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THE QUANTUM CHROMODYNAMICS VACUUM STATE ON THE LATTICE IN THE $1/N_c$ EXPANSION

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Abstract:

The physical vacuum state of QCD with quarks and gluons on the lattice is determined in the $1/N_c$ expansion up to the order $1/N_c^3$ (sixth order in perturbation theory). The vacuum expectation values of the gluon and quark fields and the slope of the quark-antiquark potential are calculated.

One way of investigating the confinement problem in QCD is to make the number of colours (N_c) large [1]. Expansions in powers of $1/N_c$ may give insight into the properties of the physical ($N_c = 3$) theory provided 1/3 is "small enough" for the series to be convergent. (For recent reviews see [2,3].)

The interesting large N_c limit in the continuum theory is $\lambda = g^2 N_c$ fixed (g = coupling constant) and $N_c \to \infty$, when the planar graphs dominate [1]. In the lattice version of the theory [4] the same limit can also be considered [5,6]. The problem is that the theory is non-trivial even at $N_c = \infty$ because an infinite number of diagrams are still contributing and it cannot be solved unless some further drastic simplifications are introduced (for attempts in this direction see e.g. [7,8]). Another possibility in lattice QCD is the $1/N_c$ expansion for fixed g [9,10]. The $N_c = \infty$ limit is then trivial and therefore it can be solved. It shows confinement similarly to the strong coupling expansion at $N_c = 3$ [4,11]. The non-trivial features of the dynamics come in through the higher orders of the $1/N_c$ expansion. (For other recent attempts to use the $N_c \to \infty$ limit see also [12-14]).

The purpose of the present paper is to include light, dynamical quarks in addition to the previously considered gluonic sector [9,10]. We also discuss the possibility of introducing constituent masses for quarks and gluonic "boxitons" in the unperturbed Hamiltonian H_0 of the $1/N_c$ expansion. These are free parameters which can be choosen phenomenologically for finite orders in $1/N_c$. It turns out that for non-zero values of the constituent masses (and for finite lattice spacing) the $1/N_c$ series becomes convergent also in the weak coupling region.

Before introducing the quarks let us briefly show how the constituent mass of boxitons changes the $1/N_c$ expansion in the purely gluonic sector. In Refs. [9,10] we wrote the Hamiltonian of the gluonic sector in the form

$$H = \frac{g^2 N_c}{a} \left(H_0 + \frac{1}{N_c} V \right) \equiv H_E - H_A \quad , \qquad (1)$$

where H_0 is the "unperturbed" Hamiltonian and V is the interaction Hamiltonian. $g^2 N_c a^{-1} H_0$ was choosen to be the dominating piece of the colour-electric part H_E of the Hamiltonian. From the definition of H_E [10] :

$$H_0 = \frac{n_\ell}{4} \; ; \quad V = ag^{-2}H - \frac{1}{4} N_c n_\ell \quad , \tag{2}$$

where n_ℓ is the number of links in the gluon-loop configuration characterizing the physical state. Now we include in H_0 a term proportional to the number of plaquettes n_p and compensate it in V :

$$H_0 = \frac{n_\ell}{4} + \frac{n_p \varkappa_\Box}{g^4 N_c^2} \quad ;$$
$$V = ag^{-2}H - \frac{1}{4} N_c n_\ell - \varkappa_\Box g^{-4} N_c^{-1} n_p \quad . \tag{3}$$

(The number of plaquettes in the configuration is, of course, zero in the mathematical vacuum. It is increased by 1 by the action of the colour-magnetic part H_A of H and left unchanged by the colour electric part. In Ref. [10], for instance, we had $n_p = 0, 1, 2, 3.$) The free parameter $\varkappa_\Box$ is the "constituent boxiton mass" measured in lattice units:

$$\varkappa_\Box \equiv am_\Box \quad . \tag{4}$$

Performing the perturbation calculation with the modified H_0 in Eq. (3) means a partial summation (reordering) of the perturbation series of the old H_0. The new energy denominators have, namely, the form

$$\frac{1}{\Delta E} = \frac{1}{\Delta E(\varkappa_\Box = 0) + n_p \varkappa_\Box g^{-4} N_c^{-2}} \quad . \tag{5}$$

Expanding these, for large N_c , in powers of $1/N_c$ we obtain the old series. The old and new series have, however, completely different behaviour for small g (or $\lambda = g^2 N_c$) and fixed N_c and $\varkappa_\Box$. For $g \to 0$ the energy denominator in Eq. (5) can be expanded like

$$\frac{1}{\Delta E} = \frac{g^4 N_c^2}{n_p \varkappa_\Box} + \ldots \quad . \tag{6}$$

The old series contain, from the matrix elements of V , inverse powers of g. As it can be easily shown, however, these are compensated by the positive powers of g coming from Eq. (6). As a result, for finite $\varkappa_\Box \neq 0$ the physical quantities have a finite limit and a convergent expansion in terms of g at $g = 0$. The situation is the same also concerning the constituent quark masses $\varkappa_c$ introduced below. Although this fact seems to us interesting we must emphasize that it does not solve immediately the problem of the analytic continuation to zero lattice spacing when both a and g tend to zero. Namely, asymptotic freedom says that g^2 goes to zero only logarithmically for $a \to o$, therefore $g^4 a^{-1} \to \infty$ and, for finite $m_\Box$, also $g^4 \varkappa_\Box^{-1} \to \infty$. This spoils the convergence of the $1/N_c$ series. The reason of the divergence is in this form, that the constituent masses and hence also the hadron masses are very small compared to the cut-off a^{-1}. A possible way out is, as it can be seen from Eq. (6), that the relevant values of n_p grow roughly as a^{-1}. This means, however, very high orders in the $1/N_c$ expansion unless the high orders can be somehow approximated by the low order formula at some larger n_p .

From the phenomenological point of view the constituent masses $\varkappa_{c,\Box}$ are free parameters to be determined experimentally. At very high order in $1/N_c$ we expect that physical results are not very much sensitive to the values of $\varkappa_{c,\Box}$ because the whole series ("infinite order") has to be independent of them.

In order to put quark fields on the lattice we follow the prescription given by Wilson [15-16]. The quark part of the Hamiltonian for N_f flavours ($f = 1, 2, \ldots, N_f$) is the following [17-18] :

$$H_Q = a^{-1} \sum_{f,k} \{(\varkappa_f + 3)(\bar{A}_{kf} A_{kf} - B_{kf} \bar{B}_{kf}) -$$
$$-\frac{1}{2} \sum_r (\bar{A}_{k(r)f}\, U[\overleftarrow{r}k]\, A_{kf} + \bar{B}_{k(r)f}\, U[\overleftarrow{r}k]\, B_{kf} + \tag{7}$$
$$+ \bar{B}_{k(r)f}\, \gamma_r\, U[\overleftarrow{r}k]\, A_{kf} + \bar{A}_{k(r)f}\, \gamma_r\, U[\overleftarrow{r}k]\, B_{kf})\} \; .$$

Here we used the notation $k(r) \overleftarrow{r} k \equiv \overleftarrow{r} k$, introduced in [9,10] , for the link starting at the lattice site k in the direction r and ending at the site $k(r)$. $U[\overleftarrow{r}k]$ is the $SU(N_c)$ string operator belonging to the link $\overleftarrow{r}k$. $\varkappa_f$ is the "current quark mass" for flavour f measured in lattice units:

$$\varkappa_f \equiv am_f \quad , \tag{8}$$

γ_r ($r = 1, 2, 3$) are Hermitean Dirac-matrices and A, $\bar{B} = B^+\gamma_4$ ($\bar{A}$, B) denote annihilation (creation) operators at the lattice points for quarks and antiquarks, respectively. They satisfy the usual anticommutation relations:

$$\{ A_{kf\alpha} , \bar{A}_{k'f'\alpha'} \} = \delta_{kk'} \delta_{ff'} \Lambda_{+\alpha\alpha'} \; ;$$
$$\{ B_{kf\alpha}, \bar{B}_{k'f'\alpha'} \} = -\delta_{kk'} \delta_{ff'} \Lambda_{-\alpha\alpha'} \; , \tag{9}$$

where $\Lambda_\pm = \frac{1}{2}(1 \pm \gamma_4)$. ($\alpha$ and α' stand for Dirac indices.) The mathematical vacuum $|o\rangle$ is annihilated by A and $\bar{B}$:

$$A\,|o\rangle = \bar{B}\,|o\rangle = 0 \; . \tag{10}$$

The first term of the quark Hamiltonian (7) is equal to $a^{-1}(\varkappa_f + 3)$-times the number of quarks plus the number of antiquarks. Therefore, even in the case of zero current quark masses ($\varkappa_f = 0$) the states containing extra quark-antiquark pairs are not degenerate with the ones without the $q\bar{q}$ pairs. As a consequence, we can use non-degenerate perturbation theory for the calculation of the lowest energy states. (On the other hand, the term with $(\varkappa_f + 3)$ breaks the chiral symmetry at zero quark masses, which can be restored only in the $a \to o$ limit [15,16].)

In the present paper we are interested in the effects due to the light quarks. Therefore, we take for simplicity $\varkappa_f = 0$ and $N_f = 3$ (for u, d and s quarks) and we neglect heavy flavours altogether. In this case the Hamiltonian can be written as follows

$$H \equiv H_E - H_A + H_Q = \frac{g^2 N_c}{a} \left(H_o + \frac{1}{N_c} V_1 + \sqrt{\frac{N_f}{N_c}} V_2\right) \; . \tag{11}$$

The unperturbed Hamiltonian H_o is the sum of the purely gluonic part in (3) and the part containing the "constituent quark mass" $\varkappa_c \equiv am_c$:

$$H_o = \frac{n\ell}{4} + \frac{\varkappa_\square n_p}{g^4 N_c^2} + \frac{2\varkappa_c n_q}{g^2 N_c} \; . \tag{12}$$

Here n_q is the number of quarks (equal to the number of antiquarks) in the physical state. The interaction parts $V_{1,2}$ contain the rest of H. In particular, the piece V_2 consists of the terms in H_Q with the Dirac matrices γ_r. From the diagonal term with $(\varkappa_f + 3)$ in Eq. (7) a part is put in H_o and the rest is in V_1 :

$$V_1 = 2g^{-2} \, (3 + \varkappa_f - \varkappa_c)\, n_q + \cdots \; . \tag{13}$$

For this we shall use later on the short notation

$$X_f \equiv 3 + \varkappa_f - \varkappa_c = 3 - \varkappa_c \equiv X \; . \tag{14}$$

As can be seen in Eq. (11), the interaction part of the Hamiltonian has a part behaving like N_c^{-1} at $N_c \to \infty$ and another part proportional to $\sqrt{N_f/N_c}$. This means that when light quarks are present ($N_f \neq 0$) then the perturbation expansion goes, in fact, according to $N_c^{-1/2}$. In order to calculate e.g. up to the order $1/N_c^3$ we need the sixth order in perturbation theory.

The construction of physical states is in principle similar to the quarkless case [9,10] . There are, of course, additional states containing quark-antiquark pairs. In order to guarantee gauge invariance the $q\bar{q}$ pairs must be connected by an open "string" of $U[\overleftarrow{r}k]$ operators multiplied together along some path on the lattice. For the calculation of the "physical vacuum"

(i.e. lowest energy) state in perturbation theory we need the states created by the repeated action of the interaction Hamiltonian on the mathematical vacuum $|o\rangle$. The simplest such state created from $|o\rangle$ by V_2 is

$$| Q \rangle \equiv \sum_{fkr} \mathcal{A}_{k(r)f}\, \gamma_r\, U[\overleftarrow{r}k]\, B_{kf}\, |o\rangle \quad . \tag{15}$$

Note that the $q\bar{q}$-pair appearing here is the lattice version of the $L = 1$, $S = 1$ quark pair with vacuum quantum numbers advocated in the quark pair creation models of hadron decays [19-21].

Up to the sixth order we also need the state $| A \rangle \equiv | 1 \rangle$ with one plaquette introduced in [9,10] , the states $| Q\, \mu\, Q \rangle$ ($\mu = 0, \pm 1$) with two $q\bar{q}$-pairs , the states $| A\, \mu\, Q \rangle$ ($\mu = 0, \pm 1$) with one plaquette and one $q\bar{q}$-pair, the states $| Q\, \lambda\, Q\, \mu\, Q\, \nu \rangle$ ($\mu\, \nu\, \lambda$ = 000, 100, -100, 111, 1-1-1) with three $q\bar{q}$-pairs and the state $| Q^* \rangle$ with an "excited" $q\bar{q}$-pair. The numbers λ, μ, ν give always the number and orientation of common links of the $q\bar{q}$-pairs or plaquettes. The "excited" $q\bar{q}$-pair Q^* is produced from the pair Q by the action of the second term in H_Q. The quark and antiquark is separated in it by two links:

$$| Q^* \rangle = \sum_{fk} \sum_{-r \neq r'} \mathcal{A}_{k(rr')f}\, \tfrac{1}{2}\, (\gamma_r + \gamma_{r'})\, U\, [\overleftarrow{r}' k(r) \overleftarrow{r} k]\, B_{kf}\, | 0 \rangle \quad . \tag{16}$$

For the calculation of the norms of the states and the matrix elements it is sufficient to use the rules given in Ref. [9] and, for the quarks, the anti-commutators (9) together with Eq. (10).

We determined the physical vacuum state $| v \rangle$ obtained from the mathematical vacuum $| o \rangle$ by perturbation theory, up to sixth order. The vacuum energy density with respect to the mathematical vacuum turns out to be:

$$\varepsilon_V = -6\, N_c a^{-4} g^2 \{ \frac{2N_f k_a}{N_c g^4} + \frac{k_r}{N_c^2 g^8} - \frac{16 X N_f k_a^2}{N_c^2 g^6} + \frac{2N_f k_a^2}{N_c^3 g^4} + \frac{48 N_f k_a^2 k_c}{N_c^3 g^8} + (128 X^2 - 88) \frac{N_f k_a^3}{N_c^3 g^8} + \dots \} \quad . \tag{17}$$

Here and below the following notations are used :

$$k_a = (1 + \frac{8\varkappa_c}{N_c g^2})^{-1} \quad ;$$
$$k_e = (1 - \frac{8\varkappa_c}{N_c g^2})^{-1} \quad ;$$
$$k_c = (1 + \frac{4\varkappa_c}{N_c g^2})^{-1} \quad ; \tag{18}$$
$$k_r = (1 + \frac{\varkappa_\square}{N_c^2 g^4})^{-1} \quad .$$

Note that these are all equal to 1 if the constituent masses $\varkappa_{c,\square}$ are zero.

It is an important property of the $1/N_c$ expansion that it does not give rise to infrared singularities. That is, the physical quantities remain finite if the size of the box of periodicity V (for which we impose periodic boundary conditions) tends to infinity. We could not find a general proof of this remarkable property but it was born out of the calculation in all cases we considered.

The interesting physical property of the vacuum is the expectation value of the gluon field squared [22] :

$$< v \mid F_{a\mu\nu} F_a^{\mu\nu} \mid v > \equiv - 4 < v \mid L_{YM} \mid v > \cong 0.48 g^{-2} \mathrm{GeV}^4 \quad . \quad (19)$$

For this "gluon condensate" we obtain (from the definition given on the lattice in Ref. [9]):

$$< v \mid g^2 L_{YM} \mid v > a^4 = - 6N_c + 12 N_f k_a^2 + \frac{6k_r(2 + k_r)}{N_c g^4} - \frac{192 N_f X k_a^3}{N_c g^2} -$$

$$- \frac{12 N_f k_a^2 (1 - 2k_a)}{N_c^2} + \frac{144 N_f k_a^2}{N_c^2 g^4} (2k_c^2 + 4k_a k_c - 11k_a^2 + 16 X^2 k_a^2) + \cdots \quad . \quad (20)$$

The vacuum expectation value of the quark density

$$\bar{q}q \equiv V^{-1} \sum_k \bar{A}_{kf} A_{kf} \quad (21)$$

is given by

$$< v \mid \bar{q}q \mid v > a^3 = \frac{48 k_a^2}{N_c g^4} \{ 1 - \frac{16 k_a X}{N_c g^2} + \frac{2k_a}{N_c^2} +$$

$$+ \frac{12}{N_c^2 g^4} (k_c^2 + 4k_a k_c - 11 k_a^2 + 16 k_a^2 X^2) + \ldots \} \quad . \quad (22)$$

The value of the "quark condensate" $< v \mid q\bar{q} \mid v >$ can be deduced from current algebra and PCAC [23,24]:

$$< v \mid \bar{q}q \mid v > = \frac{m_\pi^2 F_\pi^2}{4(m_u + m_d)} \cong 0.006 \ \mathrm{GeV}^3 \quad . \quad (23)$$

(Here we took $F_\pi = 130$ MeV and $m_u + m_d = 15$ MeV for the current quark masses.)

The slope of the quark-antiquark potential can also be calculated if a pair of external quark-antiquark sources is put in the quantum state. The sources are introducing extra strings in the state. If these strings end on dynamical quarks (created by $\bar{A}$ and B) then we call the state of the sources "screened". If, however, the extra string connects the external sources then the sources are "unscreened". (See also the similar discussion of external gluon sources in Ref. [10].) The lowest energy unscreened state in the $N_c \to \infty$ limit is, when the string connecting the sources is a straight line and there are no dynamical quark pairs or gluon loops present. For the calculation of the $q\bar{q}$-potential we start the perturbation expansion from this lowest unscreened state and take the energy difference compared to the physical vacuum without external sources. In the $N_c \to \infty$ limit the notion of a potential makes sense, bacause the quark pair creation (and hence the screening of external sources) is a small perturbation suppressed like $1/\sqrt{N_c}$.

Of course, the screened states come in virtually in the perturbation theory and actually decrease the slope of the linear potential. The result for the slope of the $q\bar{q}$-potential A^{-2} is:

$$A^{-2}a^2 = g^2N_c \{ \frac{1}{4} - \frac{1}{4N_c^2} - \frac{8N_f k_a^2(k_a-k_e)}{N_c^3 g^4} + \cdots \} . \qquad (24)$$

Experimentally, from the charmonium spectrum we have [25]:

$$A^{-2} \cong 0.18 \text{ GeV}^2 . \qquad (25)$$

The experimental values in Eqs. (19, 23, 25) can be fitted at $N_c = 3$ by the following numbers:

$$a = 3.14 \text{ GeV}^{-1} = 0.62 \text{ fermi} ;$$

$$g = 1.91 ; \qquad \alpha_s = \frac{g^2}{4\pi} = 0.29 ; \qquad (26)$$

$$m_c = 0.544 \text{ GeV} .$$

The values of a and g are not much different from the ones given in [10] for the pure gluonic sector ($a = 3.5$ GeV^{-1}, $g = 1.8$). The constituent mass of the light quarks m_c is somewhat larger than expected. It would be very interesting to see whether the hadronic spectrum is consistent with this value in the $1/N_c$ expansion. The constituent boxiton mass $m_\square$ is rather inrelevant for the quantities we are calculating as it appears only in k_r (18) multiplied by a small coefficient. For definiteness we have put it equal to zero in Eq.(26) but values up to $m_\square < 2 - 3$ GeV do not change the results appreciably. It has to be noted that the higher order terms in Eqs. (19, 23, 25) are not very small at the values in Eq.(26), typically of the order of 20-40%. This means that higher orders in $1/N_c$ may still modify somewhat the situation.

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