping to a pre-stable log curve with one node in such a way that

$$P_{q_1} = \mathbb{N}^2, \quad P_{\eta_1} = P_{q_2} = P_{\eta_2} = \mathbb{N},$$

with only non-trivial generization maps $P_{q_1} \to P_{\eta_i}$ the two projections $\mathbb{N}^2 \to \mathbb{N}$. Take $u_{q_1}(a, b) = a + b$, $u_{q_2}(c) = 2c$. Then

$$a_{q_1}(1, 0) = (1, 0, 1, 0), \quad a_{q_1}(0, 1) = (0, -1, 1, 0), \quad a_{q_2}(1) = (1, -1, 0, 2).$$

The subgroup of $\prod_\eta P_\eta^{\mathrm{gp}} \times \prod_q \mathbb{Z} = \mathbb{Z}^4$ generated by these elements contains $(0, 0, 2, -2) = a_{q_1}(1, 1) - a_{q_2}(1)$, but it does not contain $(0, 0, 1, -1)$. Hence saturation in the definition of R is necessary to make $(\prod_\eta P_\eta^{\mathrm{gp}} \times \prod_q \mathbb{Z})/R$ torsionfree.

Remark 1.17. Another useful way of thinking about Q is in the dual space:

$$Q^\vee = \left\{ ((V_\eta)_\eta, (e_q)_q) \in \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_q \mathbb{N} \,\middle|\, \forall q : V_{\eta_2} - V_{\eta_1} = e_q u_q \right\}.$$

In particular, this avoids any saturation issues. Note that following the discussion in §1.3 any $((V_\eta), (e_q)) \in Q^\vee$ that does not lie in any proper face gives rise to a stable log map over the standard log point on the level of ghost sheaves. In fact, this is how the authors first found Q.

Proposition 1.18. *If* $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u}) \neq \emptyset$ *then it has as initial object the tuple* $(Q, \overline{\mathcal{M}}_C, \psi, \varphi)$ *from Construction 1.15.*

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[4]We realized this saturation issue only after studying [Ch] in greater detail.

Proof. We first prove the universal property in an enlarged category without the assumption that $Q^\times = \{0\}$ and the morphisms of monoids sharp. Let $(Q', \overline{\mathcal{M}}'_C, \psi', \varphi') \in \mathrm{Ob}\left(\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u})\right)$. Let $\rho'_q \in Q'$ be the element defining $\overline{\mathcal{M}}'_{C,\bar{q}}$. Then because of (1.8) for $(Q', \overline{\mathcal{M}}'_C, \psi', \varphi')$ and since Q' is torsion-free and saturated the map

$$(1.15) \qquad \prod_\eta P_\eta \times \prod_q \mathbb{N} \xrightarrow{\;\prod_\eta \varphi'_{\bar{\eta}} \times \prod_q \rho'_q\;} Q'$$

factors over Q. Tracing the image of the generator of the q-th copy of $\mathbb{N}$ shows that the induced map $Q \to Q'$ maps ρ_q to ρ'_q. Moreover, by the very definition of this factorization it is compatible with $\varphi_{\bar{\eta}} : P_\eta \to Q$ and $\varphi'_{\bar{\eta}} : P_\eta \to Q'$. This proves existence of a morphism $(Q, \overline{\mathcal{M}}_C, \psi, \varphi) \to (Q', \overline{\mathcal{M}}'_C, \psi', \varphi')$.

For uniqueness note that by compatibility of ρ_q, ρ'_q and of $\varphi_{\bar{\eta}}$, $\varphi'_{\bar{\eta}}$ any such morphism would lift to the homomorphism stated in (1.15).

It remains to show that $(Q, \overline{\mathcal{M}}_C, \psi, \varphi) \in \underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u})$, that is, that $Q^\times = 0$ and all sheaf homomorphisms are sharp. Since $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u}) \neq \emptyset$ there is at least one morphism $(Q, \overline{\mathcal{M}}_C, \psi, \varphi) \to (Q', \overline{\mathcal{M}}'_C, \psi', \varphi')$ as constructed above. Now because $(Q', \overline{\mathcal{M}}'_C, \psi', \varphi') \in \mathrm{Ob}(\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u}))$, for any η and q the compositions

$$P_\eta \xrightarrow{\;\varphi_{\bar{\eta}}\;} Q \longrightarrow Q', \quad \mathbb{N} \xrightarrow{\;\cdot \rho_q\;} Q \longrightarrow Q'$$

are sharp. This proves that ψ and φ are indeed sharp homomorphisms. Thus also the composition

$$\prod_\eta P_\eta \times \prod_q \mathbb{N} \longrightarrow Q \longrightarrow Q'$$

is sharp and hence, by surjectivity of the first arrow up to saturation, also $Q \to Q'$ is sharp and $Q^\times = \{0\}$. Finally $\overline{\mathcal{M}}_C \to \overline{\mathcal{M}}'_C$ is sharp because at a node the homomorphism is defined by the product of the homomorphisms at the adjacent generic points. $\qquad \square$

Definition 1.19. A stable log map $(C/W, \mathbf{x}, f)$ is called *basic* if for any geometric point $\overline{w} \to \underline{W}$ the induced object of $\underline{\mathrm{GS}}(\underline{f^*_{\overline{w}}}\overline{\mathcal{M}}_X, \mathbf{u})$ is universal. Here $\mathbf{u}$ is the type of $(C/W, \mathbf{x}, f)$ at $\overline{w}$.

Remark 1.20. Following up on Remark 1.17 the tropical interpretation of the basicness condition is as follows. If $\overline{w} : \mathrm{Spec}\,\kappa \to \underline{W}$ is a geometric point then $\mathrm{Int}\,\overline{\mathcal{M}}^\vee_{W,\overline{w}}$ can be identified with the set of tropical data of the pull-backs to $(\mathrm{Spec}\,\kappa, \mathbb{N})$ via a log enhancement of $\overline{w}$. Thus in the proper tropical situations of Discussion 1.12 the real cone generated by $\overline{\mathcal{M}}^\vee_{W,\overline{w}}$ is the moduli space of tropical curves of the given type; the faces parametrize degenerate curves with some edges of length 0.

As a first property we show that basicness is an open condition:

Proposition 1.21. *Let $(C/W, \mathbf{x}, f)$ be a stable log map to a log scheme X. Then*

$$\Omega := \left\{ \overline{w} \in |\underline{W}| \,\big|\, \{\overline{w}\} \times_{\underline{W}} (C/W, \mathbf{x}, f) \text{ is basic} \right\}$$

is an open subset of $|\underline{W}|$.[5]

Proof. Since basicness is a condition on morphisms of fine sheaves, Ω is constructible.[6] It remains to show that Ω is closed under generization. So let $\overline{w}_1 \in \Omega$, $\overline{w}_2 \in |\underline{W}|$ and $\overline{w}_1 \in \mathrm{cl}(\overline{w}_2)$. We need to show $\overline{w}_2 \in \Omega$. Since basicness is stable under strict base change we may first replace $\underline{W}$ by $\mathrm{Spec}\,(\mathcal{O}_{W,\overline{w}_1})$ and then by $\mathrm{cl}(\overline{w}_2)$ with the induced reduced scheme structure, to reduce to the case $\underline{W} = \mathrm{Spec}\,R$ for a strictly Henselian local domain R, and with $\overline{w}_1$ and $\overline{w}_2$ the closed point 0 and generic point $\mathrm{Spec}\,K$, K the quotient field of R. Denote by $\kappa = R/\mathfrak{m}$ the residue field of R, and endow $\mathrm{Spec}\,\kappa$ and $\mathrm{Spec}\,K$ with the log structures induced by the embeddings into $W = (\mathrm{Spec}\,R, \mathcal{M}_R)$.

Now we have two relevant stable log maps over fields, the closed fibre

$$(C_0/(\mathrm{Spec}\,\kappa, Q), \mathbf{x}_0, f_0) := \mathrm{Spec}\,\kappa \times_{\underline{W}} (C/W, \mathbf{x}, f),$$

which is basic by assumption, and the generic fibre

$$(C_K/(\mathrm{Spec}\,K, Q_K), \mathbf{x}_K, f_K) := \mathrm{Spec}\,K \times_{\underline{W}} (C/W, \mathbf{x}, f),$$

a stable log map over some log point $(\mathrm{Spec}\,K, Q_K)$. We use the standard notations for the points and the monoids of the closed fibre, while for the generic fibre we add hats. Note that apart from the usual generization maps between points on the same fibre we also have generization homomorphisms from the closed to the generic fibre,

$$P_\eta \longrightarrow P_{\hat{\eta}}, \quad P_p \longrightarrow P_{\hat{p}}, \quad P_q \longrightarrow P_{\hat{q}},$$

where $P_q \longrightarrow P_{\hat{q}}$ only exists for those nodes $q \in \underline{C}_0$ that are contained in the closure of a node of $\underline{C}_K$.

For each generic point $\eta \in \underline{C}_0$ with generization $\hat{\eta} \in \underline{C}_K$ the homomorphism $\underline{f}^* \overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$ defines a commutative square

$$
\begin{array}{ccc}
P_\eta & \longrightarrow & Q \\
\downarrow & & \downarrow \\
P_{\hat{\eta}} & \longrightarrow & Q_K.
\end{array}
$$

Moreover, if $q \in \mathrm{cl}(\hat{q})$ then ρ_q maps to $\rho_{\hat{q}}$. We thus obtain a commutative diagram

(1.16)
$$
\begin{array}{ccc}
P := \prod_\eta P_\eta \times \prod_q \mathbb{N} & \xrightarrow{\ \theta\ } & Q \\
\tilde{\psi} \downarrow & & \downarrow \psi \\
\hat{P} := \prod_{\hat{\eta}} P_{\hat{\eta}} \times \prod_{\hat{q}} \mathbb{N} & \xrightarrow{\ \hat{\theta}\ } & Q_K,
\end{array}
$$

with θ the map defining Q up to saturation and the vertical arrows the generization epimorphisms. If q is a node not in the closure of some $\hat{q}$ then $\tilde{\psi}$ maps this copy of $\mathbb{N}$ to 0.

---

[5] In writing $\{w\} \times_{\underline{W}} (C/W, \mathbf{x}, f)$ we view $(C/W, \mathbf{x}, f)$ as an object over $\underline{W}$. In particular, this fibre product is a log curve over $\mathrm{Spec}\,\kappa(w)$ endowed with the log structure making the inclusion into W strict.

[6] Alternatively, the following arguments indeed show that basicness holds on subsets of $\underline{W}$ which admit a chart $Q \to \mathcal{M}_W$ inducing an isomorphism $Q \simeq \overline{\mathcal{M}}_{W,\overline{w}}$ for some $\overline{w} \in |\Omega|$ (cf. [Ch], Proposition 3.5.2).

As a generization homomorphism ψ induces an isomorphism $S^{-1}Q/(S^{-1}Q)^\times \to Q_K$ with $S = \psi^{-1}(0)$ a face of Q. Similarly, $\tilde\psi$ induces an isomorphism $\tilde S^{-1}P/(\tilde S^{-1}P)^\times \to \hat P$. By sharpness of θ and $\hat\theta$ we have the relation

$$(1.17) \qquad \tilde S = \tilde\psi^{-1}(0) = \tilde\psi^{-1}\big(\hat\theta^{-1}(0)\big) = \theta^{-1}\big(\psi^{-1}(0)\big) = \theta^{-1}(S).$$

Recall from Construction 1.15 that Q is the saturation of the image of P in P^{gp}/R, where R is the saturation of the subgroup of P^{gp} generated by certain elements $a_q(m)$, for all nodes q and all $m \in P_q$. By the compatibility of types with generization (Lemma 1.10) $\tilde\psi^{\mathrm{gp}}$ maps R to the analogous subgroup $\hat R \subset \hat P$. Since by (1.16) $\hat\theta$ is surjective up to saturation, it remains to show that up to saturation $\hat\theta$ is the quotient of $\hat P$ by $\hat R$. Thus let $\hat m_1, \hat m_2 \in \hat P$ with $\hat\theta(\hat m_1) = \hat\theta(\hat m_2)$. Let m_1, m_2 be lifts of $\hat m_1, \hat m_2$ to P. Then $\theta(m_1) - \theta(m_2) \in S^{\mathrm{gp}}$ (viewed in Q^{gp}) for

$$\psi(\theta(m_1)) = \hat\theta(\hat m_1) = \hat\theta(\hat m_2) = \psi(\theta(m_2)).$$

Thus there exist $h_i \in \theta^{-1}(S)$ such that $\theta(m_1 + h_1) = \theta(m_2 + h_2)$. Using $\theta^{-1}(S) = \tilde S$ from (1.17) we may replace m_i by $m_i + h_i$ to achieve $\theta(m_1) = \theta(m_2)$. But then $m_1 - m_2 \in R$, and hence

$$\tilde\psi(m_1) - \tilde\psi(m_2) \in \tilde\psi^{\mathrm{gp}}(R) = \hat R,$$

finishing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following proposition establishes a universal property for basic stable log maps. This shows in particular that we do not lose any generality in imposing basicness. This result is not needed for the construction of log Gromov-Witten invariants, but is included for reassurance. It is also referred to in the comparison with Jun Li's moduli space in Corollary 6.4.

Proposition 1.22. *Any stable log map arises as the pull-back from a basic stable log map with the same underlying ordinary stable map. Both the basic stable log map and the morphism are unique up to unique isomorphism.*

Proof. Let $(\pi : C \to W, \mathbf{x}, f)$ be the given stable log map defining morphisms of log structures $\pi^\flat : \underline\pi^*\mathcal{M}_W \to \mathcal{M}_C$ and $f^\flat : \underline f^*\mathcal{M}_X \to \mathcal{M}_C$. We consider the category of morphisms of log structures $\mathcal{M}'_W \to \mathcal{M}_W$ on $\underline W$ and $\mathcal{M}'_C \to \mathcal{M}_C$, $\underline\pi^*\mathcal{M}'_W \to \mathcal{M}'_C$, $\underline f^*\mathcal{M}_X \to \mathcal{M}'_C$ on $\underline C$, compatible with $\pi^\flat$ and $f^\flat$ in the obvious way. The statement is equivalent to saying that this category has an initial object.

If $\mathcal{M}_1 \to \mathcal{M}_2$ is a morphism of fine log structures on a scheme Y then from the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_Y^\times & \longrightarrow & \mathcal{M}_1^{\mathrm{gp}} & \longrightarrow & \overline{\mathcal{M}}_1^{\mathrm{gp}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{O}_Y^\times & \longrightarrow & \mathcal{M}_2^{\mathrm{gp}} & \longrightarrow & \overline{\mathcal{M}}_2^{\mathrm{gp}} & \longrightarrow & 0
\end{array}
$$

it follows that $\mathcal{M}_1 = \mathcal{M}_2 \times_{\overline{\mathcal{M}}_2} \overline{\mathcal{M}}_1$. In particular, our category of diagrams of log structures on $\underline W$ and on $\underline C$ with fixed target log structures is equivalent to the induced category of

diagrams of ghost sheaves. It thus suffices to show that this latter category has an initial object.

For geometric points this is the statement of Proposition 1.18. Now let a collection of types for geometric fibres be given that is compatible with generization, as defined by the geometric fibres of the given stable log map $(C/W, \mathbf{x}, f)$. Then Diagram (1.16) in the proof of Proposition 1.21 shows that the fibrewise defined diagrams of ghost sheaves are compatible with generization. Hence they define the desired initial object in the category of diagrams of ghost sheaves, as needed. $\qquad\square$

Another remarkable property of basic stable log maps is that they do not admit non-trivial automorphisms that are the identity on the underlying ordinary stable maps. Stack-theoretically this means that the forgetful map from the stack of basic stable log maps to the stack of ordinary stable maps is representable (Proposition 2.7). The statement is also useful for checking that the stack of stable log maps has a separated diagonal (Proposition 2.5).

Proposition 1.23. *An automorphism $\varphi : C/W \to C/W$ of a basic stable log map $(\pi : C \to W, \mathbf{x}, f)$ with $\underline{\varphi} = \mathrm{id}_{\underline{C}}$ is trivial.*

Proof. An automorphism of C/W is an automorphism $\varphi^\flat$ of the log structure $\mathcal{M}_C$ of C descending to an automorphism $\psi = (\mathrm{id}, \psi^\flat)$ of $(\underline{W}, \mathcal{M}_W)$. It is an automorphism of $(C/W, \mathbf{x}, f)$ if $\underline{\varphi}(\mathbf{x}) = \mathbf{x}$ and if it commutes with the morphism $f^\flat : \underline{f}^* \mathcal{M}_X \to \mathcal{M}_C$. Since $\underline{\varphi} = \mathrm{id}_{\underline{C}}$ the latter condition means

(1.18) $$\varphi^\flat \circ f^\flat = f^\flat.$$

We claim that it suffices to show $\psi^\flat = \mathrm{id}_{\mathcal{M}_W}$. In fact, let $U \subset \underline{C}$ be the complement of the set of special points (images of marked points or criticial points of $\underline{\pi}$). Because $\pi^\flat : \pi^* \mathcal{M}_W \to \mathcal{M}_C$ is an isomorphism on U, $\psi^\flat = \mathrm{id}_{\mathcal{M}_W}$ implies $\varphi^\flat|_U = \mathrm{id}$. But by Theorem 1.1, $\mathcal{M}_C$ has no section with support on the set of special points. Hence $\varphi^\flat = \mathrm{id}_{\mathcal{M}_C}$.

Now on the level of ghost sheaves a basic stable log map is determined by the underlying ordinary stable map and the type. Moreover, there is no morphism between stable log maps of different types. This shows that $\underline{\varphi} = \mathrm{id}_{\underline{C}}$ implies $\overline{\varphi^\flat} = \mathrm{id}_{\overline{\mathcal{M}}_C}$. By strictness of $\pi : C \to W$ away from the special points this imples $\overline{\psi^\flat} = \mathrm{id}_{\overline{\mathcal{M}}_W}$ Thus $\psi^\flat$ has the form

$$\psi^\flat(m) = h(\overline{m}) \cdot m$$

for a homomorphism $h : \overline{\mathcal{M}}_W \to \mathcal{O}_{\underline{W}}^\times$. Now if $\overline{w} \to \underline{W}$ is a geometric point, then by basicness, up to saturation $\overline{\mathcal{M}}_{W, \overline{w}}$ is generated by the fibres P_η of $\underline{f}^* \overline{\mathcal{M}}_X$ at the generic points $\eta \in \underline{C}_{\overline{w}}$, and by one copy of $\mathbb{N}$ for each singular point of $\underline{C}_{\overline{w}}$. Strictness of $\pi|_U$ together with (1.18) implies that h is trivial on the part of $\overline{\mathcal{M}}_W$ generated by P_η. The factors of $\mathbb{N}$ are generated by the image of the basic log structure $\pi^* \mathcal{M}_W^o \to \mathcal{M}_C^o$ on the pre-stable curve $\underline{C}/\underline{W}$. But $\mathcal{M}_C^o$ has no non-trivial automorphism inducing the identity on $\underline{C}$ and on $\overline{\mathcal{M}}_C^o$. This is a direct consequence of the existence of basic log structures for pre-stable curves, see the appendix. Hence $h = 1$ also on this part. Finally note that a homomorphism from a fine monoid to

a group is trivial if and only if it is trivial on its saturation. Hence h is trivial and thus $\psi^\flat = \mathrm{id}_{\mathcal{M}_W}$ as remained to be shown. $\qquad\square$

2. Algebraicity

2.1. **The stack of stable log maps.** We are now ready to define the stack of stable log maps. We continue with the convention that X is a log scheme over S, with the log structures defined on the Zariski sites. Remember also that all our log structures are defined over S, that is, come with a morphism from the pull-back of $\mathcal{M}_S$ that is compatible with $\mathcal{M}_S \to \mathcal{O}_{\underline{S}}$. We endow $(\mathrm{Sch}/\underline{S})$ with the étale Grothendieck topology.

Definition 2.1. The *stack of stable log maps to X (over S)* is the category $\tilde{\mathscr{M}}(X) = \tilde{\mathscr{M}}(X/S)$ (Definition 1.5) together with the forgetful morphism $\tilde{\mathscr{M}}(X) \to (\mathrm{Sch}/\underline{S})$ mapping $(C/W, \mathbf{x}, f)$ to $\underline{W}$. The full subcategory of *basic* stable log maps is denoted $\mathscr{M}(X)$.

Since the morphisms in $\tilde{\mathscr{M}}(X)$ and $\mathscr{M}(X)$ are given by cartesian diagrams of log smooth curves over the underlying base schemes, $\tilde{\mathscr{M}}(X) \to (\mathrm{Sch}/\underline{S})$ and $\mathscr{M}(X) \to (\mathrm{Sch}/\underline{S})$ are fibred groupoids. As is customary, for any $a \in \tilde{\mathscr{M}}(X)$ over $\underline{W} \in (\mathrm{Sch}/\underline{S})$ and $\varphi : \underline{V} \to \underline{W}$ we choose one morphism in $\tilde{\mathscr{M}}(X)$ covering φ and denote it by $\varphi^* a \to a$.

Once we prove that $\tilde{\mathscr{M}}(X)$ is an algebraic stack, Proposition 1.21 shows that $\mathscr{M}(X)$ is also algebraic, for it is an an open substack of $\tilde{\mathscr{M}}(X)$. We therefore restrict attention to $\tilde{\mathscr{M}}(X)$ for most of this section.

Lemma 2.2. $\tilde{\mathscr{M}}(X)$ *is a stack.*

Proof. We verify Axioms (i) and (ii) in [LaMB], Definition 3.1.

(i) We have to check the sheaf axioms for morphisms between two objects in $\tilde{\mathscr{M}}(X)$ over the same base scheme. This amounts to the following. Let $a_i \in \tilde{\mathscr{M}}(X)_{\underline{W}}$, $i = 1, 2$, be two stable log maps with the same base scheme $\underline{W}$. Let $h : \tilde{\underline{W}} \to \underline{W}$ be an étale cover, $\tilde{\psi} : h^* a_1 \to h^* a_2$ a morphism over $\tilde{\underline{W}}$ and $\mathrm{pr}_i : \tilde{\underline{W}} \times_{\underline{W}} \tilde{\underline{W}} \to \tilde{\underline{W}}$ for $i = 1, 2$ the projections. With $\pi := h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$ there are two morphisms

$$\mathrm{pr}_i^* \tilde{\psi} : \pi^* a_1 \longrightarrow \pi^* a_2, \quad i = 1, 2.$$

The sheaf axiom says that if these two morphisms agree then there exists a unique morphism $\psi : a_1 \to a_2$ with $\tilde{\psi} = h^* \psi$. For the underlying morphisms of schemes this follows from faithfully flat descent ([SGA1], VIII, Theorem 5.2). Since on the domains we work with log structures in the étale topology the refinement to morphisms of log schemes is a tautology.

(ii) This axiom deals with descent for objects in $\tilde{\mathscr{M}}(X)$. On the underlying domain of the stable log map this follows by the sheaf property of the stack of pre-stable curves $\mathbf{M}$ (see the appendix). Then the underlying morphisms to $\underline{X}$ descend as in (i). Again the refinement to morphisms of log spaces is a tautology. $\qquad\square$

The rest of this section is devoted to proving algebraicity of $\tilde{\mathscr{M}}(X)$. Denote by $\mathscr{M} = \mathscr{M}_S$ the log stack over S of (not necessarily basic) pre-stable marked log curves. In the

appendix we recall the folklore result that $\mathscr{M}$ is an algebraic stack locally of finite type over $\underline{S}$ (Proposition A.3). There is a forgetful morphism of stacks

$$\tilde{\mathscr{M}}(X) \longrightarrow \mathscr{M},$$

mapping a stable log map $(C/W, \mathbf{x}, f)$ to the pre-stable marked log curve $(C/W, \mathbf{x})$. Note that this functor is faithful, since a morphism of stable log maps is given by a morphism on the domains. Another forgetful morphism is to the algebraic stack of ordinary stable maps $\mathbf{M}(\underline{X})$ [BeMa]:

$$\tilde{\mathscr{M}}(X) \longrightarrow \mathbf{M}(\underline{X}).$$

In the next subsection we will prove the following.

Proposition 2.3. $\tilde{\mathscr{M}}(X) \to \mathscr{M} \times \mathbf{M}(\underline{X})$ *is representable and locally of finite presentation.*

A direct consequence is the main result of this section.

Theorem 2.4. $\tilde{\mathscr{M}}(X)$ *is an algebraic stack locally of finite type over $\underline{S}$.*

Proof. By [LaMB], Proposition 4.5,(ii) algebraicity of $\tilde{\mathscr{M}}(X)$ follows from Proposition 2.3. It is locally of finite type over $\underline{S}$ since so are $\mathscr{M}$ and $\mathbf{M}(\underline{X})$. $\square$

As is the case with $\mathscr{M}$, the stack $\tilde{\mathscr{M}}(X)$ is only an algebraic stack in the sense of [Ol2], p.750. This definition drops the separatedness of the diagonal morphism (quasi-separatedness) from the definition in [LaMB]. In contrast, the open substack $\mathscr{M}(X)$ of $\tilde{\mathscr{M}}(X)$ does have a separated diagonal, so is an algebraic stack in the sense of [LaMB]:

Proposition 2.5. *The diagonal morphism* $\Delta_{\mathscr{M}(X)/\underline{S}} : \mathscr{M}(X) \longrightarrow \mathscr{M}(X) \times \mathscr{M}(X)$ *is separated.*

Proof. The statement amounts to the following: An automorphism of a stable log map $(C/W, \mathbf{x}, f)$ over an integral scheme $\underline{W}$ that is generically the identity is trivial. This is clearly true for the underlying ordinary stable map, and lifts to log stable maps by virtue of Proposition 1.23. $\square$

As a corollary of Theorem 2.4 we obtain algebraicity of the open substack $\mathscr{M}(X)$ of $\tilde{\mathscr{M}}(X)$ of basic stable log maps. Moreover, $\mathscr{M}(X)$ carries a canonical log structure, that is, a factorization

$$\mathscr{M}(X) \longrightarrow (\mathrm{Log}/S) \longrightarrow (\mathrm{Sch}/\underline{S}),$$

where (Log/S) is the category of fine saturated log schemes over S with strict morphisms.

Corollary 2.6. $\mathscr{M}(X)$ *is an algebraic log stack with separated diagonal and locally of finite type over $\underline{S}$.*

Proof. For the log structure define the functor $\mathscr{M}(X) \longrightarrow (\mathrm{Log}/S)$ by sending a stable log map $(C/W, \mathbf{x}, f)$ to the logarithmic base W. $\square$

Assuming Proposition 2.3 we can establish at this point some further properties of $\mathscr{M}(X)$.

Proposition 2.7. *The forgetful morphism $\mathscr{M}(X) \to \mathbf{M}(\underline{X})$ of algebraic stacks is representable.*

Proof. By Corollaire 8.1.2 in [LaMB] we have to show that the diagonal morphism

$$\mathscr{M}(X) \longrightarrow \mathscr{M}(X) \times_{\mathbf{M}(\underline{X})} \mathscr{M}(X)$$

is a monomorphism. This amounts to the statement about automorphisms of basic stable log maps verified in Proposition 1.23. □

Corollary 2.8. *The algebraic stack $\mathscr{M}(X)$ is a Deligne-Mumford stack.*

Proof. An algebraic stack representable over a Deligne-Mumford stack is itself a Deligne-Mumford stack. In fact, if $X \to \mathfrak{X}$ is an étale presentation of an algebraic stack and $\mathfrak{Y} \to \mathfrak{X}$ is representable, then $Y := X \times_{\mathfrak{X}} \mathfrak{Y}$ is also an étale surjection, and Y is an algebraic space by representability. □

2.2. **Representability of spaces of log morphisms.** We now prove Proposition 2.3. To avoid excessive underlining, in this subsection we change our convention and denote schemes or algebraic spaces by unadorned letters and use a dagger for log schemes, as in $X^{\dagger} = (X, \mathcal{M}_X)$. We have to show that if W is a scheme[7] and $W \to \mathscr{M} \times \mathbf{M}(X)$ is a morphism then the fibre product $W \times_{\mathscr{M} \times \mathbf{M}(X)} \tilde{\mathscr{M}}(X)$ is an algebraic space locally of finite type over W. Explicitly, this means the following. The morphism $W \to \mathscr{M} \times \mathbf{M}(X)$ amounts to giving a pair $(\mathcal{W}^{\dagger}, f)$ with $\mathcal{W}^{\dagger} = (C^{\dagger}/W^{\dagger}, \mathbf{x})$ a pre-stable marked log curve and $f : C \to X$ a morphism of schemes making $(C/W, \mathbf{x}, f)$ an ordinary stable map. For $V \to W$, the fibre category of $W \times_{\mathscr{M} \times \mathbf{M}(X)} \tilde{\mathscr{M}}(X)$ over V can be taken as the category with objects morphisms of log structures

$$(f^* \mathcal{M}_X)_V \longrightarrow (\mathcal{M}_C)_V$$

over $S^{\dagger}$ and with only the identity as morphisms. Here the index V means pull-back via the base change morphism $C_V := V \times_W C \to C$. Triviality of the automorphisms in this fibre category is due to the fact that $\tilde{\mathscr{M}}(X) \to \mathscr{M} \times \mathbf{M}(X)$ is faithful. Thus the question is about the representability of the functor of morphisms between two given log structures over $S^{\dagger}$ along the fibres of the proper morphism $C \to W$.

Abstracting, now let $\pi : Y \to W$ be a proper morphism of schemes and let $\alpha_i : \mathcal{M}_i \to \mathcal{O}_Y$, $i = 1, 2$, be two fine saturated log structures on Y. Consider the functor

(2.1) $\qquad\qquad \mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2) : (\mathrm{Sch}/W) \longrightarrow (\mathrm{Sets})$

that on objects is defined by

$$V \longmapsto \big\{ \varphi : (\mathcal{M}_1)_V \to (\mathcal{M}_2)_V \text{ morphism of log structures} \big\}.$$

---

[7]We follow the usual convention to confuse a scheme and its associated stack

Then the statement that $W \times_{\mathscr{M} \times \mathbf{M}(X)} \tilde{\mathscr{M}}(X)$ is an algebraic space[8] essentially is a special case of the following proposition.

Proposition 2.9. $\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)$ *is represented by an algebraic space locally of finite type over W.*

The proof is after some preparations at the end of this subsection. This then finishes the proof of Proposition 2.3 in the case $S = \mathrm{Spec}\,\Bbbk$ with the trivial log structure. In the general case we have in addition two morphisms of log structures $\psi_i : \pi^* \mathcal{M}_S \to \mathcal{M}_i$, $\pi : C \to W$ the projection, and need to restrict to those $\varphi : (\mathcal{M}_1)_V \to (\mathcal{M}_2)_V$ compatible with ψ_i. But by Proposition 2.9 composition with ψ_1 defines a morphism of algebraic spaces

$$\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2) \longrightarrow \mathrm{LMor}_{Y/W}(\pi^* \mathcal{M}_S, \mathcal{M}_2).$$

Now $W \times_{\mathscr{M} \times \mathbf{M}(X)} \tilde{\mathscr{M}}(X)$ arises as the fibre product with the morphism

$$W \longrightarrow \mathrm{LMor}_{Y/W}(\pi^* \mathcal{M}_S, \mathcal{M}_2)$$

defined by ψ_2, and is hence represented by an algebraic space. This finishes the proof of Proposition 2.3 also in the general case.

Remark 2.10. One problem in showing representability of $\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)$ is that it is non-separated, essentially because the induced map $\overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}_2$ can not be determined by its restriction to an open dense subset. As an example (cf. [Ol2], Remark 3.12) consider the log structure $\mathcal{M}$ on $\mathbb{A}^1 = \mathrm{Spec}\,\Bbbk[x]$ with chart

$$\mathbb{N}^2 \longrightarrow \Bbbk[x], \quad (a, b) \longmapsto x^{a+b}.$$

The map $\mathbb{N}^2 \to \mathbb{N}^2$, $(a, b) \mapsto (b, a)$ induces a non-trivial automorphism of $\mathcal{M}$ that restricts to the identity on $\mathbb{A}^1 \setminus \{0\}$.

To find an étale cover of the algebraic space representing $\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)$ in Proposition 2.9 we thus first restrict the map $\overline{\varphi}$. To this end let $\overline{w} \to W$ be a geometric point and

$$\overline{\varphi}_{\overline{w}} : (\overline{\mathcal{M}}_1)_{\overline{w}} \to (\overline{\mathcal{M}}_2)_{\overline{w}}$$

be a choice of $\overline{\varphi}$ over one geometric fibre $Y_{\overline{w}}$ of $Y \to W$. Now since $\overline{\mathcal{M}}_i$ are fine sheaves the choice at a geometric point $\overline{x} \to Y$ determines $\overline{\varphi}$ at any generization $\overline{y}$ of $\overline{x}$. Moreover, if $\overline{y}$ specializes to some other point $\overline{z}$ such that the generization map $\overline{\mathcal{M}}_{i,\overline{z}} \to \overline{\mathcal{M}}_{i,\overline{y}}$ is an isomorphism then $\overline{\varphi}$ is also determined at $\overline{z}$. Iterating the generization-specialization process we are lead to the following definition.

---

[8]As for algebraic stacks we have to drop the condition of quasi-separatedness from the definition of algebraic spaces ([Kt], Ch.II, Definition 1.1).

Definition 2.11. Let $\overline{\mathcal{M}}$ be a fine sheaf on a scheme Y and let $A \subset |Y|$ be a set of geometric points. We say $\overline{x} \in |Y|$ has *property* (A_{gen}) *with respect to* $\overline{\mathcal{M}}$ if there exists a sequence $\overline{y}_1, \overline{z}_1, \ldots, \overline{y}_r, \overline{z}_r \in |Y|$ for some r with the following properties.

$$
\begin{aligned}
&\overline{y}_1 \in A, \quad \overline{z}_r = \overline{x}, \\
&\overline{y}_i \in \mathrm{cl}(\overline{z}_i), && i = 1, \ldots, r, \\
&\overline{y}_i \in \mathrm{cl}(\overline{z}_{i-1}) && \text{and} \quad \overline{\mathcal{M}}_{\overline{y}_i} \to \overline{\mathcal{M}}_{\overline{z}_{i-1}} \text{ is an isomorphism, } i = 2, \ldots, r.
\end{aligned}
$$
$\square$

Thus the giving of $\overline{\varphi}$ on a closed subset $A \subset |Y|$ then determines $\overline{\varphi}$ also on the subset

$$
U_A := \bigl\{ \overline{x} \in |Y| \,\big|\, \overline{x} \text{ fulfills } (A_{\mathrm{gen}}) \bigr\}
$$

of $|Y|$. Note that by definition U_A is closed under generization. Since $\overline{\mathcal{M}}$ is a fine sheaf it is also immediate that U_A is a constructible subset of $|Y|$, and hence $U_A \subset |Y|$ is open.

Since the statement of Proposition 2.9 is local in W and by properness of $Y \to W$ we may assume any point of $|Y|$ fulfills (A_{gen}) for $A = Y_{\overline{w}}$ with respect both to $\overline{\mathcal{M}}_1$ and to $\overline{\mathcal{M}}_2$, that is, $U_{Y_{\overline{w}}} = Y$. Then for any $V \to W$ there is at most one $\overline{\varphi} : (\overline{\mathcal{M}}_1)_V \to (\overline{\mathcal{M}}_2)_V$ compatible with $\overline{\varphi}_{\overline{w}}$ under sequences of generization maps. Let us call such $\overline{\varphi}$ (or a lift φ to a morphism of log structures) *compatible with* $\overline{\varphi}_{\overline{w}}$, and similarly for any $A \subset |Y|$. Note that $\overline{\varphi}_{\overline{w}}$ may not extend to Y, but it may do so after certain base changes.

We first treat the representability problem locally on Y, that is, for $Y = W$.

Lemma 2.12. *Let $Y = W$ and suppose that there exists a closed subset $A \subset Y$ such that any $\overline{x} \in |Y|$ fulfills (A_{gen}) with respect to both $\overline{\mathcal{M}}_i$ (Definition 2.11). Let*

$$
\overline{\varphi}_A : (\overline{\mathcal{M}}_1)_A \longrightarrow (\overline{\mathcal{M}}_2)_A
$$

be a homomorphism of sheaves of monoids. Then the functor

$$
\mathrm{LMor}_Y^{\overline{\varphi}_A} : (Y' \xrightarrow{f} Y) \longmapsto \bigl\{ \varphi : (Y', f^*\mathcal{M}_2) \to (Y', f^*\mathcal{M}_1) \,\big|\, \varphi \text{ is compatible with } \overline{\varphi}_A \bigr\}
$$

is represented by a scheme $\mathrm{LMor}_Y^{\overline{\varphi}_A}$ of finite type and affine over Y.

Proof. It is sufficient to prove the statement on an étale open cover of Y, since we can then use descent for affine morphisms ([SGA1], VIII, Theorem 2.1) to obtain a scheme over Y. Thus we can assume that we in fact have charts $\psi_i : P_i \to \Gamma(Y, \mathcal{M}_i)$ for the two log structures. We can also assume that $\overline{\varphi}_A$ is induced by a homomorphism of monoids $\overline{\varphi}_A : P_1 \to P_2$.

Let $p_1, \ldots, p_n \in P_1$ be a generating set for P_1 as a monoid. Consider the sheaf of finitely generated $\mathcal{O}_Y$-algebras

$$
\mathcal{F}_Y := \mathcal{O}_Y[P_1^{\mathrm{gp}}] / \langle \alpha_1(\psi_1(p_i)) - z^{p_i} \alpha_2(\psi_2(\overline{\varphi}_A(p_i))) \,|\, 1 \le i \le n \rangle.
$$

Then the desired scheme is $\mathrm{LMor}_Y^{\overline{\varphi}_A} := \mathbf{Spec}\, \mathcal{F}_Y$.

To see that this is the correct scheme, suppose $f : Y' \to Y$ is given. We wish to show that giving a commutative diagram of schemes

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ g\ } & \mathbf{Spec}\,\mathcal{F}_Y \\
 & {\scriptstyle f}\!\!\searrow & \downarrow \\
 & & Y
\end{array}
$$

is equivalent to giving a log morphism $\varphi : (Y', f^*\mathcal{M}_2) \to (Y', f^*\mathcal{M}_1)$ which is the identity on Y' and lifts $f^*(\overline{\varphi}_A) : f^*\overline{\mathcal{M}}_1 \to f^*\overline{\mathcal{M}}_2$. Of course giving g is equivalent to giving a section of $(\mathbf{Spec}\,\mathcal{F}_Y) \times_Y Y'$ over Y'. But

$$(\mathbf{Spec}\,\mathcal{F}_Y) \times_Y Y' =$$
$$\mathbf{Spec}\,\mathcal{O}_{Y'}[P_1^{\mathrm{gp}}]/\big\langle f^*\big(\alpha_1(\psi_1(p_i))\big) - z^{p_i} f^*\big(\alpha_2(\psi_2(\overline{\varphi}_A(p_i)))\big) \,\big|\, 1 \le i \le n \big\rangle,$$

and the latter scheme is $\mathbf{Spec}\,\mathcal{F}_{Y'}$ associated to the data $(Y', f^*\mathcal{M}_1)$, $(Y', f^*\mathcal{M}_2)$ with charts $f^*(\psi_i) = f^\flat \circ \psi_i : P_i \to \Gamma(Y', f^*\mathcal{M}_i)$. Thus, without loss of generality, we can assume that $Y = Y'$ and f is the identity.

Now giving $\varphi : (Y, \mathcal{M}_2) \to (Y, \mathcal{M}_1)$ lifting $\overline{\varphi}_A$ is equivalent to specifying $\varphi^\flat$. In order for $\varphi^\flat$ to lift $\overline{\varphi}_A$, there must be a map $\eta : P_1 \to \Gamma(Y, \mathcal{O}_Y^\times)$ with the property that for all $p \in P_1$,

$$\varphi^\flat(\psi_1(p)) = \eta(p) \cdot \psi_2(\overline{\varphi}_A(p)).$$

Giving η completely determines $\varphi^\flat$. In addition, $\varphi^\flat$ is a homomorphism of monoids if and only if η is a homomorphism, and since η takes values in the group $\mathcal{O}_Y^\times$, specifying $\varphi^\flat$ is equivalent to specifying a section of $\mathbf{Spec}\,\mathcal{O}_Y[P_1^{\mathrm{gp}}]$. Indeed, giving a section of $\mathbf{Spec}\,\mathcal{O}_Y[P_1^{\mathrm{gp}}]$ over Y is the same as giving a morphism $Y \to \mathrm{Spec}\,\Bbbk[P_1^{\mathrm{gp}}]$, which in turn is the same as giving an element of $\mathrm{Hom}(P_1, \Gamma(Y, \mathcal{O}_Y^\times))$.

Secondly, since $\varphi^* = \mathrm{id}$, we must have $\alpha_1 = \alpha_2 \circ \varphi^\flat$, so for each $p \in P_1$, we must have

$$\alpha_1(\psi_1(p)) = \alpha_2(\varphi^\flat(\psi_1(p))) = \eta(p) \cdot \alpha_2(\psi_2(\overline{\varphi}_A(p))).$$

If this holds for each p_i, it holds for all p. Thus a section of $\mathbf{Spec}\,\mathcal{O}_Y[P_1^{\mathrm{gp}}]$ over Y determines a morphism of log structures if and only if it lies in the subscheme determined by the equations

$$\alpha_1(\psi_1(p_i)) - z^{p_i}\alpha_2(\psi_2(\overline{\varphi}_A(p_i))),$$

demonstrating the result. $\square$

Lemma 2.13. *Let $Y \to W$ be a projective, separated morphism of schemes. Let $\overline{w} \to W$ be a geometric point, and assume any $\overline{x} \in |Y|$ fulfills (A_{gen}) with respect to both $\overline{\mathcal{M}}_i$ for $A = Y_{\overline{w}}$ (Definition 2.11). Then for a homomorphism $\overline{\varphi}_A : (\overline{\mathcal{M}}_1)_{\overline{w}} \to (\overline{\mathcal{M}}_2)_{\overline{w}}$ of sheaves of monoids the functor*

$$\mathrm{LMor}_{Y/W}^{\overline{\varphi}_A} : (V \to W) \longmapsto \big\{ \varphi : (\mathcal{M}_1)_V \to (\mathcal{M}_2)_V \,\big|\, \varphi \text{ is compatible with } \overline{\varphi}_A \big\}$$

is represented by a scheme $\mathrm{LMor}_{Y/W}^{\overline{\varphi}_A}$ of finite type over W.

Proof. Let $Z = \mathrm{LMor}_Y^{\overline{\varphi}_A}$. By Lemma 2.12, $\mathrm{LMor}_{Y/W}^{\overline{\varphi}_A}$ is isomorphic to the functor

$$(V \to W) \longmapsto \{\text{sections of } Z_V \to Y_V\}.$$

This is precisely the functor of sections $\prod_{Y/W}(Z/Y)$ discussed in [Gr], p.221-19 (see also [Nt]), and it is represented by an open subscheme of $\mathrm{Hilb}_{Z/W}$. Furthermore, if $\mathcal{L}$ is a relatively ample line bundle on Y, then as $Z \to Y$ is affine, the pull-back of $\mathcal{L}$ to Z is also (trivially) relatively ample over W. We use this ample line bundle to define Hilbert polynomials. Any section of $Z_V \to Y_V$ then must have the same Hilbert polynomial with respect to $\mathcal{L}$ as the Hilbert polynomial of $\mathcal{L}$ on Y_V, so in fact $\Pi_{Y/W}(Z/Y)$ defines an open subscheme of $\mathrm{Hilb}_{Z/W}^P$, for P this fixed Hilbert polynomial. Thus $\Pi_{Y/W}(Z/Y)$ is represented by a scheme of finite type over W. $\square$

We are now in position to prove Proposition 2.9.

Proof of Proposition 2.9. Since $\mathcal{M}_1, \mathcal{M}_2$ are defined in the étale topology, $\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)$ is clearly a sheaf in the étale topology. It remains to prove the local representability statement of [Kt], Ch.II, Definition 1.1,b, for $\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)$. For any geometric fibre $A = Y_{\overline{w}}$ take an étale neighbourhood $U = U(\overline{w}) \to W$ of $\overline{w}$ such that Y_U fulfills the assumptions of Lemma 2.13. Then for any $\overline{\varphi}_{\overline{w}}$ we have a scheme $\mathrm{LMor}_{Y_U/U}^{\overline{\varphi}_{\overline{w}}}$ of finite type over W. We claim that the natural functor

$$Z := \coprod_{\overline{\varphi}_{\overline{w}}} \mathrm{LMor}_{Y_U/U}^{\overline{\varphi}_{\overline{w}}} \longrightarrow \mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)$$

is schematic and an étale surjection. This means explicitly that if V is a scheme and $V \to \mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)$ is a morphism of stacks then the fibre product in the sense of 2-categories $V \times_{\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)} Z$ is represented by a scheme, and the projection to V is an étale surjection. We claim that for one $\mathrm{LMor}_{Y_U/U}^{\overline{\varphi}_{\overline{w}}}$ the representing scheme is an open subset of $V \times_W U$. In fact, the functor $V \to \mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)$ says that we fix a morphism of log structures $\varphi : (\mathcal{M}_1)_V \to (\mathcal{M}_2)_V$. Now a functor

$$\psi : T \longrightarrow V \times_{\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)} \mathrm{LMor}_{Y_U/U}^{\overline{\varphi}_{\overline{w}}}$$

from a scheme T is nothing but (i) a morphism $T \to V$ and (ii) a morphism $T \to \mathrm{LMor}_{Y_U/U}^{\overline{\varphi}_{\overline{w}}}$ such that (iii) the compositions with the morphisms to $\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)$ coincide. Note that (i) provides a pull-back $\varphi_T : (\mathcal{M}_1)_T \to (\mathcal{M}_2)_T$ of φ, (ii) gives a morphism of schemes $T \to U$ and a morphism of log structures $\varphi' : (\mathcal{M}_1)_T \to (\mathcal{M}_2)_T$ with $\overline{\varphi'}$ induced by $\overline{\varphi}_{\overline{w}}$, and (iii) says $\varphi' = \varphi_T$. Thus ψ is nothing but a factorization of the composition $T \to V \to W$ through U along with the information that $\overline{\varphi}_T$ is induced by $\overline{\varphi}_{\overline{w}}$. Given φ the latter condition defines an open subset of $V \times_W U$. Thus ψ is canonically identified with a homomorphism from T to an open subset of $V \times_W U$. This proves the claim. Note that since $U \to W$ is étale so is the projection $V \times_W U \to V$. Finally, surjectivity of $V \times_{\mathrm{LMor}_{Y/W}(\mathcal{M}_1, \mathcal{M}_2)} Z \to V$ follows from the fact that we took the union over all $\overline{\varphi}_{\overline{w}}$. $\square$

3. Boundedness

The aim of this section is to identify parts of $\mathscr{M}(X)$ that are of finite type. The main results are Theorem 3.9 in §3.1 and Theorem 3.14 in §3.2.

3.1. Finiteness of combinatorial types.

In Definition 1.9,(2) we defined the type of a stable log map over a geometric point. It is given by the dual intersection graph $\Gamma_{\underline{C}}$ of the domain and data $\mathbf{u} = \big((u_p)_p, (u_q)_q\big)$. Morally the $u_p : P_p \to \mathbb{N}$ tell the order of contact with the toric divisors in a local chart for the log structure on X. Since by log smoothness these orders stay locally constant in families of stable log maps, they are part of the data distinguishing connected components of $\mathscr{M}(X)$.

Definition 3.1. A *class* β of stable log maps to X consists of the following.

(i) The data $\underline{\beta}$ of an underlying ordinary stable map, that is, the genus g of $\underline{C}$, the number k of marked points, and data A bounding the degree, e.g. as described in [BeMa], p.12.[9]

(ii) Strict closed embeddings $Z_1, \ldots, Z_r \subset X$ together with sections $s_i \in \Gamma(Z_i, (\overline{\mathcal{M}}_{Z_i}^{\mathrm{gp}})^*)$.

A stable log map $(C/W, \mathbf{x}, f)$ *is of class* β if the underlying ordinary stable map is of type (g, k, A), and if for any i we have $\mathrm{im}(\underline{f} \circ x_i) \subset \underline{Z}_i$ and for any geometric marked point $\overline{w} \to \underline{W}$ the map

$$(3.1) \qquad \overline{\mathcal{M}}_{Z_i, \underline{f}(x_i(\overline{w}))} = (\underline{f}^* \overline{\mathcal{M}}_X)_{x_i(\overline{w})} \xrightarrow{\overline{f}^\flat} \overline{\mathcal{M}}_{C, x_i(\overline{w})} = \overline{\mathcal{M}}_{W,\underline{w}} \oplus \mathbb{N} \xrightarrow{\mathrm{pr}_2} \mathbb{N}$$

equals the germ of s_i at $\underline{f}(x_i(\overline{w}))$.

The substack of $\mathscr{M}(X)$ of stable log maps of class β is denoted $\mathscr{M}(X, \beta)$.   □

Note that the composition (3.1) is denoted u_p in other parts of the text.

Remark 3.2. The matching condition $\mathrm{im}(\underline{f} \circ x_i) \subset \underline{Z}_i$ clearly defines a closed algebraic substack of $\mathcal{M}(X)$, while the remaining conditions are open. In particular, $\mathscr{M}(X, \beta)$ is also an algebraic stack locally of finite type over $\underline{S}$.

A necessary condition for boundedness of $\mathscr{M}(X, \beta)$ is that only finitely many types of stable log maps to X of class β occur. Unfortunately, we have been unable to prove this in complete generality; so far we have only been able to prove finiteness given certain assumptions on X. On the other hand, we also couldn't find an X for which this finiteness does not hold. Thus we believe that the following definition is in fact empty, at least locally over $\mathbf{M}(\underline{X})$.

Definition 3.3. A class β of stable log maps is called *combinatorially finite* if the set of types of stable log maps of class β is finite.

---

[9]In [BeMa] A is defined by the degree function on the cone of isomorphism classes of ample invertible sheaves; if $\Bbbk \subset \mathbb{C}$ one might prefer prescribing a class in the singular homology group $H_2(\underline{X}_{\mathbb{C}}, \mathbb{Z})$ of the associated complex variety $\underline{X}_{\mathbb{C}}$.

Thus in general, if one wishes to deal with log Gromov-Witten invariants for an X for which we do not prove finiteness below, one will have to check finiteness for that X. However, we think that the cases discussed below will cover most, if not all, applications of log Gromov-Witten theory.

We fix in this section an ordinary stable map $\underline{f} : \underline{C} \to \underline{X}$ over $\operatorname{Spec} \kappa$, and we consider all possible types of liftings of such a map to $f : C \to X$ over the standard log point $(\operatorname{Spec} \kappa, \mathbb{N})$.

We introduce several conditions a log scheme can satisfy which will be useful for proving boundedness.

Definition 3.4. We say a sheaf of monoids $\overline{\mathcal{M}}$ on (the Zariski site of) of a scheme $\underline{Y}$ is *almost generated* if the maps

$$\operatorname{Hom}(\overline{\mathcal{M}}_y, \mathbb{R}_{\geq 0}) \to \operatorname{Hom}(\Gamma(\underline{Y}, \overline{\mathcal{M}}), \mathbb{R}_{\geq 0})$$

are injective for all $y \in \underline{Y}$. We say a log scheme Y is *almost generated* if $\overline{\mathcal{M}}_Y$ is almost generated.

We say a sheaf of monoids $\overline{\mathcal{M}}$ on a scheme $\underline{Y}$ is *quasi-generated* if

$$\operatorname{Hom}(\overline{\mathcal{M}}_y, \mathbb{R}) \to \operatorname{Hom}\left(\Gamma(\underline{Y}, \overline{\mathcal{M}}^{\mathrm{gp}}), \mathbb{R}\right)$$

are injective for all $y \in \underline{Y}$. We say a log scheme Y (with a log structure on the Zariski site) is *quasi-generated* if $\overline{\mathcal{M}}_y$ is quasi-generated.

Remark 3.5. 1) If $\overline{\mathcal{M}}$ is a sheaf of fine monoids then being almost generated is equivalent to saying that for any $y \in \underline{Y}$ the image of the restriction map $\Gamma(\underline{Y}, \overline{\mathcal{M}}) \to \overline{\mathcal{M}}_y$ spans $\overline{\mathcal{M}}_y^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$. In fact, write $P = \overline{\mathcal{M}}_y$ and $Q = \operatorname{im}\left(\Gamma(\underline{Y}, \overline{\mathcal{M}}) \to \overline{\mathcal{M}}_y\right) \subset P$. If $\overline{\mathcal{M}}$ is almost generated then $\operatorname{Hom}(P, \mathbb{R}_{\geq 0}) \to \operatorname{Hom}(Q, \mathbb{R}_{\geq 0})$ is injective. But then, since P and Q are fine monoids, also the induced map of groupifications

$$\operatorname{Hom}(P, \mathbb{R}) = \operatorname{Hom}(P, \mathbb{R}_{\geq 0})^{\mathrm{gp}} \longrightarrow \operatorname{Hom}(Q, \mathbb{R}_{\geq 0})^{\mathrm{gp}} = \operatorname{Hom}(Q, \mathbb{R})$$

is injective. Thus Q spans $P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$.

Conversely, if $\operatorname{Hom}(P, \mathbb{R}) \to \operatorname{Hom}(Q, \mathbb{R})$ is injective then so is the restriction to $\operatorname{Hom}(P, \mathbb{R}_{\geq 0})$.

2) Similarly, a sheaf of fine monoids $\overline{\mathcal{M}}$ is quasi-generated iff for any $y \in \underline{Y}$ the image of $\Gamma(\underline{Y}, \overline{\mathcal{M}}^{\mathrm{gp}}) \to \overline{\mathcal{M}}_y^{\mathrm{gp}}$ spans $\overline{\mathcal{M}}_y^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$.

Examples 3.6. (1) The condition that X be Deligne-Faltings arises in the work of Abramovich and Chen [ACGM], [Ch]. This means that there is a surjection $\underline{\mathbb{N}}^r \to \overline{\mathcal{M}}_X$ for some r. The condition of X being almost generated is strictly weaker. For example, suppose $\underline{X}$ is a surface which is singular only at a point $p \in \underline{X}$, where $\underline{X}$ has an A_{e-1} singularity, $e \geq 2$. Let $D = D_1 \cup D_2$ be a divisor in $\underline{X}$ with D_1, D_2 irreducible and $D_1 \cap D_2 = \{p, q\}$, where $q \in \underline{X}$ is a smooth point of $\underline{X}$. Assume that near p we have $\underline{X}$ locally given by the equation $xy = t^e$, and $D_1 \cup D_2$ is locally given by $t = 0$. Let $\underline{X}$ be given the divisorial log structure induced by D. Then one checks easily that $\Gamma(\underline{X}, \overline{\mathcal{M}}_X) \cong \overline{\mathcal{M}}_{X,p} = S_e$, the submonoid of $\mathbb{N}^2$ generated by

$(e, 0)$, $(0, e)$, and $(1, 1)$. In particular, $\Gamma(\underline{X}, \overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X,q} = \mathbb{N}^2$ is not surjective. Thus X cannot be Deligne-Faltings. However, one checks easily that X is almost generated.

On the other hand, Deligne-Faltings log structures are always almost generated, as a surjection $P = \mathbb{N}^r \to \overline{\mathcal{M}}_x$ yields an injection $\mathrm{Hom}(\overline{\mathcal{M}}_x, \mathbb{R}_{\geq 0}) \to \mathrm{Hom}(P, \mathbb{R}_{\geq 0})$.

(2) It is not clear that being quasi-generated is weaker than being almost generated. However, in some common situations, it is easier to check. In fact, it suffices to find a group M and a map $M \to \Gamma(\underline{Y}, \overline{\mathcal{M}}^{\mathrm{gp}})$ such that for all $y \in \underline{Y}$ the induced map

$$\mathrm{Hom}(\overline{\mathcal{M}}_y, \mathbb{R}) \to \mathrm{Hom}(M, \mathbb{R})$$

is injective. For example, if X is a toric variety with the divisorial log structure defined by the toric divisors and M is the character lattice, there is a natural surjection $\underline{M} \to \overline{\mathcal{M}}_X^{\mathrm{gp}}$.

(3) If X is almost generated (quasi-generated), and $\underline{f} : \underline{C} \to \underline{X}$ is an ordinary stable map of curves, then the pull-back log structure $\underline{f}^* \mathcal{M}_X$ on $\underline{C}$ is almost generated (quasi-generated).

(4) Suppose we are given an ordinary stable map $\underline{f} : \underline{C} \to \underline{X}$ and the dual intersection graph of $\underline{C}$ is a tree, for example if $g(C) = 0$. Then the pull-back log structure $\underline{f}^* \mathcal{M}_X$ is almost generated. In fact, even better, the map $\Gamma(\underline{C}, \underline{f}^* \overline{\mathcal{M}}_X) \to (\underline{f}^* \overline{\mathcal{M}}_X)_x$ is surjective for every $x \in \underline{C}$, so the pull-back log structure is Deligne-Faltings.

Indeed, with these assumptions, the sheaf $\underline{f}^* \overline{\mathcal{M}}_X$ is entirely determined by the stalks P_η and P_x for special points x, along with uniquely determined generization maps $\chi_{\eta,x} : P_x \to P_\eta$ for every distinguished point x. To specify a section of $\underline{f}^* \overline{\mathcal{M}}_X$, we just need to specify elements $s_x \in P_x$ for all $x \in \underline{C}$ such that $\chi_{\eta,x}(s_x) = s_\eta$ for $x \in D$, $D := \mathrm{cl}(\eta) \subset \underline{C}$. Now picking a point $x \in D$, and $s_x \in P_x$, set $s_\eta = \chi_{\eta,x}(s_x)$. For every other point $x' \in D$, $x' \neq x$, the generization map $\chi_{\eta,x'}$ is always surjective, so we can choose $s_{x'} \in P_{x'}$ with $\chi_{\eta,x'}(s_{x'}) = s_\eta$. Some of these points x' will be double points, hence allowing us to define $s_{\eta'}$ for other generic points η'. Continuing in this fashion, using the fact there are no cycles in the dual intersection graph of $\underline{C}$, gives us a section of $\underline{f}^* \overline{\mathcal{M}}_X$ whose germ at x is the given s_x.

Our arguments hinge on the following well-known finiteness result:

Proposition 3.7. *Fixing a graph Γ, a lattice N, and weight vectors $u_{(v,E)} \in N$ for every non-compact edge E of Γ, there are only a finite number of types of tropical curves with target $N_\mathbb{R}$ (as in Definition 1.11) with this given Γ and $u_{(v,E)}$.*

Proof. This is a weaker result than [NiSi], Proposition 2.1. $\square$

Theorem 3.8. *Let $(\underline{C}/\mathrm{Spec}\,\kappa, \mathbf{x}, \underline{f})$ be an ordinary stable map and suppose $\overline{\mathcal{M}} := \underline{f}^* \overline{\mathcal{M}}_X$ is almost generated. Then there are only a finite number of types of log curves with the given underlying ordinary stable map.*

Proof. Let

$$M = \Gamma(\underline{C}, \overline{\mathcal{M}})^{\mathrm{gp}}.$$

Let $N = \mathrm{Hom}(M, \mathbb{Z})$, $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. Clearly M is spanned by the submonoid $\Gamma(\underline{C}, \overline{\mathcal{M}}) \subset M$. Thus also the dual submonoid $\Gamma(\underline{C}, \overline{\mathcal{M}})^\vee \subset N$ spans the dual space N, and such dual

submonids are also sharp. In particular, $\Gamma(\underline{C}, \overline{\mathcal{M}})^\vee$ coincides with $K \cap N$ for some strictly convex rational polyhedral cone K in $N_\mathbb{R}$.

Observe that since we can pull-back any stable log map to a standard log point it is enough to bound the types of tropical curves over standard log points. We thus consider now a stable log map $(C/(\operatorname{Spec}\kappa, \mathbb{N}), \mathbf{x}, f)$ over the standard log point. As in §1.3 this determines data V_η, e_q, u_p, u_q. The intersection graph $\Gamma_{\underline{C}}$ of $\underline{C}$ along with the data $((u_p), (u_q))$ is the type of $(C/(\operatorname{Spec}\kappa, \mathbb{N}), \mathbf{x}, f)$ by pull-back. Similar to Discussion 1.12 we now look at the associated tropical curve. By the definition of almost generated, we obtain for all $x \in \underline{C}$ inclusions

$$\operatorname{Hom}(P_x, \mathbb{R}_{\geq 0}) \hookrightarrow K.$$

Thus $V_\eta \in P_\eta^\vee$, $u_p \in P_p^\vee$ live naturally in K and in fact in the monoid $\Gamma(\underline{C}, \overline{\mathcal{M}})^\vee \subset N$. We also have, for each irreducible component $D = \operatorname{cl}(\eta)$ of $\underline{C}$, a map $\tau_\eta^X : \Gamma(\tilde{D}, g^*\overline{\mathcal{M}}) \to \mathbb{Z}$, where $g : \tilde{D} \to \underline{C}$ is the normalization of D followed by inclusion into $\underline{C}$, see (1.10). We then have a composition of the pull-back map on sections with τ_η^X:

$$M \to M_\eta := \Gamma(\tilde{D}, g^*\overline{\mathcal{M}}) \xrightarrow{\tau_\eta^X} \mathbb{Z}.$$

This composition determines an element of N which we also denote by τ_η^X.

To build a tropical curve from this data add to $\Gamma_{\underline{C}}$ a number of unbounded edges: for each vertex v_η we attach an unbounded edge, E_η, to v_η. We then define $h : \Gamma_{\underline{C}} \to N_\mathbb{R}$ by

$$h(v_\eta) = V_\eta;$$

each edge E_q with endpoints v_{η_1} and v_{η_2} is mapped to the line segment joining V_{η_1} and V_{η_2}; and each edge E_p with endpoint v_η is mapped to the ray with endpoint V_η with direction u_p. Finally, we map the ray E_η to the ray with endpoint V_η in the direction defined by τ_η^X. (If $\tau_\eta^X = 0$, the edge is contracted).

To give this the structure of a tropical curve, one also needs to assign integral weight vectors to each flag (v, E) of $\Gamma_{\underline{C}}$. If $E = E_q$, we assign the vector $\pm u_q$, with the sign chosen so that $\pm u_q$ points away from $h(v)$. If $E = E_p$, we associate the vector u_p, and if $E = E_\eta$, we associate the vector τ_η^X. These associated vectors are integral tangent vectors to the image of the corresponding edge. Then the tropical curve balancing condition is just the statement that for a given η,

$$\tau_\eta^X + \sum_x u_x = 0,$$

where the sum is over all special points $x \in \operatorname{cl}(\eta)$. But this is precisely the image of the equation of Proposition 1.14 in $M_\eta^\vee$, under the map $M_\eta^\vee \to N$. Thus h defines a balanced tropical curve in $N_\mathbb{R}$.

Summing the balancing condition over all vertices gives a global balancing condition involving all unbounded edges. Namely, in N, we have

$$\sum_\eta \tau_\eta^X + \sum_p u_p = 0,$$

where η runs over all generic points of $\underline{C}$ and p runs over all marked points. Now $\sum_\eta \tau_\eta^X \in N$ is given, completely specified by the original stable map $\underline{f} : \underline{C} \to \underline{X}$ and independent of $f^\flat$, and $u_p \in \Gamma(\underline{C}, \overline{\mathcal{M}})^\vee \subset N$, which as observed, is strictly convex. Thus there is only a finite number of possibilities for writing $-\sum_\eta \tau_\eta^X$ as a sum of such u_p's. This shows finiteness of the choices of the u_p's.

For given $(u_p)_p$ and τ_η^X, Proposition 3.7 shows there are only a finite number of possible combinatorial types of this tropical curve. This means there are only a finite number of possibilities for the vectors $u_q \in N$. On the other hand, these vectors are images of $u_q \in (P_q^{\mathrm{gp}})^*$. Since $(P_q^{\mathrm{gp}})^*$ injects into N, we conclude we only have a finite number of allowable types. $\square$

We have a slightly weaker result in the quasi-generated case, where we need to fix the u_p's, that is, the class.

Theorem 3.9. *In the situation of Theorem 3.8 suppose that $\overline{\mathcal{M}} = \underline{f}^* \mathcal{M}_X$ is only quasi-generated. Then if the u_p's are fixed, there are only a finite number of types of log curves with the given underlying ordinary stable map.*

Proof. The argument is essentially the same as the proof of the previous theorem, this time taking $M = \Gamma(\underline{C}, \overline{\mathcal{M}}^{\mathrm{gp}})$. Then via the same construction, one obtains a tropical curve in $N_\mathbb{R}$. However, one no longer has all u_p living in a strictly convex cone in $N_\mathbb{R}$, so we cannot use this to control the vectors u_p. However if we assume the u_p are given, then Proposition 3.7 still gives the needed finiteness. $\square$

We next address what can be accomplished in more general situations. In general, a natural target space for a tropical curve might be a space we call the *tropicalization* of X, (analogous to the notion of the tropical part of an exploded manifold of Parker's work [Pa]), defined as follows. Given a log scheme X with log structure in the Zariski topology, we set

$$\mathrm{Trop}(X) := \left(\coprod_{x \in \underline{X}} \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}) \right) \bigg/ \sim,$$

where the disjoint union is over all scheme-theoretic points of $\underline{X}$ and the equivalence relation is generated by the identifications of faces given by dualizing generization maps $\overline{\mathcal{M}}_{X,x} \to \overline{\mathcal{M}}_{X,x'}$ when x is specialization of x'. One then obtains for each x a map

$$i_x : \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}) \to \mathrm{Trop}(X).$$

For general X, $\mathrm{Trop}(X)$ may not be particularly well-behaved, as the equivalence relation might yield strange self-identifications of faces. It is sometimes useful to impose a condition which allows us to avoid such a possibility:

Definition 3.10. Let X be an fs log scheme. We say X is *monodromy free* if for any geometric point $x \in \underline{X}$, i_x is injective on the interior of any face of $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0})$.

In general, $\mathrm{Trop}(X)$ is a good range space for the tropical curves arising from log maps. Specifically, given a stable log map $(C/(\mathrm{Spec}\,\kappa,\mathbb{N}),\mathbf{x},f)$ over a standard log point, we map v_η to $i_{f(\bar\eta)}(V_\eta)$, where now V_η is viewed as an element of $\mathrm{Hom}(P_\eta,\mathbb{R}_{\geq 0})$. We map an edge E_q to the image of the line segment in $\mathrm{Hom}(P_q,\mathbb{R}_{\geq 0})$ joining the images of the endpoints of E_q, as usual. We map E_p to a ray with endpoint $i_{f(\bar\eta)}(V_\eta)$ in the direction $i_{f(\bar\eta)}(u_p)$, if p is in the closure of η.

Remark 3.11. A more conceptual way to view this construction is to observe that Trop is a covariant functor: given a morphism of Zariski log schemes $f : X \to Y$, for $x \in X$ we obtain a map $f^\flat : \overline{\mathcal{M}}_{Y,f(x)} \to \overline{\mathcal{M}}_{X,x}$ and hence a map $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x},\mathbb{R}_{\geq 0}) \to \mathrm{Hom}(\overline{\mathcal{M}}_{Y,f(x)},\mathbb{R}_{\geq 0})$. This is compatible with the equivalence relations defining $\mathrm{Trop}(X)$ and $\mathrm{Trop}(Y)$. Indeed, if x is a specialization of x', $f(x)$ is a specialization of $f(x')$, and hence inclusions of faces are compatible with the induced maps on cones.

The above description of the tropical curve associated to $(C/(\mathrm{Spec}\,\kappa,\mathbb{N}),\mathbf{x},f)$ can be viewed as follows. We have a map $\mathrm{Trop}(\pi) : \mathrm{Trop}(C) \to \mathrm{Trop}(\mathrm{Spec}\,\kappa,\mathbb{N}) = \mathbb{R}_{\geq 0}$, and $\mathrm{Trop}(f) : \mathrm{Trop}(\pi)^{-1}(1) \to \mathrm{Trop}(X)$ is easily seen to coincide with the tropical curve described above.

More generally, suppose we have a basic stable log map $(C/(\mathrm{Spec}\,\kappa,Q),\mathbf{x},f)$ over a point. Then we obtain a family of tropical curves parameterized by $\mathrm{Hom}(Q,\mathbb{R}_{\geq 0})$. In particular, the tropical curves corresponding to pull-backs of this stable log map to standard log points are given by the restriction of $\mathrm{Trop}(f)$ to $\mathrm{Trop}(\pi)^{-1}(q)$, for $q \in \mathrm{Int}(Q^\vee)$.

While this quite generally provides a tropical curve, it can be difficult to use the balancing condition in the often strange space $\mathrm{Trop}(X)$ in a useful way to prove finiteness.

Here is a special case that we state without proof, where it is still possible to prove finiteness. This is likely to be a good model for the type of application we have in mind, in which one considers varieties degenerating to unions of reasonably simple varieties.

Theorem 3.12. *Let $\underline{X}$ be a scheme with $D \subset \underline{X}$ a divisor inducing a divisorial log structure on $\underline{X}$. Suppose that this makes X an fs log scheme, log smooth over $\mathrm{Spec}\,\kappa$, and suppose furthermore:*

(1) *For each irreducible component $\underline{Y}$ of D, the restriction of the log structure of X to $\underline{Y}$ is almost generated. Furthermore, for $y \in \underline{Y}$, the inclusion $\mathrm{Hom}(\overline{\mathcal{M}}_{X,y},\mathbb{R}_{\geq 0}) \to \mathrm{Hom}(\Gamma(\underline{Y},\overline{\mathcal{M}}_X),\mathbb{R}_{\geq 0})$ is an inclusion of faces.[10]*

(2) *X is monodromy free.*

Then given an ordinary stable map $\underline{f} : \underline{C} \to \underline{X}$ and a collection of $u_p \in P_p^\vee$ for $p \in \underline{C}$ marked points, there are only a finite number of possible types of log curves with the given underlying ordinary stable map. $\square$

We were also able to prove boundedness without any further hypotheses in the case of genus one, also stated here without proof.

---

[10]Note this holds, for example, if the log structure on Y is Deligne-Faltings.

Theorem 3.13. *Let $(\underline{C}/\operatorname{Spec}\kappa, \underline{x}, \underline{f})$ be an ordinary stable map and suppose the dual inter-section graph $\Gamma_{\underline{C}}$ has genus at most one. Then there are only a finite number of types of log curves with the given underlying ordinary stable map.*   □

3.2. **Stable log maps of constant type and boundedness.** Here is the main result of this section.

Theorem 3.14. *For any combinatorially finite class β of stable log maps to X, the algebraic stack $\mathscr{M}(X, \beta)$ is of finite type over $\underline{S}$.*

By Corollary 2.6 we already know that $\mathscr{M}(X) \to \underline{S}$ is locally of finite type. Moreover, the stack of ordinary stable maps of fixed class $\mathbf{M}(\underline{X}, \underline{\beta})$ is of finite type over $\underline{S}$. So to finish the proof of Theorem 3.14 it remains to show that for any morphism $\underline{W} \to \mathbf{M}(\underline{X})$ from a quasi-compact scheme $\underline{W}$ the fibre product $\underline{W} \times_{\mathbf{M}(\underline{X})} \mathscr{M}(X, \beta)$ is quasi-compact. By [LaMB], Corollaire 5.6.3 this is equivalent to showing that the topological space of geometric points $\left|\underline{W} \times_{\mathbf{M}(\underline{X})} \mathscr{M}(X, \beta)\right|$ is quasi-compact. We do this by a stratawise approach.

Definition 3.15. A *weak covering* of a topological space Z is a collection of subsets $\{A_i\}_{i \in I}$ with the following property: For any $z \in Z$ there exists $i \in I$ with $\operatorname{cl}(z) \cap A_i \neq \emptyset$.

Lemma 3.16. *Let Z be a topological space weakly covered by finitely many quasi-compact subsets $A_i \subset Z$, $i \in I$. Then Z is quasi-compact.*

Proof. Let $U_j \subset Z$, $j \in J$, be open subsets covering Z. By quasi-compactness of the A_i, for any $i \in I$ there exist finitely many U_j covering A_i. Hence, since I is finite, there exists a finite subset $J' \subset J$ with $A_i \subset \bigcup_{j \in J'} U_j$, for any i. We claim $Z = \bigcup_{j \in J'} U_j$. In fact, let $z \in Z$. Then by the weak covering assumption there exists $i \in I$ with $\operatorname{cl}(z) \cap A_i \neq \emptyset$. Thus by the choice of $J' \subset J$ there exists $j \in J'$ with $\operatorname{cl}(z) \cap U_j \neq \emptyset$, and then $z \in U_j$ for $U_j \subset Z$ is open. Thus already finitely many of the U_j cover Z.   □

Our strata A_i will be defined by taking locally trivial families of ordinary stable maps.

Definition 3.17. An ordinary stable map $(\underline{\pi} : \underline{C} \to \underline{W}, \mathbf{x}, \underline{f})$ over an integral scheme $\underline{W}$ is called *combinatorially constant* if the following conditions are satisfied, where we write $\mathcal{M} := \underline{f}^* \mathcal{M}_X$ as usual.
  (1) If $g : \underline{\tilde{C}} \to \underline{C}$ is the normalization, then the composition $\underline{\pi} \circ g$ is a smooth map, and there are pairwise different sections y_q of $\underline{\pi}$ with $\bigcup_q \operatorname{im}(y_q) = \operatorname{crit}(\underline{\pi})$.
  (2) Each irreducible component $\underline{C}_\eta \subset \underline{C}$ is geometrically connected and there is a section σ_η of $\underline{\pi}|_{\underline{C}_\eta}$ with $\operatorname{im}(\sigma_\eta)$ disjoint from $\operatorname{im}(\mathbf{x})$ and from $\operatorname{crit}(\underline{\pi})$.
  (3) For any i, q the sheaves $x_i^* \overline{\mathcal{M}}$ and $y_q^* \overline{\mathcal{M}}$ are constant, and there are charts of log structures on $\underline{W}$
$$P_\eta \longrightarrow \sigma_\eta^* \mathcal{M}, \quad \textstyle\prod_q \underline{\mathbb{N}} \longrightarrow \mathcal{M}_W^0$$
  inducing isomorphisms $P_\eta \simeq \sigma_\eta^* \overline{\mathcal{M}}$ and $\prod_q \underline{\mathbb{N}} \simeq \overline{\mathcal{M}}_W^0$. Here $\mathcal{M}_W^0$ is the basic log structure for the pre-stable curve $\underline{C}/\underline{W}$.   □

Note that (3) defines isomorphisms $\sigma_\eta^* \mathcal{M} \simeq \underline{P}_\eta \times \mathcal{O}_{\underline{W}}^\times$ and $\mathcal{M}_W^0 \simeq \prod_q \underline{\mathbb{N}} \times \mathcal{O}_{\underline{W}}^\times$. Note also that for a combinatorially constant stable map $(\underline{C}/\underline{W}, \mathbf{x}, \underline{f})$ the existence of the sections σ_η and Stein factorization imply that the geometric fibres of $\underline{\pi} \circ g|_{\underline{C}_\eta}$ are connected. Thus all geometric fibres of $\underline{\pi}$ have the same dual intersection graph Γ as the generic fibre, and $\underline{C}$ is obtained by gluing together families of smooth, connected curves along pairs of sections. Moreover, a type $\mathbf{u} = \big((u_p), (u_q)\big)$ at the generic fibre induces a type for the fibre over every geometric point $\overline{w} \to \underline{W}$. By abuse of notation we call $\mathbf{u}$ a *type for* $(\underline{C}/\underline{W}, \mathbf{x}, \underline{f})$.

Let $\underline{W} \to \mathbf{M}(\underline{X})$ be the morphism defined by a combinatorially constant stable map $\mathfrak{f} = (\underline{C}/\underline{W}, \mathbf{x}, \underline{f})$. Then $\underline{W} \times_{\mathbf{M}(\underline{X})} \mathscr{M}(X)$ is isomorphic as a stack to the (non-full) subcategory $\mathscr{M}(X, \mathfrak{f}) \subset \mathscr{M}(X)$ with objects stable log maps with underlying ordinary stable map obtained by pull-back from $\mathfrak{f}$, and with morphisms on the underlying schemes induced by the identity on $\mathfrak{f}$. Now the type of a stable log map in $\mathscr{M}(X, \mathfrak{f})$ is locally constant. Hence we have a decomposition into disjoint open substacks

$$(3.2) \qquad\qquad \underline{W} \times_{\mathbf{M}(\underline{X})} \mathscr{M}(X) \simeq \coprod_{\mathbf{u}} \mathscr{M}(X, \mathfrak{f}, \mathbf{u})$$

according to the type $\mathbf{u}$ for $\mathfrak{f}$.

The following lemma shows that provided $\underline{W}$ is reduced, the log structure on the base of a stable log map in $\mathscr{M}(X, \mathfrak{f}, \mathbf{u})$ is locally constant.

Lemma 3.18. *Let W be a reduced scheme with a fine log structure $\mathcal{M}$ such that $\overline{\mathcal{M}}$ is a constant sheaf. Then étale locally, $\mathcal{M} \simeq \mathcal{O}_{\underline{W}}^\times \times \overline{\mathcal{M}}$.*

Proof. Let Q be the stalk of $\overline{\mathcal{M}}$. Then a chart for $\mathcal{M}$ takes the form, étale locally on $\underline{W}$,

$$Q \longrightarrow \mathcal{O}_{\underline{W}} \qquad m \longmapsto \begin{cases} 1 & m = 0; \\ 0 & m \neq 0. \end{cases}$$

Indeed, a chart defining $\underline{W}$ must take every non-zero element of Q to a function which is not invertible at any point of its domain. But as $\underline{W}$ is reduced, the only function which is non-invertible at every point of an open set of $\underline{W}$ is 0. Thus locally $\mathcal{M}$ takes the given form. $\qquad\square$

Proposition 3.19. *Let $\mathfrak{f} = (\underline{C}/\underline{W}, \mathbf{x}, \underline{f})$ be a combinatorially constant ordinary stable map over an integral, qausi-compact scheme $\underline{W}$. Then for any type $\mathbf{u}$ for $\mathfrak{f}$ the stack $\mathscr{M}(X, \mathfrak{f}, \mathbf{u})$ is quasi-compact.*

Proof. Let Q be the basic monoid defined by $\mathbf{u}$. Let $\mathcal{M}_1 := \mathcal{M}_{\underline{W}}^0$ be the basic log structure for the pre-stable curve $\underline{C}/\underline{W}$, $\mathcal{M}_2 := Q \times \mathcal{O}_{\underline{W}}^\times$ the constant log structure and

$$\overline{\varphi} : \overline{\mathcal{M}}_1 = \prod_q \underline{\mathbb{N}} \longrightarrow \underline{Q} = \overline{\mathcal{M}}_2$$

the homomorphism coming from the definition of Q. To take care of the domains of the relevant stable log maps we now look at $\underline{Y} := \mathrm{LMor}_{\underline{W}}^{\overline{\varphi}}$ from Lemma 2.12. By the universal

property of $\mathcal{M}_W^0$ this scheme classifies log smooth structures on $\underline{C_V}/\underline{V}$ with trivialized base log structure $Q \times \mathcal{O}_{\underline{V}}^\times$. Let $C_{\underline{Y}}/\underline{Y}$ be the universal log smooth curve.

It remains to lift the pull-back $\underline{f}_{\underline{Y}}$ of $\underline{f}$ to a log morphism. This is done by $\underline{Z} := \mathrm{LMor}_{\underline{C}_Y/\underline{Y}}^{\overline{\psi}}$, where now the two log structures are $\mathcal{M}_1 := \underline{f}_{\underline{Y}}^* \mathcal{M}_X$ and $\mathcal{M}_2 := \mathcal{M}_{C_{\underline{Y}}}$ on $\underline{C_Y}$. The map $\overline{\psi} : \overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}_2$ is again fixed by basicness. We then obtain a universal basic stable log map $(C_{\underline{Z}}/Z, \mathbf{x}_{\underline{Z}}, f_{\underline{Z}})$. Since $\underline{Z} \to \underline{W}$ is of finite type, $\underline{Z}$ is quasi-compact by quasi-compactness of $\underline{W}$.

Now the morphism $\underline{Z} \to \mathscr{M}(X, \mathfrak{f}, \mathbf{u})$ of algebraic stacks thus obtained is an epimorphism. In fact, by Lemma 3.18 locally on $\underline{W}$ the log structure $\mathcal{M}_W$ of a stable log map $(C/W, \mathbf{x}, f) \in \mathrm{Ob}\big(\mathscr{M}(X, \mathfrak{f}, \mathbf{u})\big)$ is isomorphic to the constant log structure $Q \times \mathcal{O}_{\underline{W}}^\times$, and hence $(C/W, \mathbf{x}, f)$ is locally the image of the pull-back of the universal basic stable log map over $\underline{Z}$. Thus the induced map of space of geometric points $|\underline{Z}| \to |\mathscr{M}(X, \mathfrak{f}, \mathbf{u})|$ is a continuous surjection, and quasi-compactness of $|\mathscr{M}(X, \mathfrak{f}, \mathbf{u})|$ follows from quasi-compactness of $|\underline{Z}|$. $\qquad\square$

Lemma 3.20. *Let $(\underline{C}/\underline{W}, \mathbf{x}, \underline{f})$ be an ordinary stable map with $\underline{W}$ quasi-compact. Then there exist finitely many locally closed, integral subschemes $\underline{W}_i \subset \underline{W}$ weakly covering $\underline{W}$ and étale surjections $\tilde{\underline{W}}_i \to \underline{W}_i$ such that the pull-back of $(\underline{C}/\underline{W}, \mathbf{x}, \underline{f})$ to $\tilde{\underline{W}}_i$ is combinatorially constant.*

Proof. By quasi-compactness it suffices to construct the $\underline{W}_i$ locally. Standard arguments for families of pre-stable curves provide the $\underline{W}_i$ fullfilling Conditions (1) and (2) in Definition 3.17. Then use the fact that $\mathcal{M}$ is a fine sheaf to also achieve (3). $\qquad\square$

We are now in position to prove Theorem 3.14.

Proof of Theorem 3.14. By the discussion at the beginning of this subsection we have to show that if $\underline{W} \to \mathscr{M}(X, \beta)$ is a morphism from a quasi-compact scheme $\underline{W}$ then $\big|\underline{W} \times_{\mathbf{M}(\underline{X})} \mathscr{M}(X, \beta)\big|$ is quasi-compact. By the quasi-compactness criterion Lemma 3.16 together with Lemma 3.20 we can assume that $\underline{W}$ is integral and that the ordinary stable map $\mathfrak{f} = (\underline{C}/\underline{W}, \mathbf{x}, \underline{f})$ defined by $\underline{W} \to \mathbf{M}(\underline{X})$ is combinatorially constant. Let $\mathbf{u}_1, \ldots, \mathbf{u}_s$ be the finite list of types $\mathbf{u}$ of stable log maps in class β according to Definition 3.3. From (3.2) we have the decomposition

$$\big|\underline{W} \times_{\mathbf{M}(\underline{X})} \mathscr{M}(X, \beta)\big| = \coprod_{\mu=1}^s \big|\mathscr{M}(X, \mathfrak{f}, \mathbf{u}_\mu)\big|.$$

Now according to Proposition 3.19 each component on the right-hand side is quasi-compact. Hence $\underline{W} \times_{\mathbf{M}(\underline{X})} \mathscr{M}(X, \beta)$ is quasi-compact, as had to be shown. $\qquad\square$

4. Stable reduction

In this section we prove a stable reduction theorem for basic stable log maps. Throughout, R is a discrete valuation ring over our base scheme $\underline{S}$ with maximal ideal $\mathfrak{m}$, residue field $R/\mathfrak{m} = \kappa$ and quotient field K. We assume K is endowed with a fine, saturated log structure

over $\mathcal{M}_S$, hence defining a log point $(\mathrm{Spec}\, K, Q_K)$ over S for a toric monoid Q_K. The closed point in $\mathrm{Spec}\, R$ is denoted 0.

Theorem 4.1. *Assume that $\underline{X} \to \underline{S}$ is proper. Let $\big(\pi_K : C_K \to (\mathrm{Spec}\, K, Q_K), \mathbf{x}_K, f_K\big)$ be a basic stable log map to X over the log point $(\mathrm{Spec}\, K, Q_K)$. Then possibly after replacing K by a finite extension $\tilde{K}$ and R by its integral closure in $\tilde{K}$ and pulling back the stable log map via $(\mathrm{Spec}\, \tilde{K}, Q_K) \to (\mathrm{Spec}\, K, Q_K)$, the following holds: There exists a log structure $\mathcal{M}_R$ on R over S together with a strict morphism $(\mathrm{Spec}\, K, Q_K) \to (\mathrm{Spec}\, R, \mathcal{M}_R)$, and a basic stable log map $(C/(\mathrm{Spec}\, R, \mathcal{M}_R), \mathbf{x}, f)$ to X such that the restriction to $\mathrm{Spec}\, K$ is isomorphic to $(C_K/(\mathrm{Spec}\, K, Q_K), \mathbf{x}_K, f_K)$.*

Moreover, such an extension as a basic stable log map is unique up to unique isomorphism.

Corollary 4.2. *For any combinatorially finite class β of stable log maps, $\mathscr{M}(X, \beta)$ is proper over $\underline{S}$.*

Proof. Recall that a morphism of algebraic stacks is proper if it is separated, of finite type and universally closed ([LaMB], Definition 7.11). Separatedness follows by the uniqueness part in the theorem from the valuative criterion [LaMB], Proposition 7.8. Theorem 3.14 established that $\mathscr{M}(X, \beta) \to \underline{S}$ is of finite type. In view of the existence part in Theorem 4.1 universal closedness then follows from the valuative criterion [LaMB], Theorem 7.10.   □

The proof of Theorem 4.1 is divided into three steps presented in the following subsections.

4.1. **Extension on the level of ghost sheaves.**

By the stable reduction theorem for ordinary stable maps [FuPa] we may assume $(\underline{C}_K/K, \mathbf{x}_K, \underline{f}_K)$ is the restriction to K of a marked stable map $(\underline{\pi} : \underline{C} \to \mathrm{Spec}\, R, \mathbf{x}, \underline{f})$ over R. This step may involve a base change, but note that any two such extensions are uniquely isomorphic after a joint dominating base change. In particular, the central fibre $(\underline{C}_0/\mathrm{Spec}\, \kappa, \mathbf{x}_0, \underline{f}_0)$ is uniquely determined. The objective in this subsection is to derive the extension as a stable log map on the level of ghost sheaves. In particular, we need to identify the type of the central fibre.

The central fibre being a stable curve over a field we use our usual convention that η, q, p denote generic points, nodes and marked points of $\underline{C}_0$, respectively. For the corresponding objects on $\underline{C}_K$ we use an index K. Thus the sheaf $\underline{f}^*\overline{\mathcal{M}}_X$ is determined by the usual generization maps $\chi_{\eta,p} : P_p \longrightarrow P_\eta$, $\chi_{\eta,q} : P_q \longrightarrow P_\eta$ on $\underline{C}_0$, together with the following generization maps from $\underline{C}_0$ to $\underline{C}_K$:

$$\chi_{\eta,K} : P_\eta \longrightarrow P_{\eta_K}, \quad \chi_{p,K} : P_p \longrightarrow P_{p_K}, \quad \chi_{q,K} : P_q \longrightarrow P_{q_K}.$$

Note that the closures of nodes $q_K \in \underline{C}_K$ may only give a subset of the nodes of $\underline{C}_0$; for the other nodes of $\underline{C}_0$ we only have the composition $\chi_{\eta,K} \circ \chi_{\eta,q} : P_q \to P_{\eta_K}$. Let us refer to the latter as *isolated nodes*.

Proposition 4.3. *There exists a type $\mathbf{u} = (u_x)_{x \in \underline{C}_0}$ for stable log maps with underlying ordinary stable map $(\underline{C}_0/\mathrm{Spec}\, \kappa, \mathbf{x}_0, \underline{f}_0)$ such that the central fibre of any extension to $\mathrm{Spec}\, R$*

of $(C_K/(\operatorname{Spec} K, Q_K), \mathbf{x}_K, f_K)$ as a basic stable log map has type $\mathbf{u}$. Moreover, $\mathbf{u}$ defines the extension on the level of ghost sheaves uniquely up to unique isomorphism.

Proof. Let $(C/(\operatorname{Spec} R, \mathcal{M}_R), \mathbf{x}, f)$ be an extension of $(C_K/(\operatorname{Spec} K, Q_K), \mathbf{x}_K, f_K)$. We are going to show that this extension is determined uniquely on the level of ghost sheaves by data computable without knowing the extension.

By basicness we have a commutative diagram

$$
\begin{array}{ccc}
\prod_\eta P_\eta \times \prod_q \mathbb{N} & \longrightarrow & Q \\
\downarrow & & \downarrow \\
\prod_{\eta_K} P_{\eta_K} \times \prod_{q_K} \mathbb{N} & \longrightarrow & Q_K,
\end{array}
$$

defining the basic monoids Q, Q_K for the central and the generic fibres, respectively. The left-hand vertical arrow in the diagram is defined as the product of $\prod \chi_{\eta,K}$ and the projection to the non-isolated nodes. The horizontal arrows are quotients by subgroups $R_{\mathbf{u}} \subset \prod_\eta P_\eta^{\mathrm{gp}} \times \prod_q \mathbb{Z}$ and $R_{\mathbf{u}_K} \subset \prod_{\eta_K} P_{\eta_K}^{\mathrm{gp}} \times \prod_{q_K} \mathbb{Z}$, determined by the respective types $\mathbf{u}$ and $\mathbf{u}_K$, and followed by saturation, see Construction 1.15.

Recall also that the upper horizontal map together with the types u_p of the marked points determine the maps of ghost sheaves $\underline{\pi}^* \overline{\mathcal{M}}_R \to \overline{\mathcal{M}}_C$, $\underline{f}^* \overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$ on the central fibre, and similarly for the lower horizontal arrow and the generic fibre. Commutativity of the diagram then establishes the generization maps between the relevant sheaves, and hence determines the extension on the level of ghost sheaves. Thus it suffices to find the type $\mathbf{u} = \big((u_p)_p, (u_q)_q\big)$ with $R_{\mathbf{u}}$ mapping to $R_{\mathbf{u}_K}$.

If $p \in \underline{C}_0$ is a marked point there exists a unique marked point $p_K \in \underline{C}_K$ with $p \in \operatorname{cl}(p_K)$. By the structure of log curves (Theorem 1.1,ii) u_p must then equal the composition

$$
P_p \xrightarrow{\chi_{p,K}} P_{p_K} \xrightarrow{u_{p_K}} \mathbb{N},
$$

where the second map is given by the type of the generic fibre. Hence u_p depends only on the generic fibre and $\underline{f}$.

Similarly, for a non-isolated node q let q_K be the node of $\underline{C}_K$ with $q \in \operatorname{cl}(q_K)$. Then the defining equation (1.8) for u_q,

$$
\varphi_{\overline{\eta}_2}\big(\chi_{\eta_2,q}(m)\big) - \varphi_{\overline{\eta}_1}\big(\chi_{\eta_1,q}(m)\big) = u_q(m) \cdot \rho_q,
$$

with $\varphi := \overline{f^\flat}$ and $m \in P_q$, generizes to the defining equation for u_{q_K}:

$$
\varphi_{\overline{\eta}_{2,K}}\big(\chi_{\eta_{2,K},q_K}(m_K)\big) - \varphi_{\overline{\eta}_{1,K}}\big(\chi_{\eta_{1,K},q_K}(m_K)\big) = u_{q_K}(m_K) \cdot \rho_{q_K},
$$

where $m_K := \chi_{q,K}(m)$. Hence $u_q = u_{q_K} \circ \chi_{q,K}$ is also determined a priori by the generic fibre.

So far we have just reversed the reasoning in the proof of Proposition 1.21. The isolated nodes, however, require different arguments. Recall that u_q was defined by the homomorphism

$$
\varphi_{\overline{q}} : P_q = \overline{\mathcal{M}}_{X,\underline{f}(q)} \longrightarrow \overline{\mathcal{M}}_{C,\overline{q}} = Q \oplus_{\mathbb{N}} \mathbb{N}^2, \quad m \longmapsto \big(m', (a, b)\big)
$$

as $u_q(m) = b - a$ (for the correct orientation). Now while the composition with generization to η_K would map any $(0, (a, b))$ to $0 \in Q_K$, we can retrieve $b - a$ by working on the level

of log structures. In fact, choosing a compatible chart on an étale neighbourhood of q, the generization map has the form

$$(Q \oplus_{\mathbb{N}} \mathbb{N}^2) \times \mathcal{O}_{\underline{C},\overline{q}}^{\times} \simeq \mathcal{M}_{C,\overline{q}} \longrightarrow \mathcal{M}_{C,\overline{\eta}_K} \simeq Q_K \times \mathcal{O}_{\underline{C},\overline{\eta}_K}^{\times}$$
$$((m', (a, b)), h) \longmapsto (m'_K, z^a w^b \underline{\pi}^* g(m') h),$$

with $z = 0$, $w = 0$ defining the two branches of $\underline{C}_0$ inside $\underline{C}$ at q, and $g(m') \in K^{\times}$. Assume that $\underline{C}$ has an A_{e-1}-singularity at q. Then letting $\mathrm{ord}_{\eta_1}, \mathrm{ord}_{\eta_2}$ be the discrete valuations of these branches it holds $\mathrm{ord}_{\eta_1}(z) = \mathrm{ord}_{\eta_2}(w) = e$, and hence $e \cdot (b - a) = (\mathrm{ord}_{\eta_2} - \mathrm{ord}_{\eta_1})(z^a w^b h)$ and $\mathrm{ord}_{\eta_1}(\underline{\pi}^*(g(m'))) = \mathrm{ord}_{\eta_2}(\underline{\pi}^*(g(m')))$. Thus $e \cdot u_q$ equals the following composition:

$$(4.1) \qquad P_q \longrightarrow \mathcal{M}_{X,\underline{f}(q)} \xrightarrow{f^{\flat}} \mathcal{M}_{C,\overline{q}} \longrightarrow \mathcal{M}_{C,\overline{\eta}_K} \simeq Q_K \times \mathcal{O}_{\underline{C},\overline{\eta}_K}^{\times} \xrightarrow{\mathrm{pr}_2} \mathcal{O}_{\underline{C},\overline{\eta}_K}^{\times} \xrightarrow{\mathrm{ord}_{\eta_2} - \mathrm{ord}_{\eta_1}} \mathbb{Z}.$$

The first map is a choice of chart for $\underline{f}^* \mathcal{M}_X$ at q. Finally observe that the composition $\mathcal{M}_{X,\underline{f}(q)} \to \mathcal{M}_{C,\overline{q}} \to \mathcal{M}_{C,\overline{\eta}_K}$ equals $\mathcal{M}_{X,\underline{f}(q)} \to \mathcal{M}_{X,\underline{f}(\eta_K)} \xrightarrow{f_K^{\flat}} \mathcal{M}_{C,\overline{\eta}_K}$. Hence u_q is completely determined also for an isolated node q by the generic fibre and by the extension $\underline{f}$ of $\underline{f}_K$, already known to be unique up to base change. $\qquad \square$

4.2. The log structure on the base.

We now want to show that the extension of the log structure on the base is uniquely defined. The decisive tool is the identification of "fibrewise constant" subsheaves $\mathcal{M}(\eta)$ of $\underline{f}^* \mathcal{M}_X$ as follows. For a generic point $\eta \in \underline{C}_0$ let $U(\eta) \subset \underline{C}$ be the set of generizations in $\underline{C}$ of non-special points in the closure of η in $\underline{C}_0$. Said differently, to construct $U(\eta)$, take the closure of η in $\underline{C}$ and subtract the set of special points of $\underline{C}$ and the irreducible components of $\underline{C}_0$ not containing η. Then $U(\eta)$ is a smooth open neighbourhood of η and $\underline{f}^* \overline{\mathcal{M}}_X|_{U(\eta)}$ has only two interesting stalks, at η and at its generization η_K, see Remark 1.8. Hence $\underline{f}^* \overline{\mathcal{M}}_X|_{U(\eta)}$ is completely determined by the generization map $P_\eta \to P_{\eta_K}$. In particular, $\underline{f}^* \overline{\mathcal{M}}_X|_{U(\eta)}$ is globally generated with global section space equal to P_η. Note also that each $x \in U(\eta)$ is in the closure of η_K, so we may restrict germs of sections to η_K. The notation for the restriction of m is $m|_{\eta_K}$.

Definition 4.4. The *fibrewise constant subsheaf* $\mathcal{M}(\eta) \subset \underline{f}^* \mathcal{M}_X|_{U(\eta)}$ is the sheaf in the Zariski topology defined by the condition

$$m \in \mathcal{M}(\eta)_x \quad :\Longleftrightarrow \quad f_K^{\flat}(m|_{\eta_K}) \in \mathrm{im}\left(\Gamma(\mathcal{M}_K) \xrightarrow{\pi_K^{\flat}} \mathcal{M}_{C,\overline{\eta}_K}\right). \qquad \square$$

Since $C_K \to (\mathrm{Spec}\, K, Q_K)$ is strict at $\overline{\eta}_K$, the map $\pi_K^{\flat}$ is injective and there is a canonical map

$$\beta_\eta : \mathcal{M}(\eta)_\eta \longrightarrow \Gamma(\mathcal{M}_K), \quad m \longmapsto (\pi_K^{\flat})^{-1}\left(f_K^{\flat}(m|_{\eta_K})\right).$$

Note the composition of β_η with the structure homomorphism $\Gamma(\mathcal{M}_K) \to K$ factors over a map

$$\alpha_\eta : \mathcal{M}(\eta)_\eta \longrightarrow R.$$

In fact, if $m \in \mathcal{M}(\eta)_\eta$ maps to $a \in K$ then $\pi^*(a)$ is in the image of the structure homomorphism $\mathcal{M}(\eta)_\eta \to \mathcal{O}_{\underline{C},\eta}$ and hence a is regular at 0.

Calling $\mathcal{M}(\eta)$ fibrewise constant is justified by the fact that the restrictions of $\mathcal{M}(\eta)$ to $\underline{C}_K \cap U(\eta)$ and to $\underline{C}_0 \cap U(\eta)$ are constant sheaves with fibres isomorphic to $P_{\eta_K} \times K^\times$ and to $P_\eta \times R^\times$, respectively. For our purposes only the stalk $\mathcal{M}(\eta)_\eta$ is really relevant and it suffices to prove the following weaker statement.

Lemma 4.5. $\mathcal{M}(\eta)_\eta \simeq P_\eta \times R^\times$.

Proof. By strictness of $C_K \to (\operatorname{Spec} K, Q_K)$ at $\overline{\eta}_K$ there exists an isomorphism $\mathcal{M}_{C,\overline{\eta}_K} \simeq Q_K \times \mathcal{O}^\times_{\underline{C},\overline{\eta}_K}$ such that $\pi^\flat_K(\Gamma(\mathcal{M}_K)) = Q_K \times K^\times$. Let $\theta : P_\eta \to \mathcal{M}_{X,\underline{f}(\eta)}$ be a chart, hence inducing an isomorphism $(f^*\mathcal{M}_X)_\eta \simeq P_\eta \times \mathcal{O}^\times_{\underline{C},\eta}$. Note that $\mathcal{O}^\times_{\underline{C},\overline{\eta}_K} \simeq \mathbb{Z} \times \mathcal{O}^\times_{\underline{C},\overline{\eta}}$ since R is a discrete valuation ring. We can thus define $g : P_\eta \to \mathcal{O}^\times_{\underline{C},\overline{\eta}}$ by the composition

$$P_\eta \xrightarrow{\theta} \mathcal{M}_{X,\underline{f}(\eta)} \xrightarrow{f^\flat(\,\cdot\,|_{\eta_K})} \mathcal{M}_{C_K,\overline{\eta}_K} \simeq Q_K \times \mathcal{O}^\times_{\underline{C},\overline{\eta}_K} \xrightarrow{\operatorname{pr}_2} \mathcal{O}^\times_{\underline{C},\overline{\eta}_K} \simeq \mathbb{Z} \times \mathcal{O}^\times_{\underline{C},\overline{\eta}} \xrightarrow{\operatorname{pr}_2} \mathcal{O}^\times_{\underline{C},\overline{\eta}}.$$

Note that $g(m)$ measures the defect of $\theta(m)$ to be fibrewise constant. Thus

$$g^{-1} \cdot \theta : P_\eta \longrightarrow \mathcal{M}_{X,\underline{f}(\eta)}$$

factors over $\mathcal{M}(\eta)_\eta$. From the analogous property of θ the constructed homomorphism $P_\eta \to \mathcal{M}(\eta)_\eta$ is right-inverse to

$$\mathcal{M}(\eta)_\eta \to \mathcal{M}_{X,\underline{f}(\eta)} \to \overline{\mathcal{M}}_{X,\underline{f}(\eta)} = P_\eta, \quad m \longmapsto \overline{m}.$$

To prove the conclusion it remains to show that if $m_1, m_2 \in \mathcal{M}(\eta)_\eta$ fulfill $\overline{m}_1 = \overline{m}_2$ there exists a unique $h \in R^\times$ with $m_2 = h \cdot m_1$. By the definition of the ghost sheaf there certainly exists $h \in \mathcal{O}^\times_{\underline{C},\eta}$ such that $m_2 = h \cdot m_1$ holds in $\mathcal{M}_{X,\underline{f}(\eta)}$. But then also $f^\flat_K(m_2|_{\eta_K}) = h \cdot f^\flat_K(m_1|_{\eta_K})$ holds in $\mathcal{M}_{C,\overline{\eta}_K}$. Since both m_1, m_2 are fibrewise constant it follows $h \in K^\times$. Hence $h \in K^\times \cap \mathcal{O}^\times_{\underline{C},\eta} = R^\times$ as claimed. $\qquad\square$

To motivate our construction of the log structure on R note that given $\mathcal{M}_R$, the product

(4.2) $$\mathcal{M}_{R,0} \longrightarrow Q \times_{Q_K} \Gamma(\mathcal{M}_K),$$

of the quotient $\mathcal{M}_{R,0} \to \overline{\mathcal{M}}_{R,0} = Q$ and the generization map exhibits $\mathcal{M}_{R,0}$ canonically as a submonoid of $Q \times_{Q_K} \Gamma(\mathcal{M}_K)$. The point is that we can determine this image without knowledge of $\mathcal{M}_{R,0}$. To this end let $\alpha_0 : \mathcal{M}^0_R \to \mathcal{O}_{\operatorname{Spec} R}$ be the basic log structure on $\operatorname{Spec} R$ for the pre-stable curve $(\underline{C}/\operatorname{Spec} R, \mathbf{x})$. The universal property of this log structure produces a homomorphism

$$\beta_0 : \mathcal{M}^0_{R,0} \longrightarrow \mathcal{M}^0_{R,(0)} \longrightarrow \Gamma(\mathcal{M}_K).$$

Here $(0) \in \operatorname{Spec} R$ denotes the generic point. From the definition of Q there are also canonical homomorphisms

$$\psi_\eta : \mathcal{M}(\eta)_\eta \longrightarrow P_\eta \longrightarrow Q, \quad \psi_0 : \mathcal{M}^0_{R,0} \longrightarrow \prod_q \mathbb{N} \longrightarrow Q.$$

Define $\beta := (\prod_\eta \beta_\eta) \cdot \beta_0$ and $\psi := (\sum_\eta \psi_\eta) + \psi_0$. Clearly, $\psi_\eta \times \beta_\eta$ and $\psi_0 \times \beta_0$ have image in $Q \times_{Q_K} \Gamma(\mathcal{M}_K) \subset Q \times \Gamma(\mathcal{M}_K)$. Thus $\psi \times \beta$ maps $(\prod_\eta \mathcal{M}(\eta)_\eta) \times \mathcal{M}_{R,0}^0$ to $Q \times_{Q_K} \Gamma(\mathcal{M}_K)$. Now define $\mathcal{M}_R$ by the stalk

$$(4.3) \qquad \mathcal{M}_{R,0} := \mathrm{im}(\psi \times \beta)^{\mathrm{sat}} \subset Q \times_{Q_K} \Gamma(\mathcal{M}_K)$$

at 0 and by the projection to $\Gamma(\mathcal{M}_K)$ as the generization map to the generic point. For the structure homomorphism $\alpha_{R,0} : \mathcal{M}_{R,0} \to R$ to be compatible with generization it must be taken as composition

$$\mathcal{M}_{R,0} \longrightarrow Q \times_{Q_K} \Gamma(\mathcal{M}_K) \xrightarrow{\mathrm{pr}_2} \Gamma(\mathcal{M}_K) \longrightarrow K,$$

which is compatible with $(\prod_\eta \alpha_\eta) \times \alpha_0$ and hence has image in R.

Here is the key technical result for establishing that $\mathcal{M}_R$ has the requested properties.

Lemma 4.6. $\beta^{\mathrm{gp}}\big(\ker(\psi^{\mathrm{gp}})\big) = R^\times$, where $R^\times \subset \Gamma(\mathcal{M}_K)$ is canonically embedded via the inverse of the structure homomorphism $\Gamma(\mathcal{M}_K) \to K$.

Proof. Each β_η as well as β_0 are equivariant with respect to the multiplication action of $R^\times$, while ψ_η and ψ_0 are invariant under this action. Hence it is enough to show $\beta^{\mathrm{gp}}(\ker(\psi^{\mathrm{gp}})) \subset R^\times$.

An element $m = \big((m_\eta)_\eta, m_0\big) \in \big(\prod_\eta \mathcal{M}(\eta)_\eta^{\mathrm{gp}}\big) \times (\mathcal{M}_{R,0}^0)^{\mathrm{gp}}$ lies in $\ker \psi^{\mathrm{gp}}$ iff $\overline{m} \in \prod_\eta P_\eta^{\mathrm{gp}} \times \prod_q \mathbb{Z}$ lies in the relation subgroup $R_{\mathbf{u}}$ defining Q. In particular, $\psi(\overline{m})$ generizes to $0 \in Q_K$, and hence $\beta^{\mathrm{gp}}(\ker(\psi^{\mathrm{gp}})) \subset K^\times$. It also suffices to check the statement for $\overline{m}$ equal to one of the generators $a_q(\overline{m}_q)$, $\overline{m}_q \in P_q$, of $R_{\mathbf{u}}$, as defined in Construction 1.15. Thus we have to show that if $m = \big((m_\eta)_\eta, m_0\big) \in \big(\prod_\eta \mathcal{M}(\eta)_\eta^{\mathrm{gp}}\big) \times (\mathcal{M}_{R,0}^0)^{\mathrm{gp}}$ maps to some $a_q(\overline{m}_q)$ then $\mathrm{ord}_0(\beta^{\mathrm{gp}}(m)) = 0$, where ord_0 is the discrete valuation of K. Note from the definition of $a_q(\overline{m}_q)$ that $\beta_\eta(m_\eta) \in R^\times$ except possibly for $\eta = \eta_1, \eta_2$, the generic points of the branches of $\underline{C}_0$ at q. Similarly, all entries of $\overline{m}_0 \in \prod_q \mathbb{Z}$ vanish except at entry q, which equals $u_q(\overline{m}_q)$. By working on an étale neighbourhood of q we may assume $\underline{C} \to \mathrm{Spec}\, R$ is locally given by $\mathrm{Spec}\big(R[z,w]/(zw - t_q)\big)$ with $t_q \in R$. Let $m_q \in \mathcal{M}_{X,\underline{f}(q)}$ be a lift of $\overline{m}_q$.

There are two cases, depending on q being an isolated node or not. Let us first assume q is isolated. Then $t_q \neq 0$ and $\beta_0(m_0) = h \cdot t_q^{u_q(\overline{m}_q)}$ for some $h \in R^\times$. Denote by $\eta_K \in \underline{C}_K$ the generic point with $\eta_1, \eta_2 \in \mathrm{cl}(\eta_K)$ as before. Let $\mathcal{M}_{C_K, \overline{\eta}_K} \simeq Q_K \times \mathcal{O}_{\underline{C}, \overline{\eta}_K}^\times$ be an isomorphism induced by a distinguished chart as in Theorem 1.1. In particular, $\pi_K^\flat : \Gamma(\mathcal{M}_K) \to \mathcal{M}_{C_K, \overline{\eta}_K}$ is then given by the natural inclusion $Q_K \times K^\times \to Q_K \times \mathcal{O}_{\underline{C}_K, \overline{\eta}_K}^\times$. As in (4.1) consider the composition

$$\kappa_q : \mathcal{M}_{X, \underline{f}(q)} \longrightarrow \mathcal{M}_{X, \underline{f}(\eta_K)} \xrightarrow{f_K^\flat} \mathcal{M}_{C_K, \overline{\eta}_K} \simeq Q_K \times \mathcal{O}_{\underline{C}, \overline{\eta}_K}^\times \xrightarrow{\mathrm{pr}_2} \mathcal{O}_{\underline{C}, \overline{\eta}_K}^\times.$$

Since $\overline{m}_q$ generizes to $\overline{m}_{\eta_1}$ and to $-\overline{m}_{\eta_2}$, respectively, the generizations of m_q at η_1, η_2 differ from m_{η_1} and $m_{\eta_2}^{-1}$ only by functions invertible at η_i. Therefore

$$\begin{aligned} \mathrm{ord}_0(\beta(m)) &= \mathrm{ord}_0\big(\beta_{\eta_1}(m_{\eta_1}) \cdot \beta_{\eta_2}(m_{\eta_2}) \cdot \beta_0(m_0)\big) \\ &= \mathrm{ord}_{\eta_1}\big(\kappa_q(m_q)\big) - \mathrm{ord}_{\eta_2}\big(\kappa_q(m_q)\big) + \mathrm{ord}_0(t_q^{u_q(\overline{m}_q)}). \end{aligned}$$

Now by the discussion of the map (4.1) at the end of §4.1 we know

$$e \cdot u_q(\overline{m}_q) = \operatorname{ord}_{\eta_2}\big(\kappa_q(m_q)\big) - \operatorname{ord}_{\eta_1}\big(\kappa_q(m_q)\big),$$

and $e = \operatorname{ord}_0(t_q)$. Thus $\operatorname{ord}_0(\beta(m)) = 0$ as claimed.

For a non-isolated node q let q_K be the node of $\underline{C}_K$ with $q \in \operatorname{cl}(q_K)$. Now $t_q = 0$ and a distinguished chart for C_K at q_K takes the form

$$\mathcal{M}_{C_K,\overline{q}_K} \simeq (Q_K \oplus_{\mathbb{N}} \mathbb{N}^2) \times \mathcal{O}^{\times}_{\underline{C}_K,\overline{q}_K}.$$

Again denote by $\kappa_q : \mathcal{M}_{X,\underline{f}(q)} \to \mathcal{O}^{\times}_{\underline{C}_K,\overline{q}_K}$ the composition $f^{\flat}(.\,|_{q_K})$ with the projection induced by the chart, and similarly let $\kappa_K : \Gamma(\mathcal{M}_K) = Q \times K^{\times} \to K^{\times}$ be the projection. Then

(4.4) $$\pm \operatorname{ord}_0\big(\kappa_K(\beta_{\eta_i}(m_{\eta_i}))\big) = \operatorname{ord}_{\eta_i}\big(\kappa_q(m_q)\big) = \operatorname{ord}_{\eta_i}(h),$$

with $h := \kappa_q(m_q) \in \mathcal{O}^{\times}_{\underline{C}_K,\overline{q}_K}$ and opposite signs for $i = 1, 2$. Now h is invertible at q_K and, since κ_q factors over a chart for $\mathcal{M}_{C_K}$, the zero and polar locus of h is contained in $\underline{C}_0$. Hence we can write $h = g_q g_K$ with $g_q \in \mathcal{O}^{\times}_{\underline{C},\overline{q}}$, $g_K \in K^{\times}$, and thus $\operatorname{ord}_{\eta_1}(h) = \operatorname{ord}_0(g_K) = \operatorname{ord}_{\eta_2}(h)$. This shows

$$\kappa_K(\beta_{\eta_1}(m_1) \cdot \beta_{\eta_2}(m_2)) \in R^{\times}.$$

As for m_0 note that there is a lift to $\mathcal{M}^0_{R,0}$ of the q-th generator of $\overline{\mathcal{M}}^0_{R,0} = \prod_q \mathbb{N}$ mapping to $(0,(1,1),1) \in (Q_K \oplus_{\mathbb{N}} \mathbb{N}^2) \times \mathcal{O}^{\times}_{\underline{C}_K,\overline{q}_K}$ in our chart. Hence in our chart $\pi_K^{\flat}\big(\beta_0(m_0)\big)$ takes the form $\big((0,(a,a)),h\big)$ for $a = u_q(\overline{m}_q) \in \mathbb{N} \setminus \{0\}$ and $h \in \mathcal{O}^{\times}_{\underline{C},\overline{q}}$. Therefore also $\kappa_K(\beta_0(m_0)) \in R^{\times}$. Taken together we obtain $\beta(m) \in R^{\times}$ as claimed. □

Proposition 4.7. *The extension* $\alpha : \mathcal{M}_R \to \mathcal{O}_{\operatorname{Spec} R}$ *of* $\mathcal{M}_K$ *defined by* $\mathcal{M}_{R,0} \subset Q \times_{Q_K} \Gamma(\mathcal{M}_K)$ *is a log structure on* R *with* $\overline{\mathcal{M}}_{R,0} = Q$. *Moreover, if* $\mathcal{M}'_R$ *is the log structure on* $\operatorname{Spec} R$ *for an extension of* $(C_K/(\operatorname{Spec} K, Q_K), \mathbf{x}_K, f_K)$ *to a basic stable log map over* R, *then the image of* $\mathcal{M}'_{R,0}$ *under the canonical map to* $Q \times_{Q_K} \Gamma(\mathcal{M}_K)$ *equals* $\mathcal{M}_{R,0}$.

Proof. By Lemma 4.5 the maps $\mathcal{M}(\eta)_\eta \to \overline{\mathcal{M}}_{X,\underline{f}(\eta)} = P_\eta$ are surjective, as is $\mathcal{M}^0_{R,0} \to \prod_q \mathbb{N}$. Hence $\psi : \big(\prod_\eta \mathcal{M}(\eta)_\eta\big) \times \mathcal{M}^0_{R,0} \longrightarrow Q$ is surjective up to saturation by the definition of Q. Since $\mathcal{M}_{R,0} = \operatorname{im}(\psi \times \beta)^{\operatorname{sat}}$ the projection to the first factor

$$\kappa : \mathcal{M}_{R,0} \longrightarrow Q \times_{Q_K} \Gamma(\mathcal{M}_K) \longrightarrow Q$$

is also surjective. Since κ is also $R^{\times}$-invariant we obtain a surjection $\mathcal{M}_{R,0}/R^{\times} \to Q$. Conversely, if $m \in \mathcal{M}_{R,0}$ maps to a non-zero element in Q then $\alpha_{R,0}(m) \in R$ is not invertible. Indeed, let $\big((m_\eta), m_0\big) \in \prod_\eta \mathcal{M}(\eta)_\eta \times \mathcal{M}^0_{R,0}$ be a lift of (a power of) m. Then by the definition of ψ we must have $\psi_0(m_0) \neq 0$ or $\psi_\eta(m_\eta) \neq 0$ for some η. But since α_η and α_0 are induced by charts of log structures, one of $\alpha_0(m_0)$ or $\alpha_\eta(m_\eta)$ is not invertible. Hence also $\alpha_{R,0}(m)$ is not invertible. This shows $\alpha_{R,0}^{-1}(R^{\times}) \subset \mathcal{M}_{R,0} \cap \big(\{0\} \times \Gamma(\mathcal{M}_K)\big)$. Thus Lemma 4.6 says that $\alpha_{R,0}$ maps $\alpha_{R,0}^{-1}(R^{\times})$ isomorphically to $R^{\times} \subset K^{\times}$. Taken together this shows that $\mathcal{M}_R \to \mathcal{O}_R$ is a log structure on R with $\overline{\mathcal{M}}_{R,0} = Q$.

For the uniqueness statement it suffices to produce a factorization of $\psi \times \beta$ over $\mathcal{M}'_{R,0}$:

$$\left(\prod_\eta \mathcal{M}(\eta)_\eta\right) \times \mathcal{M}^0_{R,0} \longrightarrow \mathcal{M}'_{R,0} \longrightarrow Q \times_{Q_K} \Gamma(\mathcal{M}_K).$$

In fact, then $\mathcal{M}_{R,0} \subset \mathrm{im}\left(\mathcal{M}'_{R,0} \to Q \times_{Q_K} \Gamma(\mathcal{M}_K)\right)$ and equality follows by the equality of ghost sheaves via basicness. The factorization is immediate on $\mathcal{M}^0_{R,0}$ by the universal property of this log structure. For $\mathcal{M}(\eta)_\eta$ recall first that $\Gamma(U(\eta), \underline{f}^* \overline{\mathcal{M}}_X) = P_\eta$. Thus for each $\overline{m} \in P_\eta$ we obtain an $\mathcal{O}^\times_{U(\eta)}$-torsor $\mathcal{L}_{\overline{m}} \subset (\underline{f}^* \mathcal{M}_X)|_{U(\eta)}$. Then the extension $f^\flat$ of $f^\flat_K$ induces an isomorphism of $\mathcal{L}_{\overline{m}}$ with an $\mathcal{O}^\times_{U(\eta)}$-torsor in $\mathcal{M}'_C|_{U(\eta)}$. But $(\underline{C}, \mathcal{M}'_C) \to (\mathrm{Spec}\, R, \mathcal{M}'_R)$ is strict on $U(\eta)$, and hence this torsor is trivial and equal to the pull-back of an $\mathcal{O}^\times_R$-torsor in $\mathcal{M}'_R$. Moreover, if $m \in \Gamma(U(\eta), \mathcal{L}_{\overline{m}})$ maps to the pull-back of a section of $\mathcal{M}'_R$ over $\mathrm{Spec}\, K$ then there exists a unique $m' \in \Gamma(\mathcal{M}'_R)$ with $f^\flat(m) = \pi^\flat(m')$. The map $m \mapsto m'$ defines the desired factorization $\mathcal{M}(\eta)_\eta \to \mathcal{M}'_{R,0} \to Q \times_{Q_K} \Gamma(\mathcal{M}_K)$. $\qquad\square$

4.3. Extension of the log morphism.

Our log structure $\mathcal{M}_R$ comes with a morphism $\mathcal{M}^0_R \to \mathcal{M}_R$, which by the universal property of $\mathcal{M}^0_R$ extends the given structure of a log smooth curve on $\underline{C}_K \to \mathrm{Spec}\, K$ to $\underline{C} \to \mathrm{Spec}\, R$, in the category (Log/S). Moreover, by uniqueness of $\mathcal{M}_{R,0} \subset Q \times_{Q_K} \Gamma(\mathcal{M}_K)$ this extension is uniquely isomorphic to the log structure on the domain $\underline{C}$ of any extension of $(C_K/(\mathrm{Spec}\, K, Q_K), \mathbf{x}_K, f_K)$ as a basic stable log map. Denote by $\varphi : \underline{f}^* \overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$ the extension of $\overline{f^\flat_K}$ constructed in §4.1.

Now if $U \subset \underline{C}$ is an open subset and $\overline{m} \in \Gamma(U, \underline{f}^* \overline{\mathcal{M}}_X)$ we obtain two $\mathcal{O}^\times_U$-torsors in $\underline{f}^* \mathcal{M}_X|_U$ and in $\mathcal{M}_C|_U$, respectively. If $\mathcal{L}_{\overline{m}}$ and $\mathcal{L}_{\varphi(\overline{m})}$ are the corresponding line bundles the question is if the isomorphism

$$f^\flat_K|_{\mathcal{L}_{\overline{m}}} : \mathcal{L}_{\overline{m}}|_{\underline{C}_K \cap U} \longrightarrow \mathcal{L}_{\varphi(\overline{m})}|_{\underline{C}_K \cap U}$$

extends to U, thus defining $f^\flat|_{\mathcal{L}_{\overline{m}}}$ uniquely. Note that since $\underline{C}$ is Cohen-Macaulay any such extension is unique, and an extension exists iff it exists in codimension one, that is, at the generic points $\eta \in \underline{C}_0$. But if $m \in \mathcal{M}_{X, \underline{f}(\eta)}$ there exists $h \in \mathcal{O}^\times_{\underline{C}, \eta}$ with $m' := h \cdot m \in \mathcal{M}(\eta)_\eta$. Moreover, $\mathcal{M}_{C, \overline{\eta}} = (\underline{\pi}^* \mathcal{M}_{R,0})_{\overline{\eta}}$. Using the definition of $\mathcal{M}_{R,0}$ we therefore see that

$$f^\flat_K(m) = h^{-1} \cdot f^\flat_K(m') = h^{-1} \cdot \pi^\flat_K\left((\psi_\eta \times \beta_\eta)(m')\right)$$

extends over η as $h^{-1} \cdot \pi^\flat(m'')$ with $m'' := (\psi_\eta \times \beta_\eta)(m') \in \mathcal{M}_{R,0}$. This proves the unique existence of an extension f of f_K to $\underline{C}$, the last step in our proof of Theorem 4.1. $\qquad\square$

5. Log Gromov-Witten invariants

We are now in position to define log Gromov-Witten invariants by constructing a virtual fundamental class on $\mathscr{M}(X)$. This is quite standard by now. We follow the method of Behrend and Fantechi [Be],[BeFa], adapted by Kim to the log setting [Ki] using Olsson's log cotangent complex [Ol3]. A low tech approach based on a global version of Artin's obstruction theory [Ar], close to the original approach of Li and Tian [LiTi] and avoiding cotangent complexes completely is also possible [Si1],[Si3], but this would be less economic.

We now require $X \to S$ to be log smooth and quasi-projective. Let $\mathscr{U} \to \mathscr{M}$ be the universal curve over the log stack of (not necessarily basic) pre-stable curves $\mathscr{M}$. Let $\mathscr{M}(X) = \mathscr{M}(X/S)$ be the stack of basic stable log maps over S,

$$\pi : \mathscr{V} := \mathscr{M}(X) \times_{\mathscr{M}} \mathscr{U} \longrightarrow \mathscr{M}(X)$$

the universal curve and

$$f : \mathscr{V} \longrightarrow X$$

the evaluation morphism, both considered as 1-morphisms of algebraic log stacks. There is a composition of natural maps

$$Lf^* L^{\bullet}_{X/S} \longrightarrow L^{\bullet}_{\mathscr{V}/S} \longrightarrow L^{\bullet}_{\mathscr{V}/\mathscr{U}} \simeq L\pi^* L^{\bullet}_{\mathscr{M}(X)/\mathscr{M}}.$$

Note this is a sequence of log cotangent complexes, which depend on the log structures. The isomorphism is due to compatibility with flat base base change, see [Ol3], (1.1 (iv)).[11] Tensoring with the relative dualizing sheaf ω_π and using the fact that $L\pi^! L^{\bullet}_{\mathscr{M}(X)/\mathscr{M}} \simeq L\pi^* L^{\bullet}_{\mathscr{M}(X)/\mathscr{M}} \overset{L}{\otimes} \omega_\pi$, adjunction now defines a morphism

$$R\pi_*(Lf^* L^{\bullet}_{X/S} \overset{L}{\otimes} \omega_\pi) \longrightarrow L^{\bullet}_{\mathscr{M}(X)/\mathscr{M}}.$$

By log smoothness $L^{\bullet}_{X/S} = [\Omega^1_{X/S}]$ is represented by a locally free sheaf. Hence by duality, the left-hand side equals

$$E^{\bullet} := \left(R\pi_*[f^* \Theta_{X/S}] \right)^{\vee}.$$

According to [Be], Proposition 5, $E^{\bullet}$ can be represented by a two-term complex of locally free sheaves, at least assuming quasi-compactness. Moreover, by the strictness of $\mathscr{M}(X) \to \mathscr{M}$ the log cotangent complex $L^{\bullet}_{\mathscr{M}(X)/\mathscr{M}}$ equals the ordinary cotangent complex $L^{\bullet}_{\underline{\mathscr{M}}(X)/\underline{\mathscr{M}}}$, where as usual the underlining means forgetting the log structures, that is, taking the underlying ordinary stacks. Thus by [Ol3], Theorem 5.9, the morphism

$$E^{\bullet} \longrightarrow L^{\bullet}_{\mathscr{M}(X)/\mathscr{M}} = L^{\bullet}_{\underline{\mathscr{M}}(X)/\underline{\mathscr{M}}}$$

defines a resolution of a perfect obstruction theory in the sense of [BeFa], Definition 4.4 (see the criterion of [BeFa], Theorem 4.5,3). Thus by [BeFa], §5, we now have a *virtual fundamental class* $[\![\mathscr{M}(X/S)]\!]$, as a well-defined Chow class on $\mathscr{M}(X/S)$.

6. EXAMPLES

Example 6.1. Let $\underline{X}$ be a non-singular variety and $D \subset \underline{X}$ a smooth divisor, and define X as the log scheme with the divisorial log structure $\mathcal{M}_{(X,D)}$. This is the case of relative Gromov-Witten invariants considered by [LiRu], [IoPa], [Ga], [Li1].

Consider first the case where $\underline{C}$ is a smooth curve and $\underline{f} : \underline{C} \to \underline{X}$ is an ordinary stable map such that $\underline{f}^{-1}(D)$ consists of a finite number of points. We would like to understand

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[11]This compatibility is stated in loc.cit. only for arbitrary base change applied to a log flat morphism, but mutatis mutandis the proof goes through for the analogous case of a log flat base change of an arbitrary morphism. We thank M. Olsson for confirming this point.

when this can be lifted to a stable log map. We know that if $\underline{f}$ lifts to a stable log map, then $\underline{f}^*\overline{\mathcal{M}}_X$ can only jump at marked and double points, by Remark 1.8, and hence every point of $\underline{f}^{-1}(D)$ must be marked. So we should consider a situation

$$\underline{f} : \big(\underline{C}, (x_1, \ldots, x_d, y_1, \ldots, y_p)\big) \longrightarrow \underline{X}$$

with $\underline{f}^{-1}(D) = \{x_1, \ldots, x_d\}$.

What is the possible type of a log lifting? Since $\underline{C}$ has no double points, the only relevant information is the choice of $u_{x_i} \in \mathbb{N}^\vee = \mathbb{N}$, where $\mathbb{N} = \overline{\mathcal{M}}_{X, f(x_i)}$. (The u_{y_i}'s are necessarily zero as they lie in the zero monoid.) Furthermore, $P_\eta = 0$ for η the generic point of $\underline{C}$, so once the type is chosen, necessarily $Q = 0$. Thus $\mathcal{M}_C$ is just the divisorial log structure on $\underline{C}$ associated with the divisor $x_1 + \cdots + y_p$.

The only constraint on the choice of u_{x_i} comes from the balancing condition. More precisely, note that the torsor coming from $n \in \Gamma(\underline{X}, \overline{\mathcal{M}}_X) = \mathbb{N}$ corresponds to the line bundle $\mathcal{O}_{\underline{X}}(-nD)$. Thus in the notation of (1.10), $\tau_\eta^X : \Gamma(\underline{C}, \underline{f}^*\overline{\mathcal{M}}_X) = \mathbb{N}^d \to \mathbb{Z}$ is given by $\tau_\eta^X(n_1, \ldots, n_d) = -\sum \mu_i n_i$, where $\mu_i > 0$ is the order of tangency of D with $\underline{C}$ at the point x_i. On the other hand, the map $\tau_\eta^C : \Gamma(\underline{C}, \overline{\mathcal{M}}_C) = \mathbb{N}^{d+p} \to \mathbb{Z}$ is given by $\tau_\eta^C(n_1, \ldots, n_d, m_1, \ldots, m_p) = -\sum n_i - \sum m_j$. Since the map $\varphi : \Gamma(\underline{C}, \underline{f}^*\overline{\mathcal{M}}_X) \to \Gamma(\underline{C}, \overline{\mathcal{M}}_C)$ induced by $\overline{f^\flat}$ is given by

$$\varphi(n_1, \ldots, n_p) = (u_{x_1} n_1, \ldots, u_{x_d} n_d, 0, \ldots, 0),$$

the only way that $\tau_\eta^C \circ \varphi$ can coincide with τ_η^X is if $u_{x_i} = \mu_i$ for $1 \le i \le d$. Thus we see that the elements $u_{x_i} \in \mathbb{N}^\vee$ can be interpreted as imposing the orders of tangency.

Note that once $u_{x_i} = \mu_i$ for each i, there is a unique map $\underline{f}^*\mathcal{M}_X \to \mathcal{M}_C$ induced by $\underline{f}^* : f^{-1}\mathcal{O}_{\underline{X}} \to \mathcal{O}_{\underline{C}}$, as in this case the structure maps for the log structures embed $\underline{f}^*\mathcal{M}_X$ and $\mathcal{M}_C$ in $\mathcal{O}_{\underline{C}}$. Thus the open substack of the moduli space of stable log maps corresponding to curves considered in this example coincides with the corresponding substack (not necessarily open now) of the moduli of ordinary stable maps consisting of stable maps as above with the correct orders of tangencies, and the log structure is trivial. $\qquad\square$

Example 6.2. Continuing with the case of a pair $(\underline{X}, D)$ as in the previous example, it is not difficult to obtain interesting examples for the monoid Q. Consider the case that $\underline{X} = \mathbb{P}^1$, $D = \{0\}$, and consider the limiting situation in Figure 6.1. The figure on the left shows a rational curve with marked points p_1, p_2, p_3 mapping $2:1$ to $\mathbb{P}^1$, with order of tangencies to D at the three marked points being $0, 0$ and 2 respectively. Such a situation can degenerate to the stable marked curve $\underline{f}$ pictured to the right, in which all but the two horizontal components map to 0. By semi-stable reduction for stable log maps, there is a limiting stable log map, with some associated monoid Q. Let us determine what the type of this curve is and what Q is.

Let D_i, $i = 1, 2, 3$ be the irreducible components of the degenerate domain curve containing the points p_i, $i = 1, 2, 3$, respectively. Let D_4 and D_5 be the two horizontal components, with $D_4 \cap D_1 \ne \emptyset$ and $D_5 \cap D_2 \ne \emptyset$. Denote by η_i the generic point of D_i. Because $\underline{f}(D_i) = \{0\}$,

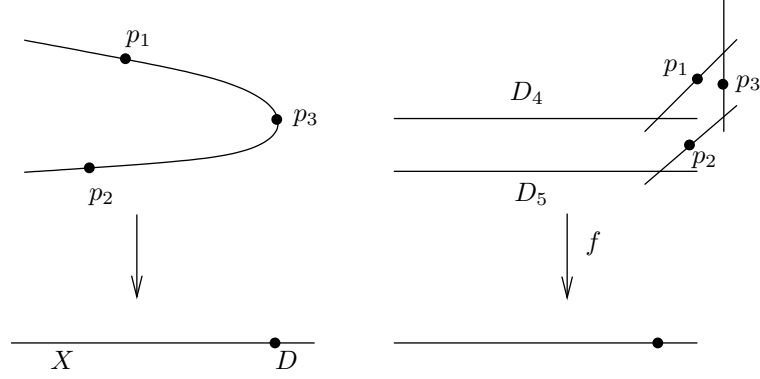FIGURE 6.1.

$i = 1, 2, 3$, we have $\tau^X_{\eta_i} = 0$ for these i, but $\tau^X_{\eta_4} = \tau^X_{\eta_5} : \mathbb{N} \to \mathbb{Z}$ is given by $n \mapsto -n$, as in Example 6.1. We have the four double points

$$q_1 = D_4 \cap D_1, \quad q_2 = D_1 \cap D_3, \quad q_3 = D_2 \cap D_3, \quad q_4 = D_5 \cap D_2.$$

To determine the type of the central fibre, one can use the method of the proof of Proposition 4.3. In particular, necessarily $u_{p_1} = u_{p_2} = 0 \in \mathbb{N}$ and $u_{p_3} = 2 \in \mathbb{N}$. However, to determine the u_{q_i}, it is easier to use the balancing condition Proposition 1.14 to observe that there is only one choice of type for the central fibre with these u_{p_i}. Indeed, suppose we give the central fibre the structure of a stable log map over the standard log point as in §1.3. This means in particular we have data $V_{\eta_1}, \ldots, V_{\eta_5}$ with $V_{\eta_1}, V_{\eta_2}, V_{\eta_3} \in \mathbb{N}^\vee$ and $V_{\eta_4}, V_{\eta_5} = 0$, as well as positive integers $e_{q_1}, \ldots, e_{q_4}$. Suppose the u_{p_i}'s are as given above. If we define the u_{q_i}'s by the choice of signs

$$u_{q_1} = (V_{\eta_1} - V_{\eta_4})/e_{q_1}, \quad u_{q_2} = (V_{\eta_3} - V_{\eta_1})/e_{q_2}$$
$$u_{q_3} = (V_{\eta_3} - V_{\eta_2})/e_{q_3}, \quad u_{q_4} = (V_{\eta_2} - V_{\eta_5})/e_{q_4}$$

then by Proposition 1.14 necessarily $u_{q_i} = 1 \in \mathbb{N}^{\mathrm{gp}}$ for all i. Then the monoid Q associated to this type is $(P_{\eta_1} \oplus P_{\eta_2} \oplus P_{\eta_3} \oplus \mathbb{N}^4)/R_{\mathbf{u}}$, where $R_{\mathbf{u}}$ is generated by

$$a_{q_1}(1) = (1, 0, 0, -1, 0, 0, 0),$$
$$a_{q_2}(1) = (-1, 0, 1, 0, -1, 0, 0),$$
$$a_{q_3}(1) = (0, -1, 1, 0, 0, -1, 0),$$
$$a_{q_4}(1) = (0, 1, 0, 0, 0, 0, -1).$$

By eliminating the last four components using these relations, $\mathbb{Z}^7/R_{\mathbf{u}}$ can be identified with $\mathbb{Z}^3$, with the monoid $\mathbb{N}^7$ having image in $\mathbb{Z}^3$ being generated by $e_1 = (1, 0, 0)$, $e_2 = (-1, 0, 1)$, $e_3 = (0, -1, 1)$ and $e_4 = (0, 1, 0)$. Thus Q is the monoid generated by e_1, e_2, e_3, e_4 subject to the relation $e_1 + e_2 = e_3 + e_4$. This is the monoid defining the quadric cone in $\mathbb{A}^4$.

Note the choice of a map $Q \to \mathbb{N}$ yields a tropical curve in $(\Gamma(\underline{C}, \underline{f}^*\overline{\mathcal{M}}_X)^{\mathrm{gp}})^*$, as depicted in Figure 6.2. □
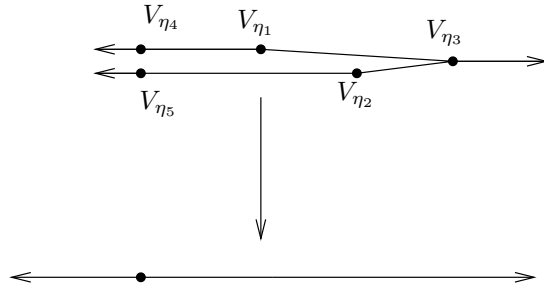
FIGURE 6.2. The tropical curve associated to f. Here the range is $\mathrm{Hom}(\Gamma(\underline{C}, \underline{f}^*\overline{\mathcal{M}}_X)^{\mathrm{gp}}, \mathbb{R})$.

In the situation considered above of a smooth divisor $D \subset \underline{X}$, it is worth comparing the moduli spaces that we construct and those constructed by Jun Li. Conjecturally, the Gromov-Witten invariants defined using these moduli spaces will coincide, though we will make no attempt to prove this here. On the other hand, the moduli spaces themselves are demonstrably not isomorphic.

We will sketch here the construction of a map between these moduli spaces, assuming familiarity with Li's notion of stable relative maps.

Let $\mathcal{M}_X$ be the divisorial log structure on $\underline{X}$ defined by D, β a class of stable log maps, $X = (\underline{X}, \mathcal{M}_X)$, and let $\mathscr{M}(X, \beta)$ be as usual the stack of basic stable log maps of class β. Let $\mathbf{M}(\underline{X}/D, \beta)$ be Jun Li's moduli space of stable relative maps. We wish to construct a morphism of stacks $\Psi : \mathbf{M}(\underline{X}/D, \beta) \to \mathscr{M}(X, \beta)$.

Recall a family of relative stable maps over a base scheme $\underline{W}$ is given by the following data. For each $n \geq 0$, there is a pair $(\underline{X}[n], D[n])$ constructed from $(\underline{X}, D)$ where $\underline{X}[n]$ is defined over $\mathbb{A}^n$ and $D[n] \subset \underline{X}[n]$ is a divisor. Then a family of relative stable maps is a diagram

(6.1)
$$
\begin{array}{ccc}
(\underline{C}, \mathbf{x}) & \xrightarrow{\underline{f}} & \underline{X}[n] \\
\downarrow & & \downarrow{\scriptstyle g} \\
\underline{W} & \longrightarrow & \mathbb{A}^n
\end{array}
$$

which is a family of ordinary stable maps in $\underline{X}[n]$ satisfying certain conditions (predeformability and finiteness of a certain notion of automorphism group), and has specified tangencies with $D[n]$. In §1.3 of [Li2], Li observes that $\underline{X}[n]$ and $\mathbb{A}^n$ carry canonical log structures. The log structure on $\mathbb{A}^n$ is the divisorial log structure associated to the divisor B given by $t_1 \cdots t_n = 0$, and the one on $\underline{X}[n]$ is induced by $g^{-1}(B) \cup D[n]$. Furthermore, there is a natural projection $\underline{\Theta} : \underline{X}[n] \to \underline{X}$ such that $\underline{\Theta}^{-1}(D) \subset g^{-1}(B) \cup D[n]$, hence giving rise to a log morphism $\Theta : X[n] \to X$. Li then constructs log structures on $\underline{C}$ and $\underline{W}$ making (6.1) into a commutative diagram of log schemes. Now $C \to W$ is a log smooth curve with marked points along $\underline{f}^{-1}(D[n])$.

To give a morphism of stacks $\mathbf{M}(\underline{X}/D, \beta) \to \mathscr{M}(X, \beta)$, we need to associate to a diagram (6.1) a stable log map $(\bar{C}/W, \mathbf{x}, \bar{f})$ to X. This will then give the desired morphism of stacks

via Proposition 1.22. The only difficulty is that the composition $\Theta \circ f$ need not be stable. We just need to show the following proposition:

Proposition 6.3. *There is a commutative diagram of log schemes*

$$
\begin{array}{ccc}
C & \xrightarrow{\;f\;} & X[n] \\
\pi \downarrow & \;\;\psi \;\;\;\; \Theta & \\
W & \xleftarrow{\;\bar{\pi}\;} \bar{C} \xrightarrow{\;\bar{f}\;} & X
\end{array}
$$

so that $(\bar{C}/W, \mathbf{x}, \bar{f})$ is a stable log map.

Proof. It is standard that such a diagram exists at the level of schemes, with $\underline{\bar{f}}$ a stabilization of $\underline{\Theta} \circ \underline{f}$ ([BeMa], Theorem 3.6). The map $\underline{\psi}$ contracts in every fibre of $\underline{\pi}$ every $\mathbb{P}^1$ component with only two special points on which $\underline{\Theta} \circ \underline{f}$ is constant. We need to understand these morphisms at the log level.

Step 1. *Review of $g : X[n] \to X$.* The pair $(\underline{X}[n], D[n])$ is constructed inductively: $(\underline{X}[0], D[0])$ is the pair $(\underline{X}, D)$. Then the pair $(\underline{X}[n], D[n])$ is obtained by blowing up $\underline{X}[n-1] \times \mathbb{A}^1$ along $D[n-1] \times \{0\}$, and $D[n]$ is the proper transform of $D[n-1] \times \mathbb{A}^1$. If D is given by the vanishing of a regular function $w_0 = 0$, (which can always be accomplished locally on $\underline{X}$) then in fact $\underline{X}[n]$ can be described as the subscheme of $\underline{X} \times (\mathbb{P}^1)^n \times \mathbb{A}^n$ given by the equations

$$w_0 z_1 = w_1 t_1, \quad w_1 z_2 = w_2 z_1 t_2, \quad \ldots, \quad w_{n-1} z_n = z_{n-1} w_n t_n,$$

where z_i, w_i are homogeneous coordinates on the i-th copy of $\mathbb{P}^1$ and $t_1, \ldots, t_n$ are coordinates on $\mathbb{A}^n$. This is covered by affine open subsets $\underline{X}_i$, $1 \le i \le n+1$, where $\underline{X}_i$ is given by $z_1 = \cdots = z_{i-1} = 1 = w_i = \cdots = w_n$. Thus $w_{i-1} z_i = t_i$ on $\underline{X}_i$ for $i \le n$. The log structure of $X[n]$ restricted to $\underline{X}_i$ for $i \le n$ has a chart

$$\mathbb{N}^n \oplus_{\mathbb{N}} \mathbb{N}^2 \to \mathcal{O}_{X_i}$$

where the map $\mathbb{N} \to \mathbb{N}^n$ is $1 \mapsto e_i$, the i-th generator of $\mathbb{N}^n$, and the chart is

$$\left(\sum_{i=1}^{n} a_i e_i, (a, b) \right) \longmapsto \left(\prod_{i=1}^{n} t_i^{a_i} \right) w_{i-1}^a z_i^b.$$

The chart on X_{n+1} is just $\mathbb{N}^n \oplus \mathbb{N} \to \mathcal{O}_{X_{n+1}}$ given by $(\sum a_i e_i, a) \mapsto w_n^a \prod t_i^{a_i}$.

Recall that $X[n]$ fibres over $\mathbb{A}^n$ with coordinates $t_1, \ldots, t_n$, and that the critical locus of this map consists of n distinct subvarieties $D_1, \ldots, D_n$, with D_i sitting over $t_i = 0$. In the above description, $D_i \subset X_i$ is given by $w_{i-1} = z_i = 0$. Also $D[n] \subset X_{n+1}$ is given by $w_n = 0$.

The map $\Theta : X[n] \to X$ is the projection to X. We describe this as a log morphism as follows. Let s_{w_0} be the section of $\mathcal{M}_X$ corresponding to the function w_0 vanishing only along D. In general, if we have a chart $P \to \mathcal{O}_Y$ for a log structure, for $m \in P$ we denote by s_m the corresponding section of the associated log structure, so that any section of the associated

log structure is of the form $h \cdot s_m$ for some $m \in P$, $h \in \mathcal{O}_Y^\times$. Then $\Theta : X_i \to X$ is given by (for $i \leq n$)

$$s_{w_0} \longmapsto s_{(e_1 + \cdots + e_{i-1}, (1,0))}$$

since on $\underline{X}_i$, $w_0 = w_{i-1} t_1 \cdots t_{i-1}$, while $\Theta : X_{n+1} \to X$ is given by

$$s_{w_0} \longmapsto s_{(e_1 + \cdots + e_n, 1)}.$$

Step 2. *Review of $C \to W$.* Fix a geometric point $\overline{w} \in |W|$. Let $\mathbf{D}$ be the set of *distinguished* double points of $\underline{C}_{\overline{w}}$, that is, double points mapping to $\bigcup_{\ell=1}^n D_\ell$, with $\mathbf{D}_\ell \subset \mathbf{D}$ the subset mapping to D_ℓ. Let $\mathbf{U}$ be the set of undistinguished double points. For each $q \in \mathbf{D}_\ell$, let μ_q denote the order of tangency of either branch of $C_{\overline{w}}$ at q with D_ℓ. Let N_ℓ be the free monoid $\mathbb{N}\rho_\ell$ generated by ρ_ℓ if $\mathbf{D}_\ell = \emptyset$; otherwise N_ℓ is the monoid generated by $\{\rho_q \,|\, q \in \mathbf{D}_\ell\}$ modulo the relations $\mu_q \rho_q = \mu_{q'} \rho_{q'}$ for each $q, q' \in \mathbf{D}_\ell$. Then[12]

$$\overline{\mathcal{M}}_{W,\overline{w}} = \bigoplus_{\ell=1}^n N_\ell \oplus \bigoplus_{q \in \mathbf{U}} \mathbb{N}\rho_q.$$

For every point $q \in \mathbf{D}_\ell$, one can find an étale open neighbourhood U_q of $q \in \underline{C}$ such that $\underline{f}$ maps U_q into $\underline{X}_\ell$. Furthermore, U_q is of the form $\operatorname{Spec} A[x_q, y_q]/(x_q y_q - t_q)$ where étale locally $W = \operatorname{Spec} A$, and $t_q \in A$. As observed in [Li2], Simplification 1.7, we can choose U_q and coordinates x_q, y_q so that

$$\underline{f}^*(w_{\ell-1}) = x_q^{\mu_q}, \quad \underline{f}^*(z_\ell) = y_q^{\mu_q}.$$

Then $\underline{f}^*(t_\ell) = t_q^{\mu_q}$.

Similarly, for each $q \in \mathbf{U}$, we can describe a neighbourhood U_q of q in $\underline{C}$ as

$$U_q \cong \operatorname{Spec} A[x_q, y_q]/(x_q y_q - t_q)$$

for some $t_q \in A$.

Given these choices, Li puts log structures on C and W as follows. There is a chart for the log structure of W,

$$\sigma : \overline{\mathcal{M}}_{W,\overline{w}} \longrightarrow A$$

given by

$$\rho_q \longmapsto t_q, \quad \rho_\ell \longmapsto \underline{h}^* t_\ell,$$

where $\underline{h} : \underline{W} \to \mathbb{A}^n$. A chart for the log structure on C for the neighbourhood U_q is

$$\psi_q : \overline{\mathcal{M}}_{W,\overline{w}} \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \mathcal{O}_{U_q}$$

$$(\alpha, (a,b)) \longmapsto \underline{\pi}^*(\sigma(\alpha)) x_q^a y_q^b.$$

Here the map $\mathbb{N} \to \overline{\mathcal{M}}_{W,\overline{w}}$ is $1 \mapsto \rho_q$.

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[12] Jun Li does not include the contributions from the undistinguished nodes; this is incorrect and these must be included in order for $C \to W$ to be log smooth. However, these contributions will play no further role in the discussion.

If p is any marked point of $\underline{C}$ with $\underline{f}(p) \in D[n]$, then in a suitable neighbourhood $U_p = \operatorname{Spec} A[x_p]$ of p, with chart $\overline{\mathcal{M}}_{W,\overline{w}} \oplus \mathbb{N} \to A[x_p]$ given as usual as $(\alpha, a) \mapsto \pi^*(\sigma(\alpha)) x_p^a$. We have $\underline{f}^*(w_n) = x_p^\mu \cdot h_p$ for a unit h_p, which will be assumed to be 1 via a suitable choice of U_p and x_p.

One checks these induced log structures do not depend on any choices and glue uniquely, yielding a well-defined $C \to W$.

Step 3. *The log morphism $f : C \to X[n]$.* For $q \in \mathbf{D}_i$, $f : (U_q, \mathcal{M}_C|_{U_q}) \to X_i$ is given by

$$s_{(\sum a_\ell e_\ell, (a,b))} \longmapsto s_{(\sum a_\ell \rho'_\ell, (\mu_q a, \mu_q b))}$$

where $\rho'_\ell = \rho_\ell$ if $\mathbf{D}_\ell$ is empty and is equal to $\mu_q \rho_q$ for any $q \in \mathbf{D}_\ell$ otherwise. If p is a marked point of $\underline{C}$ with $\underline{f}(p) \in D[n]$, then f is given on $(U_p, \mathcal{M}_C|_{U_p}) \to X_{n+1}$ by

$$s_{(\sum a_\ell e_\ell, a)} \longmapsto s_{(\sum a_\ell \rho'_\ell, \mu_p a)}.$$

This is sufficient to completely specify f.

Step 4. *The map $C \to \bar{C}$.* The log structure on $\underline{\bar{C}}$ can be described as follows. First, if $\bar{C}^o \subset \underline{\bar{C}}$ is the largest open set for which $\underline{\psi}^{-1}(\bar{C}^o) \to \bar{C}^o$ is an isomorphism, $\bar{C}^o$ has the same log structure as $\psi^{-1}(\bar{C}^o)$.

Next, let $\overline{y} \in |\bar{C}|$ be a geometric point not in $\bar{C}^o$, with $\overline{w} = \overline{\pi}(\overline{y})$. There are two cases: $\overline{y}$ either is not, or is, a double point of $\underline{\bar{C}}_{\overline{w}}$. In both cases, $\psi^{-1}(\overline{y})$ is a chain $\underline{C}_1 \cup \cdots \cup \underline{C}_m$ of rational curves, with $q_i = \underline{C}_i \cap \underline{C}_{i+1}$ a double point, and $\underline{C}_{\overline{w}}$ has another component $\underline{C}_0$ with $\underline{C}_0 \cap \underline{C}_1 = q_0$ also a double point. However, if $\overline{y}$ is not a double point, then there is a point $p_m \in \underline{C}_m$ which is a log marked point, while if $\overline{y}$ is a double point, $\underline{C}_{\overline{w}}$ has a component $\underline{C}_{m+1}$ and a double point $q_m = \underline{C}_m \cap \underline{C}_{m+1}$.

In what follows, let m' be $m-1$ if $\overline{y}$ is not a double point, and m if $\overline{y}$ is a double point.

If $\overline{y}$ is not a double point, then there is an étale open neighbourhood of $\overline{y}$ of the form $U := \operatorname{Spec} A[x]$, where $x = 0$ is the image of the section $\underline{\psi} \circ x_i$, with x_i the marked point of $\underline{C}$ corresponding to p_m.

If $\overline{y}$ is a double point, then one can show that there is an étale open neighbourhood of $\overline{y}$ of the form $U := \operatorname{Spec} A[x, y]/(xy - \prod_{i=0}^m t_{q_i})$.

In both cases, one can describe $U \times_{\underline{\bar{C}}} C$. In the case that $\overline{y}$ is not a double point, then this scheme is given by the equations

$$xu_1 = v_1 t_{q_0}, \quad v_1 u_2 = u_1 v_2 t_{q_1}, \quad \ldots, \quad v_{m-1} u_m = u_{m-1} v_m t_{q_{m-1}}$$

in $\mathbb{A}^1 \times (\mathbb{P}^1)^m \times \operatorname{Spec} A$, with homogeneous coordinates on the i-th $\mathbb{P}^1$ being u_i, v_i. If $\overline{y}$ is a double point, then this scheme is given by the equations

$$xu_1 = v_1 t_{q_0}, \quad v_1 u_2 = u_1 v_2 t_{q_1}, \quad \ldots, \quad v_{m-1} u_m = u_{m-1} v_n t_{q_{m-1}}, \quad yv_m = u_m t_{q_m}$$

in $\mathbb{A}^2 \times (\mathbb{P}^1)^m \times \operatorname{Spec} A$.

We can cover $U \times_{\underline{\bar{C}}} C$ with Zariski open subsets U_i for $0 \le i \le m$, where U_i is the set where $u_1 = \cdots = u_i = 1 = v_{i+1} = \cdots = v_m$. Note U_i is an open neighbourhood of q_i, $U_i \cong \operatorname{Spec} A[v_i, u_{i+1}]/(v_i u_{i+1} - t_{q_i})$ if $i \le m'$, and $U_m \cong \operatorname{Spec} A[v_m]$ if $\overline{y}$ is not a double point.

We have to be slightly careful with the coordinates x and y; these can't be chosen arbitrarily. To do so we have to relate these to the map $\underline{f}$. At this point, we shall make a simplifying assumption that $\overline{w} \in W$ maps to $0 \in \mathbb{A}^n$. This can always be achieved locally on W by decreasing n, and general case can be dealt with by the reader with some extra book-keeping.

Possibly after reversing the order of $\underline{C}_0, \ldots, \underline{C}_{m+1}$ in the case that $\overline{y}$ is a double point, we can assume that there is some positive integer ℓ_0 such that $\underline{f}(q_i) \in D_{\ell_0+i}$ for $i \le m'$, and $\underline{f}(p_m) \in \underline{X}_{n+1} \cap D[n]$ if $\overline{y}$ is not a double point. In the latter case, $\ell_0 = n - m + 1$. The map $\underline{f} : U_i \to \underline{X}[n]$ factors through $\underline{X}_{\ell_0+i}$, and necessarily yields

$$
\begin{aligned}
i = 0 : & \quad \underline{f}^*(w_{\ell_0-1}) = h_{0,1}x^\mu, \quad \underline{f}^*(z_{\ell_0}) = h_{0,2}u_1^\mu \\
0 < i < m : & \quad \underline{f}^*(w_{\ell_0+i-1}) = h_{i,1}v_i^\mu, \quad \underline{f}^*(z_{\ell_0+i}) = h_{i,2}u_{i+1}^\mu \\
i = m : & \quad \underline{f}^*(w_{\ell_0+m-1}) = h_{m,1}v_m^\mu, \quad \underline{f}^*(z_{\ell_0+m}) = h_{m,2}y^\mu,
\end{aligned}
$$

the last line if $\overline{y}$ is a double point. Here $\mu = \mu_{q_i}$ for any i. If $\overline{y}$ is not a double point, then $U_m \to \underline{X}[n]$ factors through X_{n+1}, with

$$f^*(w_n) = h_{m,1}v_m^\mu.$$

Now in fact we can assume $h_{0,2} = h_{m,1} = 1$, $h_{i,k} = 1$ for $0 < i < m$, $k = 1, 2$. This is because $\underline{f}$ on each component $\underline{C}_i$, $1 \le i \le m$, is just a μ-fold cover of $\mathbb{P}^1$ totally ramified at 0 and ∞, so that the above listed $h_{i,k}$'s are constant, and then after applying a suitable change of coordinates on $\underline{C}$, one can assume these constants are 1. Further, by making a change of coordinates for x (and y), one can assume $h_{0,1} = 1$ (and $h_{m,2} = 1$ in the double point case).

In particular, one can then assume that the open sets $U_{q_0}, \ldots, U_{q_{m-1}}, U_{p_m}$ if $\overline{y}$ is not a double point ($U_{q_0}, \ldots, U_{q_m}$ if $\overline{y}$ is a double point) are taken to be the open sets $U_0, \ldots, U_m$. Thus we know what the chart for the log structure on U_i is, by Step 2. This is given on U_i for $i \le m'$ by charts $\overline{\mathcal{M}}_{W,\overline{w}} \oplus_{\mathbb{N}} \mathbb{N}^2 \to \mathcal{O}_{U_i}$ given by

$$(\alpha, (a, b)) \longmapsto \underline{\pi}^*(\sigma(\alpha))v_i^a u_{i+1}^b,$$

and on U_m for $\overline{y}$ not a double point by a chart $\overline{\mathcal{M}}_{W,\overline{w}} \oplus \mathbb{N} \to \mathcal{O}_{U_m}$ given by

$$(\alpha, a) \longmapsto \underline{\pi}^*(\sigma(\alpha))v_m^a.$$

Using this particular choice of the coordinate x (and y if $\overline{y}$ is a double point), we can put a log structure on U as follows. If $\overline{y}$ is not a double point, we have a chart $\overline{\mathcal{M}}_{W,\overline{w}} \oplus \mathbb{N} \to \mathcal{O}_U$ given by

$$(\alpha, a) \longmapsto \overline{\underline{\pi}}^*(\sigma(\alpha)) \cdot x^a.$$

If $\overline{y}$ is a double point, we have a chart

$$
\overline{\mathcal{M}}_{W,\overline{w}} \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \mathcal{O}_U
$$
$$
(\alpha, (a, b)) \longmapsto \overline{\underline{\pi}}^*(\sigma(\alpha))x^a y^b,
$$

with the map $\mathbb{N} \to \overline{\mathcal{M}}_{W,\overline{w}}$ given by $1 \mapsto \sum_{i=1}^m \rho_{q_i}$. It is straightforward to check that this log structure is independent of choices and this description gives compatible log structures for

different choices of the point $\overline{y}$. Hence one obtains a log structure on $\underline{\overline{C}}$ clearly making $\overline{\pi}$ log smooth.

We can specify the log morphism $\psi : U_i \to U$ as follows. If $\overline{y}$ is not a double point, this map is defined as

$$s_{(\alpha,a)} \longmapsto \begin{cases} s_{(\alpha + a\sum_{j<i}\rho_{q_j},(a,0))} & i < m; \\ s_{(\alpha + a\sum_{j<m}\rho_{q_j},a)} & i = m. \end{cases}$$

On the other hand, if $\overline{y}$ is a double point, then for every i, we take the map

$$s_{(\alpha,(a,b))} \longmapsto s_{(\alpha + a\sum_{j<i}\rho_{q_j} + b\sum_{j>i}\rho_{q_j},(a,b))}.$$

Using the explicit description for the charts on the sets U_i, one checks that the morphisms $U_i \to U$ agree on overlaps and hence give a morphism $U \times_{\overline{C}} C \to U$. Furthermore, these morphisms are compatible for different choices of open neighbourhoods U of different points $\overline{y}$, giving the log morphism $\psi : C \to \overline{C}$.

Step 5. *The map $\overline{f} : \overline{C} \to X$.* We now define the log map $\overline{f} : U \to X$ by

$$s_{w_0} \longmapsto \begin{cases} s_{(\sum_{\ell=1}^{\ell_0-1}\rho'_\ell,\mu)} \in \mathcal{M}_U & \text{if } \overline{y} \text{ is not a double point;} \\ s_{(\sum_{\ell=1}^{\ell_0-1}\rho'_\ell,(\mu,0))} \in \mathcal{M}_U & \text{if } \overline{y} \text{ is a double point,} \end{cases}$$

where ρ'_ℓ is defined in Step 3. One checks from the description of all the maps above that this yields the desired commutative diagram, at least where these maps are now defined. Further, one checks that all maps are independent of choices and glue, hence giving the desired global commutative diagram. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 6.4. *There is a morphism of stacks $\Psi : \mathbf{M}(\underline{X}/D,\beta) \to \mathscr{M}(X,\beta)$.*

Example 6.5. Let us return to the situation of Example 6.2, comparing our picture with the expanded degeneration picture. In our picture, the degenerate curve f corresponds to one point in the moduli space. On the other hand, there are three combinatorially distinct possible limits in the expanded degeneration picture, as depicted in Figure 6.3. In Cases I and II, one must insert additional irreducible components into the limit curve C: these components are contracted in our picture, as they are not stable components. Roughly speaking, these two cases correspond, in our language, to curves over $(\operatorname{Spec}\kappa, \mathbb{N})$ with $V_{\eta_1} < V_{\eta_2}$ and $V_{\eta_1} > V_{\eta_2}$ respectively. In Case III, however, the limit domain is the same as ours, but note there is still some actual moduli. Indeed, the first bubble component of the range has four special points: the two double points of the range contained in this component, and the images of the marked points P_1 and P_2. The cross-ratio of these four points provides a one-parameter moduli space, a copy of $\mathbb{G}_m$. Cases I and II can be viewed as limit points in this one-parameter moduli space. So the map $\mathbf{M}(\underline{X}/D,\beta) \to \mathscr{M}(X,\beta)$ contracts a $\mathbb{P}^1$ to a point.

This suggests that in fact $\mathbf{M}(\underline{X}/D,\beta)$ is, in this case, only birational onto its image in $\mathscr{M}(X,\beta)$. We conjecture this is the case in general. In this case, this morphism gives a small resolution of an ordinary double point in the three-dimensional moduli space of basic stable

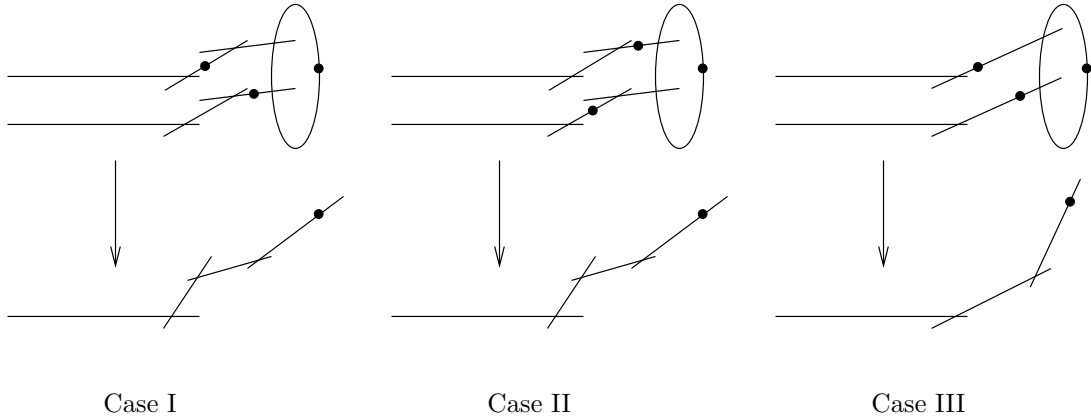Case I   Case II   Case III

FIGURE 6.3.

log maps of the type being considered here. Nevertheless, one may conjecture that the log Gromov-Witten invariants defined here coincide with relative Gromov-Witten invariants as defined by Jun Li. We shall leave it to others to prove such a conjecture.

Example 6.6. Next let us consider $\underline{X} = \mathbb{P}^2$ with $D \subset \underline{X}$ a union of two distinct lines L_1, L_2. This is a normal crossings divisor, and is the first case that the classical form of relative Gromov-Witten invariants does not cover. Consider the moduli space of degree two stable log rational curves in $\mathbb{P}^2$ with four marked points p_{ij}, $1 \le i,j \le 2$, with p_{ij} a point with tangency of order 1 with L_i and tangency of order 0 with L_{3-i}. The generic case just consists of a conic intersecting D at four distinct smooth points of D. For such curves, the existence of the double point in D is irrelevant, and the analysis of Example 6.1 still applies.

Now consider a limiting curve in which the image of the conic degenerates to a reducible conic F with $F \cap D = L_1 \cap L_2$, that is, the double points of F and D coincide. By stable reduction, there is a limiting stable map $\underline{f}$, which in the simplest case is given in Figure 6.4. Furthermore, by stable log reduction, this gives a stable log map. Let us determine its type.

First note that $\Gamma(\underline{C}, \underline{f}^*\overline{\mathcal{M}}_X) = \mathbb{N}^2$. Let the irreducible components of $\underline{C}$ be D_1, D_2, D_3 with D_3 the contracted component. Then necessarily $u_{p_{1j}} = (1,0)$ and $u_{p_{2j}} = (0,1)$ for $j = 1,2$. One can think of this as saying that even though the p_{ij}'s map into the intersection of the irreducible components, $u_{p_{ij}}$ remembers which component the curve should be tangent to at p_{ij}.

As in Example 6.2, we can determine the u_q's by considering the structure of a stable log map over the standard log point. Necessarily $V_{\eta_1} = V_{\eta_2} = 0$. Letting $q_i = D_i \cap D_3$, $i = 1,2$, there is no choice but for $u_{q_i} = (1,1)$ (with the proper choice of order) in order to achieve the balancing condition at V_{η_3}. One sees in this case that $Q = \mathbb{N}$, with an element of $Q^\vee$ specifying $e_{q_1} = e_{q_2}$.

Example 6.7. Let $\underline{X}$ be a complete toric variety and let $\partial \underline{X} \subset \underline{X}$ be the toric boundary, the union of toric divisors on $\underline{X}$. Let $\underline{X}$ be given the divisorial log structure induced by
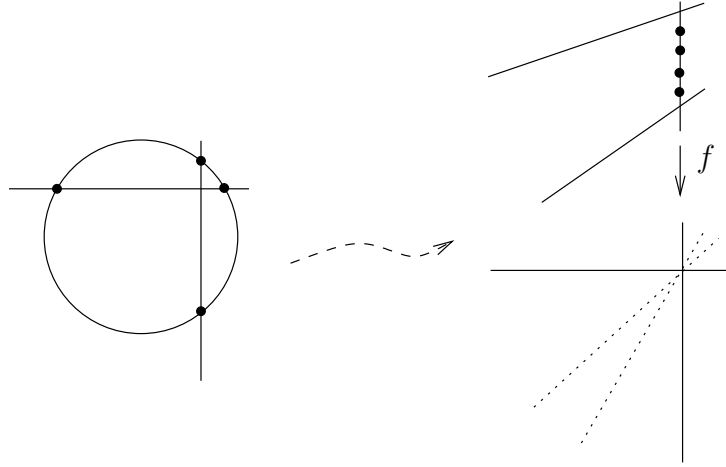
FIGURE 6.4. The conic on the left degenerates to the reducible curve on the right; the dotted lines are the image of the stable map.

this divisor. In general, we do not expect that $\overline{\mathcal{M}}_X$ will be generated by global sections; however, it is easy to see that if N is the character lattice of the torus acting on X, then there is a surjection $\underline{N} \to \overline{\mathcal{M}}_X^{\mathrm{gp}}$, as each stalk of $\overline{\mathcal{M}}_X^{\mathrm{gp}}$ is a quotient of N. In particular, given a stable log map $(C/(\mathrm{Spec}\,\kappa, \mathbb{N}), \mathbf{x}, f)$ with target X, by Discussion 1.12 one obtains a balanced tropical curve $h : \Gamma_{\underline{C}} \to M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R})$. In general, this tropical curve contains one unbounded edge for each irreducible component $D = \mathrm{cl}(\eta)$ of $\underline{C}$ in the direction of τ_η^X, or rather, the image of τ_η^X in $M = \mathrm{Hom}(N, \mathbb{Z})$. More specifically, given the component D and its normalization $g : \tilde{D} \to D$, one has a composition

$$ N \longrightarrow \Gamma(\tilde{D}, g^* \underline{f}^* \overline{\mathcal{M}}_X^{\mathrm{gp}}) \xrightarrow{\tau_\eta^X} \mathbb{Z}, $$

which yields the image of τ_η^X in M. Note, however, that the composed map $N \to \Gamma(\underline{X}, \overline{\mathcal{M}}_X^{\mathrm{gp}}) \to$ Pic X is the zero map, as the divisor of zeroes and poles of a monomial z^n is linearly equivalent to zero. Hence the image of τ_η^X in M is in fact zero.

Thus the construction of Discussion 1.12 yields an ordinary tropical curve in $M_{\mathbb{R}}$ whose only unbounded edges necessarily correspond to marked points of the curve $\underline{C}$, and the balancing condition of Proposition 1.14 gives the usual tropical balancing condition in $M_{\mathbb{R}}$. Furthermore, there is a relationship between this tropical curve and the fan Σ in $M_{\mathbb{R}}$ for X. Indeed, suppose that for an irreducible component $D = \mathrm{cl}(\eta)$ of $\underline{C}$, the toric stratum of X containing $\underline{f}(\eta)$ corresponds to a cone $\tau \in \Sigma$. Then $h(v_\eta) \in \tau$. Indeed, we have the composition

$$ N \longrightarrow \Gamma(\underline{C}, \underline{f}^* \overline{\mathcal{M}}_X^{\mathrm{gp}}) \twoheadrightarrow P_\eta^{\mathrm{gp}}, $$

with $P_\eta = (\tau^\vee \cap N)/(\tau^\vee \cap N)^\times$. Thus we have the dual map

$$ P_\eta^\vee \hookrightarrow (P_\eta^{\mathrm{gp}})^* \longrightarrow M, $$

which identifies $P_\eta^\vee$ with $\tau \cap M$. As $h(v_\eta)$ is the image of $V_\eta \in P_\eta^\vee$ under this map, one sees $h(v_\eta) \in \tau$.
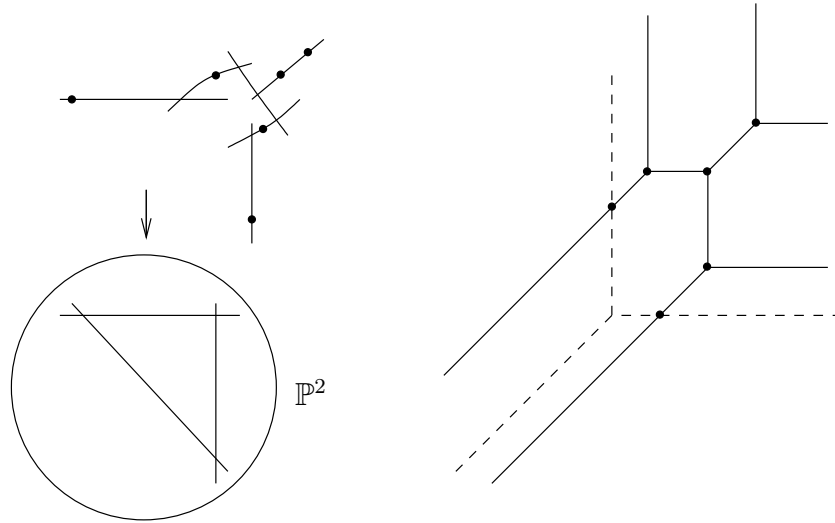
FIGURE 6.5. The figure on the left depicts the underlying stable map, with marked points as indicated. All components are contracted but the left-hand and lower components, which map to coordinate lines. The right-hand figure shows a possible corresponding tropical curve, with the dotted lines indicating the fan for $\mathbb{P}^2$.

Note further that the tropical curve $h : \Gamma_{\underline{C}} \to M_{\mathbb{R}}$ in fact determines the type of the curve, as the maps $P_x^\vee \to M$ are injective for any $x \in |\underline{C}|$. However, given an ordinary stable map $\underline{f} : \underline{C} \to \underline{X}$, it is not true that any type of tropical curve $h : \Gamma_{\underline{C}} \to M_{\mathbb{R}}$ with $h(v_\eta)$ lying in the correct cone of Σ is an allowable type of log map. Indeed, the balancing condition holding in $M_{\mathbb{R}}$ does not imply the balancing condition of Proposition 1.14.

Figure 6.5 gives an example of a stable map $\underline{f} : \underline{C} \to \mathbb{P}^2$ and a corresponding possible tropical curve. It is not difficult to see that $Q = \mathbb{N}^5$ here.

APPENDIX A. THE LOG STACK OF PRESTABLE CURVES

In this appendix we will sketch the argument that the stack $\mathscr{M} = \mathscr{M}_S$ of pre-stable log smooth curves defined over $S = (\underline{S}, \mathcal{M}_S)$ is an algebraic log stack locally of finite type over S. We note that this will be an algebraic stack only in the weaker sense of [Ol2], in the sense that the diagonal morphism will not be separated, due to the phenomenon of Example 2.10 inherent in log moduli problems.

We begin by considering the stack $\mathbf{M}$ of ordinary pre-stable curves over $\underline{S}$, that is, for a scheme V,

$$\mathbf{M}(V) = \{(C/V, \mathbf{x}) \text{ is a pre-stable curve}\}.$$

Denote by $\mathbf{M}_{g,k}$ the Deligne-Mumford stack of stable curves of genus g with k marked points [DeMu], [Kd].

Lemma A.1. $\mathbf{M}$ *is an algebraic stack, locally of finite type over* $\underline{S}$.

Proof. The fact that the diagonal $\Delta : \mathbf{M} \to \mathbf{M} \times_S \mathbf{M}$ is representable follows from the fact that given two pre-stable curves $(C_1/V, \mathbf{x}_1)$, $(C_2/V, \mathbf{x}_2)$, the isomorphism functor

$$\mathrm{Isom}_V \big((C_1/V, \mathbf{x}_1), (C_2/V, \mathbf{x}_2) \big)$$

is representable by a closed subscheme of $\mathrm{Isom}_V(C_1, C_2)$, which in turn is representable by an open subscheme of the Hilbert scheme of $C_1 \times_V C_2$ by [Gr]. In particular, Δ is separated and quasi-compact.

To see that $\mathbf{M}$ has a smooth cover, denote by $\mathbf{M}^o_{g,k}$ the open sub*scheme* of $\mathbf{M}_{g,k}$ whose points parameterize stable marked curves with no non-trivial automorphisms. Consider the map

$$p : \coprod_{g,k,\ell \geq 0} \mathbf{M}^o_{g,k+\ell} \to \mathbf{M}$$

which takes a curve $(C/V, x_1, \ldots, x_{k+\ell})$ to the curve $(C/V, x_1, \ldots, x_k)$. Let C^o be the complement of the critical points of $C \to V$ and let $(C^o)^\ell$ denote $C^o \times_V \cdots \times_V C^o$ (ℓ copies). Given any map $V \to \mathbf{M}$ corresponding to a pre-stable curve $(C/V, x_1, \ldots, x_k)$, one sees easily that $V \times_{\mathbf{M}} \coprod_{g,k,\ell} \mathbf{M}^o_{g,k+\ell}$ is represented by an open subscheme of $\coprod_{\ell \geq 0}(C^o)^\ell$, and hence p is smooth. On the other hand, p is clearly surjective, since given any pre-stable curve over a separably closed field, one can always add enough marked points so that its automorphism group becomes trivial.

Since $\mathbf{M}_{g,k}$ is of finite type for each g and k, this shows $\mathbf{M}$ is an algebraic stack locally of finite type. $\qquad\square$

Now let us incorporate log structures. Given a pre-stable curve $(\underline{C}/\underline{W}, \mathbf{x})$, the argument of [Kf], p. 227ff, constructs canonical log structures and a log morphism $(\underline{C}, \mathcal{M}_C) \to (\underline{W}, \mathcal{M}_W)$, with the log structure on $\underline{W}$ called *basic*. While the result is stated there for stable curves, stability is not used. The point of the basicness property is the following:

Proposition A.2. *Given $(C/W, \mathbf{x})$ basic in the above sense, and a pre-stable marked log curve $(D/Z, \mathbf{y})$ with maps $\underline{\alpha} : \underline{Z} \to \underline{W}$ and $\underline{\beta} : \underline{D} \to \underline{C}$ inducing an isomorphism $\underline{D} \to \underline{C} \times_{\underline{W}} \underline{Z}$, there exist unique maps $\alpha : Z \to W$ and $\beta : D \to C$ with underlying scheme morphisms $\underline{\alpha}$, $\underline{\beta}$ such that the diagram*

$$
\begin{array}{ccc}
D & \xrightarrow{\ \beta\ } & C \\
\downarrow & & \downarrow \\
Z & \xrightarrow[\ \alpha\ ]{} & W
\end{array}
$$

is cartesian.

Proof. This is the content of Proposition 2.1 and Theorem 2.1 of [Kf]. The result as stated does not rely on stability of the curves. $\qquad\square$

In particular, basicness is stable under base change, that is, given $\underline{W}' \to \underline{W}$, the basic log structure on $(\underline{C} \times_{\underline{W}} \underline{W}'/\underline{W}', \mathbf{x})$ is the pull-back of the basic log structure on $(\underline{C}/\underline{W}, \mathbf{x})$. This

endows $\mathbf{M}$ with a log structure,

$$\mathbf{M} \longrightarrow (\mathrm{Log}/S) \longrightarrow (\mathrm{Sch}/\underline{S}),$$

thus generalizing (1.2).

Finally we want to allow arbitrary log structures on the base. For this we use Olsson's deep result that the stack $\mathcal{L}og_S$ of fine log schemes over S is algebraic (but not with a separated diagonal) and locally of finite type over $\underline{S}$ ([Ol2], Theorem 1.1). Similarly, there is an algebraic stack $\mathcal{L}og_S^{\bullet\to\bullet}$ of schemes $\underline{T}$ over $\underline{S}$ together with a morphism of fine log structures $\mathcal{M}_1 \to \mathcal{M}_2$ on T ([Ol3], Example 2.1; this is a direct consequence of [Ol2], Proposition 5.9). The forgetful functor to the first log structure $\mathcal{M}_1$ defines a morphism

$$\mathcal{L}og_S^{\bullet\to\bullet} \longrightarrow \mathcal{L}og_S$$

of algebraic stacks. Clearly, $\mathbf{M} \to (\mathrm{Log}/S)$ in fact defines a morphism of algebraic stacks $\mathbf{M} \to \mathcal{L}og_S$. Now the fibre product

$$\mathcal{L}og_S^{\bullet\to\bullet} \times_{\mathcal{L}og_S} \mathbf{M}$$

is isomorphic to the stack $\mathscr{M}$ of pre-stable log curves (with arbitrary log structures on the base). In fact, an object over $\underline{W} \in (\mathrm{Sch}/\underline{S})$ consists of a *basic* pre-stable log curve $(C/(\underline{W}, \mathcal{M}_W^0), \mathbf{x})$ and a morphism of fine log structure $\mathcal{M}_W \to \mathcal{M}_W^0$. Pulling back thus defines a log curve over $(\underline{W}, \mathcal{M}_W)$, and Proposition A.2 says that this functor defines an isomorphism of stacks from the fibre product to $\mathscr{M}$. Summarizing, we have the following result.

Proposition A.3. *The stack $\mathscr{M}$ of pre-stable marked log curves over fine log schemes over S is an algebraic log stack locally of finite type over $\underline{S}$.* $\qquad\square$

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