Universal Central Extensions of Gauge Algebras and Groups

Bas Janssens
University of Utrecht
mail@bjadres.nl

Christoph Wockel
University of Hamburg
christoph@wockel.eu

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Abstract

We show that the canonical central extension of the group of sections of a Lie group bundle over a compact manifold, constructed in [NW09], is universal. In doing so, we prove universality of the corresponding central extension of Lie algebras in a slightly more general setting.

I Setting of the Problem

Let $\mathfrak{K} \to M$ be a finite-dimensional, locally trivial bundle of Lie algebras. A cocycle on its Lie algebra of sections can then be constructed as follows. For any Lie algebra $\mathfrak{k}$, the derivations $\operatorname{der}(\mathfrak{k})$ act naturally on its second symmetric tensor power $S^2(\mathfrak{k})$ by $\mathrm{d} \cdot (x \otimes_s y) = \mathrm{d}(x) \otimes_s y + x \otimes_s \mathrm{d}(y)$, and we denote the quotient by $V(\mathfrak{k}) := S^2(\mathfrak{k}) / \operatorname{der}(\mathfrak{k}) \cdot S^2(\mathfrak{k})$. The symmetric bilinear form $\kappa \colon \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ given by

$$\kappa(x, y) := [x \otimes_s y]$$

satisfies $\kappa(\mathrm{d}(x), y) + \kappa(x, \mathrm{d}(y)) = 0$ for all $\mathrm{d} \in \operatorname{der}(\mathfrak{k})$ and is universal with this property. Now let ∇ be a Lie connection on $\mathfrak{K}$, i.e. one that satisfies $\nabla[\xi, \eta] = [\nabla\xi, \eta] + [\xi, \nabla\eta]$ for all sections $\xi, \eta \in \Gamma(\mathfrak{K})$. Then ∇ induces a flat connection $\mathtt{d}_\nabla$ on $V(\mathfrak{K})$ by $\mathtt{d}_\nabla[\xi \otimes_s \eta] = [\nabla\xi \otimes_s \eta] + [\xi \otimes_s \nabla\eta]$, where $V(\mathfrak{K})$ is the vector bundle one obtains by applying $\mathfrak{k} \mapsto V(\mathfrak{k})$ fibrewise. The connection $\mathtt{d}_\nabla$ does not depend on ∇, as any two Lie connections differ by a pointwise derivation, which acts trivially on $V(\mathfrak{K})$. We therefore omit the subscript and simply write $\mathtt{d}$. Using the identity $\mathtt{d}\,\kappa(\xi, \eta) = \kappa(\nabla\xi, \eta) + \kappa(\xi, \nabla\eta)$ and the compatibility of ∇ with the Lie bracket, it is not hard to check that

$$\omega_\nabla \colon \Gamma_c(\mathfrak{K}) \times \Gamma_c(\mathfrak{K}) \to \overline{\Omega}^1_c(M, V(\mathfrak{K})), \quad (\eta, \xi) \mapsto [\kappa(\eta, \nabla\xi)] \tag{1}$$

defines a Lie algebra cocycle, where $\overline{\Omega}^1_c(M, V(\mathfrak{K}))$ denotes $\Omega^1_c(M, V(\mathfrak{K})) / \mathtt{d}\,\Omega^0_c(M, V(\mathfrak{K}))$, and the subscript c denotes compact support.

The classes $[\omega_\nabla]$ in $H^2(\Gamma_c(\mathfrak{K}), \overline{\Omega}^1_c(M, V(\mathfrak{K})))$ are naturally parameterised by the quotient $\Omega^1(M, \operatorname{der}(\mathfrak{K}))/\Omega^1(M, \mathfrak{K})$, as the Lie connections constitute an affine space over $\Omega^1(M, \operatorname{der}(\mathfrak{K}))$, and $\nabla - \nabla' = \operatorname{ad}_\Xi$ for some $\Xi \in \Omega^1(M, \mathfrak{K})$ implies that $(\omega_\nabla - \omega_{\nabla'})(\xi, \eta) = [\kappa(\xi, [\Xi, \eta])] = -[\kappa(\Xi, [\xi, \eta])]$ is a coboundary. In particular, the class $[\omega_\nabla] = [\omega]$ is canonical if the fibres are semi-simple, because in that case $\operatorname{der}(\mathfrak{K}) = \mathfrak{K}$. We endow our spaces of smooth forms and sections with the usual LF-topology (cf. [Mai02]), and denote the continuous linear maps and cohomologies by $\operatorname{Hom}_{\mathrm{ct}}$ and H^*_{ct} respectively. The first result of this paper is now the following

Proposition I.1. *If $\mathfrak{K}$ is a finite-dimensional Lie algebra bundle with semi-simple fibres, then $[\omega] \in H^2_{\mathrm{ct}}(\Gamma_c(\mathfrak{K}), \overline{\Omega}^1_c(M, V(\mathfrak{K})))$ is (weakly) universal, i.e. the map*

$$\operatorname{Hom}_{\mathrm{ct}}(\overline{\Omega}^1_c(M, V(\mathfrak{K})), X) \to H^2_{\mathrm{ct}}(\Gamma_c(\mathfrak{K}), X), \quad \varphi \mapsto [\varphi \circ \omega] \tag{2}$$

is bijective for each topological vector space X, considered as a trivial $\Gamma_c(\mathfrak{K})$-module.

Note that by Proposition II.4 and [Nee02b, Lem. 1.12] the associated central extension of locally convex Lie algebras is also universal in the stronger sense of [Nee02b].

The proof of the previous proposition will be carried out in Section II. Note that for semi-simple Lie algebras, κ is the universal symmetric invariant bilinear form [NW09, App. B]. It is equal to the Killing form for all central simple real Lie algebras, but not for e.g. the real simple Lie algebras $\mathfrak{sl}_n(\mathbb{C})$. The motivating example for the above proposition is the gauge algebra $\Gamma(\mathrm{ad}(P))$ of a principal fibre bundle $P \to M$, whence the title of this paper.

We now formulate the corresponding result for Lie groups rather than Lie algebras. Let $\mathcal{K} \to M$ be a finite-dimensional, locally trivial bundle of Lie groups, and set $\mathfrak{K} := L(\mathcal{K})$. Unlike in the case of Lie algebras, we will assume the base manifold M to be compact and connected. Consequently, the fibres of $\mathcal{K}$ will all be isomorphic to a single Lie group K with Lie algebra $\mathfrak{k}$. We will assume that $\mathfrak{k}$ is semi-simple, and that $\pi_0(K)$ is finitely generated. The latter implies that $\mathrm{Aut}(K)$ carries a natural Lie group structure [Bou98, Ch. III, §10] modelled on $\mathfrak{k} = \mathrm{der}(\mathfrak{k})$. We may thus consider $\mathcal{K}$ to be associated to its principal $\mathrm{Aut}(K)$-frame bundle $\mathrm{Fr}(\mathcal{K}) := \cup_{x\in M} \mathrm{Iso}(\mathcal{K}_x, K)$, placing us in the setting considered in [NW09]. The group $\Gamma(\mathcal{K})$ of smooth sections then carries naturally the structure of a Fréchet-Lie group with Lie algebra $\Gamma(\mathfrak{K})$, see [NW09, App. A]. The connected component $\Gamma(\mathcal{K})_0$ possesses a central extension, which we will show to be universal. It is constructed as follows.

First we have to ensure a technical condition in order to use the more detailed results from [NW09]. For this we consider the homomorphism $\rho\colon \pi_1(M) \to \mathrm{GL}(V(\mathfrak{k}))$, given by composing the connecting homomorphism $\delta\colon \pi_1(M) \to \pi_0(\mathrm{Aut}(K))$ of the fibration $\mathrm{Fr}(\mathcal{K}) \to M$ with the natural representation $\pi_0(\mathrm{Aut}(K)) \to \mathrm{GL}(V(\mathfrak{k}))$. ($\mathrm{Aut}(K)$ acts naturally on $V(\mathfrak{k})$, and $\mathrm{Aut}(K)_0$ acts trivially because $\mathrm{der}(\mathfrak{k})$ does so.)

If now $\mathbb{D}_{\mathcal{K}} := \rho(\pi_1(M))$ is finite, then by [NW09, Cor. 4.18] the period group Π_ω associated to ω is discrete, and by [Nee02a, Th. 7.9] there exists a central extension

$$\overline{\Omega}^1(M, V(\mathfrak{K}))/\Pi_\omega \to \widehat{\Gamma(\mathcal{K})_0} \xrightarrow{q} \widetilde{\Gamma(\mathcal{K})_0}$$

(i.e. a short exact sequence of locally convex Lie groups with central kernel that is a smooth principal bundle, cf. [Nee02a]) where $c\colon \widetilde{\Gamma(\mathcal{K})_0} \to \Gamma(\mathcal{K})_0$ denotes the simply connected cover. In order so see that $c \circ q$ also defines a central extension we observe that $\Gamma(\mathcal{K})_0$ is a covering group of $\Gamma(\mathrm{Ad}(\mathrm{Fr}(\mathcal{K})))_0$. In fact, the exact sequence $Z(K) \to K \to \mathrm{Inn}(K) \subseteq \mathrm{Aut}(K)$ induces a (fibrewise) exact sequence

$$\mathcal{Z}(\mathcal{K}) \to \mathcal{K} \to \mathcal{I}nn(\mathcal{K}),$$

which leads to a covering $Z(K) \cong \Gamma(\mathcal{Z}(\mathcal{K})) \to \Gamma(\mathcal{K})_0 \to \Gamma(\mathcal{I}nn(\mathcal{K}))_0$ since $Z(K)$ is discrete (note that $\mathcal{I}nn(\mathcal{K})$ is obtained from $\mathcal{K}$ by a push-forward along the morphism $K \to \mathrm{Inn}(K)$ of $\mathrm{Aut}(K)$-spaces). Since K is semi-simple, we have that $\mathrm{Inn}(K)$ is an open subgroup of $\mathrm{Aut}(K)$, and thus $\Gamma(\mathrm{Ad}(\mathrm{Fr}(\mathcal{K})))_0 = \Gamma(\mathcal{I}nn(\mathcal{K}))_0$. It has been shown in [NW09, Cor. 22] that the adjoint action of $\Gamma(\mathrm{ad}(\mathrm{Fr}(\mathcal{K})))$ integrates to an action of $\Gamma(\mathrm{Ad}(\mathrm{Fr}(\mathcal{K})))_0$, and since $\Gamma(\mathcal{K})_0 \to \Gamma(\mathrm{Ad}(\mathrm{Fr}(\mathcal{K})))_0$ is a covering, the adjoint action of $\Gamma(\mathfrak{K}) \cong \Gamma(\mathrm{ad}(\mathrm{Fr}(\mathcal{K})))$ integrates to an action of $\Gamma(\mathcal{K})_0$ on $\overline{\Omega}^1(M, V(\mathfrak{K})) \oplus_\omega \Gamma(\mathfrak{K})$. From [Nee02a, Rem. 7.14] it now follows that

$$Z \to \widehat{\Gamma(\mathcal{K})_0} \xrightarrow{c\,\circ\, q} \Gamma(\mathcal{K})_0 \tag{3}$$

with $Z := \ker(c \circ q) \cong (\overline{\Omega}^1_c(M, V(\mathfrak{K}))/\Pi_\omega) \times \pi_1(\Gamma(\mathcal{K})_0)$ is a central extension, which turns out to be universal:

Theorem I.2. *Let $\mathcal{K} \to M$ be a finite-dimensional Lie group bundle over a compact and connected manifold M, such that its typical fibre K is semi-simple and has finitely generated $\pi_0(K)$. If, moreover, $\mathbb{D}_{\mathcal{K}}$ is finite, then the central extension (3) is universal for abelian Lie groups modelled on Mackey-complete locally convex spaces.*

This means that for each central extension $A \to \widehat{G} \to \Gamma(\mathcal{K})_0$ of locally convex Lie groups that is a smooth principal bundle and such that $L(A)$ is Mackey-complete, there exists a unique morphism

$\varphi\colon \widehat{\Gamma(\mathcal{K})}_0 \to \widehat{G}$ such that the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\varphi|_Z"] & \widehat{\Gamma(\mathcal{K})}_0 \arrow[r] \arrow[d, "\varphi"] & \Gamma(\mathcal{K})_0 \arrow[d, equal] \\ A \arrow[r] & \widehat{G} \arrow[r] & \Gamma(\mathcal{K})_0 \end{tikzcd}$$

commutes.

In general it is not easy to decide whether the group $\mathbb{D}_{\mathcal{K}}$ is finite. In order to make the above theorem applicable, we list here some conditions that ensure the finiteness of $\mathbb{D}_{\mathcal{K}}$.

- If the image of the connecting homomorphism $\delta\colon \pi_1(M) \to \pi_0(\mathrm{Aut}(K))$ of the fibration $\mathrm{Fr}(\mathcal{K}) \to M$ is finite, then $\mathbb{D}_{\mathcal{K}}$ is finite. This is in particular the case if $\mathcal{K}$ is trivial, reproducing the universal central extension of $\Gamma(\mathcal{K})_0 \cong C^\infty(M,K)_0$ of [MN03].

- If $\mathfrak{k}$ is compact and simple (or, more generally, central simple real), then the Killing form is universal and $\mathrm{Aut}(K)$-invariant. Thus the homomorphism $\pi_0(\mathrm{Aut}(K)) \to \mathrm{GL}(V(\mathfrak{k}))$ vanishes, so that $\mathbb{D}_{\mathcal{K}}$ and $V(\mathfrak{K})$ are trivial.

- If $\pi_0(K)$ and $Z(K_0)$ are both finite, then so is $\pi_0(\mathrm{Aut}(K))$, and therefore $\mathbb{D}_{\mathcal{K}}$. This can be seen as follows. The image of the natural homomorphism $\mathrm{Aut}(K) \to \mathrm{Aut}(\mathfrak{k}) \times \mathrm{Aut}(\pi_0(K))$ is both open and closed, and its kernel is naturally identified with the group of crossed homomorphisms from $\pi_0(K)$ to $Z(K_0)$. Since both the kernel and the image have finitely many connected components (recall that $\pi_0(\mathrm{Aut}(\mathfrak{k}))$ corresponds to the diagram automorphisms of $\mathfrak{k}$), $\mathrm{Aut}(K)$ must have finitely many components as well.

Note that the concepts of universality and weak universality of cocycles and central extensions that are used in our main references [Nee02b] and [Mai02] differ slightly, but coincide in the case of *perfect* Lie algebras. We will see in the next section that $\Gamma(\mathfrak{K})$ is in fact perfect. We will give precise references for the equality of these concepts at each stage where an ambiguity might occur.

Our results prove quite some occurrences on the claimed universality of certain central extensions of gauge groups and algebras, see for instance [LMNS98] or [MW04]. Note also that a common mistake is made in some treatments of this subject by considering the Killing form as universal invariant bilinear form κ, which is not always justified (see above).

II Universality of the Lie Algebra Cocycle

This Section is devoted to the proof of Proposition I.1, in which a pivotal role will be played by fact that our Lie algebra cocycles are essentially local in nature. We will state and prove this in a slightly more general setting where the fibres of our Lie algebra bundle are allowed to be infinite-dimensional. More precisely, we will require $\mathfrak{K} \to M$ to be a locally trivial bundle of locally convex topological Lie algebras, with a base M that is finite-dimensional but not necessarily compact. We then equip its space of compactly supported sections $\Gamma_c(\mathfrak{K}) = \varinjlim \Gamma_K(\mathfrak{K})$ with the usual inductive limit topology.

Throughout this section, X will denote an arbitrary topological vector space, considered as a trivial module for the Lie algebra in question.

Definition II.1. A (not necessarily continuous) 2-cocycle on $\Gamma_c(\mathfrak{K})$ with values in X is called *diagonal* if $\psi(\eta,\xi) = 0$ whenever $\mathrm{supp}(\eta) \cap \mathrm{supp}(\xi) = \emptyset$. ■

Combined with Proposition II.4, the following Lemma shows that all continuous 2-cocycles are diagonal if the fibres of $\mathfrak{K}$ are topologically perfect, and that all 2-cocycles (also the non-continuous ones) on $\Gamma_c(\mathfrak{K})$ are diagonal if the fibres of $\mathfrak{K}$ are finite-dimensional and perfect.

Lemma II.2. *If $\Gamma_c(\mathfrak{K}|_U)$ is topologically perfect for each open $U \subseteq M$, then every continuous cocycle is local. If, moreover, each $\Gamma_c(\mathfrak{K}|_U)$ is perfect, then every cocycle is local.*

Proof. Suppose that ξ and η in $\Gamma_c(\mathfrak{K})$ have disjoint support and set $U := M \setminus \operatorname{supp}(\xi)$. Since ξ and η have disjoint support, we have that $\eta|_U \in \Gamma_c(\mathfrak{K}|_U)$. By assumption, we can write

$$\eta|_U = \varinjlim_i \eta_i,$$

where $(\eta_i)_{i\in I}$ is a convergent Cauchy net and $\eta_i = \sum_j [\mu_{j,i}, \nu_{j,i}]$ is a finite sum of commutators in $\Gamma_c(\mathfrak{K}|_U)$. In case that $\Gamma_c(\mathfrak{K}|_U)$ is perfect, we may assume that I is finite.

We now set $\mu'_{j,i}$ to be the continuous extension of $\mu_{j,i}$ by zero, and likewise we define $\nu'_{j,i}$ by extending $\nu_{j,i}$. Observe that we have in particular $[\xi, \mu'_{j,i}] = [\xi, \nu'_{j,i}] = 0$. This now implies

$$\begin{aligned}\psi(\xi,\eta) &= \psi(\xi, \varinjlim_i \sum_j [\mu'_{j,i}, \nu'_{j,i}]) = \varinjlim_i \sum_j \psi(\xi, [\mu'_{j,i}, \nu'_{j,i}]) \\ &= \varinjlim_i \sum_j \big(\psi(\mu'_{j,i}, [\xi, \nu'_{j,i}]) + \psi(\nu'_{j,i}, [\mu'_{j,i}, \xi])\big) = \varinjlim_i \sum_j \big(\psi(\mu'_{j,i}, 0) + \psi(\nu'_{j,i}, 0)\big) = 0\end{aligned}$$

for each cocycle in the case of finite I and for each continuous cocycle in the case of arbitrary I. ∎

Remark II.3. The previous proof also works for the Lie algebra of compactly supported vector fields $\mathrm{Vec}_c(M)$ (cf. [Jan10, Cor. 7.4] and [SP54]), and can even be generalised (cf. [Ame75]) to $\mathrm{Vec}(M)$. Now both Lie algebras satisfy the conditions from Lemma II.2, so that their second Lie algebra cohomology is diagonal (cf. [GF70, Cor. 6.3]). Results such as Peetre's theorem [Pee60] tell one that continuity and diagonality are not worlds apart, which raises the interesting question of whether $H^2(\mathrm{Vec}(M),\mathbb{R})$ is in fact isomorphic to the *continuous* cohomology $H^2_{\mathrm{ct}}(\mathrm{Vec}(M),\mathbb{R})$. The latter has been calculated explicitly (cf. [GF70] and [Fuk86]). ∎

Proposition II.4. *If the fibres of $\mathfrak{K}$ are topologically perfect, then so is $\Gamma_c(\mathfrak{K})$. If the fibres of $\mathfrak{K}$ are finite-dimensional perfect Lie algebras, then $\Gamma_c(\mathfrak{K})$ is even perfect. The same conclusions hold in particular for $\Gamma_c(\mathfrak{K}|_U)$ with arbitrary open $U \subseteq M$.*

Proof. Since each $\xi \in \Gamma_c(\mathfrak{K})$ is compactly supported, it can be written as $\xi = \sum_{i=1}^N \xi_i$ with each ξ_i having compact support in a trivialising open subset U_i. In order to substantiate the first claim, it thus suffices to show that $C^\infty_c(U_i,\mathfrak{k})$ is topologically perfect if $\mathfrak{k}$ is topologically perfect. We clearly have

$$[C^\infty_c(U_i)\otimes\mathfrak{k}, C^\infty_c(U_i)\otimes\mathfrak{k}] = C^\infty_c(U_i)\otimes[\mathfrak{k},\mathfrak{k}].$$

Now $C^\infty_c(U_i)\otimes[\mathfrak{k},\mathfrak{k}]$ is dense in $C^\infty_c(U_i)\,\widehat{\otimes}\,\overline{\mathfrak{k}}$ ($\overline{\mathfrak{k}}$ denoting the uniform completion of $\mathfrak{k}$), and since $C^\infty_c(U_i)\,\widehat{\otimes}\,\overline{\mathfrak{k}} \cong C^\infty_c(U_i,\overline{\mathfrak{k}})$ [Gro55, Chap. II, p. 81], $C^\infty_c(U_i)\otimes[\mathfrak{k},\mathfrak{k}]$ is dense, considered as a subspace of $C^\infty_c(U_i,\mathfrak{k})$. Thus $[C^\infty_c(U_i,\mathfrak{k}), C^\infty_c(U_i,\mathfrak{k})]$ is dense in $C^\infty_c(U_i,\mathfrak{k})$.

If $\mathfrak{k}$ is a finite-dimensional, perfect Lie algebra, then all spaces considered above are in fact equal and we have

$$[C^\infty_c(U_i,\mathfrak{k}), C^\infty_c(U_i,\mathfrak{k})] = [C^\infty_c(U_i)\otimes\mathfrak{k}, C^\infty_c(U_i)\otimes\mathfrak{k}] = C^\infty_c(U_i)\otimes[\mathfrak{k},\mathfrak{k}] = C^\infty_c(U_i,\mathfrak{k}). \qquad \blacksquare$$

Corollary II.5. *If the fibres of $\mathfrak{K}$ are topologically perfect, then $\mathcal{S}_{\mathrm{ct}}(U) = H^2_{\mathrm{ct}}(\Gamma_c(\mathfrak{K}|_U), X)$ constitutes a monopresheaf of vector spaces. If the fibres are finite-dimensional perfect Lie algebras, then the same applies to $\mathcal{S}(U) = H^2(\Gamma_c(\mathfrak{K}|_U), X)$.*

Proof. A monopresheaf is a presheaf that satisfies the 'local identity' axiom, but not necessarily the 'gluing' axiom. If $V \subseteq U$, then the restriction map $\rho_{VU}\colon \mathcal{S}(U) \to \mathcal{S}(V)$ is defined as $\rho_{VU}([\psi_U]) = [\psi_V]$ with $\psi_V(\xi,\eta) := \psi_U(\widetilde{\xi},\widetilde{\eta})$, where $\widetilde{\xi}$ and $\widetilde{\eta}$ denote the extensions by zero of ξ and η from V to U. In particular, ψ_V is continuous if ψ_U is so. The class $[\psi_V]$ does not depend on the choice of $\psi_U \in [\psi_U]$. Indeed, if $\psi_U - \psi'_U = \delta\,\beta_U$, then $\psi_V(\xi,\eta) - \psi'_V(\xi,\eta) = \beta_U([\widetilde{\xi},\widetilde{\eta}]) = \beta_U(\widetilde{[\xi,\eta]}) = \delta\,\beta_V([\xi,\eta])$, where again β_V is continuous if β_U is. The presheaf property $\rho_{WV} \circ \rho_{VU} = \rho_{WU}$ is clear from the definition.

The fact that this presheaf is actually a monopresheaf will now follow from Lemma II.2 and Proposition II.4, which tell us that the cocycles considered are diagonal. Let $\{V_i\}_{i\in I}$ be an open cover of U. For the 'local identity' axiom to hold, we must prove that $[\psi_U]$ vanishes if its restriction $[\psi_i]$ (with $\psi_i := \psi_{V_i}$) vanishes for all $i \in I$. Replace $\{V_i\}_{i\in I}$ by a subcover that intersects every compactum in only finitely many V_i, and equip it with a partition of unity $\{\lambda_i\}_{i\in I}$. Let $\psi_i = \delta\,\beta_i$. We then define

$$\beta_U(\chi) := \sum_{i\in I} \beta_i(\lambda_i\chi)$$

for $\chi \in \Gamma_c(\mathfrak{K}|_U)$, the sum containing only finitely many nonzero terms because $\mathrm{supp}(\chi)$ intersects only finitely many V_i. Clearly β_U is continuous if all the β_i are. We prove that $\psi_U = \delta\beta_U$. By diagonality of the cocycle ψ_U, we can write $\psi_U(\lambda_i\xi,\eta) = \psi_U(\lambda_i\xi,\lambda'_i\eta)$, where λ'_i is some function with support contained in V_i that satisfies $\lambda'_i \equiv 1$ on a neighbourhood of $\mathrm{supp}(\lambda_i)$. Indeed, $\psi_U(\lambda_i\xi,\eta-\lambda'_i\eta) = 0$ because $\lambda_i\xi$ and $\eta - \lambda'_i\eta$ have disjoint support. We can therefore write

$$\psi_U(\xi,\eta) = \sum_{i\in I}\psi_U(\lambda_i\xi,\lambda'_i\eta) = \sum_{i\in I}\psi_i(\lambda_i\xi,\lambda'_i\eta) = \sum_{i\in I}\beta_i(\lambda_i\lambda'_i[\xi,\eta]) = \beta_U([\xi,\eta])\,,$$

where in the last step we used $\lambda_i\lambda'_i = \lambda_i$. Thus $[\psi_U] = [\delta\beta_U] = 0$, as required. ■

Proposition II.6. *The assignment $\mathcal{F}(U) = \mathrm{Hom}(\overline{\Omega}{}^1_c(U,V(\mathfrak{K})),X)$ constitutes a sheaf of vector spaces. The same goes for $\mathcal{F}_{\mathrm{ct}}(U) = \mathrm{Hom}_{\mathrm{ct}}(\overline{\Omega}{}^1_c(U,V(\mathfrak{K})),X)$.*

Recall that $\overline{\Omega}{}^1_c(U,V(\mathfrak{K}))$ was defined as $\Omega^1_c(U,V(\mathfrak{K}))/\,\mathtt{d}\,\Omega^0_c(U,V(\mathfrak{K}))$. For two vector spaces X and Y, $\mathrm{Hom}(X,Y)$ denotes the space of linear maps from X to Y. If X and Y happen to be topological vector spaces, then $\mathrm{Hom}_{\mathrm{ct}}(Y,X)$ is the space of continuous linear maps.

Proof. Let $W \subseteq V$, with W and V open in M. The restriction $\rho_{WV}\colon \mathcal{F}(V) \to \mathcal{F}(W)$ is the dual of the map $\iota_{VW} : \overline{\Omega}{}^1_c(W,V(\mathfrak{K})) \to \overline{\Omega}{}^1_c(V,V(\mathfrak{K}))$ defined as follows. Take $[\omega_W] \in \Omega^1_c(W,V(\mathfrak{K}))$, choose a representative ω_W, extend it by zero to $\omega_V \in \Omega^1_c(V,V(\mathfrak{K}))$ and take its class $[\omega_V] \in \overline{\Omega}{}^1_c(V,V(\mathfrak{K}))$. The result does not depend on the choice of representative. Indeed, if $\omega_W - \omega'_W = \mathtt{d}\,\gamma_W$ with $\gamma_W \in \Omega^0_c(W,V(\mathfrak{K}))$, then one can extend γ_W by zero to $\gamma_V \in \Omega^0_c(V,V(\mathfrak{K}))$ to find $\omega_V - \omega'_V = \mathtt{d}\,\gamma_V$. We can therefore define $\iota_{VW}([\omega_W]) = [\omega_V]$. The fact that $\mathcal{F}(U) := \mathrm{Hom}(\overline{\Omega}{}^1_c(U,V(\mathfrak{K})),X)$ with $\rho_{WV}\colon\mathcal{F}(V)\to\mathcal{F}(W)$ is a presheaf, $\rho_{XW}\circ\rho_{WV} = \rho_{XV}$, follows from the fact that $\check{\mathcal{F}}(U) := \overline{\Omega}{}^1_c(U,V(\mathfrak{K}))$ with $\iota_{VW}\colon\check{\mathcal{F}}(W)\to\check{\mathcal{F}}(V)$ is a precosheaf, $\iota_{VW}\circ\iota_{WX} = \iota_{VX}$. An analogous statement holds for $\mathcal{F}_{\mathrm{ct}}$ because the maps ι_{WV} are continuous.

In order to show that $\mathcal{F}$ is a sheaf, it suffices to show that $\check{\mathcal{F}}$ is a cosheaf. For this, one needs to check ([Bre68, Prop. 1.3]) the following 3 statements.

1. For all open $V, W \subseteq M$, we have $\iota_{V\cup W,V}\check{\mathcal{F}}(V) + \iota_{V\cup W,W}\check{\mathcal{F}}(W) = \check{\mathcal{F}}(V\cup W)$.
2. If $\iota_{V\cup W,V}[\omega_V] = \iota_{V\cup W,W}[\omega_W]$ for $[\omega_V] \in \check{\mathcal{F}}(V)$ and $[\omega_W] \in \check{\mathcal{F}}(W)$, then there exists an element $[\omega_{V\cap W}] \in \check{\mathcal{F}}(V\cap W)$ such that $[\omega_V] = \iota_{V,V\cap W}[\omega_{V\cap W}]$ and $[\omega_W] = \iota_{W,V\cap W}[\omega_{V\cap W}]$.
3. For every system $\{V_i\}_I$ directed upwards by inclusion, the natural map $\lim\limits_{\rightarrow}\check{\mathcal{F}}(V_i) \to \check{\mathcal{F}}(\cup_I V_i)$ is an isomorphism.

Statement 1 is true because one can use a partition of unity $\{\lambda_V,\lambda_W\}$ to write $\omega_{V\cup W} = \lambda_V\omega_{V\cup W} + \lambda_W\omega_{V\cup W}$.

We proceed to prove statement 2. Suppose that $\iota_{V\cup W,V}[\omega_V] = \iota_{V\cup W,W}[\omega_W]$. Then $\omega_V = \omega_W + \mathtt{d}\,\gamma_{V\cup W}$, with $\mathrm{supp}(\gamma_{V\cup W}) \subseteq V\cup W$. Write $\gamma_{V\cup W} = \gamma_V - \gamma_W$, with $\mathrm{supp}(\gamma_V)\subseteq V$ and $\mathrm{supp}(\gamma_W)\subseteq W$. Then $\omega_V - \mathtt{d}\,\gamma_V = \omega_W - \mathtt{d}\,\gamma_W$, so that the support of both is contained in $V\cap W$. Statement 2 then holds with $[\omega_{V\cap W}] = [\omega_V - \mathtt{d}\,\gamma_V] = [\omega_W - \mathtt{d}\,\gamma_W]$ in $\check{\mathcal{F}}(V\cap W)$.

Finally, we verify statement 3 by first observing that since the support of any $\omega \in \Omega^1_c(\cup_I V_i)$ is compact, it is contained in some $V_{i_1} \cup \ldots \cup V_{i_l}$. This shows surjectivity. To verify injectivity, we observe that if $[\omega] = 0$ in $\check{\mathcal{F}}(\cup_I V_i)$, then $\omega = \mathtt{d}\gamma$ for $\gamma \in \Omega^0_c(\cup_I V_i, V(\mathfrak{K}))$. Since $\mathrm{supp}(\gamma)$ is compact, we have $\gamma \in \Omega^0_c(V_{i_1} \cup \ldots \cup V_{i_l}, V(\mathfrak{K}))$. Thus $[\omega] = 0$ in $\check{\mathcal{F}}(V_{i_1} \cup \ldots \cup V_{i_l})$ and, consequently, in $\varinjlim \check{\mathcal{F}}(V_i)$. ∎

From now on, we restrict attention to the case where the fibres of $\mathfrak{K}$ are finite-dimensional, semi-simple Lie algebras. We have exhibited the sheaf $\mathcal{F}_{\mathrm{ct}}(U) = \mathrm{Hom}_{\mathrm{ct}}(\overline{\Omega}^1_c(U, V(\mathfrak{K})), X)$ and the monopresheaf $\mathcal{S}_{\mathrm{ct}}(U) = H^2_{\mathrm{ct}}(\Gamma_c(\mathfrak{K}|_U), X)$. The canonical class $[\omega]$ then induces a morphism

$$\mu_U \colon \mathcal{F}_{\mathrm{ct}}(U) \to \mathcal{S}_{\mathrm{ct}}(U)$$

of presheaves. For $\xi, \eta \in \Gamma_c(\mathfrak{K}|_U)$, we have $\omega(\eta, \xi) = [\kappa(\eta, \nabla\xi)]$ in $\overline{\Omega}^1_c(U, V(\mathfrak{K}))$, and the morphism μ_U is then simply defined as $\mu_U \varphi = [\varphi \circ \omega]$.

If $U \subset M$ is a trivialising neighbourhood, then a local trivialisation $\mathfrak{K}|_U \cong U \times \mathfrak{k}$ induces isomorphisms $\mathcal{F}_{\mathrm{ct}}(U) \cong \mathrm{Hom}_{\mathrm{ct}}(\overline{\Omega}^1_c(U, V(\mathfrak{k})), X)$ and $\mathcal{S}_{\mathrm{ct}}(U) \cong H^2_{\mathrm{ct}}(C^\infty_c(U, \mathfrak{k}), X)$. The map μ_U then takes the shape $\varphi \mapsto [\varphi \circ \omega_{U,\mathfrak{k}}]$ with $\omega_{U,\mathfrak{k}}$ the cocycle $\omega_{U,\mathfrak{k}}(f, g) = [\kappa(f, dg)]$. Indeed, $\nabla\xi$ corresponds with $dg + [A, g]$ in the local trivialisation ($\mathrm{der}(\mathfrak{k}) \cong \mathfrak{k}$ for semi-simple Lie algebras) so that $\kappa(\eta, \nabla\xi)$ corresponds with $\kappa(f, dg + [A, g])$. This differs from $\kappa(f, dg)$ by a mere coboundary $-\kappa(A, [f, g])$. The following theorem shows that μ_U is an isomorphism for sufficiently small U.

Theorem II.7. *If $\mathfrak{k}$ is a finite-dimensional semi-simple Lie algebra and U is a finite-dimensional manifold, then the cocycle*

$$\omega_{U,\mathfrak{k}} \colon C_c(U, \mathfrak{k}) \times C^\infty_c(U, \mathfrak{k}) \to \Omega^1(U, V(\mathfrak{k}))/dC^\infty(U, \mathfrak{k}), \quad (f, g) \mapsto [\kappa(f, dg)] \tag{4}$$

is (weakly) universal. This means that the linear map

$$\mathrm{Hom}_{\mathrm{ct}}((\Omega^1_c(U, V(\mathfrak{k}))/dC^\infty_c(U, \mathfrak{k})), X) \to H^2_{\mathrm{ct}}(C^\infty_c(U, \mathfrak{k}), X), \quad \varphi \mapsto [\varphi \circ \omega_{U,\mathfrak{k}}]$$

is an isomorphism.

Note that since $C^\infty_c(M, \mathfrak{k})$ is not unital we cannot use [Mai02, Th. 16] directly, as claimed in [Mai02, Cor. 18].

Proof. The combination of [Mai02, Th. 11] and [Mai02, Th. 16] shows that

$$\mathrm{Hom}((\Omega^1_c(U, V(\mathfrak{k}))/dC^\infty_c(U, \mathfrak{k})), X) \to H^2_{\mathrm{ct}}(C^\infty_c(U, \mathfrak{k}) \rtimes \mathfrak{k}, X), \quad \varphi \mapsto [\varphi \circ \omega_{U,\mathfrak{k}}]$$

is an isomorphism. It remains to be shown that the canonical inclusion $i \colon C^\infty_c(U, \mathfrak{k}) \to C^\infty_c(U, \mathfrak{k}) \rtimes \mathfrak{k}$ induces an isomorphism $H^2_{\mathrm{ct}}(i)$. We first note that we can extend each cocycle ω on $C^\infty_c(U, \mathfrak{k})$ to $C^\infty_c(U, \mathfrak{k}) \rtimes \mathfrak{k}$ if we interpret $\mathfrak{k}$ as constant functions. In other words, we set $\omega(x, y) = 0$ for $x, y \in \mathfrak{k}$, and $\omega(f, x) := \omega(f, \lambda \cdot x)$ for $f \in C^\infty_c(U, \mathfrak{k})$, $x \in \mathfrak{k}$ and $\lambda \in C^\infty_c(U)$ with $\lambda \equiv 1$ on $\mathrm{supp}(f)$. Since ω is diagonal, this does not depend on the choice of λ. The extension is again a cocycle $\omega(x, [f, g]) + \text{cyclic} = \omega(\lambda \cdot x, [f, g]) + \text{cyclic} = 0$ and $\omega(f, [x, y]) + \text{cyclic} = \omega(f, [\lambda \cdot x, \lambda' \cdot y]) + \text{cyclic} = 0$ for $x, y \in \mathfrak{k}$, $f, g \in C^\infty_c(U, \mathfrak{k})$, and λ equal to 1 on $\mathrm{supp}(f)$ (and $\mathrm{supp}(g)$), λ' equal to 1 on $\mathrm{supp}(\lambda)$. This shows that $H^2_{\mathrm{ct}}(i)$ is surjective.

If $[\omega] \in \ker(H^2_{\mathrm{ct}}(i))$, then $\omega(f, g) = \lambda([f, g])$ for $f, g \in C^\infty_c(U, \mathfrak{k})$ and $\lambda \colon C^\infty_c(U, \mathfrak{k}) \to X$ linear and continuous. If we extend λ by 0 to $\mathfrak{k}$, then $\omega' = \omega - \lambda \circ [\,\cdot\,, \cdot\,]$ is a cocycle that vanishes on $C^\infty_c(U, \mathfrak{k}) \times C^\infty_c(U, \mathfrak{k})$. We have $\omega'([f, g], x) + \omega'([g, x], f) + \omega'([x, f], g) = 0$ and as the last two terms vanish, so does the first. Since $C^\infty_c(U, \mathfrak{k})$ is perfect, ω' vanishes on $C^\infty_c(U, \mathfrak{k}) \times \mathfrak{k}$ and on $\mathfrak{k} \times C^\infty_c(U, \mathfrak{k})$, and factors through a cocycle on $\mathfrak{k} \times \mathfrak{k}$, which is a coboundary by Whitehead's Lemma. Thus $H^2_{\mathrm{ct}}(i)$ is also injective. ∎

Summarising, we have defined a morphism $\mu \colon \mathcal{F}_{\mathrm{ct}} \to \mathcal{S}_{\mathrm{ct}}$ from a sheaf to a monopresheaf that is an isomorphism on sufficiently small subsets U of M, at least in the case that the fibres of $\mathfrak{K}$ are semi-simple. According to the following standard proposition, the monopresheaf $\mathcal{S}_{\mathrm{ct}}$ must then in fact be a sheaf, and the morphism μ an isomorphism of sheaves. This means that in particular $\mu_M \colon \mathcal{F}_{\mathrm{ct}}(M) \to \mathcal{S}_{\mathrm{ct}}(M)$ is an isomorphism. In other words, the map $\varphi \mapsto [\varphi \circ \omega]$ is an isomorphism

$$\mathrm{Hom}_{\mathrm{ct}}(\overline{\Omega}^1_c(M, V(\mathfrak{K})), X) \cong H^2_{\mathrm{ct}}(\Gamma_c(\mathfrak{K}), X)\,, \tag{5}$$

proving Proposition I.1.

Proposition II.8. *Let $\mathcal{F}$ be a sheaf, $\mathcal{S}$ a monopresheaf (i.e. a presheaf that satisfies the local identity axiom), and let $\mu \colon \mathcal{F} \to \mathcal{S}$ be a morphism of presheaves such that each $x \in M$ has an open neighbourhood V such that $\mu_W \colon \mathcal{F}(W) \to \mathcal{S}(W)$ is an isomorphism for any open $W \subseteq V$. Then $\mathcal{S}$ is a sheaf, and μ an isomorphism.*

Proof. We show that $\mu_U \colon \mathcal{F}(U) \to \mathcal{S}(U)$ is an isomorphism for arbitrary open $U \subseteq M$. Fix an open cover $\{V_i\}_{i\in I}$ of U such that $\mu_W \colon \mathcal{F}(W) \to \mathcal{S}(W)$ is an isomorphism for all $W \subseteq V_i$. First of all, we show that μ_U is injective. Suppose that $\mu_U(f_U) = 0$ in $\mathcal{S}(U)$. Then certainly $\rho_{V_iU}\mu_U(f_U) = \mu_{V_i}\rho_{V_iU}(f_U) = 0$ for all $i \in I$, and since μ_{V_i} is an isomorphism we have $f_{V_i} := \rho_{V_iU}(f_U) = 0$. But if $f_{V_i} = 0$ for all $i \in I$, then f_U must be 0 by the 'local identity' axiom for $\mathcal{F}$. Next, we show that μ_U is surjective. Given $s_U \in \mathcal{S}(U)$, we construct an $f_U \in \mathcal{F}(U)$ such that $\mu_U(f_U) = s_U$. Set $s_i := \rho_{V_iU}(s_U)$, so $\rho_{V_{ij}V_i}(s_i) = \rho_{V_{ij}V_j}(s_j)$ by the presheaf property of $\mathcal{S}$. (We write $V_{ij} = V_i \cap V_j$.) Set $f_i := \mu^{-1}_{V_i}(s_i)$ and observe $\mu_{V_{ij}}\rho_{V_{ij}V_i}(f_i) = \rho_{V_{ij}V_i}(s_i) = \rho_{V_{ij}V_j}(s_j) = \mu_{V_{ij}}\rho_{V_{ij}V_j}(f_j)$. Since $\mu_{V_{ij}}$ is an isomorphism, $\rho_{V_{ij}V_i}f_i = \rho_{V_{ij}V_j}f_j$. By the gluing property of $\mathcal{F}$, the f_i then extend to an $f_U \in \mathcal{F}(U)$ with $\rho_{V_iU}f_U = f_i$. Since $\rho_{V_iU}\mu_U(f_U) = \mu_{V_i}\rho_{V_iU}(f_U) = s_i$ for all $i \in I$, we must have $\mu_U(f_U) = s_U$ by the 'local identity' axiom for $\mathcal{S}$. ■

Note that the last proposition not only yields Proposition I.1 by eqn. (5), but it also shows that $\mathcal{S}_{\mathrm{ct}}(U) = H^2_{\mathrm{ct}}(\Gamma_c(\mathfrak{K}|_U), X)$ actually constitutes a sheaf.

III Universality of the Gauge Group Extension

In this section, we will use the Recognition Theorem from [Nee02b] in order to prove Theorem I.2. Recall [Nee02b, Def. 4.8] that for a connected locally convex Lie group H, the group $D(\widetilde{H})$ is defined as the kernel of the morphism $\widetilde{H} \to \mathrm{ab}(\mathfrak{h})$ (with $\mathfrak{h} := L(H)$) induced by the Lie algebra morphism $\mathfrak{h} \to \mathrm{ab}(\mathfrak{h})$.

Theorem III.1. *([Nee02b, Th. 4.13]) Let $Z \to \widehat{H} \to H$ be a central extension of Lie groups such that H is a connected and locally convex Lie group, $\mathfrak{z} := L(Z)$ is Mackey complete,*

1. *the induced Lie algebra extension $\mathfrak{z} \to \widehat{\mathfrak{h}} \to \mathfrak{h}$ is weakly $\mathbb{R}$-universal,*
2. *$\widehat{H}$ is simply connected and*
3. *$\pi_1(H) \subseteq D(\widetilde{H})$.*

If the derived extension $\mathfrak{z} \to \widehat{\mathfrak{h}} \to \mathfrak{h}$ is universal for a Mackey complete space $\mathfrak{a}$ (in the sense of [Nee02b, Def. 1.9]), then $\widehat{H}$ is weakly universal for each regular abelian Lie group A with $L(A) = \mathfrak{a}$.

Note that if $\mathfrak{h}$ is perfect, we have $D(\widetilde{H}) = \widetilde{H}$ and the last condition is automatically satisfied.

Proof. (of Theorem I.2) By [Nee02b, Lem. 4.5] it suffices to check that the Lie group extension (3) is weakly universal since $\Gamma(\mathfrak{K})$ is perfect by Proposition II.4. Now Proposition I.1 and [Nee02b, Lem. 1.12] show that the induced Lie algebra extension (2) is actually universal even in the stronger sense of [Nee02b]. By the Recognition Theorem it thus remains to check that $\widehat{\Gamma(\mathcal{K})_0}$ is simply connected. We first note that by [Nee02a, Rem. 7.14] we have $Z \cong \overline{\Omega}^1_c(M, V(\mathfrak{K}))/\Pi_\omega \times \pi_1(\Gamma(\mathcal{K})_0)$. If we consider the long exact homotopy sequence

$$\pi_2(\Gamma(\mathcal{K})_0) \xrightarrow{\delta_2} \pi_1(Z) \to \pi_1(\widehat{\Gamma(\mathcal{K})_0}) \to \pi_1(\Gamma(\mathcal{K})_0) \xrightarrow{\delta_1} \pi_0(Z),$$

then δ_2 is surjective by [Nee02a, Prop. 5.11] and δ_1 is an isomorphism by construction. Thus $\widehat{\Gamma(\mathcal{K})_0}$ is simply connected. ■

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References

[Ame75] Ichiro Amemiya. Lie algebra of vector fields and complex structure. *J. Math. Soc. Japan*, 27(4):545–549, 1975.

[Bou98] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 1–3*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[Bre68] Glen E. Bredon. Cosheaves and homology. *Pacific J. Math.*, 25:1–32, 1968.

[Fuk86] Dmitry B. Fuks. *Cohomology of infinite-dimensional Lie algebras*. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986. Translated from the Russian by A. B. Sosinskiĭ.

[GF70] Izrail M. Gel'fand and Dmitry B. Fuks. Cohomologies of the Lie algebra of formal vector fields. *Izv. Akad. Nauk SSSR Ser. Mat.*, 34:322–337, 1970.

[Gro55] Alexandre Grothendieck. Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.*, 1955(16):140, 1955.

[Jan10] Bas Janssens. *Transformation & Uncertainty*. PhD thesis, University of Utrecht, 2010. Available as arXiv:1011.3035.

[LMNS98] Andrei Losev, Gregory Moore, Nikita Nekrasov, and Samson Shatashvili. Central extensions of gauge groups revisited. *Selecta Math. (N.S.)*, 4(1):117–123, 1998.

[Mai02] Peter Maier. Central extensions of topological current algebras. In *Geometry and analysis on finite- and infinite-dimensional Lie groups (Będlewo, 2000)*, volume 55 of *Banach Center Publ.*, pages 61–76. Polish Acad. Sci., Warsaw, 2002.

[MN03] Peter Maier and Karl-Hermann Neeb. Central extensions of current groups. *Math. Ann.*, 326(2):367–415, 2003.

[MW04] Stephan Mohrdieck and Robert Wendt. Integral conjugacy classes of compact Lie groups. *Manuscripta Math.*, 113(4):531–547, 2004.

[Nee02a] Karl-Hermann Neeb. Central extensions of infinite-dimensional Lie groups. *Ann. Inst. Fourier (Grenoble)*, 52(5):1365–1442, 2002.

[Nee02b] Karl-Hermann Neeb. Universal central extensions of Lie groups. *Acta Appl. Math.*, 73(1-2):175–219, 2002.

[NW09] Karl-Hermann Neeb and Christoph Wockel. Central extensions of groups of sections. *Ann. Glob. Anal. Geom.*, 36(4):381–418, 2009.

[Pee60] Jaak Peetre. Réctification (*sic.*) à l'article "Une caractérisation abstraite des opérateurs différentiels". *Math. Scand.*, 8:116–120, 1960.

[SP54] M. E. Shanks and Lyle E. Pursell. The Lie algebra of a smooth manifold. *Proc. Amer. Math. Soc.*, 5:468–472, 1954.