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Abstract

We examine surjective maps which preserve a fixed bounded distance in both directions on some classical dual polar spaces.

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1 Introduction

This paper is a continuation paper of [9] of the study of bounded distance preserving mappings in the geometries of matrices. We recall that the classical kinds of geometry of matrices studied by Hua and Wan [14] are: The geometry of rectangular matrices, symmetric matrices, Hermitian matrices, and alternate matrices. The matrices are also called the *points* of the geometry. Two matrices x, y of the same kind are called *adjacent* if the rank of $x-y$ equals one, except for alternate matrices; two alternate matrices x, y are adjacent if $\operatorname{rank}(x-y) = 2$. The adjacency relation turns the point set of a matrix space into the set of vertices of a graph. In the *fundamental theorem* of the geometry of matrices, any bijection φ for which φ and φ^{-1} preserve adjacency, i.e., any isomorphism between the related graphs, is determined. We refer to the book of Wan [14] for a wealth of results and references.

In the space of rectangular matrices over a commutative field and the space of Hermitian matrices over a commutative field with characteristic $\neq 2$, Ming-Huat Lim and Joshua Juat-Huan Tan [10] characterize the isomorphisms as

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surjective mappings φ of the space which satisfy

$$d(x,y) \leq k \quad \Leftrightarrow \quad d(x^\varphi, y^\varphi) \leq k \tag{1}$$

for some integer $k \in \{2, \ldots, n-1\}$, where n denotes the maximal rank of matrices in the space. The main idea in their paper is to consider the set $S^{\perp_k} := \{x \in \mathcal{P} \mid d(x,y) \leq k \text{ for all } y \in S\}$ for a nonempty subset $S \subset \mathcal{P}$, where $\mathcal{P}$ is the matrix space. They show that for any $a \neq b \in \mathcal{P}$ with $d(a,b) \leq k$, the following two equivalent properties hold:

$$\{a,b\}^{\perp_k \perp_k} = \{a,b\} \quad \Leftrightarrow \quad 1 < d(a,b) \leq k, \tag{2}$$

$$\left|\{a,b\}^{\perp_k \perp_k}\right| \geq 3 \quad \Leftrightarrow \quad d(a,b) = 1. \tag{3}$$

Recently, Ming-Huat Lim [11] also determine surjective mappings of the Grassmann space satisfying (1).

In the paper [9] we find out five elementary conditions. In any graph with diameter more than two, which satisfies these five elementary conditions, the equivalent properties (2) and (3) hold. Thus any bijection of the graph, which satisfies (1) is an adjacency-isomorphism. We recall the five elementary conditions:

Let G be a (finite or infinite) graph. The set of vertices of G will be denoted by $\mathcal{P}$. Two vertices $x, y \in \mathcal{P}$ are *adjacent* if $\{x,y\}$ is an edge. The distance between two vertices $x, y \in \mathcal{P}$ is written as $d(x,y)$. Then $x, y \in \mathcal{P}$ are adjacent if and only if $d(x,y) = 1$. If G is connected, the triangle inequality holds:

$$d(x,z) \leq d(x,y) + d(y,z) \qquad \forall x, y, z \in \mathcal{P}.$$

The *diameter* of G, denoted by $\operatorname{diam}(G)$, is the maximal distance between two vertices in G. It may be infinite. We study graphs G satisfying the following five conditions.

(A1) The graph G is connected.

(A2) For any vertices $x, y \in \mathcal{P}$ and any integer k with $d(x,y) \leq k \leq \operatorname{diam}(G)$ there is a vertex $z \in \mathcal{P}$ with

$$d(x,z) = d(x,y) + d(y,z) = k.$$

(A3) For all vertices $x, y, z \in \mathcal{P}$ with $d(x,y) = d(x,z) + d(z,y)$, there is a vertex $w \in \mathcal{P}$ with

$$d(w,x) = d(y,z), \ d(w,y) = d(x,z), \text{ and } d(w,z) = d(x,y).$$

(A4) For any $1 \leq k \leq \operatorname{diam}(G)$ and any vertices $x \neq y, z \in \mathcal{P}$ with $d(z,x) = d(z,y) = k$ there is a vertex $w \in \mathcal{P}$ with

$$d(w,z) = 1, \ d(w,x) = k-1, \text{ and } d(w,y) \geq k.$$

Furthermore, for any vertices $x, y, z \in \mathcal{P}$ with $d(x,y) = 3$, $d(z,x) = 2$ and $d(z,y) = 2$ there exists a vertex $w \in \mathcal{P}$ with

$$d(w,z) = 3, \ d(w,x) = 1, \ \text{and } d(w,y) \leq 3.$$

(A5) For any vertices $a, b \in \mathcal{P}$ with $d(a,b) = 1$ there exists a vertex $p \in \mathcal{P} \backslash \{a,b\}$ satisfying

$$d(x,p) \leq \max\{d(x,a), d(x,b)\},$$

for any vertex $x \in \mathcal{P}$.

These five conditions ensure that the properties (2) and (3) hold:

Lemma 1.1. *[9, Lemma 2.3] Let G be a graph which satisfies the conditions (A1)-(A5) and $2 < \operatorname{diam}(G)$. Let $1 < k < \operatorname{diam}(G)$ be an integer. Then for any $a \neq b \in \mathcal{P}$ with $0 < d(a,b) \leq k$,*

$$\left|\{a,b\}^{\perp_k \perp_k}\right| \geq 3 \quad \Leftrightarrow \quad d(a,b) = 1.$$

We have the following theorem.

Theorem 1.1. *[9, Theorem 2.1] Let G, G' be two graphs with graph theoretical distances d, d', respectively, which satisfy the above properties (A1)–(A5) and $2 \leq \operatorname{diam}(G)$. Let $1 \leq k < \operatorname{diam}(G)$ be an integer. If $\varphi : \mathcal{P} \to \mathcal{P}'$ is a surjection which satisfies*

$$d(x,y) \leq k \quad \Leftrightarrow \quad d'(x^\varphi, y^\varphi) \leq k \qquad \forall x, y \in \mathcal{P}.$$

Then φ is an isomorphism.

This result is applied in the paper [9] to the graphs arising on the spaces of rectangular matrices, symmetric, Hermitian matrices under two restrictions, alternate matrices, and Grassmann spaces.

In the following we study the conditions (A1)–(A5) for the projective spaces of matrices, namely

- the projective space of *symmetric matrices – a symplectic dual polar space*,
- the projective space of *Hermitian matrices – a unitary dual polar space*, and
- the projective space of *alternate matrices – an orthogonal dual polar space.*

The projective space of *rectangular matrices* (the *Grassmann space*) has been studied in the paper [9].

2 Polarity

In this section we shortly introduce the dual polar spaces which are of interest in this article. The terminology of *semi-bilinear form* and its connection to duality and polarity are described in Baer [1].

Let $n \geq 2$ be an integer and let $\mathcal{D}$ be a division ring which possesses an involution $^{-}$, i.e., an anti-automorphism of $\mathcal{D}$ whose square equals the identity map id of $\mathcal{D}$. Let $\mathcal{F} := \{a \in \mathcal{D} \mid a = \overline{a}\}$ be the set of all fixed elements of $\mathcal{D}$ under $^{-}$. Let V be the left $2n$-dimensional vector space $\mathcal{D}^{2n}$. Define a non-degenerate semi-bilinear form $(u, v) := uK\overline{v}^t$, where $u = (u_1, \ldots, u_{2n})$, $v = (v_1, \ldots, v_{2n}) \in V$ and K is a $2n \times 2n$ matrix with entries in $\mathcal{D}$, which satisfies

- $K \in GL_{2n}(\mathcal{D})$. This implies the semi-bilinear form is non-degenerate, i.e. $(u, v) = 0$, for all $v \in V$, then $u = 0$.
- $\overline{K}^t = \epsilon K$, where $\epsilon = 1$ or -1. This implies the reflexivity: $(u, v) = 0$ if and only if $(v, u) = 0$.

This semi-bilinear form induces a duality $\perp$ on V, where the dual subspace of $U \leq V$ is defined by $U^{\perp} := \{v \in V \mid (v, u) = 0\, \forall\, u \in U\}$. We write $v \perp u$, if $(v, u) = 0$. There is a dimension formula $\dim U + \dim U^{\perp} = \dim V$, for any subspace U of V. A subspace U of V is called *totally isotropic* if $U \leq U^{\perp}$. If $U = U^{\perp}$ then U is called *self-dual*. It is obvious that

$$U^{\perp\perp} = U \qquad (*)$$

for all subspaces U. A duality on V with the property $(*)$ is also called a *polarity* on V. The self-dual subspaces are just the totally isotropic subspaces with dimension n.

For any subspaces U_1 and U_2 we have the following basic properties:

$$U_1 < U_2 \quad \Leftrightarrow \quad U_2^{\perp} < U_1^{\perp} \tag{4}$$

$$(U_1 + U_2)^{\perp} = U_1^{\perp} \cap U_2^{\perp} \tag{5}$$

$$(U_1 \cap U_2)^{\perp} = U_1^{\perp} + U_2^{\perp} \tag{6}$$

If $\dim U_1 = \dim U_2$, then

$$U_1 \cap U_2^{\perp} = \{0\} \quad \Leftrightarrow \quad U_2 \cap U_1^{\perp} = \{0\} \tag{7}$$

Proposition 2.1. *(Cf. [5, chapter 2, section 3].) For any subspaces U, X of V, which satisfy $U < U^{\perp}$ and $X = X^{\perp}$, then $W := U + (U^{\perp} \cap X)$ is self-dual. Moreover $U + X = W + X$ and $U^{\perp} \cap X = W \cap X$.*

Proposition 2.2. *Let X, W be any self-dual subspaces of V, with $\dim(W+X) = n + s$, $0 \leq s \leq n$. Then for any subspace U of W, $W = U + (U^{\perp} \cap X)$ if, and only if, $\dim(U + X) = n + s$.*

Proof. Let U be a subspace of W. Then U is totally isotropic and $U+(U^{\perp}\cap X)$ is self-dual. If $W = U + (U^{\perp} \cap X)$, then from Proposition 2.1, $\dim(W+X) = \dim(U+X) = n+s$. Conversely, if $\dim(U+X) = n+s = \dim(W+X)$, then $U+X = W+X$ equivalently $U^{\perp}\cap X = W\cap X < W$. Hence $U+(U^{\perp}\cap X) < W$. $U+(U^{\perp}\cap X)$ and W both are self-dual and n-dimensional, we have then $U+(U^{\perp}\cap X) = W$ $\square$

Proposition 2.3. *Let X, Y, Z be n-dimensional subspaces in V, which satisfy $\dim(X+Z)+\dim(Z+Y) = n+\dim(X+Y)$. Then using dimension arguments we have $X\cap Y < Z < X+Y$.*

Lemma 2.1. *Let X and Y be any self-dual subspaces of V with $\dim(X+Y) = n+s+t$, where $1 \le s, 0 \le t$, $s+t \le n$ are integers. Then any self-dual subspace W which satisfies $\dim(W+X) = n+s$ and $\dim(W+Y) = n+t$ if, and only if, there is an s-dimensional subspace $U < Y$, with $U\cap X = \{0\}$, and $W = U + (U^{\perp}\cap X)$*

Proof. Let W be a self-dual subspace which satisfies $\dim(W+X) = n+s$ and $\dim(W+Y) = n+t$. Then by Proposition 2.3, we have $X\cap Y < W < X+Y$. Let U be an s-dimensional subspace $U < Y$, with $U+(X\cap Y) = W\cap Y$. Then $U < W$ and $U\cap X = \{0\}$. This implies $\dim(U+X) = n+s = \dim(W+X)$. From Proposition 2.2 we have $W = U+(U^{\perp}\cap X)$.
Conversely, let $U < Y$ be an s-dimensional subspace with $U\cap X = \{0\}$. Then $U < Y = Y^{\perp} < U^{\perp}$ is totally isotropic and $W := U+(U^{\perp}\cap X)$ is self-dual with $\dim(W+X) = \dim(U+X) = n+s$, from Proposition 2.1. Furthermore we have $\dim(W+Y) = \dim(U+(U^{\perp}\cap X)+Y) = \dim((U^{\perp}\cap X)+Y) = \dim(U^{\perp}\cap X)+\dim Y-\dim((U^{\perp}\cap X)\cap Y) = \dim(W\cap X)+\dim Y-\dim(X\cap Y) = (n-s)+n-(n-s-t) = n+t$. $\square$

Lemma 2.2. *Let X, Y and Z be any self-dual subspaces of V with $\dim(X+Y) = n+s+t$, $\dim(Z+X) = n+s$ and $\dim(Z+Y) = n+t$, where $1 \le s, 0 \le t$, $s+t \le n$ are integers. Then there is a self-dual subspace W, which satisfies $\dim(W+X) = n+t$, $\dim(W+Y) = n+s$, and $\dim(W+Z) = n+s+t$.*

Proof. Let X, Y and Z be self-dual subspaces satisfy the hypotheses. Then from the Lemma 2.1 there is an s-dimensional subspace $U_1 < Y$ with $U_1\cap X = \{0\}$, such that $Z = U_1+(U_1^{\perp}\cap X)$. Since $U_1^{\perp}\cap X = Z\cap X$ is $(n-s)$-dimensional, there is an s-dimensional subspace $U_2 < X$ with $U_2\cap U_1^{\perp} = \{0\}$, such that $X = U_2+(U_1^{\perp}\cap X) = U_2+(Z\cap X)$. Define $W := U_2+(U_2^{\perp}\cap Y)$. Since $U_2\cap Y < U_2\cap U_1^{\perp} = \{0\}$, we have by Lemma 2.1, that $\dim(W+X) = n+t$ and $\dim(W+Y) = n+s$. Furthermore, from (7) we have $U_1\cap W < U_1\cap U_2^{\perp} = \{0\}$. Together with $U_1 < Y$, $\dim U_1 = s$ and $\dim W\cap Y = n-s$, we have $Y = U_1+(W\cap Y)$. This implies $X+Y = (U_2+(Z\cap X))+(U_1+(W\cap Y)) < W+Z < X+Y$. Hence $W+Z = X+Y$ and $\dim(W+Z) = \dim(X+Y) = n+s+t$. $\square$

Remark: It was proved in [2, Theorem 2] for near polygons with quads (also known as dense near polygons), that for any two points x, y with distance i, and any geodesic $(x = x_0, x_1, \ldots, x_i = y)$, there is a geodesic $(y = y_0, y_1, \ldots, y_i = x)$ such that distance between x_j and y_j is i, for all $0 \leq j \leq i$.

Consider a point-line geometry related to a fixed polarity $\perp$ with self-dual subspaces $U = U^\perp$ as *points* and the sets of all self-dual subspaces which contain a common $(n-1)$-dimensional subspace as *lines*. Such point-line geometry is a classical dual polar space [4]. Two self-dual subspaces are called *adjacent*, if they are distinct and collinear, i.e., their intersection is $(n-1)$-dimensional. The adjacency relation turns the point set into the set of vertices of a graph, we call it *collinearity graph*. In the following we will study three types of dual polar spaces which are closely related to the geometries of matrices. Let I_n denote the $n \times n$ identity matrix.

- $K := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, $\overline{} = \mathrm{id}$, $\mathcal{D} = \mathcal{F}$ is a commutative field.

  The corresponding geometry is called the *projective space of symmetric matrices* and is a *symplectic dual polar space*.

- $K := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, $\overline{} \neq \mathrm{id}$, $\mathcal{D}$ is a division ring.

  The corresponding geometry is called the *projective space of Hermitian matrices* and is a *unitary dual polar space*.

- $K := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, $\overline{} = \mathrm{id}$, $\mathcal{D} = \mathcal{F}$ is a commutative field, $\mathrm{ch}(\mathcal{F}) \neq 2$.

  The corresponding geometry is called the *projective space of alternate matrices* and is an *orthogonal dual polar space*.

We would like to describe the relation between dual polar spaces and the projective spaces of matrices. Let W be an n-dimensional subspace of V. A matrix representation of W is an $n \times 2n$ matrix over $\mathcal{D}$, whose row vectors form a basis of W. A matrix representation is unique up to a left multiplication with a nonsingular $n \times n$ matrix over $\mathcal{D}$. We write a matrix representation of W in the block form

$$(A \mid B), \tag{8}$$

where both A and B are $n \times n$ matrices. Then W is self-dual if and only if

$$(A \mid B)\, K \left(\overline{A} \mid \overline{B}\right)^t = 0 \tag{9}$$

if and only if

$$A\overline{B}^t = B\overline{A}^t, \tag{10}$$

where $K = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, or

$$AB^t = -BA^t, \tag{11}$$

where $K = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, and $\overline{} = \mathrm{id}$.

The conditions (10),(11) are independent of the choice of the matrix representation. For W self-dual we call $(A \mid B)$ *homogeneous coordinates* of W. If rank $A = n$, then $\big(I_n \mid A^{-1}B\big)$ are also homogeneous coordinates of W, and $A^{-1}B$ is an $n \times n$ symmetric matrix, Hermitian matrix, or alternate matrix, with respect to K and $^-$. Conversely, for every $n \times n$ symmetric, Hermitian or alternate matrix H, $(I_n \mid H)$ are homogeneous coordinates of a self-dual subspace respectively. The graph-theoretic distance between two self-dual subspaces W_1, W_2 with homogeneous coordinates $(A_1 \mid B_1)$, $(A_2 \mid B_2)$ is $d(W_1, W_2) = \operatorname{rank}\big((A_1 \mid B_1)\, K\, \big(\overline{A_2} \mid \overline{B_2}\big)^t\big) = \dim(W_1 + W_2) - n$, cf. [14, Proposition 5.44, 5.48, 6.43, 6.47]. Define $\mathcal{PS}_n(\mathcal{F})$ to be the symplectic dual polar space associated to $K = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $^- = \mathrm{id}$. Let $\infty := (0 \mid I_n) \in \mathcal{PS}_n(\mathcal{F})$. Define $D_k(\infty) := \{W \in \mathcal{PS}_n(\mathcal{F}) \mid d(W, \infty) = k\}$ for all $k = 0, \ldots, n$. Then $D_n(\infty) = \{(I_n \mid S) \in \mathcal{PS}_n(\mathcal{F}) \mid S \in \mathcal{S}_n(\mathcal{F})\}$. Hence

- the symplectic dual polar space is also called *projective space of* $n \times n$ *symmetric matrices* over $\mathcal{F}$, [14], analogously,
- the unitary dual polar space with respect to $K = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $^-$ is called *projective space of* $n \times n$ *Hermitian matrices* over $\mathcal{D}$ respective $^-$, denoted by $\mathcal{PH}_n(\mathcal{D})$, and
- the orthogonal dual polar space with respect to $K = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ and $^- = \mathrm{id}$ is called *projective space of* $n \times n$ *alternate matrices* over $\mathcal{F}$, denoted by $\mathcal{PK}_n(\mathcal{F})$.

3 Projective geometry of Hermitian and symmetric matrices

In this section we will study the projective space of symmetric matrices and Hermitian matrices together. Let $K = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Let $n \geq 2$ be an integer and $\mathcal{D}$ an arbitrary division ring which possesses an involution $^-$. Denote the centre of $\mathcal{D}$ by $Z(\mathcal{D})$ and $\mathcal{F} := \{a \in \mathcal{D} \mid a = \overline{a}\}$. We assume additionally

$$\mathrm{ch}(\mathcal{D}) \neq 2, \qquad \text{when } \mathcal{D} \text{ is commutative and } {}^- = \mathrm{id}\,. \tag{12}$$

Let V be the $2n$-dimensional left vector space over $\mathcal{D}$. The projective space of Hermitian matrices $\mathcal{PH}_n(\mathcal{D})$ with respect to $\mathcal{D}$ and K contains all self-dual subspaces of V. They are n-dimensional subspaces W with homogeneous coordinates $(A \mid B)$ satisfying $(A \mid B)\, K\, \big(\overline{A} \mid \overline{B}\big)^t = 0$. The projective space of symmetric matrices is included as the case that $\mathcal{D}$ is commutative and $^-$ is the identity map. We call the self-dual subspaces of V *points* of the space $\mathcal{PH}_n(\mathcal{D})$. Two points are *adjacent*, if their intersection is $(n-1)$-dimensional. The adjacency on $\mathcal{PH}_n(\mathcal{D})$ can be considered as the adjacency relation of a graph with the set of vertices $\mathcal{PH}_n(\mathcal{D})$. We denote the graph arising on

$\mathcal{PH}_n(\mathcal{D})$ as $\Gamma(\mathcal{PH}_n(\mathcal{D}))$. It was proved in [14, Proposition 5.44, 5.48, 6.43, 6.47] and [6, Proposition 5.9.7, 5.9.10] that the distance between two points W_1, W_2 satisfies $d(W_1, W_2) = \dim(W_1 + W_2) - n = \text{rank}(W_1 K \overline{W_2}^t)$. The set $GU_{2n}(\mathcal{D}) := \{T \in M_{2n}(\mathcal{D}) \mid TK\overline{T}^t = \lambda K, \lambda \in (\mathcal{F} \cap Z(\mathcal{D})) \setminus \{0\}\}$ together with the matrix multiplication forms a group which is called the *general unitary group*. It is a subgroup of the automorphism group of the graph $\Gamma(\mathcal{PH}_n(\mathcal{D}))$. In the case that $\mathcal{D}$ is commutative and $^- = \text{id}$, the general unitary group is called *general symplectic group* $GSp_{2n}(\mathcal{F})$.

We have some well-known properties of $GU_{2n}(\mathcal{D})$.

Proposition 3.1. *(Cf. [14, Proposition 5.43, 6.42].) $GU_{2n}(\mathcal{D})$ acts transitively on $\mathcal{PH}_n(\mathcal{D})$.*

Proposition 3.2. *(Cf. [14, Proposition 5.47, 6.46].) The set of pairs of points of $\mathcal{PH}_n(\mathcal{D})$ with same distance forms an orbit under $GU_{2n}(\mathcal{D})$.*

We denote by E_{ij} the $n \times n$ matrix over $\mathcal{D}$ whose (i, j) entry equals 1, whereas all other entries are 0. Hence from above Propositions any two points at distance k, $0 \leq k \leq n$, can be taken under $GU_{2n}(\mathcal{D})$ to $X = (I_n \mid 0)$ and $Y = \left(I_n \mid \sum_{i=1}^{k} E_{ii}\right)$.

W.L.Chow proved in [5] the fundamental theorem of the projective space of symmetric matrices $\mathcal{PS}_n(\mathcal{F})$ (the symplectic dual polar space). We rewrite the theorem of Chow in the homogeneous coordinates.

Theorem 3.1 (W.L. Chow, [5])**.** *Let $\mathcal{F}$, $\mathcal{F}'$ be arbitrary commutative fields and n, n' be integers, $n, n' \geq 2$. If there is a bijective map φ from $\mathcal{PS}_n(\mathcal{F})$ to $\mathcal{PS}_{n'}(\mathcal{F}')$ for which both the map φ and its inverse φ^{-1} preserve the adjacency, then $n = n'$ and φ is of the form*

$$(A \mid B) \mapsto (A \mid B)^{\sigma} T \tag{13}$$

for all points in $\mathcal{PS}_n(\mathcal{F})$ with homogeneous coordinates $(A \mid B)$, where $T \in GSp_{2n}(F')$, and σ is an isomorphism between $\mathcal{F}$ and $\mathcal{F}'$.

The graph $\Gamma(\mathcal{PH}_n(\mathcal{D}))$ satisfies the conditions (A1), (A2), (A3), and (A5) mentioned in the introduction. This is clear, since $\mathcal{PH}_n(\mathcal{D})$ is a thick dual polar space. *Thick* means that every line in the space contains at least three points. However, we shortly prove that these conditions hold for $\Gamma(\mathcal{PH}_n(\mathcal{D}))$.

Lemma 3.1. *The graph $\Gamma(\mathcal{PH}_n(\mathcal{D}))$ with the restriction (12) satisfies the conditions (A1)- (A5) mentioned in the introduction.*

Proof. *(A1):* $\Gamma(\mathcal{PH}_n(\mathcal{D}))$ is connected and the diameter of $\Gamma(\mathcal{PH}_n(\mathcal{D}))$ is n.

(A2): Let $X, Y \in \mathcal{PH}_n(\mathcal{D})$ be two points with distance $d(X, Y) = r$. Without loss of generality assume that $X = (I_n \mid 0)$ and $Y = (I_n \mid \sum_{i=1}^{r} E_{ii})$. For any integer k with $r \leq k \leq n$ define $Z := \left(I_n \mid \sum_{i=1}^{k} E_{ii}\right)$. Then

$d(X,Z) = k = d(X,Y) + d(Y,Z)$. So (A2) holds.

(A3): This is the Lemma 2.2.

(A4): Let $X, Y, Z \in \mathcal{PH}_n(\mathcal{D})$ be vertices with $X \neq Y$ and $d(X,Z) = d(Y,Z) =: k \geq 1$. In the case $k = 1$, let $W := X$. Now suppose $k \geq 2$.

Case 1: $k = n$. Without loss of the generality, assume $Z = (I_n \mid 0)$, $Y = (I_n \mid I_n)$, and $X = (A \mid B)$. Since $d(Z,X) = n$, we have $\operatorname{rank} B = n$, and we may assume $B = I_n$, $X = (A \mid I_n)$ where $A = \overline{A}^t$. The fact $X \neq Y$ implies $A \neq I_n$. The assumption (12) ensures that $I_n - A$ is not alternate, and there is a vector $v \in \mathcal{D}^n$ such that $v(I_n - A)\overline{v}^t \neq 0$. Obviously $x := v\ (A \mid I_n) \in X \setminus Z$. Let $U := \operatorname{span}(x)$. $(U + Z) \cap Y$ is the one-dimensional subspace $\{(\mu v, \mu v) \mid \mu \in \mathcal{D}\} < \mathcal{D}^{2n}$. Let $W := U + (U^{\perp} \cap Z)$, then $W \in \mathcal{PH}_n(\mathcal{D})$, $d(W,Z) = 1$ and $d(W,X) = n-1$, by Lemma 2.1. Suppose $d(W,Y) = n-1$, then from Lemma 2.1 there is a vector $y \in Y \setminus Z$, such that $W = \operatorname{span}(y) + (\operatorname{span}(y)^{\perp} \cap Z) = U + (U^{\perp} \cap Z)$. Hence $y \in (U+Z) \cap Y$, and there is some $\mu \in \mathcal{D} \setminus \{0\}$ with $y = (\mu v, \mu v)$. We have $(y,x) = yK\overline{x}^t = (\mu v)\overline{v}^t - (\mu v)\overline{vA}^t = \mu v(I_n - A)\overline{v}^t \neq 0$, hence $y \notin U^{\perp}$, a contradiction to $y \in W = U + (U^{\perp} \cap Z) < U^{\perp}$. So $d(W,Y) = n$, as required.

Case 2: $2 \leq k < n$.

Case 2.1: $X + Z = Y + Z$. Consider the quotient space $(X+Z)/(X \cap Z) := \{P \in \mathcal{PH}_n(\mathcal{D}) \mid X \cap Z < P < X + Z\}$. The quotient space $(X+Z)/(X \cap Z)$ contains X, Y, Z and is isomorphic to $\mathcal{PH}_k(\mathcal{D})$. The graph arising from $(X + Z)/(X \cap Z)$ contains all geodesics from any points $P, Q \in (X+Z)/(X \cap Z)$ and has diameter k. Similar to the case 1, we can find a point $W \in (X+Z)/(X \cap Z)$, with $d(W,Z) = 1$, $d(W,X) = k-1$, and $d(W,Y) = k$.

Case 2.2: $X + Z \neq Y + Z$. Choose a one-dimensional subspace $U < X \setminus (Y+Z)$. Define $W := U + (U^{\perp} \cap Z)$, then $W \in \mathcal{PH}_n(\mathcal{D})$, $d(W,Z) = 1$ and $d(W,X) = k-1$. Since $U \not< (Y+Z)$, W is not a subspace of $Y+Z$. Hence $d(W,Y) \neq k-1$ by Proposition 2.3. From the triangle inequality $k - 1 = d(Y,Z) - d(W,Z) \leq d(W,Y)$ we have $k \leq d(W,Y)$.

For the second part of (A4), let $X, Y, Z \in \mathcal{PH}_n(\mathcal{D})$ with $d(Z,X) = 2 = d(Z,Y)$ and $d(X,Y) = 3$. Then there is a one-dimensional subspace U_1 with $U_1 < Y$ and $U_1 \cap (X+Z) = \{0\}$. Define $W := U_1 + (U_1^{\perp} \cap X)$. Then from Lemma 2.1, $d(W,X) = 1$ and $d(W,Y) = 2$. Since $U_1 \cap (X+Z) = \{0\}$, we have $U_1^{\perp} + (X \cap Z) = V$ and $\dim(U_1^{\perp} \cap (X \cap Z)) = n-3$. There is a one-dimensional subspace $U_2 < X \cap Z$ with $U_2 \cap U_1^{\perp} = \{0\}$, $X = U_2 + (U_1^{\perp} \cap X)$ and $Z = U_2 + (U_1^{\perp} \cap Z)$. $W + Z = U_1 + (U_1^{\perp} \cap X) + Z = U_1 + (U_1^{\perp} \cap X) + U_2 + (U_1^{\perp} \cap Z) = U_1 + X + Z$ has dimension $n+3$. Hence $d(W,Z) = 3$.

(A5): Let $A, B \in \mathcal{PH}_n(\mathcal{D})$ be two adjacent points. Without loss of generality

assume that $A = (I_n \ \mid 0)$ and $B = (I_n \ \mid E_{11})$. Then

$$P := \left(\sum_{i=2}^{n} E_{ii} \ \mid E_{11} \right)$$

has the required property in (A5). □

From Lemma 3.1 and Theorem 1.1 we obtain:

Theorem 3.2. *Let $\mathcal{F}, \mathcal{F}'$ be fields with characteristic not equal to two. Let n, n' be integers ≥ 2 and $1 \leq k < \min\{n, n'\}$. If $\varphi : \mathcal{PS}_n(\mathcal{F}) \mapsto \mathcal{PS}_{n'}(\mathcal{F}')$ is a surjective mapping which satisfies*

$$d(X,Y) \leq k \quad \Leftrightarrow \quad d(X^\varphi, Y^\varphi) \leq k$$

for all $X, Y \in \mathcal{PS}_n(\mathcal{F})$, then φ is bijective. Both φ and φ^{-1} preserve adjacency of subspaces. Moreover $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic, $n = n'$, and φ is of the form (13).

Theorem 3.3. *Let $\mathcal{D}, \mathcal{D}'$ be division rings with involutions $^-$, $^{-'}$, both are not identity. Let n, n' be integers ≥ 2 and $1 \leq k < \min\{n, n'\}$. If $\varphi : \mathcal{PH}_n(\mathcal{D}) \mapsto \mathcal{PH}_{n'}(\mathcal{D}')$ is a surjective mapping which satisfies*

$$d(X,Y) \leq k \quad \Leftrightarrow \quad d(X^\varphi, Y^\varphi) \leq k$$

for all $X, Y \in \mathcal{PH}_n(\mathcal{D})$, then φ is bijective, $n = n'$ and both φ and φ^{-1} preserve adjacency.

The fundamental theorem of the projective geometry of Hermitian matrices describes the mapping φ in the Theorem 3.3 explicitly[1].

4 Projective Geometry of alternate matrices

Let $n \geq 2$ be an integer and let $\mathcal{F}$ be an arbitrary commutative field with characteristic not equal to two. Let $V = \mathcal{F}^{2n}$ be the $2n$-dimensional vector space over $\mathcal{F}$ equipped with a bilinear form $(x, y) = xSy^t$, where $S := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, I_n is the $n \times n$ identity matrix. This bilinear form induces an orthogonal dual polar space. Analogously to the unitary dual polar space, there is a close connection between the orthogonal dual polar space and the space of alternate $n \times n$ matrices $\mathcal{K}_n(\mathcal{F})$ (cf. [14, chapter 4.4]). We call the orthogonal dual polar space *projective space of $n \times n$ alternate matrices* over $\mathcal{F}$ and denote it by $\mathcal{PK}_n(\mathcal{F})$. The self-dual subspaces are called the *points* of $\mathcal{PK}_n(\mathcal{F})$. They are n-dimensional subspaces W with homogeneous coordinates $(A \ \mid B)$ satisfying $(A \ \mid B)\, S(A \ \mid B)^t = 0$. Two points are *adjacent*, if their intersection is $(n-1)$-dimensional. The adjacency on $\mathcal{PK}_n(\mathcal{F})$ can be considered as the adjacency

[1]However this seems to be known only under some additional assumptions on the division ring $\mathcal{D}$, see e.g. [6, 7, 14].

relation of a graph with the set of vertices $\mathcal{PK}_n(\mathcal{F})$. We denote the graph arising from $\mathcal{PK}_n(\mathcal{F})$ as $\Gamma(\mathcal{PK}_n(\mathcal{F}))$. It was proved in [14, Propositions 4.22, 4.26] that the graph-theoretic distance between two points W_1, W_2 satisfies $d(W_1, W_2) = \dim(W_1 + W_2) - n = \operatorname{rank}(W_1\, S\, {W_2}^t)$.

The set $O_{2n}(\mathcal{F}) := \{T \in M_{2n}(\mathcal{F}) \mid TST^t = S\}$ together with the matrix multiplication forms a group. It is a subgroup of the automorphism group of the graph $\Gamma(\mathcal{PK}_n(\mathcal{F}))$.

In the following we give some well-known properties of $O_{2n}(\mathcal{F})$.

Proposition 4.1. *(See e.g. [14, Proposition 4.21].) $O_{2n}(\mathcal{F})$ acts transitively on $\mathcal{PK}_n(\mathcal{F})$.*

Proposition 4.2. *(See e.g. [14, Proposition 4.25].) The set of pairs of points of $\mathcal{PK}_n(\mathcal{F})$ with same distance forms an orbit under $O_{2n}(\mathcal{F})$.*

Let X, Y be two points in $\mathcal{PK}_n(\mathcal{F})$ with distance r. Then they can be transformed under $O_{2n}(\mathcal{F})$ to $(I_n \mid 0)$ and $\left(\sum_{i=r+1}^{n} E_{ii} \mid \sum_{i=1}^{r} E_{ii}\right)$.

The graph $\Gamma(\mathcal{PK}_n(\mathcal{F}))$ satisfies the conditions (A1), (A2) and (A3), analogous to $\Gamma(\mathcal{PH}_n(\mathcal{D}))$, however it does not satisfies the conditions (A4) and (A5), because $\Gamma(\mathcal{PK}_n(\mathcal{F}))$ is a bipartite graph. The space $\mathcal{PK}_n(\mathcal{F})$ can be divided into two disjoint components, $\mathcal{PK}_n(\mathcal{F})^+$and $\mathcal{PK}_n(\mathcal{F})^-$. For any two points X and Y in $\mathcal{PK}_n(\mathcal{F})$, they are in the same component if, and only if, they are at even distance.

5 The irreducible space $\mathcal{PK}_n(\mathcal{F})^+$ of the Projective Geometry of alternate matrices

In this section we will consider the irreducible part of the space $\mathcal{PK}_n(\mathcal{F})$. Denote $O^+_{2n}(\mathcal{F}) := \{T \in M_{2n}(\mathcal{F}) \mid TST^t = S, \det T = 1\}$, it is a subgroup of $O_{2n}(\mathcal{F})$.

Define

$$\begin{aligned}
\mathcal{PK}_n(\mathcal{F})^+ &:= \{W \in \mathcal{PK}_n(\mathcal{F}) \mid \operatorname{rank} WS\,(I_n \mid 0)^t \text{ is even}\}\\
&= \{(A \mid B) \in \mathcal{PK}_n(\mathcal{F}) \mid \operatorname{rank} B \text{ is even}\},\\
\mathcal{PK}_n(\mathcal{F})^- &:= \{W \in \mathcal{PK}_n(\mathcal{F}) \mid \operatorname{rank} WS\,(I_n \mid 0)^t \text{ is odd}\}\\
&= \{(A \mid B) \in \mathcal{PK}_n(\mathcal{F}) \mid \operatorname{rank} B \text{ is odd}\}.
\end{aligned}$$

Then $\mathcal{PK}_n(\mathcal{F})^+$ contains all points with coordinates $W = (I_n \mid B)$, where B is an $n \times n$ alternate matrix. The distance between two points W_1, $W_2 \in \mathcal{PK}_n(\mathcal{F})^+ \subset \mathcal{PK}_n(\mathcal{F})$ is always even. Define the new distance on $\mathcal{PK}_n(\mathcal{F})^+$ by $d^+(W_1, W_2) := \frac{1}{2}(\dim(W_1 + W_2) - n) = \frac{1}{2} \operatorname{rank} W_1\, S\, W_2^t$. We call two points W_1, $W_2 \in \mathcal{PK}_n(\mathcal{F})^+$ *adjacent* if, and only if, $d^+(W_1, W_2) = 1$. The graph arising on $\mathcal{PK}_n(\mathcal{F})^+$ is connected with diameter $[\frac{n}{2}]$. We denote the graph with $\Gamma(\mathcal{PK}_n(\mathcal{F})^+)$. The distance d^+ of $\mathcal{PK}_n(\mathcal{F})^+$ is the same as the graph-theoretic distance. The graph $\Gamma(\mathcal{PK}_n(\mathcal{F})^+)$ satisfies the conditions (A1), (A2) and (A3). In the next we are going to prove, when n is even, that $\Gamma(\mathcal{PK}_n(\mathcal{F})^+)$ also satisfies the conditions (A4) and (A5).

Lemma 5.1. *Let $n \geq 4$ be an even integer. $\Gamma(\mathcal{PK}_n(\mathcal{F})^+)$ satisfies the condition (A4).*

Proof. Let $X, Y, Z \in \mathcal{PK}_n(\mathcal{F})^+$ be points with $X \neq Y$ and $d^+(X,Z) = d^+(Y,Z) =: k \geq 1$. In the case $k = 1$, let $W := X$. Now suppose $k \geq 2$.

Case 1: $k = \frac{n}{2}$. Without loss of generality, assume $Z = (I_n \mid 0)$, $Y = (I_n \mid K)$, and $X = (X_1 \mid X_2)$, where $K := (E_{12} - E_{21}) + (E_{34} - E_{43}) + \ldots + (E_{(n-1)n} - E_{n(n-1)})$. Since $d^+(Z,X) = \frac{n}{2}$, we have $\operatorname{rank} X_2 = n$, and we may assume $X_2 = I_n$, $X = (X_1 \mid I_n)$ where $X_1 = -X_1^t$. Since $\operatorname{ch}(\mathcal{F}) \neq 2$ and $X \neq Y$, $X_1 + K \neq 0$, there are linear independent vectors $v_1, v_2 \in \mathcal{F}^n$ such that $v_1(X_1 + K)v_2^t \neq 0$. Obviously $x_i := v_i\ (X_1 \mid I_n) \in X \setminus Z$, and $U := \operatorname{span}(x_1, x_2)$ is a two-dimensional subspace in X. $(U + Z) \cap Y$ is the two-dimensional sub vector space $\{(-(\mu_1 v_1 + \mu_2 v_2)K, (\mu_1 v_1 + \mu_2 v_2)) \mid \mu_i \in \mathcal{F}\}$. Let $W := U + (U^\perp \cap Z)$, then by Lemma 2.1, $W \in \mathcal{PK}_n(\mathcal{F})^+$, $d^+(W,Z) = 1$ and $d^+(W,X) = \frac{n}{2} - 1$. Suppose $d^+(W,Y) = \frac{n}{2} - 1$, then there is a vector $y \in Y \setminus Z$, such that $y \in W \cap Y$. Hence $y \in (U + Z) \cap Y$, and there are some $\mu_i \in \mathcal{F}$, $i = 1, 2$ with $(\mu_1, \mu_2) \neq (0, 0)$ and $y = (-(\mu_1 v_1 + \mu_2 v_2)K, (\mu_1 v_1 + \mu_2 v_2))$. Without loss of generality, assume $\mu_1 \neq 0$. We have $(-(\mu_1 v_1 + \mu_2 v_2)K, (\mu_1 v_1 + \mu_2 v_2))\, S\, (v_2 X_1, v_2)^t = \mu_1 v_1 (X + K) v_2^t \neq 0$, so $y \notin U^\perp$, a contradiction to $y \in W = U + (U^\perp \cap Z) < U^\perp$.

Case 2: $2 \leq k < \frac{n}{2}$.

Case 2.1: $X + Z = Y + Z$. The quotient space $X + Z / X \cap Z = \{P \in \mathcal{PK}_n(\mathcal{F})^+, X \cap Z < P < X + Z\}$ is isomorphic to $\mathcal{PK}_{2k}(\mathcal{F})^+$ and has diameter k. Similar to case 1, there is a point $W \in X + Z / X \cap Z$ with the required properties.

Case 2.2: $X + Z \neq Y + Z$. There is some two-dimensional subspace U in X with $U \cap (Y + Z) = \{0\}$. Define $W := U + (U^\perp \cap Z)$, then $W \in \mathcal{PK}_n(\mathcal{F})^+$, $d^+(W,Z) = 1$ and $d^+(W,X) = k - 1$. Since $U < W$ and U is not a subspace of $Y + Z$, we have W is not a subspace of $Y + Z$. Hence $d^+(W,Y) \neq k - 1$ by Proposition 2.3.

For the second part of (A4), let $X, Y, Z \in \mathcal{PK}_n(\mathcal{F})^+$ with $d^+(Z,X) = 2 = d^+(Z,Y)$ and $d^+(X,Y) = 3$. Since $\dim(X+Y) = n+6$, and $\dim(X+Z) = n+4$, there is a two-dimensional subspace U_1 with $U_1 < Y$ and $U_1 \cap (X + Z) = \{0\}$. Define $W := U_1 + (U_1^\perp \cap X)$. Then from Lemma 2.1, $d^+(W,X) = 1$ and $d^+(W,Y) = 2$. Since $U_1 \cap (X + Z) = \{0\}$, we have $U_1^\perp + (X \cap Z) = V$ and $\dim(U_1^\perp \cap (X \cap Z)) = n - 6$. There is a two-dimensional subspace $U_2 < X \cap Z$ with $U_2 \cap U_1^\perp = \{0\}$, $X = U_2 + (U_1^\perp \cap X)$ and $Z = U_2 + (U_1^\perp \cap Z)$. $W + Z = U_1 + (U_1^\perp \cap X) + Z = U_1 + (U_1^\perp \cap X) + U_2 + (U_1^\perp \cap Z) = U_1 + X + Z$ has dimension $n + 6$, hence $d^+(W,Z) = 3$. □

Lemma 5.2. *$\Gamma(\mathcal{PK}_n(\mathcal{F})^+)$ satisfies the condition (A5).*

Proof. Without loss of generality, we may assume $A = (I_n \mid 0)$ and $B = (I_n \mid E_{12} - E_{21})$. Let $P = (\sum_{i=3}^n E_{ii} \mid E_{11} + E_{22}) \in \mathcal{PK}_n(\mathcal{F})^+$. For any $W = (X \mid Y) \in \mathcal{PK}_n(\mathcal{F})^+$, we have $\operatorname{rank} WSP^t = \operatorname{rank}(x^1, x^2, y^3, \ldots, y^n)$, $\operatorname{rank} WSA^t = \operatorname{rank} Y$, and $\operatorname{rank} WSB^t = \operatorname{rank}(x^2 + y^1, -x^1 + y^2, y^3, \ldots, y^n)$, where x^i, y^j denote the column vectors of the matrices X and Y. Since rank WSP^t, rank WSA^t and rank WSB^t are all even, if rank $WSA^t <$ rank WSP^t, then $\operatorname{span}(y^1, y^2) < \operatorname{span}(y^3, \ldots, y^n)$ and $\operatorname{span}(x^1, x^2) \cap \operatorname{span}(y^3, \ldots, y^n) = \{0\}$. This implies that $\operatorname{rank} WSB^t = \operatorname{rank}(x^2 + y^1, -x^1 + y^2, y^3, \ldots, y^n) = \operatorname{rank}(x^1, x^2, y^3, \ldots, y^n) = \operatorname{rank} WSP^t$. Hence $d^+(W, P) = \frac{1}{2} \operatorname{rank} WSP^t \leq \max\{\frac{1}{2} \operatorname{rank} WSA^t, \frac{1}{2} \operatorname{rank} WSB^t\} = \max\{d^+(W, A), d^+(W, B)\}$ □

Theorem 5.1. *Let $\mathcal{F}, \mathcal{F}'$ be fields with characteristic not equal to two. Let $n, n' \geq 2$ be even integers and $1 \leq k < \min\{\frac{n}{2}, \frac{n'}{2}\}$. If $\varphi : \mathcal{PK}_n(\mathcal{F})^+ \mapsto \mathcal{PK}_{n'}(\mathcal{F}')^+$ is a surjective mapping which satisfies*

$$d^+(X, Y) \leq k \quad \Leftrightarrow \quad d^+(X^\varphi, Y^\varphi) \leq k$$

for all $X, Y \in \mathcal{PK}_n(\mathcal{F})^+$, then φ is bijective. Both φ and φ^{-1} preserve adjacency of subspaces. Moreover $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic, and $n = n'$.

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