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DISTRIBUTIONS IN THE PROCESS $e^+e^- \to \mu^+\mu^-(\gamma)$

by

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Abstract

Analytical expressions for a number of distributions in the process $e^+e^- \to \mu^+\mu^-(\gamma)$ are given. A procedure is outlined to simulate events for this reaction by using the analytical distributions. This method then provides numerically any other distribution and makes it possible to impose any experimental constraint to radiative correction calculations. Besides examples for pure QED situations, also the corrections to R and to the thrust distribution for hadronic events are discussed.

1. Introduction

The processes

$$e^+(p_+) + e^-(p_-) \to \mu^+(q_+) + \mu^-(q_-) \tag{1.1}$$

and

$$e^+(p_+) + e^-(p_-) \to \mu^+(q_+) + \mu^-(q_-) + \gamma(k) \tag{1.2}$$

are of interest for various reasons. In the first place they test the standard model [1] of unified electro-weak interactions. Secondly, the theoretical treatment of (1.1) and (1.2) can be carried over to a final state with two jets with bremsstrahlung of a photon or gluon.

For testing electro-weak interactions one concentrates on the forward-backward asymmetry of reaction (1.1). It is however unavoidable that events of reaction (1.2) contribute to this quantity. For the case where the muon events are selected on the basis of a threshold energy (E_{th}) and acollinearity angle (ζ) requirement this has been dealt with in the literature [2,3]. The differential cross-section $d\sigma/d\Omega_\mu$ for reaction (1.1) is obtained up to order α^3 and depends on the parameters E_{th} and ζ, which determine the allowed phase-space for reaction (1.2). Over the latter phase space a three dimensional integration is carried out numerically. The weak mixing angle can then be extracted from the experimental data.

Although the energy and acollinearity requirements are quite reasonable, it would be better for the analysis of the experiments to have radiative correction calculations which are applicable in more general situations. The most natural solution is to have a numerical method to simulate events of reaction (1.1) and (1.2) at the same time. The N generated events must be such that they approximate the multidifferential cross-sections for (1.1) and (1.2), when N becomes large. The events which have momenta such that they could not be seen in a specific experiment can then be omitted. Of the remaining events one knows the cross-section and therefore the radiative correction. Of course, the event generator can also be used to evaluate the cross-section for those events which would never simulate a cross-section for reaction (1.1) e.g. very acollinear or acoplanar events.

When reaction (1.1) and (1.2) are considered with quarks in the final state certain QCD relevant quantities like R or a thrust distribution for jets are extracted from the data. Since a good knowledge of these quantities for tests of QCD is important, the initial state radiation of a photon should accurately be taken into account. This is particularly so, since often the initial state QED correction is of the same order of magnitude as the physically more interesting final state QCD correction.

It turns out that the analytical knowledge of a certain number of distributions makes a numerical generation of events possible. In principle, various choices for these distributions can be made. If one is only interested in either initial state radiation or final state radiation choices for the distributions can be made for which it is easy to find analytical expressions. In the case, where one is also interested in the interference

between initial state and final state radiation, a compromise in the choice of variables has to be made. We shall calculate analytically $d\sigma / d\Omega_\mu dk$, where Ω_μ is the solid angle of the μ^+ and k is the photon energy. This distribution has the advantage that it can be directly used for a one-dimensional numerical integration over k to give the radiative corrections to $d\sigma/d\Omega_\mu$, depending on E_{th} and ζ. This is faster than the previous more dimensional integration method for the same quantity [2,3].

The outline of the paper is as follows. In section 2 the known virtual corrections and bremsstrahlung cross-section are summarized. In section 3 a number of analytic formulae for distributions and for the total cross-section are given. Section 4 describes the scheme used for the generation of events. In section 5 a number of numerical results are presented, e.g. acollinearity and acoplanarity distributions and radiative corrections to $d\sigma/d\Omega_\mu$.

2. Virtual corrections and bremsstrahlung

Expressions for the virtual and soft bremsstrahlung correction to reaction (1.1) can be found in ref. 4. The sum of both corrections is expressed in a correction δ_A, where A denotes the fact that the correction is known analytically:

$$\frac{d\sigma}{d\Omega_\mu} = \frac{d\sigma_0}{d\Omega_\mu}\left(1 + \delta_A\right) \tag{2.1}$$

where the lowest order cross-section is given by

$$\frac{d\sigma_0}{d\Omega_\mu} = \frac{\alpha^2}{4s}\left(1 + c^2\right) \tag{2.2}$$

with $c = \cos\theta$, θ being the angle between e^+ and μ^+ .

The correction δ_A can be written as

$$\begin{aligned}\delta_A = & \left(\beta_i + \beta_f + \beta_{int}\right) \ln\left(\frac{k_1}{E}\right) \\ & + \frac{2\alpha}{\pi}\left\{ \frac{13}{12}\left(\ln\frac{s}{m_e^2} + \ln\frac{s}{m_\mu^2}\right) + \frac{\pi^2}{3} - \frac{28}{9} \right. \\ & - \frac{2}{1+c^2}\left[c\left(\ln^2(\sin\tfrac{\theta}{2}) + \ln^2(\cos\tfrac{\theta}{2})\right) + \sin^2\tfrac{\theta}{2}\ln(\cos\tfrac{\theta}{2}) - \cos^2\tfrac{\theta}{2}\ln(\sin\tfrac{\theta}{2}) \right. \\ & \left.\left. + 2\ln^2(\sin\tfrac{\theta}{2}) - 2\ln^2(\cos\tfrac{\theta}{2}) - Li_2(\sin^2\tfrac{\theta}{2}) + Li_2(\cos^2\tfrac{\theta}{2}) \right] \right\} \\ = & \ \beta\ln\left(\frac{k_1}{E}\right) + \delta_{AR}\end{aligned} \tag{2.3}$$

with

$$\begin{aligned}\beta &= \beta_i + \beta_f + \beta_{int} , \\ \beta_i &= \frac{2\alpha}{\pi}\left(\ln\frac{s}{m_e^2} - 1\right) , \\ \beta_f &= \frac{2\alpha}{\pi}\left(\ln\frac{s}{m_\mu^2} - 1\right) , \\ \beta_{int} &= \frac{8\alpha}{\pi}\ln\left(\tan\frac{\theta}{2}\right) ,\end{aligned} \tag{2.4}$$

m_e and m_μ being the electron and muon mass, and Li_2 denoting the

dilogarithm. The expression for δ_A contains contributions from the vertex corrections, the vacuum polarization due to an electron and muon loop, and two box diagrams. Moreover the soft isotropically emitted bremsstrahlung up to a maximum photon energy k_1 is contained in δ_A. In the evaluation of δ_A it is assumed that

$$\sin\theta \gg (m_\mu/E) \quad \text{and} \ (m_\mu/E) \ll 1. \tag{2.5}$$

More involved expressions not using this approximation can be found in ref. 3.

The values k_1 for which δ_A is a good approximation to the radiative corrections are limited from below by the necessity of exponentiation and from above by the necessity of taking hard bremsstrahlung into account.

For too small values of k_1 δ_A becomes large and negative such that it becomes necessary to estimate higher order corrections, which is done by exponentiation of the leading log part of δ_A. One then takes as an approximative formula to the corrections

$$\frac{d\sigma}{d\Omega_\mu} = \frac{d\sigma_0}{d\Omega_\mu}\left(1 + \delta_{AR}\right)\left(\frac{k_1}{E}\right)^\beta . \tag{2.6}$$

For too large values of k_1 the factorized matrixelement for the bremsstrahlung which is used in the derivation of (2.3) is no longer a good approximation for the real (hard) bremsstrahlung. An idea of the k values for which this happens can be derived from an expression for the total cross-section for reactions (1.1) and (1.2), given in section 3. One finds for the difference between the exact total cross-section and the one evaluated on the basis of the soft photon approximation

$$\sigma_{exact} - \sigma_{soft} = \frac{2\alpha}{\pi}\sigma_0 \left(-\frac{k}{E}\ln\frac{s}{m_\mu^2} + O\left(\frac{k^2}{E^2}\right) \right) \tag{2.7}$$

where the lowest order total cross-section

$$\sigma_0 = \frac{4\alpha^2\pi}{3s} \tag{2.8}$$

is factorized out. The requirement that this difference is less than 1 % of σ_0 gives, for energies where $m_\mu/E << 1$,

$$\frac{k}{E} < 0.01 \times \frac{\pi}{2\alpha}\left(\ln\frac{s}{m_\mu^2} \right)^{-1} \tag{2.9}$$

e.g. $k/E \lesssim 0.18$ for E = 20 GeV.

To eq. (2.6) other contributions should be added, namely the vacuum polarization due to a τ-loop (δ_τ), the one due to hadrons (δ_{had}), and the Z-γ interference from the electro-weak interactions.

These contributions are

$$\delta_\tau = \frac{2\alpha}{\pi}\left(\frac{1}{3}\ln\frac{s}{m_\tau^2} - \frac{5}{9} \right) \tag{2.10}$$

where $m_\tau = 1782$ MeV

$$\delta_{had} = -2\,\mathrm{Re}\,\Pi_h(s) = \frac{s}{2\pi^2\alpha}\int_{4m_\pi^2}^{\infty}\frac{\sigma(s')}{s-s'}\,ds' \tag{2.11}$$

where the dispersion integral over the total e^+e^- hadronic cross-section has to be evaluated numerically [5].

The weak-electromagnetic interference cross-section takes the form

$$\frac{d\sigma^w}{d\Omega_\mu} = \frac{\alpha^2}{4s}\cdot 2\chi\left(g_V^2(1+c^2) + 2g_A^2 c \right) \tag{2.12}$$

where

$$\begin{aligned} \chi &= g\frac{sM_Z^2}{s-M_Z^2} \quad , \\ M_Z^2 &= \left(4g\sin^2(2\theta_w)\right)^{-1} \quad , \\ g_V &= -1 + 4\sin^2\theta_w \quad , \\ g_A &= -1 \quad , \\ g &= 4.4\ 10^{-11}\ \mathrm{MeV}^{-2} \quad . \end{aligned} \tag{2.13}$$

This weak effect can be expressed in terms of

$$\delta_w = \left(\frac{d\sigma^w}{d\Omega_\mu}\right)\Big/\left(\frac{d\sigma_0}{d\Omega_\mu}\right) \tag{2.14}$$

which has to be added to δ_{AR} in eqs. (2.3) or (2.6).

This is the only place where the weak interactions occur in the present treatment. For PETRA/PEP energies this is sufficient. For higher energies it first becomes necessary to deal with the full Z-exchange amplitude and the QED corrections (virtual and bremsstrahlung) to it, then also to take the width of the Z in the propagator into account, and finally to apply the weak virtual corrections to the QED amplitude [6].

Besides the expressions for virtual and soft bremsstrahlung corrections we need the hard bremsstrahlung cross-sections. Two choices of the kinematical variables will be convenient, one set consisting of solid angles of μ^+ and γ and the photon energy k, another set using the solid angle of the μ^+, an azimuthal angle of the photon and the energies q_+^0, q_-^0 of the μ^+ and μ^-, respectively.

We have

$$\frac{d\sigma^{B}}{d\Omega_\mu \, d\Omega_\gamma \, dk} = \frac{\alpha^3}{2\pi^2 s} \cdot \frac{|\vec{q}_+|k}{2p_+^0 - k + k\cos\theta_\gamma} A \tag{2.15}$$

and

$$\frac{d\sigma^{B}}{d\Omega_\mu \, dq_+^0 \, dq_-^0 \, d\varphi_\gamma} = \frac{\alpha^3}{2\pi^2 s} A \tag{2.16}$$

with [7]

$$\begin{aligned} A = & -\frac{m_e^2}{2s'^2}\left(\frac{t^2+u^2}{(p_-\cdot k)^2} + \frac{t'^2+u'^2}{(p_+\cdot k)^2}\right) \\ & -\frac{m_\mu^2}{2s^2}\left(\frac{t^2+u'^2}{(q_-\cdot k)^2} + \frac{t'^2+u^2}{(q_+\cdot k)^2}\right) \\ & +\frac{t^2+t'^2+u^2+u'^2}{4ss'}\left[\frac{s}{(p_+\cdot k)(p_-\cdot k)} + \frac{s'}{(q_+\cdot k)(q_-\cdot k)}\right. \\ & \left. -\frac{t}{(p_+\cdot k)(q_+\cdot k)} - \frac{t'}{(p_-\cdot k)(q_-\cdot k)} + \frac{u'}{(p_-\cdot k)(q_+\cdot k)} + \frac{u}{(p_+\cdot k)(q_-\cdot k)}\right] \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} t &= (p_+ - q_+)^2 \\ t' &= (p_- - q_-)^2 \\ u' &= (p_- - q_+)^2 \\ u &= (p_+ - q_-)^2 \\ s &= (p_+ + p_-)^2 \\ s' &= (q_+ + q_-)^2 \end{aligned} \tag{2.18}$$

In this expression for σ^B terms of $O(m_e^2/E^2)$ and $O(m_\mu^2/E^2)$ have been neglected except where denominators can take the same small values. The terms arising from initial state radiation, final state radiation and the interference of the two can be easily recognized.

3. Analytical expressions for certain distributions

In this section analytical results will be given for $d\sigma/d\Omega_\mu dk$, $d\sigma/dk$, $d\sigma/dq_+^0$ and σ_{TOT}. The first distribution is useful for reproducing the standard radiative corrections. Moreover it can be used to generate events numerically according to the procedure of section 4. Using numerically generated events one can obtain all other distributions. Nevertheless, we give here the analytic expressions for $d\sigma/dk$ and $d\sigma/dq_+^0$ in order to demonstrate the importance of hard photons and the influence of initial state radiation versus final state radiation on $d\sigma/dq_+^0$ which is of relevance for the thrust distribution of jet production. An analytical expression for σ_{TOT} gives also a good idea of the size of the bremsstrahlung correction.

For the $d\sigma/d\Omega_\mu dk$ distribution we have to integrate (2.15) over $d\Omega_\gamma$. As a coordinate system we use the $\vec{q}_+$ momentum as z-axis and $\vec{q}_+ \wedge \vec{p}_+$ as y-axis. The angles θ_γ and φ_γ are the polar and azimuthal angles of the photon momentum $\vec{k}$. For the integration it is useful to separate (2.17) into five parts, originating from the electron mass term, muon mass term, the remaining initial state radiation, final state radiation and the electron-muon interference part. The latter gives a contribution to the forward-backward asymmetry. We write accordingly

$$\frac{d\sigma}{d\Omega_\mu \, d\Omega_\gamma \, dk} = \frac{\alpha^3}{8\pi^2 s}\left(X_{me} + X_{m\mu} + X_e + X_\mu + X_{int} \right) \qquad (3.1)$$

where the functions X can be written in for the integration appropriate forms:

$$X_{me} = -\frac{4m^2 k k'}{(\hat{p}_+ \cdot k)^2}\left(\frac{2c_+^2}{y^4} - \frac{2c_+}{y^3} + \frac{1}{y^2} \right) + (c \to -c) \qquad (3.2)$$

$$X_{m\mu} = -\frac{2\mu^2 k k' c_s}{y^2}\left(\frac{1}{(\hat{q}_+ \cdot k)^2} + \frac{1}{(\hat{q}_- \cdot k)^2} \right) \qquad (3.3)$$

$$X_e = \frac{1}{(\hat{p}_+ \cdot k)}\left(\frac{16 k'^2 c_s}{y^4} - \frac{8 y_{-c} k'}{y^3} + \frac{4(1+k'^2)}{y^2} \right) + (c \to -c) - \frac{4k}{y^2} \qquad (3.4)$$

$$\begin{aligned} X_\mu = \frac{2kk'}{y^2} \cdot \frac{1}{(\hat{q}_+ \cdot k)(\hat{q}_- \cdot k)} \Big\{ & 2k^2 \sin^2\theta \sin^2\theta_\gamma \cos^2\varphi_\gamma \\ & + \left(4k^2 c \sin\theta \cos\theta_\gamma \sin\theta_\gamma + 4kx c \sin\theta_\gamma \sin\theta \right) \cos\varphi_\gamma \\ & + 4x^2 c_s + 8(1 - x - k) + 4kx \left(1 + c^2 \cos\theta_\gamma \right) \\ & + 2k^2 \left(1 + c^2 \cos^2\theta_\gamma \right) \Big\} \end{aligned} \qquad (3.5)$$

$$\begin{aligned} X_{int} = \frac{k}{y^2}\Bigg[& \left(\frac{c_-}{\hat{q}_+ \cdot k} - \frac{c_+}{\hat{q}_- \cdot k} \right)\left(\frac{4x^3 c_s - 4x^2 y_{-c} + 4x(1+k'^2)}{(\hat{p}_+ \cdot k)} - 2x(\hat{p}_- \cdot k) - 4x^2 c \right) + 4xc \\ & - \frac{16 k'^2 c_s}{y^2 (\hat{p}_+ \cdot k)} + \frac{8 y_{-c} k'}{y(\hat{p}_+ \cdot k)} - \frac{4(1+k'^2)}{(\hat{p}_+ \cdot k)} + 2(\hat{p}_- \cdot k) \Bigg] - (c \to -c) . \end{aligned} \qquad (3.6)$$

The notation ($c \to -c$) means that the previous expression should be added with c replaced by $-c$ and φ_γ by $\varphi_\gamma + \pi$. In eqs. (3.1)-(3.6) dimensionless quantities have been used, i.e. for masses and the photon energy

$$m = \frac{m_e}{E} \quad , \quad \mu = \frac{m_\mu}{E} \quad , \quad k = \frac{k^0}{E} \quad , \quad k' = 1 - k = \frac{s'}{s} \quad , \tag{3.7}$$

for 4-momenta

$$\hat{p}_\pm = p_\pm/E \quad , \quad \hat{q}_\pm = q_\pm/E \quad , \quad \hat{k} = k/E \quad . \tag{3.8}$$

For the muon energies we have

$$x = \frac{q_+^0}{E} \simeq \frac{2k'}{y} \qquad , \qquad x' = \frac{q_-^0}{E} = 2 - x - k \qquad , \tag{3.9}$$

with

$$y = 2 - k + k\cos\theta_\gamma \; .$$

The following other notations have been used

$$C_\pm = 1 \pm C \quad , \quad C_S = 1 + C^2 \quad , \quad y_{\pm C} = 2 - k \pm kC \quad . \tag{3.10}$$

It should be noted that in eq. (3.9) the muon mass has been neglected. The exact relation is

$$x = \frac{2k'(2-k) - k\cos\theta_\gamma \left[4k'^2 + \mu^2 (k^2\cos^2\theta_\gamma - (2-k)^2) \right]^{1/2}}{(2-k)^2 - k^2\cos^2\theta_\gamma} \; . \tag{3.11}$$

For $\cos\theta_\gamma = \mp 1$ this exact relation becomes simple:

$$x(\cos\theta_\gamma = \mp 1) = \tfrac{1}{2}\left(2 - k \pm k\left(1 - \frac{\mu^2}{k'}\right)^{1/2} \right) \simeq \begin{cases} 1 - \Delta \\ 1 - k + \Delta \end{cases} \tag{3.12}$$

with

$$\Delta = \frac{\mu^2 k}{4k'} \; . \tag{3.13}$$

The maximum k-value for which the photon can still be isotropically emitted is

$$k_{max}^{iso} = \frac{2 - 2\mu}{2 - \mu} \simeq 1 - \tfrac{1}{2}\mu \tag{3.14}$$

whereas the real maximum photon energy is

$$k_{max} = 1 - \mu^2 \; . \tag{3.15}$$

For $\cos\theta_\gamma = \pm 1$ eq. (3.9) is a good approximation even up to k_{max}^{iso} as follows from eq. (3.12). However, for general θ_γ values (3.9) is only a good approximation to (3.11) - within 1 % - when

$$k < 1 - 5\mu \; . \tag{3.16}$$

It should be noted that for $k > k_{max}^{iso}$ in fact there are 2 solutions for x, the one given in (3.11) and one where the sign of $\cos\theta_\gamma$ is changed. In the range (3.16) we shall perform the integrations. When using (3.9) in the propagators for the muons one should either use the real boundaries (3.12) in the integration, or neglect Δ in the boundaries (as (3.9) tells) but use the modified expressions

$$\begin{aligned} \hat{q}_-\cdot\hat{k} &= 2(1 - x + \Delta) \\ \hat{q}_+\cdot\hat{k} &= 2(x - k' + \Delta) \end{aligned} \tag{3.17}$$

where Δ has been added.

The distribution $d\sigma / d\Omega_\mu dk$

$$\frac{d\sigma}{d\Omega_\mu dk} = \frac{\alpha^3}{8\pi^2 s}\left(Z_{me} + Z_{m\mu} + Z_e + Z_\mu + Z_{int} \right) \tag{3.18}$$

where the functions Z are given in the appendix. The φ_γ integral has been performed over the full $[0, 2\pi]$ interval. The $\cos\theta_\gamma$ boundaries can still be choosen in such a way that they correspond to a specific acollinearity requirement for the muons. For the case that the full $\cos\theta_\gamma$ integration is carried out the angular distribution (3.18) is shown in fig. 1 for a set of k values. For very high k-values this distribution takes the form of the $e^+e^- \to \gamma\gamma$ angular distribution, which is understandable for hard photon emission from the initial state. As a comparison the angular distribution for the lowest order process with virtual and soft corrections (eq. (2.1)) is shown in fig. 2.

The distribution $d\sigma / dk$

In principle, this distribution can be obtained from (3.18) by integration over $\cos\theta$. The interference part gives zero and the muon part is easily obtained from integrating (A.41) and (A.42). The electron part is harder to obtain in this way. It is easier to perform first the φ_γ and c integral, giving rise to eqs. (A.44) and (A.45) and then to perform the $\cos\theta_\gamma$ integral. The resulting distributions are

$$\frac{d\sigma}{dk} = \frac{d\sigma^e}{dk} + \frac{d\sigma^\mu}{dk} \tag{3.19}$$

where

$$\frac{d\sigma^e}{dk} = \sigma_0 \beta_i \left(\frac{1}{k} + \frac{1}{2k'} - \frac{1}{2} \right) \tag{3.20}$$

$$\frac{d\sigma^\mu}{dk} = \sigma_0 \left(\frac{1}{k} - 1 + \frac{k}{2} \right)\left(\beta_f + \frac{2\alpha}{\pi} \ln k' \right) . \tag{3.21}$$

In fig. 3 the above three quantities are shown.

The total cross-section

We first calculate the contribution in the interval $[k_1, k_2]$, where k_1 is a value below which eq. (2.1) is valid and $k_2 \cong 1-5\mu$. Therefore k_1 can be neglected everywhere except in $\ln k_1$.

$$\sigma^e = \sigma_0 \beta_i \left[\ln\left(\frac{k_2}{k_1}\right) - \frac{1}{2}\ln(1-k_2) - \frac{1}{2}k_2 \right] , \qquad (3.22)$$

$$\sigma^\mu = \sigma_0 \left[\beta_f \left(\ln\left(\frac{k_2}{k_1}\right) - k_2 + \frac{1}{4}k_2^2 \right) \right.$$

$$\left. + \frac{\alpha}{\pi}\left(-2 Li_2(k_2) + \frac{1}{2}(1-k_2)(3-k_2)\ln(1-k_2) + \frac{3}{2}k_2 - \frac{1}{4}k_2^2 \right) \right] . \qquad (3.23)$$

When adding the virtual corrections to these expressions (electron loop to σ^e, muon loop to σ^μ) we find for the total cross-section with a radiated photon energy less than k_2:

$$\sigma^e(k<k_2) = \sigma_0 \left[\beta_i \left(\ln k_2 - \frac{1}{2}\ln(1-k_2) - \frac{1}{2}k_2 \right) \right. \qquad (3.24)$$

$$\left. + \frac{\alpha}{\pi}\left(\frac{13}{6}\ln\frac{s}{m_e^2} + \frac{\pi^2}{3} - \frac{28}{9} \right) \right] ,$$

$$\sigma^\mu(k<k_2) = \sigma_0 \left[\beta_f \left(\ln k_2 - k_2 + \frac{1}{4}k_2^2 \right) \right.$$

$$+ \frac{\alpha}{\pi}\left(\frac{13}{6}\ln\frac{s}{m_\mu^2} + \frac{\pi^2}{3} - \frac{28}{9} - 2 Li_2(k_2) \right.$$

$$\left.\left. + \frac{1}{2}(1-k_2)(3-k_2)\ln(1-k_2) + \frac{3}{2}k_2 - \frac{1}{4}k_2^2 \right) \right] . \qquad (3.25)$$

The total cross-section up to k_2 is given by

$$\sigma(k<k_2) = \sigma_0\left(1 + \delta_\tau + \delta_{had}\right) + \sigma^e(k<k_2) + \sigma^\mu(k<k_2) . \qquad (3.26)$$

In case one wants to neglect final state radiation one should take

$$\sigma(k<k_2) = \sigma_0\left(1 + \delta_\mu + \delta_\tau + \delta_{had}\right) + \sigma^e(k<k_2) . \qquad (3.27)$$

In the region $1 - 5\mu < k < 1 - \mu^2$ the behaviour of $d\sigma^\mu/dk$ is smooth and therefore the contribution to the total cross-section negligible. The latter can be obtained by taking $k_2 \to 1$. For the electron part this is not the case. In order to evaluate the contribution of $d\sigma^e/dk$ in this region to the total cross-section, we use the reasoning of Bonneau and Martin [8] to get the initial state radiation cross-section (cf. appendix)

$$\frac{d\sigma^e}{dk} = \beta_i \sigma_0(s') \left(\frac{1}{k} + \frac{1}{2k'} - \frac{1}{2} \right) . \qquad (3.28)$$

This equation has neglected some m^2 terms but is still exact as far as μ^2 terms are concerned. When we now do the same for $\sigma_0(s')$ we have instead of (2.8)

$$\sigma_0(s') = \frac{4\pi\alpha^2}{3s}\left(1 - \frac{\mu^2}{k'}\right)^{1/2}\left(\frac{1}{k'} + \frac{\mu^2}{2k'^2}\right) . \qquad (3.29)$$

We see that for small μ (eq. (2.5)) and k in the region (3.16) the previous distribution (3.20) is reproduced. Integrating (3.28) from k_2 to k_{max} with the help of (3.29), and adding this to (3.22) we find

$$\sigma^e = \sigma_0 \beta_i \left(\ln\frac{1}{k_1} + \frac{1}{2}\ln\frac{s}{m_\mu^2} - \frac{4}{3} \right) . \qquad (3.30)$$

Insertion of k_{max} in eq. (3.22) gives a result which differs in the constant (not k_1 and μ^2 depending) terms. It should be noted that our result disagrees with that of ref. 9, where instead of −4/3 the number −19/12 is obtained. The total muon bremsstrahlung for $k > k_1$ now reads

$$\sigma^{\mu} = \sigma_0 \left[\beta_f \left(\ln \frac{1}{k_1} - \frac{3}{4} \right) + \frac{\alpha}{\pi} \left(- \frac{\pi^2}{3} + \frac{5}{4} \right) \right] \quad . \tag{3.31}$$

Adding the virtual corrections one finds for the total cross-section for $\mu^+\mu^-$ and $\mu^+\mu^-\gamma$ final states the expression

$$\sigma_{TOT} = \sigma_0 (1 + \delta_T) = \sigma_0 (1 + \delta_e^T + \delta_\mu^T + \delta_\tau + \delta_{had}) \tag{3.32}$$

where

$$\delta_e^T = \beta_i \left(\frac{1}{2} \ln \frac{4}{\mu^2} - \frac{1}{4} \right) + \frac{2\alpha}{\pi} \left(\frac{\pi^2}{6} - \frac{17}{36} \right) \tag{3.33}$$

$$\delta_\mu^T = \frac{2\alpha}{\pi} \left(\frac{1}{3} \ln \frac{4}{\mu^2} - \frac{13}{72} \right) \tag{3.34}$$

where δ_e^T contains the electron vacuum polarization and δ_μ^T the muon vacuum polarization. Omitting the latter one would obtain from vertex correction and bremsstrahlung alone

$$\bar{\delta}_\mu^T = \frac{3\alpha}{4\pi} \tag{3.35}$$

which is a well-known result from the gluon bremsstrahlung to the total QCD cross-section (the factor 3/4 being cancelled by colour):

$$\sigma_{TOT} = R\sigma_0 \left(1 + \frac{\alpha_s}{\pi} \right) = R\sigma_0 \left(1 + \delta_{QCD} \right) . \tag{3.36}$$

At this point it is useful to illustrate numerically the implications of the total cross-section formulae.

In table I δ_T is given for various beam energies both for mupair and τ-pair production. Moreover the various contributions to δ_T are given, i.e. the leptonic and hadronic vacuum polarizations $\delta_e, \delta_\mu, \delta_\tau, \delta_{had}$, the vertex correction and bremsstrahlung for the initial state $\bar{\delta}_e^T$ and final state $\bar{\delta}_\mu^T$. The latter are (3.33) and (3.34) without δ_e, δ_μ respectively. The large difference between the total corrections for mupair and τ-pair production is entirely due to the $\frac{1}{2}\beta_i \ln(4/\mu^2)$ term in (3.33). It is clear from the table that for the correction to the total cross-section the vertex correction and bremsstrahlung for the final state can be neglected.

With these large corrections it is of interest to see which regions of phase space give the large contributions. Table II lists δ_T for mupair production for various maximum photon energies and μ^+ angular ranges. In the latter case the lowest order cross-section to which the correction applies is similarly reduced in angular range. The hard bremsstrahlung region is responsible for the large correction, as was already clear from the difference between mu and τ-pair production.

The formulae for the total mupair cross-section can also be used to obtain a rough estimate of the total correction to the hadronic cross-section. The hadronic cross-section can be taken as $R\sigma_0$, where R(s') is roughly a step-function with the following $w' = \sqrt{s'}$ intervals (0.28, 1), (1, 1.5), (1.5, 4.5), where in the first interval the ρ dominates, in the second and third R $\simeq$ 1,2 respectively. Above 4.5 GeV, R $\simeq$ 4. Since we apply the δ_T at high energies the decrease of R towards lower energies implies a reduction of δ_T with respect to the pure mupair case. The contribution of the ρ -meson is given by

$$\delta^e(\rho\text{-meson}) = \sigma_{had}\beta_i \frac{9\Gamma_e \pi}{2\alpha^2 M R(s)} . \tag{3.37}$$

For a beam energy of 10 GeV the correction δ_T to $\sigma_{had} = R\sigma_o$ is roughly 38 %. Of course an accurate numerical integration of (3.28) with the known hadronic cross-section is preferable.

For hadron production it may be of more interest to see which photon angular ranges mainly contribute to the correction. When one excludes in the bremsstrahlung expression (3.28) photons which make an angle less than 10° with the beam direction, use of eq. (A.53) shows that β_i is reduced by a factor of 4 at 10 GeV. Thus, e.g. when at a beam energy of 10 GeV δ_T is 53.7 % for mu pair production, the region $2/3 \leq k \leq k_{max}$ contributes 40 % to this. When in this region photons within 10° of the beam direction are excluded the total correction becomes 23.7 % or applied to the hadron case mentioned before, the correction of 38 % is reduced to 20 %. So, in many experimental cases the QED correction will be at least as important as the QCD correction (3.36), which amounts to about 7 %.

The momentum spectrum of the muon $d\sigma/dx$

We give here an analytical formula for the μ^+ momentum distribution caused by initial state radiation. It is obtained, just like $d\sigma/dk$, from (A.43) - (A.45), where now an integration over k has to be performed. For the true momentum spectrum the minimum k value is 1-x. If one is interested in the momentum spectrum of the most energetic muon, which is the quantity which may fake a thrust distribution if one does not discard bremsstrahlung events, then the minimum k value is 2(1-x). Since the μ^+ and μ^- momentum distributions are the same also a factor 2 in the end result arises. Thus the momentum distribution of the most energetic muon becomes

$$\begin{aligned} \frac{d\sigma}{dx} = \frac{6\alpha\sigma_o}{\pi}\Bigg[& \ln\frac{s}{m_e^2}\left\{-\frac{2(1-x)^2}{3k_m^3} + \frac{1-x}{k_m^2} - \frac{1+(1-x)^2}{k_m} + \frac{5-6x+3x^2}{6(1-x)}\right. \\ & \left. + x(x-1)\ln\frac{k_m}{2(1-x)} - \frac{1}{2}\left(x^2+(1-x)^2\right)\ln\frac{1-k_m}{2x-1}\right\} \\ & + \frac{8(1-x)^2}{3k_m^3} - \frac{4(1-x)}{k_m^2} + \frac{x^2-2x+3}{k_m} - \frac{1}{2}(1-x) - \frac{1}{3(1-x)} \\ & + \frac{1}{2}\ln\frac{1-k_m}{2x-1} - x(x-1)\ln\left[\frac{k_m}{1-k_m}\cdot\frac{2x-1}{2-2x}\right] \Bigg] \end{aligned} \tag{3.38}$$

where k_m is the maximum photon energy one is interested in. Expression (3.34) should be compared to the thrust distribution, calculated in QCD from gluon bremsstrahlung [10)]

$$\frac{1}{\sigma_o}\frac{d\sigma}{dT} = \frac{2\alpha_s}{3\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right] . \tag{3.39}$$

In fig. 4 $d\sigma/dT$ and $d\sigma/dx$ for various beam energies are presented. The maximum photon energies are choosen in such a way that s' is still large enough to produce jets ($\sqrt{s'} > 7$ GeV). In this region R(s') can be taken constant such that eq. (3.38) can be applied. Again it is seen that QCD and QED effects can be comparable in size.

4. A procedure to generate events

When one wants to generate x-values in an interval (a,b) according to a prescribed distribution f(x) there are various ways to follow. Each method should produce a number N(x) of points per interval Δx which approaches $f(x)\,\Delta x$ for a large total number of generated events.

One method would be to divide the interval (a,b) in a large number of sub-intervals Δx_i, each of such a size that $\int_i f(x)dx$ over the interval is of about the same magnitude for every interval. One could then at random generate the number i and choose a point in Δx_i.

Another way would be to generate at random points (x,y), where $x \in (a,b)$ and $y \in (f_1,f_2)$, where f_1 and f_2 are the minimum and maximum values of f in the interval (a,b). When for a generated point (x,y) $y < f(x)$ the value x is accepted, otherwise not.

A third method is to calculate analytically

$$F(z) = \int_a^z f(x)\,dx \ , \tag{4.1}$$

to generate a random number $\eta \in [0, F(b)]$ and to solve numerically

the equation

$$\eta = F(x) . \tag{4.2}$$

In the multidimensional case similar methods can be used. When a differential cross-section has narrow and high peaks in various variables, the third approach will have some advantages. The explicit integration amounts effectively to the introduction of new variables so that the distribution becomes flat. Without the explicit integration one could of course introduce directly variables which would transform away some peaks, and use then any of the two other methods. However there are more peaks than variables which can be choosen independently such that some peaks remain in the newly choosen variables.

So, ideally one should perform the five integrations of eq. (2.15) step by step and then generate each kinematical variable by using eqs. (4.1) and (4.2) in five steps. When one wants to do this for eq. (2.15) to the very end one runs into two problems. The first is that in a small region above zero photon energy eq. (2.15) is not applicable and that only the cross-section integrated over the photon energies is known (eq. (2.1) or (2.6)). The second problem is that it is not easy to integrate eq. (2.6) analytically over $c = \cos\theta$, such that the procedure with (4.2) cannot be done using analytical equations. Now it is most important to use analytical formulae for those integrals which have the strongest peaks. Those are the φ_γ and $\cos\theta_\gamma$ integrals. Therefore we adopt the following procedure.

In the phase space which is only restricted for θ (c)-values by (2.5) and k-values by (3.16) we use eq. (3.18) from a certain k value on and eq. (2.6) for very low k-values (soft region). A two dimensional k,c histogram is created numerically for which integrals like (4.1) are known.

First a k value is created, if it is in the soft region the k-value is negligible and can be set zero for experimental purposes. A c-value is generated from (2.6). Outside the soft region it is eq. (3.18) which gives the c-value numerically. With the generated k,c values it is then possible to use the primitive functions Z to generate cos θ_γ and the functions Y to generate φ_γ.

One improvement to this scheme must be made. When the k_1 value with which ends the soft region is changed the result is not the same since in the soft region the cross-section is changed by exponentiation and in the other part not. This problem would not have occurred if eq. (2.1) was used in the soft region. For small k-values the created events would then however be too few. The solution is that (3.18) should be modified in such a way that upon k integration it obtains a similar factor as eq. (2.6). Writing (3.18) formally as

$$\frac{d\sigma}{d\Omega_\mu dk} = \frac{d\sigma_o}{d\Omega_\mu}\left(\frac{\beta}{k} + \frac{d}{dk}h(k)\right) \tag{4.3}$$

the cross-section up to k_1 with exponentiation is assumed to be

$$\frac{d\sigma}{d\Omega_\mu}(k<k_1) = \frac{d\sigma_o}{d\Omega_\mu}\left(1 + \delta_{AR} + h(k_1)\right) k_1^{\beta} . \tag{4.4}$$

Differentiating we find for (4.3) now the exponentiated version

$$\frac{d\sigma^{ex}}{d\Omega_\mu dk_1} = \frac{d\sigma_o}{d\Omega_\mu}\left(\delta_{AR} + h(k_1)\right)\beta k_1^{\beta-1} + \frac{d\sigma}{d\Omega_\mu dk_1} k_1^{\beta} \tag{4.5}$$

$$\simeq \left(\frac{d\sigma}{d\Omega_\mu dk_1} + \frac{\beta\delta_{AR}}{k_1}\frac{d\sigma_o}{d\Omega_\mu}\right) k_1^{\beta} . \tag{4.6}$$

For generating events we use (4.6). The term with $h(k_1)$ is of higher order than the last term and is small in the region where exponentiation is most important. It can therefore be neglected. An evaluation would be involved. It is (4.6) which is used to generate k and c. The photon variables are then generated from $d\sigma/d\Omega_\mu dk$ alone as described before. The main purpose of this procedure is to redistribute the events for small k values, since we know that higher order corrections become necessary there. If one is only interested in hard bremsstrahlung the use of (4.6) is not necessary but using it does not affect the results. As an illustration the effect of exponentiation like in eq. (4.5) is shown for the photon spectrum in fig. 5. In fig. 6 the exponentiated form of $\sigma(k<k_1)$ is compared to the form without exponentiation.
The distribution (4.6) can also be used for a fixed scattering angle. One then generates from a one dimensional histogram the k-value. If one had choosen a different order of integration this would not have been possible.

In case one is exclusively interested in either initial or final state radiation, another order of integration is possible leading e.g. to eqs. A.43 - A.47, for which all primitive functions are known. Then the corresponding transition from (4.5) to (4.6) has not to be made and no starting histogram has to be created. When using the earlier mentioned step function approximation to R(s) it is easy to generate the momenta of a real and a massive virtual photon, which is useful to calculate the radiative correction to R. In case one is interested in hadron jets, the information of how the virtual photon decays into a $q\bar{q}$ pair is also required. The chain of integrals leading to (A.43) should then be used in order to generate the 4-momenta of the quark and antiquark. The hadrons coming from the $q\bar{q}$ pair can then be obtained by using one of the current Monte Carlo programs [11].

5. Numerical examples of radiative corrections and distributions.

Radiative correction to $d\sigma/d\Omega_\mu$

As mentioned before in section 1, we can use the expressions for $d\sigma/d\Omega_\mu dk$ to perform numerically an integration over that part of phase space, where the muon energies are larger than a prescribed value E_{th}

$$q_+^0 \, , \; q_-^0 \quad \geq \quad E_{th} \tag{5.1}$$

and where the angle between the muons lies in the region

$$\pi - \zeta \quad \leq \quad \angle(\vec{q}_+, \vec{q}_-) \quad \leq \quad \pi \, . \tag{5.2}$$

The lowest order differential cross-section $d\sigma_0/d\Omega_\mu$ then obtains a total correction δ_T, which contains virtual corrections, hadronic vacuum polarization and bremsstrahlung. The results are in agreement with those of refs. 2, 3, 12. In fig. 7 the total QED correction δ_T is given to which δ_w (2.14) should be added to obtain the total electro-weak asymmetry (fig. 8). The latter is shown for two energies, the change coming mainly from the weak interference part. It is only in this calculation in this paper that δ_w is included, everywhere else pure QED with δ_{had} is considered.

The photon energy distributions

Although we have given in section 4 an exact distribution eq. (3.19) which is plotted in fig. 3, the event generator should reproduce the same distribution. Generating 10.000 events in a phase space determined by

$$0.5^\circ \leq \theta \leq 175.5^\circ \quad ; \quad 0 \leq k \leq 0.96 \; ; \qquad (5.3)$$

the photon spectrum is that given in fig. 9. It should be noted that in contrast to eq. (3.19) the region (o, k_1) with a very small k_1 is included in the sample, as was discussed in section 4. The agreement between the exact distribution and the generated histogram is good. A similar check was made on fig. 4.

The acollinearity distribution

Using the same event sample as above one can obtain $d\sigma/d\zeta$, where ζ is the acollinearity angle (5.2). The resulting histogram is depicted in fig. 10.

The acoplanarity distribution

Defining an acoplanarity angle ψ between the two planes formed by the beam and the μ^+, μ^- respectively , a distribution like the one in fig. 11 is obtained. The angle $\pi - \psi$ is the angle between $\vec{p}_+ \wedge \vec{q}_+$ and $\vec{p}_- \wedge \vec{q}_-$.

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Appendix

In this appendix formulae are given relevant for the integration order $\varphi_\gamma, \cos\theta_\gamma$ and the order φ_γ, c. Furthermore a separate treatment of the initial state radiation only is made.

Integration over φ_γ

The functions X in eq. (3.1) are integrated over φ_γ. Denoting the integrals by

$$Y = \int X \, d\varphi_\gamma \tag{A.1}$$

we find

$$Y_{me} = -\frac{4m^2k'}{k}\left[\left(\frac{2c_+^2}{y^4} - \frac{2c_+}{y^3} + \frac{1}{y^2}\right) I_2^- + \left(\frac{2c_-^2}{y^4} - \frac{2c_-}{y^3} + \frac{1}{y^2}\right) I_2^+\right] \tag{A.2}$$

$$Y_e = \frac{1}{y^2k}\left[\left(\frac{16k'^2c_s}{y^2} - \frac{8y_{-c}k'}{y} + 4(1+k'^2)\right) I_1^- + \left(\frac{16k'^2c_s}{y^2} - \frac{8y_{+c}k'}{y} + 4(1+k'^2)\right) I_1^+ - 4k^2 I_3\right] \tag{A.3}$$

where

$$I_1^{\mp} = \frac{2}{R_{\mp}^{1/2}} \arctan\left[\frac{R_{\mp}^{1/2}}{a_{\mp} + b_{\mp}} \tan\left(\frac{\varphi_\gamma}{2}\right)\right] \tag{A.4}$$

$$I_2^{\mp} = \frac{1}{R_{\mp}}\left(-\frac{b_{\mp}\sin\varphi_\gamma}{a_{\mp} + b_{\mp}\cos\varphi_\gamma} + a_{\mp} I_1^{\mp}\right) \tag{A.5}$$

$$I_3 = \varphi_\gamma \tag{A.6}$$

$$a_{\mp} = \left(\frac{p_+^0}{|\vec{p}_+|}\right) \mp c\cos\theta_\gamma \equiv e \mp c\cos\theta_\gamma \tag{A.7}$$

$$b_{\mp} = \mp \sin\theta \sin\theta_\gamma \tag{A.8}$$

$$R_{\mp} = a_{\mp}^2 - b_{\mp}^2 = (ec \mp \cos\theta_\gamma)^2 + m^2\sin^2\theta \quad . \tag{A.9}$$

Furthermore, the muon part gives

$$Y_{m\mu} = X_{m\mu} I_3 \tag{A.10}$$

$$Y_\mu = \frac{2kk'}{y^2(\hat{q}_+k)(\hat{q}_-k)}\Big[\Big(4\alpha^2c_s + 8(1-\alpha-k) + k^2\sin^2\theta\sin^2\theta_\gamma + 2k^2(1+c^2\cos^2\theta_\gamma) + 4k\alpha(1+c^2\cos\theta_\gamma)\Big) I_3 + (\alpha + k\cos\theta_\gamma)(4kc\sin\theta\sin\theta_\gamma) I_4 + \tfrac{1}{2}k^2\sin^2\theta\sin^2\theta_\gamma I_5\Big] \tag{A.11}$$

with

$$I_4 = \sin\varphi_\gamma \tag{A.12}$$

$$I_5 = \sin 2\varphi_\gamma \,. \tag{A.13}$$

The interference term reads

$$\begin{aligned} Y_{int} = \frac{1}{y^2}\Bigg[\left(\frac{c_-}{\hat{q}_+\hat{k}} - \frac{c_+}{\hat{q}_-\hat{k}}\right)\Big[&\left(4\alpha^3 c_s - 4\alpha^2 y_{-c} + 4\alpha(1+k'^2)\right) I_1^- \\ &- 2k^2\alpha\left((1+c\cos\theta_\gamma)I_3 + \sin\theta\sin\theta_\gamma I_4\right) - 4\alpha^2 c k I_3\Big] \\ &- \left[\frac{16k'^2 c_s}{y^2} - \frac{8y_{-c}k'}{y} + 4(1+k'^2)\right] I_1^- + 4\alpha k c I_3 \\ &+ 2k^2\left[(1+c\cos\theta_\gamma) I_3 + \sin\theta\sin\theta_\gamma I_4\right] \Bigg] - (c \to -c) \,. \end{aligned} \tag{A.14}$$

The definite integrals over the interval $[0, 2\pi]$ are obtained from the above expressions by taking

$$I_1^\mp = \frac{2\pi}{R_\mp^{1/2}} \quad ; \quad I_2^\mp = \frac{2\pi a_\mp}{R_\mp^{3/2}} \quad ; \quad I_3 = 2\pi \quad ; \quad I_4 = I_5 = 0 \,. \tag{A.15}$$

<u>Integration over $\cos\theta_\gamma$</u>

The definite integrals Y with eqs. (A.15) inserted are integrated over $\cos\theta_\gamma$. The resulting integrals are denoted by Z:

$$Z = \int Y \, d\cos\theta_\gamma \,. \tag{A.16}$$

In particular

$$Z_{me} = -\frac{8\pi k'}{k}\left(\frac{2c_+^2}{y_{+c}^4} - \frac{2c_+}{y_{+c}^3} + \frac{1}{y_{+c}^2}\right)\frac{w}{R_-^{1/2}} + (c \to -c) \tag{A.17}$$

$$Z_e = \frac{8\pi}{k}\left[4k'^2 c_s f_4 - 2y_{-c}k' f_3 + (1+k'^2) f_2 + (c \to -c)\right] + \frac{8\pi}{y} \tag{A.18}$$

$$Z_{m\mu} = \frac{1}{2}\pi c_s \mu^2\left[\frac{1}{1-\alpha+\Delta} - \frac{1}{\alpha-k'+\Delta}\right] \tag{A.19}$$

$$Z_\mu = -\pi\left[c_s \frac{1+k'^2}{k}\ln\left(\frac{\alpha-k'+\Delta}{1-\alpha+\Delta}\right) - 2\alpha c_s + \frac{2k'}{\alpha}(3c^2-1)\right] \tag{A.19}$$

where

$$f_n = \frac{1}{(n-1)a_0^2}\left(-g_{n-1} + (2n-3)y_0 f_{n-1} - (n-2) f_{n-2}\right) \quad (n>1) \tag{A.21}$$

$$g_n = \frac{k R_-^{1/2}}{y^n} \tag{A.22}$$

$$f_1 = \frac{1}{a_0}\ln\left[\frac{2y\,(w + R_-^{1/2})}{(y + a_0 + kR_-^{1/2})^2}\right] \tag{A.23}$$

$$w = \cos\theta_\gamma - ec$$
$$y_0 = 2 - k + kec$$
$$a_0 = \left[\, y_0^2 + k^2 m^2 \sin^2\theta \,\right]^{1/2} . \qquad \text{(A.24)}$$

The interference term gives

$$Z_{int} = 4\pi \Big[2c_S\left(c_- Q_3 - c_+ \mathcal{R}_3\right) - 2\left(c_- Q_2 - c_+ \mathcal{R}_2\right) y_{-c}$$
$$+ 2(1+k'^2)\left(c_- Q_1 - c_+ \mathcal{R}_1\right) + \frac{k}{2}\,\mathcal{U} - \frac{2k'c}{y^2}$$
$$- \frac{k}{2}\left(c_- S_1 - c_+ T_1\right) - kc\left(c_- S_2 - c_+ T_2\right)$$
$$- 8k'^2 c_S f_4 + 4y_{-c} k' f_3 - 2(1+k'^2) f_2 \Big] - (c \to -c) \qquad \text{(A.25)}$$

where

$$Q_n = \frac{(2k')^{n-1}}{2(1+\Delta')} f_{n+1} + \frac{k'}{1+\Delta'}\, Q_{n-1} \qquad (n>1) \qquad \text{(A.26)}$$
$$Q_1 = \frac{1}{4(1+\Delta')^2}\left(2(1+\Delta') f_2 + f_1 - V \right) \qquad \text{(A.27)}$$
$$\Delta' = \Delta / k'$$
$$\mathcal{R}_n = -\frac{1}{2}(2k')^{n-1} f_{n+1} + (1+\Delta)\,\mathcal{R}_{n-1} \qquad (n>1) \qquad \text{(A.28)}$$
$$\mathcal{R}_1 = -\frac{1}{2} f_2 - \frac{1+\Delta}{4k'}\left(f_1 - W \right) \qquad \text{(A.29)}$$
$$V = \frac{1}{a_1} \ln\left[\frac{2y_1\left(w + \mathcal{R}_-^{1/2}\right)}{\left(y_1 + a_1 + k\mathcal{R}_-^{1/2}\right)^2}\right] \qquad \text{(A.30)}$$

and W is obtained from V by replacing a_1, y_1 by a_2, y_2, where

$$a_1 = \left[\, \tilde{y}_1^{\,2} + m^2 k^2 \sin^2\theta \,\right]^{1/2} \qquad \text{(A.31)}$$
$$a_2 = \left[\, \tilde{y}_2^{\,2} + m^2 k^2 \sin^2\theta \,\right]^{1/2} \qquad \text{(A.32)}$$
$$y_1 = y - 2(1+\Delta') \quad ; \quad \tilde{y}_1 = y_0 - 2(1+\Delta') \qquad \text{(A.33)}$$
$$y_2 = y - \frac{2k'}{1+\Delta} \quad ; \quad \tilde{y}_2 = y_0 - \frac{2k'}{1+\Delta} \qquad \text{(A.34)}$$
$$S_1 = -\frac{1}{2}\left[c_+ \ln(x - k' + \Delta) + \frac{x}{kk'}\left(kc_+ - 2c\right) \right] \qquad \text{(A.35)}$$
$$S_2 = -\frac{1}{2k}\left[k' \ln(x - k' + \Delta) + x + \frac{x^2}{2k'} \right] \qquad \text{(A.36)}$$
$$T_1 = \frac{1}{2k'}\left[c_- \ln(1 - x + \Delta) + \frac{x}{k}\left(kc_+ - 2c\right) \right] \qquad \text{(A.37)}$$
$$T_2 = \frac{1}{2kk'}\left[\ln(1 - x + \Delta) + x + \frac{1}{2}x^2 \right] \qquad \text{(A.38)}$$
$$\mathcal{U} = -\frac{1}{kk'}\left[\frac{1}{2}x^2\left(kc_+ - 2c\right) + 2cxk' \right] . \qquad \text{(A.39)}$$

For the definite integrals some simplifications occur. For Z_{me} the $\cos\Theta_\gamma$ dependent factor becomes 2, in Z_e we have

$$f_1(1) - f_1(-1) = -\frac{1}{a_0} \ln\left(\frac{k'm^2}{y_0^2}\right) . \qquad \text{(A.40)}$$

The muon parts become

$$Z_{m_\mu}(1) - Z_{m_\mu}(-1) = -4\pi C_S \frac{k'}{k} \tag{A.41}$$

$$Z_\mu(1) - Z_\mu(-1) = 2\pi C_S \left(\frac{1+k'^2}{k}\right) \ln\left(\frac{4k'}{\mu^2}\right) - 8\pi C^2 k \ . \tag{A.42}$$

Next, we give a result which is obtained by integration of Y over the full c-range and the azimuthal angle of the μ^+ (just a factor of 2π)

$$\frac{d\sigma}{dk\, d\cos\theta_\gamma} = \frac{2kk'}{y^2}\frac{d\sigma}{dk\,dx} = \frac{2kk'}{y^2}\frac{d\sigma}{dx\,dx'}$$

$$= \frac{\alpha^3}{8\pi^2 s}\int Y\, d\Omega_\mu \equiv \frac{\alpha^3}{8\pi^2 s}\cdot\frac{2kk'}{y^2}\, H \tag{A.43}$$

$$H_{me} = \frac{32\pi^2}{k^4}\left[2k' - x^2 - x'^2\right] \tag{A.44}$$

$$H_e = \frac{16\pi^2}{k^2 k'}\Big[\ln\frac{s}{m_e^2}\left(x^2 + x'^2 - \frac{4k'}{k^2}(xx' - k')\right)$$

$$+ \frac{1}{k^2}\left((2k^2 + 12k')(xx' - k') - 2k^2 k'\right) - k^2\Big] \tag{A.45}$$

$$H_{m\mu} = -\frac{32\pi^2}{3}\mu^2\left(\frac{1}{(\hat{q}_+\cdot k)^2} + \frac{1}{(\hat{q}_-\cdot k)^2}\right) \tag{A.46}$$

$$H_\mu = \frac{16\pi^2}{3}\cdot\frac{4}{(\hat{q}_+\cdot k)(\hat{q}_-\cdot k)}\left(x^2 + x'^2\right) \ . \tag{A.47}$$

When the photon direction is not near to the $\mu^\pm$ direction, $H_{m\mu}$ can be neglected. These formulae are in agreement with those of Ref. 9.

To conclude, we give some formulae relevant for the initial state radiation. Using now $\vec{p}_+$ as z-axis and calling the polar and azimuthal angle of the photon η and ψ and denoting k^0/E by k we have

$$\frac{d\sigma^e}{d\Omega_\gamma dk} = \frac{\alpha k}{4\pi^2}\sigma_0(s')\left[\frac{2(1+k'^2)}{(\hat{p}_+\cdot k)(\hat{p}_-\cdot k)} - \frac{m^2 k'}{(\hat{p}_+\cdot k)^2} - \frac{m^2 k'}{(\hat{p}_-\cdot k)^2} - 1\right] . \tag{A.48}$$

Integrating over the full ψ range and then integrating over $\cos\eta$ gives

$$\frac{d\sigma}{dk} = \frac{\alpha}{2\pi k}\sigma_0(s')\left[-m^2 k' J_1 - k^2 J_2 + (1+k'^2) J_3\right] \tag{A.49}$$

with

$$J_1 = \frac{1}{e - \cos\eta} - \frac{1}{e + \cos\eta}$$

$$J_2 = \cos\eta \tag{A.50}$$

$$J_3 = \ln\left(\frac{e + \cos\eta}{e - \cos\eta}\right) .$$

For the interval (-1, +1) the integrals J_i give

$$J_1 = \frac{4}{m^2} \quad ; \quad J_2 = 2 \quad ; \quad J_3 = 2\ln\frac{s}{m_e^2} \tag{A.51}$$

and therefore

$$\frac{d\sigma^e}{dk} = \frac{\alpha}{\pi}\sigma_0(s')\left(\ln\frac{s}{m_e^2} - 1\right)\frac{1+k'^2}{k} \qquad \text{(A.52)}$$

which is the same as (3.28).For a restricted angular range e.g. (-z, z) we obtain instead of (A.52)

$$\frac{d\sigma^e}{dk} = \frac{\alpha}{\pi}\sigma_0(s')\left(\frac{1+k'^2}{k}\ln\left(\frac{e+z}{e-z}\right) - kz\right). \qquad \text{(A.53)}$$

References

1. S. Glashow, Nucl. Phys. 22, 579 (1961),
   S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967),
   A. Salam,in:Elementary Particle Theory, Editor H. Svartholm (Almquist and Forlag, Stockholm, 1968).
2. F.A. Berends, K.J.F. Gaemers and R. Gastmans, Nucl. Phys. B57, 381 (1973).
3. F.A. Berends, K.J.F. Gaemers and R. Gastmans, Nucl. Phys. B63, 381 (1973).
4. F.A. Berends and R. Gastmans in Electromagnetic Interactions of Hadrons, Vol. 2, p. 471, Edited by A. Donnachie and G. Shaw (Plenum Publishing Corporation, 1978).
5. F.A. Berends and G.J. Komen, Phys. Lett. 63B, 432 (1976).
6. G. Passarino and M. Veltman, Nucl. Phys. B160, 151 (1979).
7. F.A. Berends, R. Gastmans and T.T. Wu, University of Leuven preprint KUL-TF-79/022, submitted to the 1979 International Symposium on Lepton and Photon Interactions at High Energies, 1979, Fermilab.
8. G. Bonneau and F. Martin, Nucl. Phys. B27, 381 (1971).
9. E.A. Kuraev and G.V. Meledin, Nucl. Phys. B122, 485 (1977).
10. A. de Rújula, J. Ellis, E.G. Floratos and M.K. Gaillard, Nucl. Phys. B138, 387 (1978).
11. A. Ali, E. Pietarinen, G. Kramer and J. Willrodt, DESY report 79/86.
12. G.J. Komen, Phys. Lett. 68B, 275 (1977).

contr. (%) \ energy (GeV)	10	25	50	100
δ_e	3.0	3.3	3.5	3.7
δ_μ	1.4	1.7	1.9	2.1
δ_τ	0.5	0.8	1.0	1.2
δ_{had}	4.3	5.6	6.6	7.5
$\overline{\delta_e^T}(\mu)$	44.3	57.6	68.7	80.7
$\overline{\delta_\mu^T}$	0.2	0.2	0.2	0.2
$\delta_T(\mu)$	53.7	69.2	81.9	95.4
$\overline{\delta_e^T}(\tau)$	17.8	28.8	38.1	48.3
$\delta_T(\tau)$	27.2	40.4	51.3	63.0

Table 1

Total correction to the lowest order total cross-section in % for the processes $e^+e^- \to \mu^+\mu^-$, $e^+e^- \to \tau^+\tau^-$ denoted by $\delta_T(\mu)$ and $\delta_T(\tau)$ respectively. The various contributions to the total correction are also listed.

k_{max}	E (GeV) / θ range (°)	10	25	50	100
0.95	0-180	26.1	29.7	32.4	35.0
	1-179	26.1±0.3	29.6±0.4	32.4±0.5	35.0±0.6
	10-170	25.4±0.3	29.0±0.4	31.7±0.4	34.2±0.4
	30-150	23.4±0.2	26.7±0.3	29.3±0.3	31.7±0.3
0.8	0-180	18.4	21.2	23.4	25.4
	1-179	18.8±0.3	21.2±0.3	23.3±0.3	25.4±0.3
	10-170	18.7±0.3	21.1±0.3	23.2±0.3	25.3±0.3
	30-150	18.0±0.3	20.5±0.2	22.6±0.2	24.6±0.3
0.5	0-180	10.0	12.0	13.5	14.9
	1-179	10.0±0.2	11.8±0.2	13.4±0.2	14.8±0.2
	10-170	10.2±0.2	11.9±0.2	13.4±0.2	14.8±0.2
	30-180	10.1±0.2	11.9±0.5	13.4±0.2	14.7±0.2

Table 2

Total correction to the lowest order cross-section in % for the process $e^+e^- \to \mu^+\mu^-$, as a function of the photon energy range and the μ^+ scattering angle range. The values for the full angular range are obtained from eq. (3.26), for the restricted range a numerical integration of eq. (3.18) has been performed.

Figure Captions

Fig. 1 The distribution $\sigma_0^{-1} d\sigma / d\Omega_\mu dk$ versus $\cos\theta$ for three different values of k: k = 0.05 (1), k = 0.95 (2) and k = 0.5 (3) for E = 15 GeV.

Fig. 2 The distribution $\sigma_0^{-1} d\sigma / d\Omega_\mu$ according to eq. (2.1) with k_1 = 0.01 E with E = 15 GeV.

Fig. 3 The photon spectrum $\sigma_0^{-1} d\sigma / dk$ according to eq. (3.19), solid curve, and the photon spectrum due to initial state (dotted line) and due to final state radiation (dashed line) for E = 15 GeV.

Fig. 4 The thrust distribution $\sigma_0^{-1} d\sigma/dT$ faked by initial state radiation for three beam energies E = 10, 15, 25 GeV and the QCD thrust distribution $\sigma_0^{-1} d\sigma/dT$ (dashed line). For the first distribution photon energies are allowed up to that value which gives a $\bar{q}q$ pair of an invariant mass of 7 GeV.

Fig. 5 The photon spectrum with (solid curve) and without (dashed curve) exponentiation for E = 15 GeV.

Fig. 6 The integrated photon spectrum $\sigma(k) = \sigma^e(k) + \sigma^\mu(k)$ with and without (dashed curve) exponentiation.

Fig. 7 The total QED radiative correction δ_T in % for E_{th} = 0.5 E and $\zeta = 10^\circ$ for various scattering angles θ and E = 15 GeV.

Fig. 8 The forward-backward asymmetry, now including the electro-weak interference δ_W for E = 15 and 19 GeV. The same E_{th} and ζ as in fig. 7 is used and $\sin^2\theta_W = .22$.

Fig. 9 The photon spectrum, including soft photons, as generated numerically for E = 15 GeV.

Fig. 10 The numerically generated acollinearity distribution for E = 15 GeV. ζ in degrees.

Fig. 11 The acoplanarity distribution as generated numerically for E = 15 GeV. ψ in degrees.

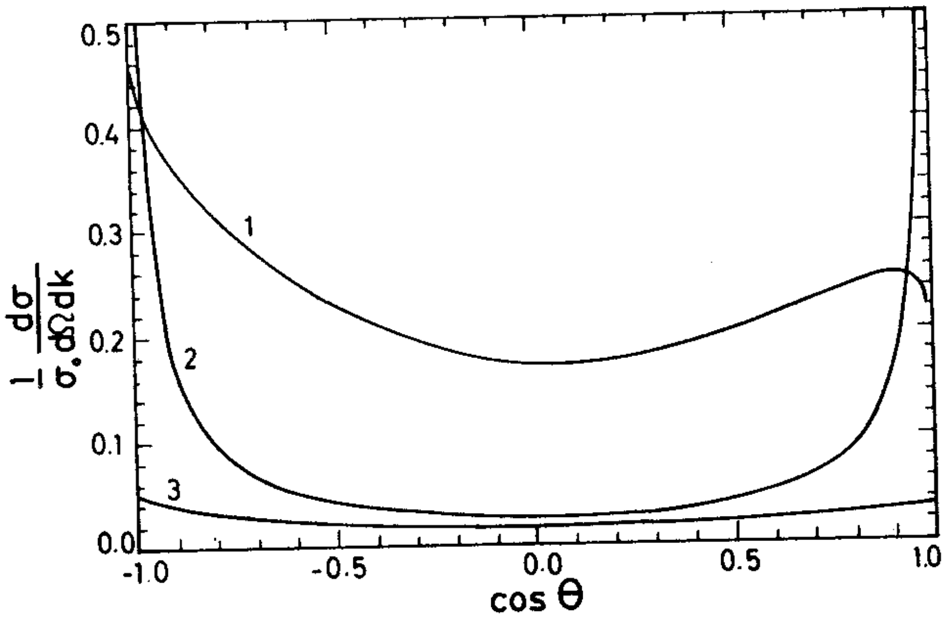

$\frac{1}{\sigma_0}\frac{d\sigma}{d\Omega dk}$
0.5
0.4
0.3
0.2
0.1
0.0
1
2
3
-1.0
-0.5
0.0
0.5
1.0
$\cos\Theta$

Fig.1

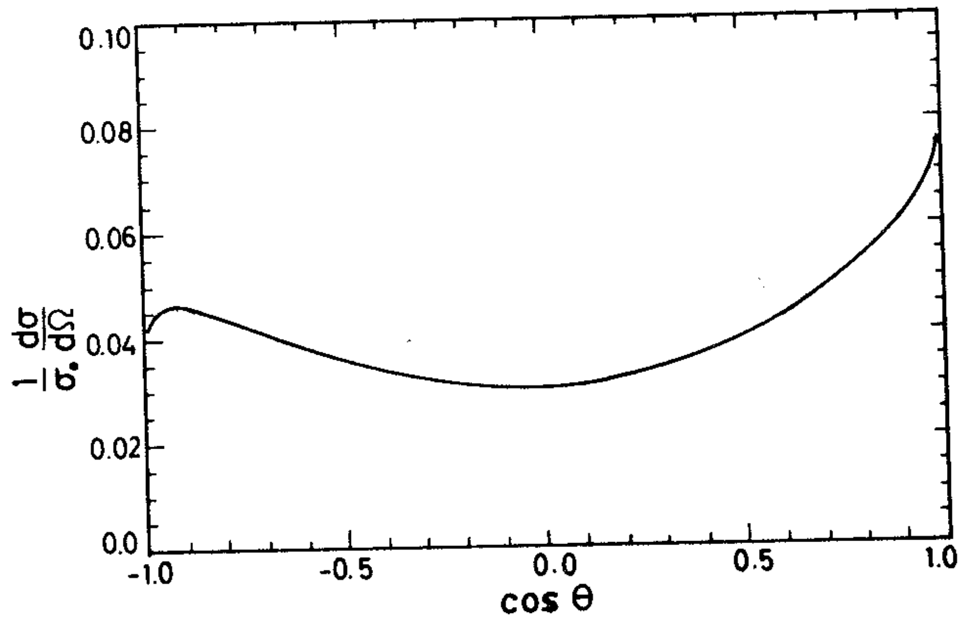
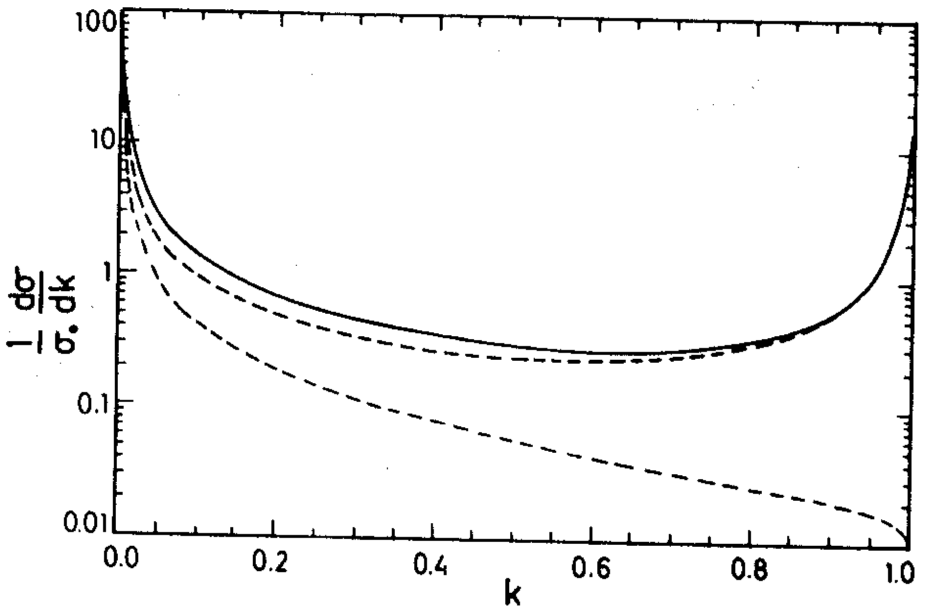
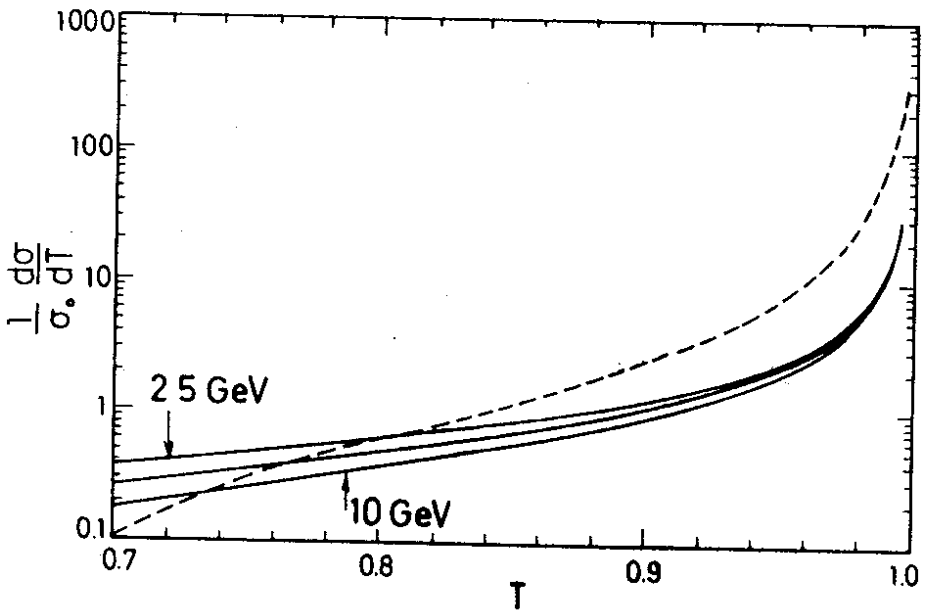

$\frac{1}{\sigma_0}\frac{d\sigma}{d\Omega}$
0.10
0.08
0.06
0.04
0.02
0.0
-1.0
-0.5
0.0
0.5
1.0
$\cos\Theta$

Fig.2



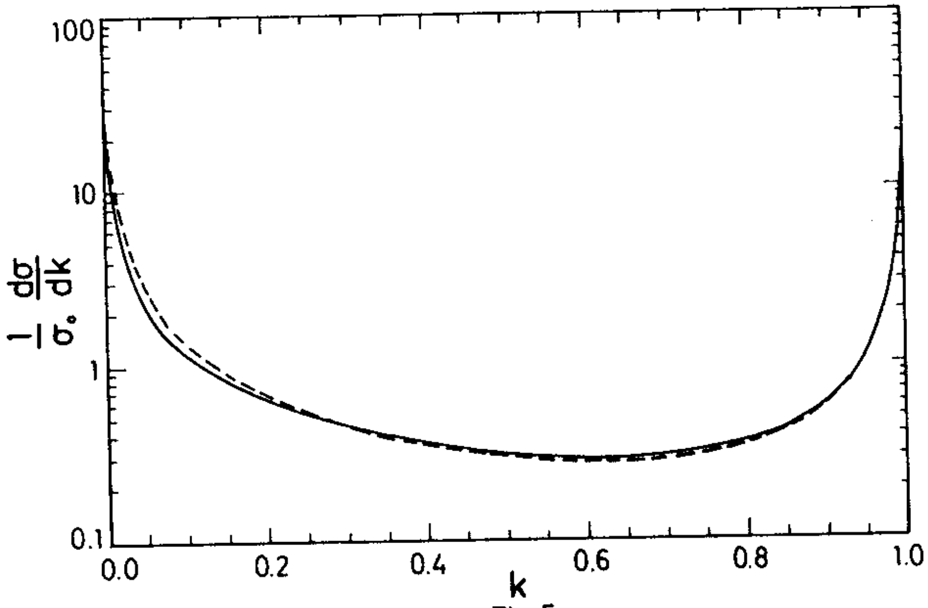
100
10
1
0.1
0.01
$\frac{1}{\sigma_0}\frac{d\sigma}{dk}$
0.0
0.2
0.4
0.6
0.8
1.0
k

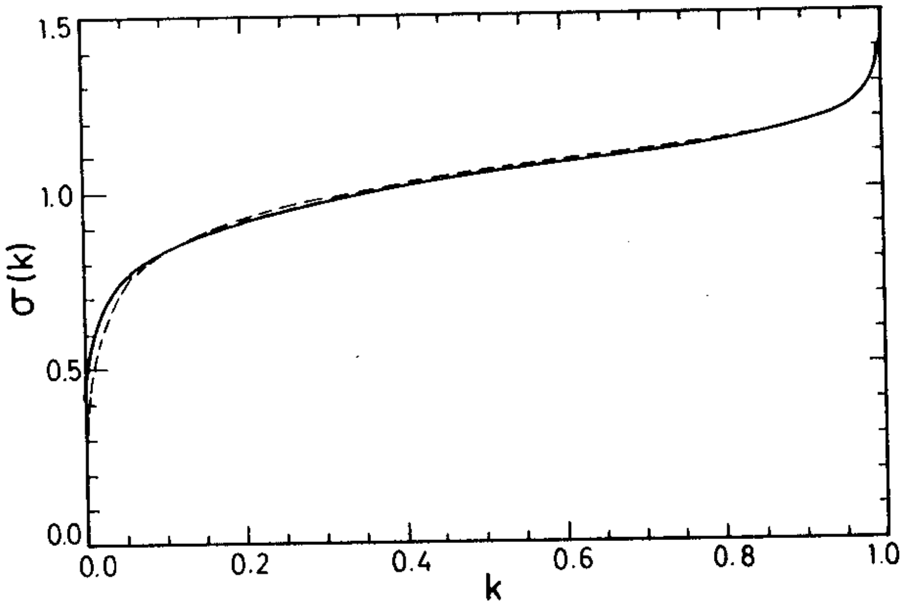
Fig. 3



1000
100
10
1
0.1
$\frac{1}{\sigma_0}\frac{d\sigma}{dT}$
2 5 GeV
10 GeV
0.7
0.8
0.9
1.0
T

Fig. 4


100
10
1
0.1
$\frac{1}{\sigma_0}\frac{d\sigma}{dk}$
0.0
0.2
0.4
0.6
0.8
1.0
k
Fig.5


1.5
1.0
0.5
0.0
$\sigma(k)$
0.0
0.2
0.4
0.6
0.8
1.0
k
Fig.6

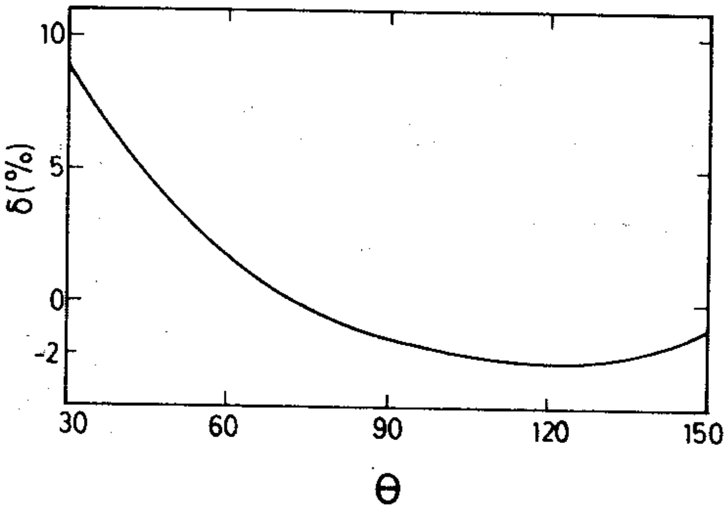

δ(%)
10
5
0
-2
30
60
90
120
150
Θ

Fig. 7

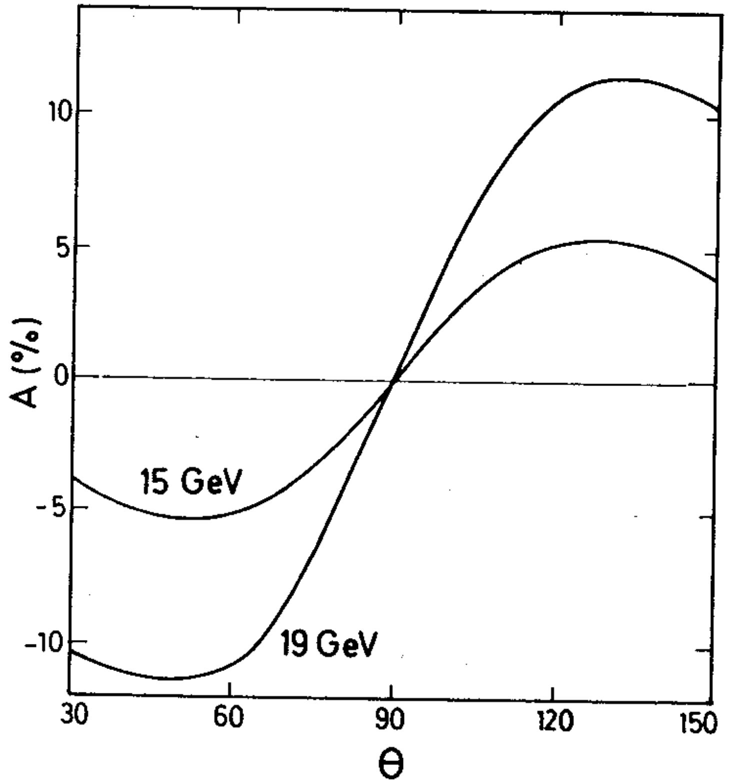
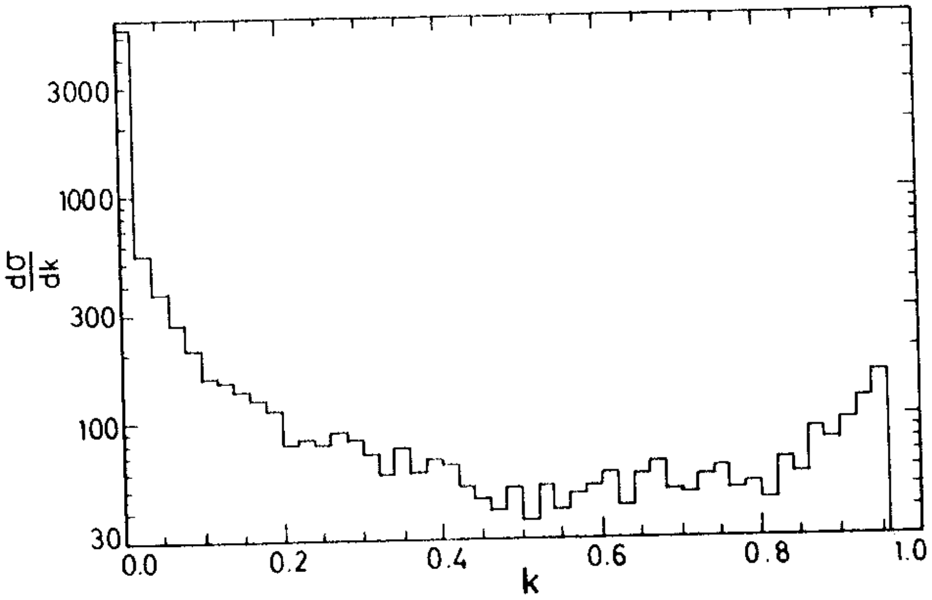
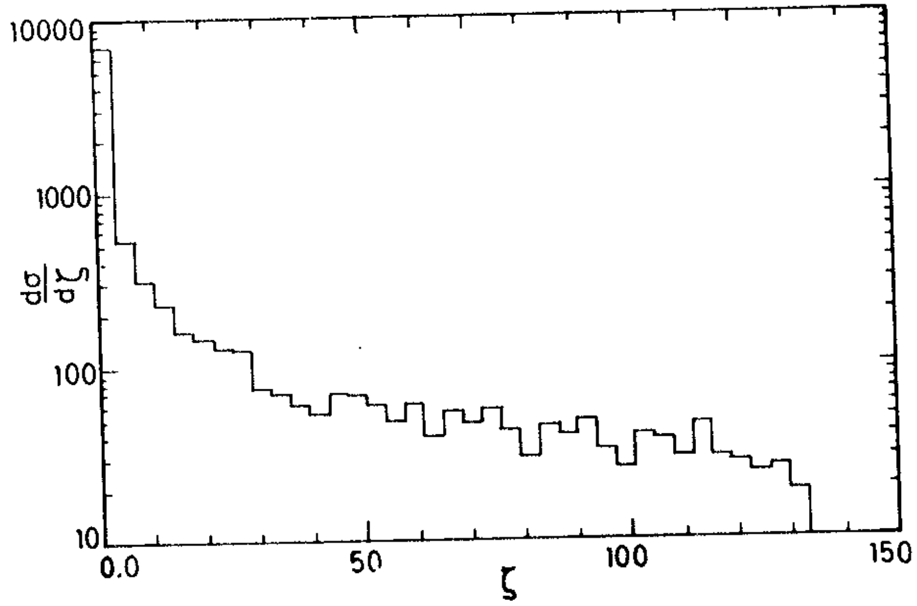

A (%)
10
5
0
-5
-10
15 GeV
19 GeV
30
60
90
120
150
Θ

Fig. 8


3000
1000
300
100
30
$\frac{d\sigma}{dk}$
0.0
0.2
0.4
0.6
0.8
1.0
k
Fig.9


10000
1000
100
10
$\frac{d\sigma}{d\zeta}$
0.0
50
100
150
ζ
Fig.10

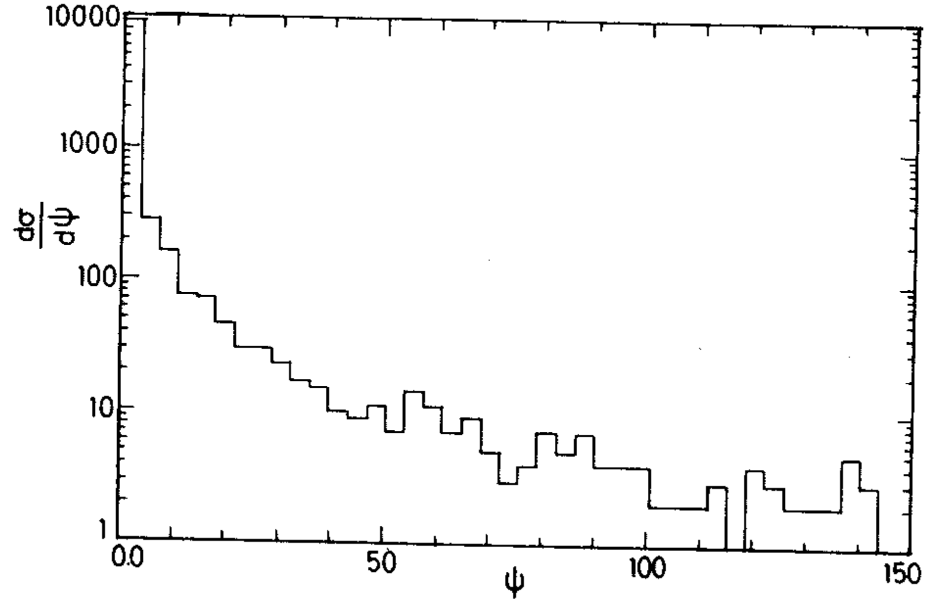


Fig. 11