EXCEPTIONAL HOLONOMY BASED ON THE HITCHIN FLOW ON COMPLEX LINE BUNDLES

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ABSTRACT. An $SU(3)$- or $SU(1,2)$-structure on a 6-dimensional manifold N_6 can be defined as a pair of a 2-form ω and a 3-form ρ. Let M_8 be an arbitrary complex line bundle over N_6. We prove that any $SU(3)$- or $SU(1,2)$-structure on N_6 with $d\omega \wedge \omega = 0$ can be uniquely extended to a Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure which is defined on a tubular neighborhood of the zero section of M_8. As an application, we prove that the known cohomogeneity-one metrics with holonomy Spin(7) on a certain complex line bundle over $SU(3)/U(1)^2$ are the only ones.

1. INTRODUCTION

The flow equations of Hitchin [14] are a method to construct manifolds with exceptional holonomy, which can also be generalized to the semi-Riemannian case [7]. In this article, we restrict ourselves to metrics with holonomy Spin(7) or $\mathrm{Spin}_0(3,4)$.

We start with a cocalibrated G_2- or G_2^*-structure ϕ on a 7-dimensional manifold N_7. Hitchin's flow equation yields a one-parameter family $(\phi_t)_{t\in[0,\epsilon)}$ of G_2- or G_2^*-structures such that $\phi_0 = \phi$ and $dt \wedge \phi_t + *\phi_t$ is a parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure.

The metrics which we obtain by this method are in general incomplete. In order to construct complete examples, we assume that N_7 degenerates for small t into a 6-dimensional manifold N_6 such that the space on which the Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure will be defined is a complex line bundle over N_6. Later on, this kind of degeneration will be defined more rigorously.

The zero section of the bundle, which we will identify with N_6, carries an $SU(3)$- or $SU(1,2)$-structure (ω_0, ρ_0) where ω_0 is a 2-form and ρ_0 is a 3-form. Many of the known cohomogeneity-one metrics with holonomy Spin(7) [1], [2], [8], [9], [10], [15], [17], [20], [21], [22] are defined on a complex line bundle and are precisely of the kind which we investigate. This is a further motivation to study degenerations of the Hitchin flow.

If we assume that all initial data on N_6 are analytic, our main result can be proven with help of the Cauchy-Kovalevskaya Theorem. We have to pay special attention to the conditions which ϕ_t has to satisfy such that the Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure on $M_8 \setminus N_6$ has a smooth extension to N_6.

It turns out that the only non-trivial condition on (ω_0, ρ_0) is $d\omega_0 \wedge \omega_0 = 0$. For any complex line bundle and any $SU(3)$- or $SU(1,2)$-structure on N_6 which satisfies this condition, we obtain a unique parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure on a tubular neighborhood of N_6.

In the literature [1], [2], [17], [20], [22], there are many cohomogeneity-one metrics with exceptional holonomy known whose principal orbit is the exceptional Aloff-Wallach space $N^{1,1}$ and whose singular orbit is $SU(3)/T^2$ where T^2 is a maximal torus of $SU(3)$. With help of our theorem, we will see that no further metrics of that kind exist. We expect that there are further applications of our theorem in the construction of parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structures which are of cohomogeneity one or have another explicit description.

2. G-structures

2.1. **G is a real form of $SL(3,\mathbb{C})$.** In order to formulate our theorem we have to introduce several G-structures. On the total space of the bundle we want to construct a Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure. In this situation, the zero section of the bundle carries a G-structure where G is a real form of $SL(3,\mathbb{C})$. Finally, the set of all points with a fixed distance from the zero section carries a G_2-structure or a G_2^*-structure. A well written introduction to all of these G-structures can be found in Cortés et al. [7]. We use similar conventions as [7] and only recapitulate the facts which we need for our considerations.

We first take a look at the case where G is a real form of $SL(3,\mathbb{C})$. The other G-structures can be build up from these ones. There are three real forms of $SL(3,\mathbb{C})$ namely $SU(3)$, $SU(1,2)$, and $SL(3,\mathbb{R})$. In the later sections, we restrict ourselves to $SU(3)$- and $SU(1,2)$-structures. Nevertheless, we introduce all three types of G-structures in this subsection, since they share many properties.

All of the G-structures from this section can be described with help of certain differential forms. We will index the differential forms by the corresponding group G in order to avoid confusion. Throughout the article we use the following convention.

Convention 2.1. Let $(v_i)_{i\in I}$ be a basis of a vector space V. We denote its dual basis by $(v^i)_{i\in I}$ and abbreviate $v^{i_1} \wedge \ldots \wedge v^{i_j}$ by $v^{i_1\ldots i_j}$.

Let $(e_i)_{i=1,\ldots,6}$ be the canonical basis of $\mathbb{R}^6$. We define the 2-forms

$$\omega_{SU(3)} := \omega_{SL(3,\mathbb{R})} := e^{12} + e^{34} + e^{56} \tag{1}$$

and

$$\omega_{SU(1,2)} := -e^{12} - e^{34} + e^{56}\,. \tag{2}$$

Moreover, we introduce the 3-forms

$$\rho_{SU(3)} := \rho_{SU(1,2)} := e^{135} - e^{146} - e^{236} - e^{245} \tag{3}$$

and

$$\rho_{SL(3,\mathbb{R})} := e^{135} + e^{146} + e^{236} + e^{245}\,. \tag{4}$$

Convention 2.2. If we consider an arbitrary 2-form in $\{\omega_{SU(3)}, \omega_{SU(1,2)}, \omega_{SL(3,\mathbb{R})}\}$ we denote it by ω. Analogously, ρ denotes a 3-form which is either $\rho_{SU(3)}$, $\rho_{SU(1,2)}$, or $\rho_{SL(3,\mathbb{R})}$.

The following lemma is proven in [7].

Lemma 2.3. *Let $G \in \{SU(3), SU(1,2), SL(3,\mathbb{R})\}$. The subgroup of all $A \in GL(6,\mathbb{R})$ which stabilize ω_G and ρ_G simultaneously is isomorphic to G.*

This lemma motivates the following definition.

Definition 2.4. Let $G \in \{SU(3), SU(1,2), SL(3,\mathbb{R})\}$, V be a 6-dimensional real vector space and (ω,ρ) be a pair of a 2-form and a 3-form on V. If there exists a basis $(v_i)_{i=1,\ldots,6}$ of V such that with respect to this basis ω can be identified with ω_G and ρ with ρ_G, (ω,ρ) is called a *G-structure*.

ω and ρ are both examples of stable forms in the sense of Hitchin [14].

Definition 2.5. Let V be a real or complex vector space and $\beta \in \bigwedge^k V^*$ with $k \in \{0,\ldots,\dim V\}$ be a k-form. β is called *stable* if the $GL(V)$-orbit of β is an open subset of $\bigwedge^k V^*$.

Lemma 2.6. *Let (ω,ρ) be a G-structure where $G \in \{SU(3), SU(1,2), SL(3,\mathbb{R})\}$. In this situation, ω and ρ are both stable forms.*

Remark 2.7. The stable forms are an open dense subset of $\bigwedge^2 \mathbb{R}^{6*}$ or $\bigwedge^3 \mathbb{R}^{6*}$. There is exactly one open $GL(6,\mathbb{R})$-orbit in $\bigwedge^2 \mathbb{R}^{6*}$ and two open orbits in $\bigwedge^3 \mathbb{R}^{6*}$. One of them is the orbit of $\rho_{SU(3)}$ and the other one is the orbit of $\rho_{SL(3,\mathbb{R})}$.

Let V be a 6-dimensional real vector space and $\bigwedge^3_s V^*$ be the set of all stable 3-forms on V. We can assign to any stable 3-form ρ a certain endomorphism J_ρ by a map

$$i : \bigwedge\nolimits_s^3 V^* \to V \otimes V^*\,. \tag{5}$$

i is a rational $GL(6,\mathbb{R})$-equivariant map and is described in detail in [7]. $i(\rho_{SU(3)})$ is the standard complex structure on $\mathbb{R}^6$ which maps e_{2i-1} to $-e_{2i}$ and e_{2i} to e_{2i-1} for all $i \in \{1,2,3\}$. $i(\rho_{SL(3,\mathbb{R})})$ is the standard para-complex structure on $\mathbb{R}^6$ which maps e_{2i-1} to e_{2i} and e_{2i} to e_{2i-1}.

If (ω,ρ) is an $SU(3)$-structure or an $SU(1,2)$-structure, J_ρ therefore is a complex structure. If (ω,ρ) is an $SL(3,\mathbb{R})$-structure, J_ρ is a para-complex structure, i.e. $J_\rho^2 = \mathrm{id}$.

As before, let V be a 6-dimensional real vector space and $\bigwedge_s^k V^*$ be the set of all stable k-forms on V. With help of another map

$$j : \bigwedge_s^2 V^* \times \bigwedge_s^3 V^* \to S^2(V^*) \tag{6}$$

we can assign to (ω,ρ) a symmetric non-degenerate bilinear form. An explicit description of j can also be found in [7]. As i, j is also a rational, $GL(6,\mathbb{R})$-equivariant map. If (ω,ρ) is an

(1) $SU(3)$-structure, $j(\omega,\rho)$ is a metric with signature $(6,0)$. In particular, $j(\omega_{SU(3)},\rho_{SU(3)})$ is the Euclidean metric on $\mathbb{R}^6$.
(2) $SU(1,2)$-structure, $j(\omega,\rho)$ is a metric with signature $(2,4)$.
(3) $SL(3,\mathbb{R})$-structure, $j(\omega,\rho)$ is a metric with signature $(3,3)$.

Convention 2.8. (1) We call J_ρ the *complex* or *para-complex structure which is associated to* ρ or shortly the *associated complex* or *para-complex structure.*
(2) We call $j(\omega,\rho)$ the *metric which is associated to* (ω,ρ) or shortly the *associated metric.* We denote it by g_6, since we will also work with metrics on 7- or 8-dimensional spaces.

The basis $(v_i)_{i=1,\dots,6}$ from Definition 2.4 is in all three cases pseudo-orthonormal, i.e. $|g_6(v_i,v_j)| = \delta_{ij}$ for all $i,j \in \{1,\dots,6\}$. We remark that the objects which we have defined are related by the formula

$$\omega(v,w) := g_6(v, J_\rho(w))\,. \tag{7}$$

Let (ω,ρ) be an $SU(3)$-structure. The stabilizer of (ω,ρ) is the same as of (ω,ρ,J_ρ,g_6). Since it is well-known that the second stabilizer is $SU(3)$, (ω,ρ) is indeed stabilized by $SU(3)$. The other statements of Lemma 2.3 can be proven analogously.

It is possible to see if a pair (ω,ρ) determines one of our G-structures without referring to a special basis.

Theorem 2.9. *Let V be a 6-dimensional real vector space and let $\omega \in \bigwedge^2 V^*$ and $\rho \in \bigwedge^3 V^*$ be stable. Moreover, let J_ρ and g_6 be defined as above. We assume that ω and ρ satisfy the equations*

(1) $\omega \wedge \rho = 0$,
(2) $J_\rho^* \rho \wedge \rho = \frac{2}{3}\omega \wedge \omega \wedge \omega$.

If in this situation

(1) g_6 *has signature* $(6,0)$ *and* J *is a complex structure,* (ω, ρ) *is an* $SU(3)$*-structure.*
(2) g_6 *has signature* $(2,4)$ *and* J *is a complex structure,* (ω, ρ) *is an* $SU(1,2)$*-structure.*
(3) g_6 *has signature* $(3,3)$ *and* J *is a para-complex structure,* (ω, ρ) *is an* $SL(3,\mathbb{R})$*-structure.*

Remark 2.10. (1) Since $J_\rho^* \rho \wedge \rho$ and $\frac{2}{3}\omega \wedge \omega \wedge \omega$ are both 6-forms, the second condition from the theorem is a normalization of the pair (ω, ρ).
(2) If (ω, ρ) is a pair of stable forms satisfying $\omega \wedge \rho = 0$ and $J_\rho^* \rho \wedge \rho = \frac{2}{3}\omega \wedge \omega \wedge \omega$, it is in fact an $SU(3)$-, $SU(1,2)$-, or $SL(3,\mathbb{R})$-structure.

The reason for the above considerations of course is to define the notion of a G-structure on a manifold.

Definition 2.11. Let M be a 6-dimensional manifold, $\omega \in \bigwedge^2 T^*M$, and $\rho \in \bigwedge^3 T^*M$. Moreover, let $G \in \{SU(3), SU(1,2), SL(3,\mathbb{R})\}$. (ω, ρ) is called a *G-structure on M* if for all $p \in M$ (ω_p, ρ_p) is a G-structure on T_pM.

Convention 2.12. Since the endomorphism field J_ρ in general has torsion, we call it the *almost* complex or para-complex structure on M.

If there already is a G-structure of the above kind on M, we can define by the following construction further G-structures on M.

Lemma 2.13. (1) *Let M be a 6-dimensional manifold carrying an $SU(3)$- or $SU(1,2)$-structure (ω, ρ). For any $\theta \in C^\infty(M)$*

$$(\omega, \cos\theta \cdot \rho + \sin\theta \cdot J_\rho^* \rho) \tag{8}$$

is another G-structure of the same type, which we denote by (ω, ρ^θ).
(2) *Let M be a 6-dimensional manifold carrying an $SL(3,\mathbb{R})$-structure (ω, ρ). For any $\theta \in C^\infty(M)$*

$$(\omega, \cosh\theta \cdot \rho - \sinh\theta \cdot J_\rho^* \rho) \tag{9}$$

is another $SL(3,\mathbb{R})$-structure, which we also denote by (ω, ρ^θ).
(3) *In both of the above cases, the metric which is associated to (ω, ρ^θ) is the same as of (ω, ρ).*

Proof. Let (ω, ρ) be an $SU(3)$- or $SU(1,2)$-structure, $p \in M$, and $(v_i)_{i=1,\dots,6}$ be a basis of T_pM with the properties from Definition 2.4. Furthermore, let

$$A_\theta := \begin{pmatrix} \cos\frac{\theta}{3} & -\sin\frac{\theta}{3} & & & & \\ \sin\frac{\theta}{3} & \cos\frac{\theta}{3} & & & & \\ & & \cos\frac{\theta}{3} & -\sin\frac{\theta}{3} & & \\ & & \sin\frac{\theta}{3} & \cos\frac{\theta}{3} & & \\ & & & & \cos\frac{\theta}{3} & -\sin\frac{\theta}{3} \\ & & & & \sin\frac{\theta}{3} & \cos\frac{\theta}{3} \end{pmatrix} \tag{10}$$

By letting A_θ act on $(v_i)_{i=1,\ldots,6}$, we obtain a new basis $(v_i')_{i=1,\ldots,6}$ of T_pM. This basis induces a new $SU(3)$- or $SU(1,2)$-structure which coincides with (ω,ρ^θ). If (ω,ρ) is an $SL(3,\mathbb{R})$-structure, we obtain (ω,ρ^θ) with help of the matrix

$$B_\theta := \begin{pmatrix} \cosh\frac{\theta}{3} & \sinh\frac{\theta}{3} & & & & \\ \sinh\frac{\theta}{3} & \cosh\frac{\theta}{3} & & & & \\ & & \cosh\frac{\theta}{3} & \sinh\frac{\theta}{3} & & \\ & & \sinh\frac{\theta}{3} & \cosh\frac{\theta}{3} & & \\ & & & & \cosh\frac{\theta}{3} & \sinh\frac{\theta}{3} \\ & & & & \sinh\frac{\theta}{3} & \cosh\frac{\theta}{3} \end{pmatrix} \tag{11}$$

Since A_θ is an element of $SO(6)$ and $SO(2,4)$ and B_θ is an element of $SO(3,3)$, the associated metric g_6 remains in both cases the same. We remark that in the case $G = SU(1,2)$ the signature of g_6 is $(-,-,-,-,+,+)$ and in the case $G = SL(3,\mathbb{R})$ it is $(+,-,+,-,+,-)$. □

Remark 2.14. (1) The stabilizer group of the pair (ω,ρ^θ) is the same as of (ω,ρ). Nevertheless, (ω,ρ^θ) and (ω,ρ) are different G-structures, except for $\theta \in \frac{2\pi}{3}\mathbb{Z}$ in the first case of the lemma. If we had defined the notion of a G-structure by a principal bundle, (ω,ρ) and (ω,ρ^θ) also would not coincide. The reason for this is that we change the frame for the G-structure by the matrix A_θ or B_θ which is not an element of the structure group.

(2) Let G be either $SU(3)$, $SU(1,2)$, or $SL(3,\mathbb{R})$. The set of all G-structures with fixed ω and g_6 is given by the G-structures from the above lemma. This is a consequence of the fact that the stabilizer of the pair (ω, g_6) is $U(3)$, $U(1,2)$, or $GL(3,\mathbb{R})$ and that the center of these groups consists of all A_θ or B_θ. We remark that the set of all G-structures with a fixed associated metric is a much larger set which we will not describe in detail.

For the proof of the main theorem we also need to define $SU(3)$-, $SU(1,2)$-, and $SL(3,\mathbb{R})$-structures on manifolds whose dimension is greater than 6.

Definition 2.15. Let M be a manifold with $\dim M \geq 6$, $\omega \in \bigwedge^2 T^*M$, and $\rho \in \bigwedge^3 T^*M$. Moreover, let $G \in \{SU(3), SU(1,2), SL(3,\mathbb{R})\}$ and let D be a 6-dimensional distribution on M. (ω,ρ) is called a *G-structure on M* if there exists another distribution D' on M such that for all $p \in M$

(1) $D_p \oplus D'_p = T_pM$,
(2) the restriction $(\omega_p|_{D_p\times D_p}, \rho_p|_{D_p\times D_p\times D_p})$ is a G-structure on D_p, and
(3) $X_p \lrcorner \omega_p = X_p \lrcorner \rho_p = 0$ for all $X_p \in D'_p$.

2.2. **G is a real form of $G_2^{\mathbb{C}}$.** With help of the G-structures from the previous subsection we are able to define the notion of a G_2- or G_2^*-structure.

Definition and Lemma 2.16. We supplement the basis $(e_i)_{i=1,\dots,6}$ of $\mathbb{R}^6$ with e_7 to a basis of $\mathbb{R}^7$. The form

(1) $\phi_{G_2} := \omega_{SU(3)} \wedge e^7 + \rho_{SU(3)}$ is stabilized by G_2.
(2) $\phi_{G_2^*,i} := \omega_{SU(1,2)} \wedge e^7 + \rho_{SU(1,2)}$ is stabilized by G_2^*.
(3) $\phi_{G_2^*,ii} := \omega_{SL(3,\mathbb{R})} \wedge e^7 + \rho_{SL(3,\mathbb{R})}$ is stabilized by G_2^*.

G_2 denotes the compact real form of the complex Lie group $G_2^{\mathbb{C}}$ and G_2^* denotes the split real form. If we consider one of the above forms on $\mathbb{R}^7$ without specifying it, we shortly denote it by ϕ.

Let V be a 7-dimensional real vector space and ϕ be a 3-form on V. If there exists a basis $(v_i)_{i=1,\dots,7}$ of V such that with respect to $(v_i)_{i=1,\dots,7}$

(1) ϕ can be identified with ϕ_{G_2}, ϕ is called a *G_2-structure*.
(2) ϕ can be identified with $\phi_{G_2^*,i}$, ϕ is called a *G_2^*-structure*.
(3) ϕ can be identified with $\phi_{G_2^*,ii}$, ϕ is also called a *G_2^*-structure*.

Remark 2.17. There are exactly two open orbits of the action of $GL(7,\mathbb{R})$ on $\bigwedge^3 \mathbb{R}^{7*}$ [19], [23]. Their union is a dense subset of $\bigwedge^3 \mathbb{R}^{7*}$. One orbit consists of all 3-forms which are stabilized by G_2 and the other one consists of all 3-forms which are stabilized by G_2^*. If ϕ is a G_2^*-structure in the sense of (2), it is therefore a G_2^*-structure in the sense of (3), too.

Any G_2- or G_2^*-structure on a vector space V determines a symmetric non-degenerate bilinear form g_7 and a volume form vol_7. Let $\bigwedge^3_s V^*$ be the set of all stable 3-forms on V. As in the previous subsection, there are explicit rational $GL(7,\mathbb{R})$-equivariant maps $\bigwedge^3_s V^* \to S^2(V^*)$ and $\bigwedge^3_s V^* \to \bigwedge^7 V^*$ which assign g_7 and vol_7 to ϕ. The explicit definition of these maps can be found in [7]. The tensors ϕ, g_7, and vol_7 are related by the formula

$$g_7(v,w)\,\mathrm{vol}_7 := -\tfrac{1}{6}(v\lrcorner\phi)\wedge(w\lrcorner\phi)\wedge\phi\,. \tag{12}$$

Analogously to Subsection 2.1, we have

Lemma 2.18. (1) *Let V be a 7-dimensional real vector space and ϕ be a stable 3-form on V. If ϕ is a G_2-structure, g_7 has signature* $(7,0)$.

In particular, g_7 is the Euclidean metric on $\mathbb{R}^7$ if ϕ coincides with ϕ_{G_2}. If ϕ is a G_2^*-structure, g_7 has signature $(3,4)$.

(2) *In the situation of Definition and Lemma 2.16, $\mathbb{R}^6$ and span(e_7) are orthogonal to each other with respect to g_7. Moreover, the restriction of g_7 to $\mathbb{R}^6$ coincides with g_6. If*
   (a) $\phi = \phi_{G_2}$, $g_7(e_7, e_7) = 1$.
   (b) $\phi = \phi_{G_2^*,i}$, $g_7(e_7, e_7) = 1$.
   (c) $\phi = \phi_{G_2^*,ii}$, $g_7(e_7, e_7) = -1$.

Remark 2.19. Since there are only two open $GL(7,\mathbb{R})$-orbits, $(7,0)$ and $(3,4)$ are the only possible signatures of g_7. Therefore, it is possible to check if a 3-form is a G_2- or G_2^*-structure without finding a basis with the properties of Definition and Lemma 2.16.

We can relate vol_7 to the 3-forms on the 6-dimensional space $\mathrm{span}(v_i)_{i=1,\ldots,6}$.

Lemma 2.20. *In the situation of Definition and Lemma 2.16, vol_7 is*

(1) $\frac{1}{4}J^*\rho \wedge \rho \wedge v^7$ *if ϕ is a G_2-structure or a G_2^*-structure in the sense of 2.16.2.*
(2) $-\frac{1}{4}J^*\rho \wedge \rho \wedge v^7$ *if ϕ is a G_2^*-structure in the sense of 2.16.3.*

In particular, vol_7 is

(1) $e^{1234567}$ *if ϕ is ϕ_{G_2} or $\phi_{G_2^*,i}$,*
(2) $-e^{1234567}$ *if ϕ is $\phi_{G_2^*,ii}$.*

Convention 2.21. We call g_7 the *metric which is associated to ϕ* and vol_7 the *volume form which is associated to ϕ* or more briefly we call them the *associated metric and volume form*. g_7 and vol_7 determine a Hodge-star operator $*$ on V. The dual 4-form $*\phi$ will be called the *associated 4-form*.

Lemma 2.22. *In the situation of Definition and Lemma 2.16, the 4-form $*\phi$ is stable and can be described as*

(1) $v^7 \wedge J_\rho^*\rho + \frac{1}{2}\omega \wedge \omega$ *if ϕ is a G_2-structure and (ω,ρ) is the $SU(3)$-structure on span$(v_i)_{i=1,\ldots,6}$.*
(2) $v^7 \wedge J_\rho^*\rho + \frac{1}{2}\omega \wedge \omega$ *if ϕ is a G_2^*-structure in the sense of 2.16.2 and (ω,ρ) is the $SU(1,2)$-structure on span$(v_i)_{i=1,\ldots,6}$.*
(3) $-v^7 \wedge J_\rho^*\rho - \frac{1}{2}\omega \wedge \omega$ *if ϕ is a G_2^*-structure in the sense of 2.16.3 and (ω,ρ) is the $SL(3,\mathbb{R})$-structure on span$(v_i)_{i=1,\ldots,6}$.*

As in the previous subsection, we can define the notion of a G_2- or G_2^*-structure on a manifold.

Definition 2.23. Let M be a 7-dimensional manifold and $\phi \in \bigwedge^3 T^*M$. Moreover, let $G \in \{G_2, G_2^*\}$. ϕ is called a *G-structure on M* if for all $p \in M$ ϕ_p is a G-structure on T_pM.

2.3. G **is a real form of** $\mathbf{Spin}^{\mathbb{C}}(7)$**.** Since the aim of this article is to construct metrics with holonomy Spin(7) or $\mathrm{Spin}_0(3,4)$, we finally have to introduce Spin(7)- and $\mathrm{Spin}_0(3,4)$-structures.

Definition and Lemma 2.24. We supplement the basis $(e_i)_{i=1,\dots,7}$ of $\mathbb{R}^7$ with e_8 to a basis of $\mathbb{R}^8$. The form

(1) $\Phi_{\mathrm{Spin}(7)} := e^8 \wedge \phi_{G_2} + *\phi_{G_2}$ is stabilized by Spin(7).
(2) $\Phi_{\mathrm{Spin}_0(3,4),i} := e^8 \wedge \phi_{G_2^*,i} + *\phi_{G_2^*,i}$ is stabilized by $\mathrm{Spin}_0(3,4)$.
(3) $\Phi_{\mathrm{Spin}_0(3,4),ii} := e^8 \wedge \phi_{G_2^*,ii} + *\phi_{G_2^*,ii}$ is stabilized by $\mathrm{Spin}_0(3,4)$.

$\mathrm{Spin}_0(3,4)$ denotes the identity component of $\mathrm{Spin}(3,4)$. If we consider one of the above forms on $\mathbb{R}^8$ without specifying it, we shortly denote it by Φ.

Let V be an 8-dimensional real vector space and Φ be a 4-form on V. If there exists a basis $(v_i)_{i=1,\dots,8}$ of V such that with respect to $(v_i)_{i=1,\dots,8}$

(1) Φ can be identified with $\Phi_{\mathrm{Spin}(7)}$, Φ is called a *Spin(7)-structure*.
(2) Φ can be identified with $\Phi_{\mathrm{Spin}_0(3,4),i}$, Φ is called a *Spin*$_0(3,4)$*-structure*.
(3) Φ can be identified with $\Phi_{\mathrm{Spin}_0(3,4),ii}$, Φ is also called a *Spin*$_0(3,4)$*-structure*.

Remark 2.25. For the same reasons as in Remark 2.17, Φ is also a $\mathrm{Spin}_0(3,4)$-structure in the sense of (3) if it is a $\mathrm{Spin}_0(3,4)$-structure in the sense of (2).

Analogously to the previous two subsections, any Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure determines a symmetric non-degenerate bilinear form g_8 and a volume form vol_8. vol_8 is given by $\frac{1}{14}\Phi \wedge \Phi$ and g_8 satisfies a slightly more complicated formula as (12), which can be found in Karigiannis [18].

Unlike ω, ρ, and ϕ, Φ is not a stable form. Nevertheless, we have similar results as in the previous two subsections.

Lemma 2.26. (1) *Let V be an 8-dimensional real vector space and Φ be a Spin(7)- or Spin*$_0(3,4)$*-structure on V. In the first case, g_8 has signature $(8,0)$ and in the second case it has signature $(4,4)$. In particular, g_8 is the Euclidean metric on $\mathbb{R}^8$ if Φ coincides with $\Phi_{Spin(7)}$.*
(2) *In the situation of Definition and Lemma 2.24, $\mathbb{R}^7$ and span(e_8) are orthogonal to each other with respect to g_8. Moreover, the restriction of g_8 to $\mathbb{R}^7$ coincides with g_7. In all three cases, we have $g_8(e_8,e_8) = 1$.*

Lemma 2.27. *In the situation of Definition and Lemma 2.24, vol$_8$ is in all three cases*

$$vol_7 \wedge v^8 \,. \tag{13}$$

Convention 2.28. As in the previous subsection, we call g_8 the *associated metric* and vol_8 the *associated volume form*.

Remark 2.29.
(1) Φ is self-dual with respect to the metric g_8 and the volume form vol_8.
(2) Any 4-form on an 8-dimensional real vector space which is stabilized by $\mathrm{Spin}(7)$ or $\mathrm{Spin}_0(3,4)$ is a $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure. However, there is no simple criterion like Theorem 2.9 which decides if a given 4-form is a $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure.

The notion of a $\mathrm{Spin}(7)$- or a $\mathrm{Spin}_0(3,4)$-structure on an 8-dimensional manifold can be defined completely analogously to Definition 2.11 and 2.23.

3. Hitchin's flow equations

One of the reasons for studying G-structures is their relation to metrics with special holonomy.

Definition 3.1.
(1) Let $G \in \{SU(3), SU(1,2), SL(3,\mathbb{R})\}$ and let (ω,ρ) be a G-structure on a 6-dimensional manifold. (ω,ρ) is called *torsion-free* if $d\omega = d\rho = 0$.
(2) Let $G \in \{G_2, G_2^*\}$ and let ϕ be a G-structure on a 7-dimensional manifold. ϕ is called *torsion-free* if $d\phi = d * \phi = 0$.
(3) Let $G \in \{\mathrm{Spin}(7), \mathrm{Spin}_0(3,4)\}$ and let Φ be a G-structure on an 8-dimensional manifold. Φ is called *torsion-free* if $d\Phi = 0$.

Lemma 3.2. *(See* [11]*,* [12]*,* [13]*) The metric which is associated to any of the torsion-free G-structures from Definition 3.1 has a holonomy group which is contained in G.*

Conversely, let G be one of the groups from Definition 3.1. Moreover, let (M,g) be a semi-Riemannian manifold such that for any $p \in M$ there exists an isomorphism $\psi : T_pM \to \mathbb{R}^k$ with $k \in \{6,7,8\}$ such that the holonomy group $Hol_p(g)$ satisfies $\psi \circ Hol_p(g) \circ \psi^{-1} \subseteq G$. Then there exists a torsion-free G-structure on M whose associated metric is g.

G_2 and $\mathrm{Spin}(7)$ are called the *exceptional holonomy groups*. Compact Riemannian manifolds with exceptional holonomy are hard to construct. However, many non-compact examples with cohomogeneity one are known [1], [2], [5], [6], [7], [8], [9], [10], [15], [16], [17], [21], [22]. Semi-Riemannian manifolds with holonomy G_2^* or $\mathrm{Spin}_0(3,4)$ are also interesting from a mathematical point of view. Explicit examples of metrics with holonomy G_2^* can be found in [7]. All of the above metrics can be obtained by a method which was developed by Hitchin [14]. Since this article is only about metrics with holonomy $\mathrm{Spin}(7)$ or $\mathrm{Spin}_0(3,4)$, we explain this method only for these cases. As in the previous section, our presentation of the issue is similar as in [7].

Definition 3.3. Let ϕ be a G_2- or G_2^*-structure on a 7-dimensional manifold. ϕ is called *cocalibrated* if $d * \phi = 0$.

Theorem 3.4. *(See [7], [14]) Let $t_0 \in \mathbb{R}$, N_7 be a 7-dimensional manifold, and $U \subseteq N_7 \times \mathbb{R}$ be an open neighborhood of $N_7 \times \{t_0\}$. Furthermore, let $G \in \{G_2, G_2^*\}$ and ϕ be a cocalibrated G-structure on N_7. Finally, let ϕ_t be a one-parameter family of 3-forms such that ϕ_t is defined on $U \cap (N_7 \times \{t\})$. We assume that ϕ_t is a solution of the initial value problem*

$$\begin{array}{rcl} \frac{\partial}{\partial t} *_7 \phi_t & = & d_7\phi_t \\ \phi_{t_0} & = & \phi \end{array} \tag{14}$$

The index "7" emphasizes that we consider $*$ and d as operators on N_7 instead of U. If U is sufficiently small, ϕ_t is a G-structure for all t with $U \cap (N_7 \times \{t\}) \neq \emptyset$. Moreover, it is cocalibrated for all t. The 4-form

$$\Phi := dt \wedge \phi_t + *_7\phi_t \tag{15}$$

is a torsion-free Spin(7)-structure if $G = G_2$ and a torsion-free $\mathrm{Spin}_0(3,4)$-structure if $G = G_2^*$. Let g_8 be the metric which is associated to Φ and g_t be the metric on $N_7 \times \{t\}$ which is associated to ϕ_t. With this notation we have

$$g_8 = g_t + dt^2 \, . \tag{16}$$

The equation $\frac{\partial}{\partial t} *_7 \phi_t = d_7\phi_t$ is called *Hitchin's flow equation* and any of its solutions ϕ_t is called a *Hitchin flow*.

Remark 3.5.
(1) Since $*_7$ depends non-linearly on ϕ_t, Hitchin's flow equation is a non-linear partial differential equation.
(2) If N_7 and ϕ are real analytic, the system (14) has a unique maximal solution which is defined on a certain open neighborhood of $N_7 \times \{t_0\}$. This is a consequence of the Cauchy-Kovalevskaya Theorem. We will therefore assume from now that N_7 and ϕ are analytic.
(3) If N_7 is compact, there exists a unique maximal open interval I with $t_0 \in I$ such that the solution of (14) is defined on $N_7 \times I$.
(4) Let $f : N_7 \to N_7$ be a diffeomorphism, I an interval with $t_0 \in I$, $U = N_7 \times I$, and ϕ_t be a solution of (14) on U. In this situation, the pull-back $f^*\phi_t$ is a solution of $\frac{\partial}{\partial t} *_7 \phi_t = d_7\phi_t$ with the initial value $\phi_{t_0} = f^*\phi$. In particular, the automorphism group of (N_7, ϕ) is preserved by Hitchin's flow equation.
(5) If N_7 is a homogeneous space G/H and ϕ is G-invariant, ϕ_t is also G-invariant for all t. In this situation, Hitchin's flow equation can be rewritten as a system of ordinary differential equations and it has a unique maximal solution on a set of type $N_7 \times I$ even if N_7 is non-compact. Conversely, let Φ be a torsion-free Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure of cohomogeneity one. The lift of Φ to the universal cover

of the union of all principal orbits can be obtained from a Hitchin flow of the above type.

4. Spin(7)-structures on complex line bundles

We assume that we are in the situation of Theorem 3.4. Let $p \in N_7$ be arbitrary. The curve $\gamma : I \to N_7 \times I$ with $\gamma(t) := (p, t)$ is a geodesic with respect to g_8. If $I \subsetneq \mathbb{R}$, $(N_7 \times I, g_8)$ thus is geodesically incomplete. One method to nevertheless construct complete solutions of Hitchin's flow equations is with help of *degenerations*.

Definition 4.1. Let N_7 be a 7-dimensional manifold, $G \in \{G_2, G_2^*\}$, and $(\phi_t)_{t\in(0,\epsilon)}$ with $\epsilon > 0$ be a one-parameter family of G-structures on N_7. We assume that $(\phi_t)_{t\in(0,\epsilon)}$ is a solution of Hitchin's flow equation and that is has a smooth extension to $t = 0$. Furthermore, we assume that N_7 is an $SO(2)$-principal bundle over a 6-dimensional manifold N_6 and that for all $t \in (0, \epsilon)$ all vertical tangent vectors X satisfy $g_7(X, X) > 0$.

Let M_8 be the $\mathbb{R}^2$-bundle over N_6 which is associated to the standard representation of $SO(2)$ and let $\pi : M_8 \to N_6$ be the projection map. If there exists a smooth Spin(7)- or $\mathrm{Spin}_0(3, 4)$-structure Φ on M_8 such that there is a differentiable map $F : N_7 \times [0, \epsilon) \to M_8$ with the following properties:

1. $F(N_7 \times \{0\}) = N_6$, where we have identified the zero section of M_8 with N_6,
2. $F(N_7 \times \{t\})$ is $SO(2)$-invariant and $\pi \circ F(N_7 \times \{t\}) = N_6$ for all $t \in (0, \epsilon)$,
3. $F|_{N_7\times(0,\epsilon)} : N_7 \times (0, \epsilon) \to M_8 \setminus N_6$ is a diffeomorphism which maps fibers of N_7 into fibers of M_8, and
4. $F^*\Phi = dt \wedge \phi_t + *_7\phi_t$,

we say that the Hitchin flow on N_7 *degenerates into* N_6.

Remark 4.2.
1. The idea behind the above definition is that N_7 is a circle bundle and that the circumference of the circle shrinks to 0 as $t \to 0$. The reason why our definition is quite technical is that we want to consider an initial value problem on $N_7 \times [0, \epsilon)$ and identify $N_7 \times \{0\}$ with the base space. Since it is not clear if the Hitchin flow can be extended to all of $[0, \infty)$, M_8 is in fact a disc bundle over N_6. However, from the topological point of view this difference is unimportant. We remark that any Spin(7)-manifold of cohomogeneity one with exactly one singular orbit of dimension 6 satisfies the conditions of Definition 4.1.
2. The fibers of M_8 carry a canonical complex structure which makes M_8 a complex line bundle over N_6.

Since in the situation of Definition 4.1 all vertical vectors are of positive length, N_6 carries an $SU(3)$- or $SU(1, 2)$-structure. This G-structure can be considered as the initial value of (14). In principle, it is possible to modify Definition 4.1 in such a way that we have an $SL(3,\mathbb{R})$-structure on N_6. In that situation, we would need an action of $SO^+(1,1)$ instead of $SO(2)$ on the fibers and M_8 would be a para-complex line bundle instead of a complex one. However, there is an important difference. The intersection of $F(N_7 \times \{t\})$ and a fiber of M_8 is not a circle but an $SO^+(1,1)$-orbit. For $t \to 0$ the orbits converge to a set of type $\{(x,y) \in \mathbb{R}^2 | x^2 - y^2 = 0,\, x \geq 0\}$. Since that set is not a point, it would require more work to prescribe suitable initial conditions for the Hitchin flow. For this reason, we restrict ourselves to the case where N_6 carries an $SU(3)$- or $SU(1,2)$-structure. In order to describe that G-structure in more detail, we define two vector fields on $M_8 \setminus N_6$.

Definition and Lemma 4.3. Let N_6, N_7, M_8, and $F : N_7 \times [0,\epsilon) \to M_8$ satisfy the conditions from Definition 4.1. We define e_φ as the vector field on $M_8 \setminus N_6$ which generates the action of $SO(2)$ on the fibers of M_8. We normalize e_φ such that its flow at the time 2π is the identity map and the flow at any time in $(0,2\pi)$ is not the identity. Moreover, we define the vector field $e_r := (dF)\left(\frac{\partial}{\partial t}\right)$ on $M_8 \setminus N_6$. We call e_r the *radial vector field.* If there exists a Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure on M_8 with the properties of Definition 4.1, e_r is the unique vertical vector field of unit length which is orthogonal to e_φ and outward directed.

In order to make g_8 smooth at N_6, we need $\lim_{t\to 0} \|e_\varphi\| = 0$. The metric on $N_7 \times \{0\}$ therefore has to be degenerate. This fact helps us to define the $SU(3)$- or $SU(1,2)$-structure on N_6.

Definition 4.4. We assume that we are in the situation of Definition 4.1. Let e^r and e^φ be the duals of e_r and e_φ with respect to g_8. Furthermore, let ω and ρ denote from now on the following differential forms.

(1) $\omega := \frac{1}{\|e_\varphi\|}\, e_\varphi \lrcorner\, (e_r \lrcorner \Phi)$,
(2) $\rho := e_r \lrcorner \Phi - \|e_\varphi\|\, e^\varphi \wedge \omega$.

Lemma 4.5. *We identify $M_8 \setminus N_6$ with $N_7 \times (0,\epsilon)$ by the map F. If Φ is a Spin(7)-structure, (ω,ρ) is an $SU(3)$-structure on the distribution span$(e_r, e_\varphi)^\perp$. Equivalently, (ω,ρ) can be described as a t-dependent $SU(3)$-structure on the distribution span$(e_\varphi)^\perp \subseteq TN_7$. Analogously, (ω,ρ) is an $SU(1,2)$-structure if Φ is a $\mathrm{Spin}_0(3,4)$-structure. Moreover, the 3-form*

$$\phi_t := \|e_\varphi\|\, \omega_t \wedge e^\varphi + \rho_t \tag{17}$$

is a G_2- or G_2^*-structure on $N_7 \times \{t\}$. $\lim_{t\to 0}(\omega_t,\rho_t)$ is an $SU(3)$- or $SU(1,2)$-structure on the distribution span$(e_\varphi)^\perp$ on $N_7 \times \{0\}$, which we denote by (ω_0,ρ_0). Finally, there exists a unique $SU(3)$- or $SU(1,2)$-structure on the zero section N_6 of M_8 such that its pull-back with respect to F is (ω_0,ρ_0).

Proof. The subgroup of Spin(7) which fixes the pair (e_7, e_8) is $SU(3)$. Moreover, Spin(7) acts transitively on the set of all orthonormal pairs in $\mathbb{R}^8$ [4]. If Φ is a Spin(7)-structure, it thus induces an $SU(3)$-structure (ω, ρ) on the distribution $\mathrm{span}(e_r, e_\varphi)^\perp$. The differential forms ω and ρ coincide with the forms from Definition 4.3. There exists an open covering $(U_i)_{i\in I}$ of N_6 such that the normal bundle of any U_i has two linearly independent sections. Therefore, Φ induces an $SU(3)$-structure on each U_i. Its pull-back with respect to F is (ω_0, ρ_0). ω_0 and ρ_0 are globally defined forms and the $SU(3)$-structure on N_6 is thus defined globally, too. Since $\mathrm{Spin}_0(3,4)$ acts transitively on the set of all orthonormal pairs whose elements have positive length, we can prove by the same arguments that (ω, ρ) is an $SU(1,2)$-structure if Φ is a $\mathrm{Spin}_0(3,4)$-structure. □

Remark 4.6. The last statement of the above lemma essentially means that we can consider the G-structure on N_6 as an initial condition for (14).

Lemma 4.7. *Let g_8 be the metric which is associated to Φ. (ω, ρ) induces a metric on span$(e_r, e_\varphi)^\perp$ which we can extend trivially to a degenerate symmetric bilinear form g_6 on $M_8 \backslash N_6$. With this notation, we have the following two relations.*

$$\Phi = \frac{1}{2}\omega \wedge \omega + \|e_\varphi\| e^\varphi \wedge J_\rho^* \rho + e^r \wedge \rho + \|e_\varphi\| e^r \wedge e^\varphi \wedge \omega \,, \tag{18}$$

$$g_8 = g_6 + e^r \otimes e^r + \|e_\varphi\|^2 e^\varphi \otimes e^\varphi \,. \tag{19}$$

Proof. Let Φ be the standard Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure on $\mathbb{R}^8$. We have

$$\Phi = \frac{1}{2}\omega \wedge \omega + e^7 \wedge J_\rho^* \rho + e^8 \wedge \rho + e^8 \wedge e^7 \wedge \omega \,. \tag{20}$$

As in the proof of Lemma 4.5 we can identify e^7 with $\|e_\varphi\| e^\varphi$ and e^8 with e^r and obtain our formula for Φ. The formula for g_8 follows from Lemma 2.18 and Lemma 2.26. □

Definition 4.8. From now on, f denotes the function $f : M_8 \to \mathbb{R}$ with

$$f(p) := \begin{cases} \|e_\varphi\| & \text{if } p \in M_8 \setminus N_6 \\ 0 & \text{if } p \in N_6 \end{cases} \tag{21}$$

In the Riemannian case, there is a simple sufficient condition for the completeness of the metric.

Lemma 4.9. *Let N_7 be a 7-dimensional compact manifold and let ϕ_t be a t-dependent G_2-structure on $N_7 \times (0, \infty)$ which satisfies Hitchin's flow equation and degenerates at $t = 0$ into an $SU(3)$-structure on a 6-dimensional manifold. In this situation, the metric which is associated to the Spin(7)-structure on the 8-dimensional manifold is complete.*

The above lemma can be proven by the same arguments as in [7] and [20], where similar conditions for the completeness can be found.

5. Proof of the main theorem

Our next step is to make Hitchin's flow equation for the form (18) explicit. In order to do this, we have to prescribe the underlying manifold and the initial data.

Convention 5.1. From now on, we assume that $(M_8, N_6, N_7, F, \omega_0, \rho_0, e^\varphi)$ is a tuple with the following properties.

(1) M_8 is a complex line bundle over a 6-dimensional manifold N_6.
(2) N_7 is an $SO(2)$- or equivalently a $U(1)$-principal bundle over N_6 and $F : N_7 \times [0, \infty) \to M_8$ is a map with the same properties as F from Definition 4.1.
(3) (ω_0, ρ_0) is an $SU(3)$- or $SU(1, 2)$-structure on N_6.
(4) e^φ is a $U(1)$-invariant 1-form on $M_8 \setminus N_6$ such that $e^\varphi(e_\varphi) = 1$ and $e^\varphi(e_r) = 0$.

Since we want to apply the Cauchy-Kovalevskaya Theorem to our situation, we assume that all of the above data are real analytic.

If M_8 is a complex line bundle over N_6 and N_6 admits an $SU(3)$- or $SU(1, 2)$-structure, at least one appropriate tuple (N_7, F, e^φ) does exist. Let J be the complex structure on the fibers of M_8. We define a $U(1)$-action on M_8 by $\cos\theta \cdot \mathrm{Id} + \sin\theta \cdot J$ with $\theta \in \mathbb{R}$. Moreover, we fix a Hermitian background metric h on M_8. The set of all points whose distance from the zero section with respect to h is 1 has a natural structure as a $U(1)$-principal bundle. If we choose N_7 as this bundle, we can construct F canonically.

Let D be a principal connection on N_7 and α be its connection form with values in $i\mathbb{R}$. α can be naturally extended to $N_7 \times [0, \infty)$. On M_8 there exists a unique 1-form whose pull-back to $N_7 \times [0, \infty)$ is $-i\alpha$. This 1-form has the same properties e^φ from Convention 5.1. The pull-back of (ω_0, ρ_0) with respect to the projection from N_7 onto N_6 defines an $SU(3)$- or $SU(1, 2)$-structure, which we also denote by (ω_0, ρ_0). The 3-form $\omega_0 \wedge (-i\alpha) + \rho_0$ is a G_2- or G_2^*-structure on N_7 and N_7 thus admits at least one such structure.

Conversely, let us assume that we have found a tuple with the properties from Convention 5.1. There exists a unique Hermitian metric h on M_8 such that $h(e_r, e_r) = 1$. Moreover, the annihilator of e^φ defines a principal connection on N_7.

We can obtain any two Hermitian metrics on M_8 from each other by rescaling the fibers by a diffeomorphism. F is therefore determined by the bundle structure of M_8 up to a fiber-preserving diffeomorphism of $M_8 \setminus N_6$.

Our next step is to deduce some helpful relations. We identify M_8 locally with $U \times \mathbb{R}^2$ such that U is an open subset of N_6 and $U(1)$ acts as $SO(2)$ on the second factor and trivially on the first one. Let $(x^i)_{1\leq i\leq 6}$ be coordinates on U. With this notation, we have

$$[e_r, e_\varphi] = [e_r, \tfrac{\partial}{\partial x^i}] = [e_\varphi, \tfrac{\partial}{\partial x^i}] = 0\,. \tag{22}$$

We denote the exterior derivative on M_8 by d_8 and the exterior derivative on N_7 simply by d. With help of the above formulas, we see that

$$d_8 e^r = 0 \quad \text{and} \quad d_8 e^\varphi(e_r, .) = d_8 e^\varphi(e_\varphi, .) = 0\,. \tag{23}$$

However, we do not necessarily have $de^\varphi = 0$. Let ϕ be a G_2- or G_2^*-structure on N_7 such that the length f of e_φ with respect to the metric which is associated to ϕ is positive. ϕ can be written as $f\,\omega\wedge e^\varphi + \rho$ where (ω,ρ) is an $SU(3)$- or $SU(1,2)$-structure on the distribution $\mathrm{Ann}(e^\varphi)$. ϕ is cocalibrated if and only if

$$\begin{aligned} d * \phi &= d(\tfrac{1}{2}\omega\wedge\omega + f\,e^\varphi \wedge J_\rho^*\rho) \\ &= (d\omega)\wedge\omega + df\wedge e^\varphi\wedge J_\rho^*\rho + f\,de^\varphi\wedge J_\rho^*\rho - f\,e^\varphi\wedge dJ_\rho^*\rho \\ &= 0 \end{aligned} \tag{24}$$

The equation

$$\mathcal{L}_X\alpha = X\lrcorner(d\alpha) + d(X\lrcorner\alpha) \tag{25}$$

for any differential form α yields

$$\begin{array}{rcl} e_\varphi\lrcorner(d\rho) &=& \mathcal{L}_{e_\varphi}\rho \\ e_\varphi\lrcorner(d\omega) &=& \mathcal{L}_{e_\varphi}\omega\,, \end{array} \tag{26}$$

We introduce the following projection map.

$$\begin{array}{rcl} \pi : \bigwedge^* T^*N_7 &\to& \{\alpha\in\bigwedge^* T^*N_7 | e_\varphi\lrcorner\alpha = 0\} \\ \pi(\alpha) &:=& \alpha - e^\varphi\wedge(e_\varphi\lrcorner\alpha) \end{array} \tag{27}$$

After separating the terms which contain an e^φ and the other ones we see that $d*\phi = 0$ can be rewritten as

$$\begin{array}{rcl} \pi(d\omega)\wedge\omega + f\,de^\varphi\wedge J_\rho^*\rho &=& 0 \\ (\mathcal{L}_{e_\varphi}\omega)\wedge\omega - \pi(df)\wedge J_\rho^*\rho - f\,\pi(dJ_\rho^*\rho) &=& 0 \end{array} \tag{28}$$

By similar arguments we see that

$$\begin{aligned} d\phi &= d(f\,\omega \wedge e^{\varphi} + \rho) \\ &= df \wedge \omega \wedge e^{\varphi} + f\,d\omega \wedge e^{\varphi} + f\,\omega \wedge de^{\varphi} + d\rho \end{aligned} \tag{29}$$

Hitchin's flow equation can be written as $\frac{\partial}{\partial t} * \phi = d\phi$ if we omit the subscript "7" of d and $*$. In our situation, it is equivalent to

$$\begin{array}{rcl} \left(\frac{\partial}{\partial t}\omega\right) \wedge \omega &=& \pi(d\rho) + f\,\omega \wedge de^{\varphi} \\ \frac{\partial}{\partial t}\left(f\,J^*_{\rho}\rho\right) &=& \mathcal{L}_{e_{\varphi}}\rho - \pi(df) \wedge \omega - f\,\pi(d\omega) \end{array} \tag{30}$$

The above system has a unique solution for any choice of the initial data.

Theorem 5.2. *Let $(M_8, N_6, N_7, F, \omega_0, \rho_0, e^{\varphi})$ be as in Convention 5.1. Moreover, let N_6 be compact, $d\omega_0 \wedge \omega_0 = 0$, and $\mathcal{L}_{e_{\varphi}}\rho_0 = c \cdot J^*_{\rho_0}\rho_0$ for a $c > 0$. As usual, we identify M_8 with $N_7 \times [0, \infty)$ and assume that $f(p, 0) = 0$ for all $p \in N_7$. In this situation, the system (30) has a unique short-time solution $(\omega_t, \rho_t, f(., t))_{t\in[0,\epsilon)}$ such that for all $t \in (0, \epsilon)$*

1. *$\phi_t := f(., t)\,\omega_t \wedge e^{\varphi} + \rho_t$ is a G_2- or G_2^*-structure, depending on whether (ω_0, ρ_0) is an $SU(3)$- or $SU(1, 2)$-structure,*
2. *$f(p, t) > 0$ for all $p \in N_7$, and*
3. *ϕ_t is cocalibrated.*

In particular, $dt \wedge \phi + *\phi$ is a Spin(7)- or $Spin_0(3, 4)$-structure.

Remark 5.3.
1. Near the zero section N_6 the length of the $U(1)$-orbits has to converge to 0. Since we want to solve the system (30) on a tubular neighborhood of N_6, we have to include the condition $f(p, 0) = 0$ in our theorem.
2. $\mathcal{L}_{e_{\varphi}}\rho_0 = c \cdot J^*_{\rho_0}\rho_0$ should be understood as an equation on $N_7 \times \{0\}$ rather than N_6, since the vector field e_{φ} is not defined on N_6. Our condition should therefore be read as "There exists an $SU(3)$- or $SU(1, 2)$-structure on N_6 such that its pull-back to $N_7 \times \{0\}$ satisfies $\mathcal{L}_{e_{\varphi}}\rho_0 = c \cdot J^*_{\rho_0}\rho_0$."
3. We choose a sufficiently small open subset U of N_6 and identify N_7 locally with $U \times S^1$. Let $\widetilde{\rho_0}$ be the restriction of ρ_0 to the set $\{(x, y) \in U \times S^1 | x \in U, y = (1, 0)\}$. In this situation, ρ_0 at the point $(x, \cos\theta, \sin\theta)$ is given by $\cos(c\theta) \cdot \widetilde{\rho_0} + \sin(c\theta) \cdot J^*_{\widetilde{\rho_0}}\widetilde{\rho_0}$. The reason for this is that the equation $\mathcal{L}_{e_{\varphi}}\rho_0 = c \cdot J^*_{\rho_0}\rho_0$ fixes ρ_0 along any fiber of N_7. ρ_0 can thus be considered as a special case of the construction from Lemma 2.13. Moreover, we see that we necessarily have $c \in \mathbb{N}$.

Proof. Let (f, ω, ρ) be a tuple of a function, a 2-form, and a 3-form which solves (30). Moreover, we assume that the initial conditions of the theorem are satisfied. Since (ω_0, ρ_0) is an $SU(3)$- or $SU(1, 2)$-structure, ω_0 and ρ_0

are both stable. Being stable is an open condition and ω and ρ are therefore stable forms on $\mathrm{Ann}(e^\varphi)$ if t is sufficiently small. Since N_6 is compact, the bound on t can be chosen independently from the point in N_6. The initial conditions $\mathcal{L}_{e_\varphi}\rho_0 = c \cdot J_{\rho_0}^*\rho_0$ and $f(p,0) = 0$ force $\frac{\partial}{\partial t} f(p,0)$ to be positive. We assume that t is so small that ω and ρ are stable and f is positive. The 3-form $f(.,t)e^\varphi \wedge \omega_0 + \rho_0$ is a stable form and therefore a G_2- or G_2^*-structure. $f(.,t)e^\varphi \wedge \omega_t + \rho_t$ is a also G_2- or G_2^*-structure and (ω_t, ρ_t) is either an $SU(3)$- or $SU(1,2)$-structure.

Our next step is to prove that (30) has a unique short-time solution. Let X be a t-dependent vector field on N_7 which is of unit length with respect to the associated metric. The norm of $X \lrcorner * \phi$ is always 4. This can be seen by choosing a basis with the properties from Definition 2.16. If we fix X as $\frac{1}{f} e_\varphi$, we see that $J_\rho^* \rho$ has to be of constant length. $\frac{\partial f}{\partial t}$ is thus 4 times the norm of the right hand side of the second equation of (30). By inserting that term into $\frac{\partial f}{\partial t}\, J_\rho^*\rho + f\, \frac{\partial}{\partial t} J_\rho^* \rho$, we find an equation for $\frac{\partial}{\partial t} J_\rho^*\rho$. In the special case $f = 0$, the second equation of (30) has only a component in the $J_\rho^*\rho$-direction and we thus have $\frac{\partial f}{\partial t} = c$ and $\frac{\partial}{\partial t} J_\rho^*\rho = 0$.

There exists a certain $GL(6)$-equivariant map $\iota : \bigwedge^4 \mathbb{R}^{6*} \to \bigwedge^2 \mathbb{R}^{6*}$ which maps stable forms into stable forms [7]. In particular, we have $\iota(\frac{1}{2}\omega \wedge \omega) = \omega$. We apply ι to the first equation of (30) and obtain an equation for $\frac{\partial}{\partial t}\omega$. Since the constructions which we have made are real analytic, the short-time existence and uniqueness follows from the Cauchy-Kovalevskaya Theorem.

The initial conditions $f(.,0) = 0$ and $d\omega_0 \wedge \omega_0 = 0$ ensure that $d * \phi_0 = 0$. From the calculation

$$\frac{\partial}{\partial t} d * \phi = d \frac{\partial}{\partial t} * \phi = d^2 \phi = 0 \tag{31}$$

it follows that ϕ_t is cocalibrated for all values of t. □

Remark 5.4. Let $(\phi_t)_{t\in[0,\epsilon)}$ be a solution of (30) with the initial data $(M_8, N_6, N_7, F, \omega_0, \rho_0, e'^\varphi)$, where e'^φ is another 1-form with the same properties as e^φ. It is possible to split ϕ_t with respect to e^φ into ρ_t and $f(.,t)\, \omega_t \wedge e^\varphi$. Since $(\omega_t, \rho_t, f(.,t))_{[0,\epsilon)}$ describes a solution of (30) with the initial data $(M_8, N_6, N_7, F, \omega_0, \rho_0,\ e^\varphi)$, we obtain no new Spin(7)- or $\mathrm{Spin}_0(3,4)$-structures for different choices of e^φ.

Not any smooth solution of (30) corresponds to a smooth 4-form on M_8. The reason for this is that the vector fields e_r and e_φ become singular along the zero section. In the following, we are going to answer the question under which conditions a Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure on $M_8 \setminus N_6$ has a smooth extension to N_6.

Let $\psi : \mathbb{R}^{\geq 0} \times \mathbb{R} \to \mathbb{R}^2$ be defined by $\psi(r,\theta) := (r\cos\theta, r\sin\theta)$. Furthermore, let $h : \mathbb{R}^{\geq 0} \times \mathbb{R} \to \mathbb{R}$ be an analytic function. There exists an analytic function $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}$ with $\widetilde{h} \circ \psi = h$ if and only if the analytic extension of h is invariant under $(r, \theta) \mapsto (-r, \theta + \pi)$. This can be seen by expanding $\widetilde{h}$ into a power series in (x, y) and replacing x by $r \cos \theta$ and y by $r \sin \theta$.

Let Ψ be the flow of the vector field e_φ at the time π. Analogously to above, f, ω, and ρ correspond to smooth objects on M_8 if their analytic continuation is invariant under $(p, t) \mapsto (\Psi(p), -t)$.

We describe these smoothness conditions at $t = 0$ in detail. The function f has to satisfy

$$f(p, t) = f(\Psi(p), -t) \, . \tag{32}$$

Since we assume that f vanishes on $N_7 \times \{0\}$, this is obviously true for $t = 0$. The conditions on ω and ρ are equivalent to

$$\Psi^* \omega_0 = \omega_0 \quad \Psi^* \rho_0 = \rho_0 \, , \tag{33}$$

where we consider ω_0 and ρ_0 as objects on $N_7 \times \{0\}$. Let $t > 0$ and let

$$\widetilde{N}_7^t := \{(v, x, y) | v \in \mathbb{R}^6, \; x^2 + y^2 = t^2\} \, . \tag{34}$$

The flat Spin(7)- or $\mathrm{Spin}_0(3, 4)$-structure on $\mathbb{R}^8$ can be considered as a degeneration of a family of G_2- or G_2^*-structures on $\widetilde{N}_7^t$ to $\mathbb{R}^6$. In this situation, e_φ is the vector field which generates the action of $SO(2)$ on the last two coordinates of $\mathbb{R}^8$. By a short calculation, we see that the $SU(3)$- or $SU(1, 2)$-structure $(\widetilde{\omega}, \widetilde{\rho})$ on the 6-dimensional distribution on $\widetilde{N}_7^t$ satisfies

$$\mathcal{L}_{e_\varphi} \widetilde{\rho} = J_{\widetilde{\rho}}^* \widetilde{\rho} \quad \text{and} \quad \mathcal{L}_{e_\varphi} \widetilde{\omega} = 0 \, . \tag{35}$$

Let Φ be a parallel Spin(7)- or $\mathrm{Spin}_0(3, 4)$-structure. Around any point p, we can choose local coordinates such that the coefficients of Φ are the same as of the flat Spin(7)- or $\mathrm{Spin}_0(3, 4)$-structure on $\mathbb{R}^8$ and that the first derivatives of all coefficients vanish at p. Therefore, it follows that (ω_0, ρ_0) has to satisfy

$$\mathcal{L}_{e_\varphi} \rho_0 = J_{\rho_0}^* \rho_0 \quad \text{and} \quad \mathcal{L}_{e_\varphi} \omega_0 = 0 \, . \tag{36}$$

If we replace the action of $A \in SO(2)$ by the action of A^{-1}, we obtain $\mathcal{L}_{e_\varphi} \rho_0 = -J_{\rho_0}^* \rho_0$. In both cases, the conditions (33) are satisfied.

Let $(\omega_t, \rho_t, f(., t))_{t \in (-\epsilon, \epsilon)}$ be a solution of (30) such that we have $f(., 0) = 0$ and (36). In that situation, $(\Psi^* \omega_{-t}, \Psi^* \rho_{-t}, f(\Psi(.), -t))_{t \in (-\epsilon, \epsilon)}$ is a solution of (30) with the same initial values. Since (30) has a unique solution for any choice of the initial data, both solutions coincide and Φ is indeed invariant under $(p,t) \mapsto (\Psi(p), -t)$.

Up to now, we have proven that ω, ρ, and f have a smooth extension to N_6. The 4-form which is determined by these objects has to coincide up to first order with the flat Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure. We therefore get additional conditions on the three objects which we have already deduced for ω and ρ. The missing condition on f can be seen as follows. The length of a $U(1)$-orbit has to be $2\pi t + O(t^2)$ for small t. This can only be satisfied if $\frac{\partial}{\partial t}|_{t=0} f(p,t) = \pm 1$ or equivalently if $\mathcal{L}_{e_\varphi}\rho_0 = \pm J^*_{\rho_0}\rho_0$, which we already assume.

We are now ready to state our main theorem.

Theorem 5.5. *Let N_6 be a compact 6-dimensional manifold and (ω_0, ρ_0) be an $SU(3)$- or $SU(1,2)$-structure on N_6 which satisfies $d\omega_0 \wedge \omega_0 = 0$. Furthermore, let M_8 be a complex line bundle over N_6 with a fixed orientation. We assume that all data are real analytic and identify N_6 with the zero section of M_8.*

If (ω_0, ρ_0) is an $SU(3)$-structure, there exists a unique parallel Spin(7)-structure Φ on a sufficiently small tubular neighborhood U of N_6 such that $\Phi \wedge \Phi$ is a positive volume form and the $SU(3)$-structure on N_6 which is induced by Φ is (ω_0, ρ_0). If (ω_0, ρ_0) is an $SU(1,2)$-structure, there exists a unique parallel $Spin_0(3,4)$-structure on U with the same properties as Φ.

Proof. Let $p \in N_6$ be arbitrary. We first consider the case where (e_φ, e_r) together with a positively oriented basis of T_pN_6 again is positively oriented. As we have discussed at the beginning of this section, it is possible to construct a degeneration F and a 1-form e^φ with the properties of Convention 5.1. Let (ω_0, ρ_0) be an arbitrary $SU(3)$- or $SU(1,2)$-structure on N_6 with $d\omega_0 \wedge \omega_0 = 0$. The pull-back of (ω_0, ρ_0) to $N_7 \times \{0\}$ is an $SU(3)$- or $SU(1,2)$-structure on $\mathrm{Ann}(e^\varphi)$ which satisfies (36) and therefore also $\frac{\partial}{\partial t}|_{t=0} f(p,t) = 1$.

The solution of the system (30) with the above initial data thus describes a Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure which is smooth along N_6. As we have shown in Remark 5.4, a different choice of e^φ would not yield a different solution of (30). If we had chosen another map $F' : N_7 \times [0,\infty) \to M_8$ with the same properties as F, we would have $F' = \psi \circ F$ where ψ is a diffeomorphism of M_8 which rescales the fibers. The corresponding solutions of (30) would therefore be related by the pull-back of ψ. With help of Theorem 5.2 we finally conclude that Φ exists and that it is unique.

In the above situation, $\Phi \wedge \Phi$ is a positive volume form. If the basis from the beginning of the proof is not positively oriented, we simply replace e_φ by $-e_\varphi$ and obtain a Φ which satisfies the conditions from the theorem and $\mathcal{L}_{e_\varphi}\rho_0 = -J^*_{\rho_0}\rho_0$. $\square$

Remark 5.6. (1) If N_6 is non-compact, a similar theorem can be proven. In that situation, U is a neighborhood on N_6 which is not necessarily tubular with respect to g_8.

(2) The metric on $\mathrm{Ann}(e^\varphi)$ which is associated to (ω_0, ρ_0) is invariant under the $U(1)$-action on $N_7 \times \{0\}$. However, the metric on M_8 which is associated to Φ does not necessarily have an isometry group of positive dimension. For example, the normal bundle of any 6-dimensional submanifold of a compact Riemannian manifold (M, g) with holonomy $\mathrm{Spin}(7)$ satisfies the conditions of Theorem 5.5. However, the isometry group of (M, g) is discrete. Since (M, g) is Ricci-flat, any Killing vector field of (M, g) would be parallel and (M, g) would thus be covered by a Riemannian product. In that situation, the holonomy would not be all of $\mathrm{Spin}(7)$.

(3) The condition $d\omega_0 \wedge \omega_0 = 0$ on an $SU(3)$- or $SU(1,2)$-structure is a weak one. In particular, it is satisfied if (ω_0, ρ_0) is half-flat. Furthermore, any 6-dimensional symplectic manifold (N_6, ω) which admits a stable 3-form with $\omega \wedge \rho = 0$ and $J_\rho^* \rho \wedge \rho = \frac{2}{3} \omega \wedge \omega \wedge \omega$ is of that type.

(4) Bielawski [3] has proven a theorem, which is similar to ours, on Ricci-flat Kähler metrics on complex line bundles. In [3], it is assumed that the metric on the base space is Kähler. In that situation, the complex line bundle has to be the canonical bundle in order to admit a Ricci-flat Kähler metric. Furthermore, the action of $U(1)$ becomes isometric and Hamiltonian.

6. Examples

In the literature [1], [2], [8], [9], [10], [15], [17], [20], [21], [22], several examples of cohomogeneity-one metrics with holonomy $\mathrm{Spin}(7)$, $SU(4)$, or $Sp(2)$ on complex line bundles are known. All of these metrics fit into the context of Theorem 5.5. In order to show how the theorem works, we take a look at one class of these metrics. Let

$$U(1)_{1,1} := \left\{ \left(\begin{array}{ccc} e^{it} & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-2it} \end{array} \right) \middle| \text{ with } t \in \mathbb{R} \right\}. \tag{37}$$

We assume that M_8 is a cohomogeneity-one manifold whose principal orbit is the exceptional Aloff-Wallach space $N^{1,1} := SU(3)/U(1)_{1,1}$. $N^{1,1}$ is a circle bundle over $SU(3)/T^2$, where T^2 consists of all diagonal matrices in $SU(3)$. M_8 shall have exactly one singular orbit of type $SU(3)/T^2$. The orbit structure fixes the topology of M_8, which has to be a complex line bundle over $SU(3)/T^2$. We choose the basis

$$
\begin{aligned}
&e_1 := E_1^2 - E_2^1 \qquad e_2 := iE_1^2 + iE_2^1 \quad e_3 := E_1^3 - E_3^1 \\
&e_4 := iE_1^3 + iE_3^1 \quad e_5 := E_2^3 - E_3^2 \qquad e_6 := iE_2^3 + iE_3^2 \\
&e_7 := \tfrac{1}{2} iE_1^1 - \tfrac{1}{2} iE_2^2 \qquad e_8 := iE_1^1 + iE_2^2 - 2iE_3^3
\end{aligned} \tag{38}
$$

of $\mathfrak{su}(3)$, where E_i^j is the 3×3-matrix with $E_i^j(e_k) = \delta_k^j \delta_i^l e_l$ and denote its dual basis by $(e^i)_{1 \leq i \leq 8}$. The tangent space of $N^{1,1}$ can be identified with $\mathfrak{m} := \mathrm{span}(e_1, \ldots, e_7)$. The extension of e_7 to a left-invariant vector field on $N^{1,1}$ is the Killing vector field of the right-action of

$$
\left\{ \left(\begin{array}{ccc} e^{it} & 0 & 0 \\ 0 & e^{-it} & 0 \\ 0 & 0 & 1 \end{array} \right) \middle| \text{ with } \ t \in \mathbb{R} \right\} . \tag{39}
$$

If we choose N_7 as $N^{1,1}$, N_6 as $SU(3)/T^2$, and e_φ as e_7, M_8 satisfies all topological conditions from Definition 4.1.

We search for $SU(3)$-invariant $SU(3)$-structures on the distribution $\mathrm{span}(e_1, \ldots, e_6)$ which are possible initial values for the system (30). If (ω_0, ρ_0) is such an initial value, we have $\mathcal{L}_{e_7} \omega_0 = 0$ and $\mathcal{L}_{e_7} \rho_0$ has to be proportional to $J_{\rho_0}^* \rho_0$. The only differential forms on $N^{1,1}$ which satisfy all these conditions are

$$
\begin{aligned}
\omega_0 &= a^2 e^{12} + b^2 e^{34} - c^2 e^{56} \\
\rho_0 &= abc \, \cos\theta \, (-e^{135} - e^{146} - e^{236} + e^{245}) \\
&\quad + abc \, \sin\theta \, (-e^{136} + e^{145} + e^{235} + e^{246}) \, ,
\end{aligned} \tag{40}
$$

where $a, b, c \in \mathbb{R} \setminus \{0\}$ and $\theta \in \mathbb{R}$ are arbitrary. The condition $d\omega_0 \wedge \omega_0 = 0$ is always satisfied. Unfortunately, we have $\mathcal{L}_{e_7} \rho_0 = -2 J_{\rho_0}^* \rho_0$ and the solutions of (30) therefore cannot describe a Spin(7)-structure which is smooth along the singular orbit. However, if we square the complex line bundle, we obtain $\mathcal{L}_{e_7} \rho_0 = -J_{\rho_0}^* \rho_0$ and the smoothness conditions are satisfied. Equivalently, we could have replaced the principal orbit by a suitable quotient of type $N^{1,1}/\mathbb{Z}_2$, since this construction yields the same complex line bundle. In that situation, Theorem 5.5 predicts for any choice of a, b, c, and θ the existence of a parallel Spin(7)-structure near the singular orbit.

It is possible to assume that $\theta = 0$. The reason for this is that the one-dimensional subgroup of $SU(3)$ which is generated by e_7 leaves ω_0 invariant and acts on ρ_0 by a change of θ. The parallel Spin(7)-structures for different values of θ can therefore be obtained from each other by the pull-back of a diffeomorphism.

The above Spin(7)-structures are of cohomogeneity one and are studied in more detail in [1], [2], [17], [20], [22]. Since there are no further suitable $SU(3)$-structures on the distribution $\mathrm{span}(e_1, \dots, e_6) \subseteq TN^{1,1}$, the following theorem follows immediately from Theorem 5.5.

Theorem 6.1. *Let M_8 be the cohomogeneity-one manifold which we have defined at the beginning of the section. The only $SU(3)$-invariant parallel Spin(7)-structures on M_8 are those which are already known in the literature.*

There are several possibilities to generalize the above examples. M_8 is not the only complex line bundle over $SU(3)/T^2$. Theorem 5.5 predicts the existence of further parallel Spin(7)-structures on all of those bundles. Moreover, it is possible to construct by our theorem parallel $\mathrm{Spin}_0(3,4)$-structures on the same spaces. However, a detailed investigation of those structures or a study of their global behaviour is beyond the scope of this paper.

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