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AT STRONG AND INTERMEDIATE COUPLING

by

Gernot Münster

*II. Institut für Theoretische Physik der Universität Hamburg*

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VORTEX FREE ENERGY AND STRING TENSION  
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Gernot Münster\*

II. Institut für Theoretische Physik der Universität Hamburg

Abstract: We present results of high temperature expansions up to order  $g^{-24}$  for the vortex free energy respectively string tension in pure lattice gauge theories with gauge groups  $SU(2)$  and  $Z_2$  in 3 and 4 dimensions. For  $SU(2)$  in 4 dimensions the result is compared with Monte Carlo calculations of Creutz and is in good agreement. An intermediate coupling region is seen, where the string tension smoothly interpolates between strong coupling and weak coupling behaviour.

Gauge field theories have been formulated on a lattice<sup>1,2</sup> in order to study the strong coupling region which is inaccessible to ordinary perturbation theory. In the last years many results have been obtained. It has been proven that lattice gauge theories confine static quarks for strong coupling<sup>3</sup>. On the other hand one would like to know if for gauge group SU(2) or SU(3) a continuum limit exists with persisting confinement property. Due to asymptotic freedom<sup>4</sup> the continuum limit of SU(2) or SU(3) lattice gauge theories is supposed to be related to the weak coupling behaviour<sup>5,6</sup>. It is expected to be a limit, where the lattice spacing  $a$  and at the same time the bare coupling  $g$  go to zero in a way determined by renormalization group equations. The problem of quark confinement in lattice gauge theories is to show that the confining phase extends down to zero coupling.

The connection between strong and weak coupling regions was studied by several authors<sup>7,8,9,10,11</sup>. Of special interest is the intermediate coupling region where a transition from strong coupling to weak coupling behaviour of e.g. the string tension or renormalization group  $\beta$ -function should take place. Monte Carlo calculations for SU(2) in 4 dimensions<sup>8</sup> indicate a rather sharp changeover. In Macks theory of quark confinement<sup>11</sup>, which is based on the assumption that confinement of static quarks is due to condensation of vortices<sup>12,13,14</sup>, the transition signals the appearance of a nonempty range of distances  $a \leq d \leq d_c(\beta)$  where perturbation theory applies. For values of  $\beta$  above it vortices need a certain thickness  $d_c(\beta) > a$  in order to condense. The center  $Z_2$  resp.  $Z_3$  of the gauge group plays an important role.

In this letter we discuss the transition between strong and weak coupling with the help of high temperature expansions. We consider Euclidean lattice gauge theories with gauge groups SU(2) and  $Z_2$ <sup>15</sup> in 3 and 4 dimensions. The gauge field  $U(b) \in \text{SU}(2)$  resp.  $\mathcal{G}(b) = \pm 1$  is attached to links  $b$  of a hypercubical lattice with lattice spacing  $a$ . The action is

$$L(U) = \frac{\beta}{2} \sum_p \text{tr } U(\dot{p}) \quad \text{for } SU(2) \quad (1 a)$$

$$L(\sigma) = \beta \sum_p \sigma(\dot{p}) \quad \text{for } Z_2 \quad (1 b)$$

where  $U(\dot{p})$  resp.  $\sigma(\dot{p})$  is the product of the group elements on the boundary of an elementary plaquette  $p$ . The parameter  $\beta$  is related to the coupling constant  $g$  by

$$\beta = 4/g^2 \quad (2)$$

The sum in (1) is over all unoriented plaquettes  $p$  of the lattice. Expectation values are formed with the path integral measure

$$d\mu = Z^{-1} \exp(L(U)) \prod_b dU(b) \quad (3 a)$$

$$\text{resp. } d\mu = Z^{-1} \exp(L(\sigma)) \prod_b d\sigma(b) \quad (3 b)$$

where  $Z$  is the partition function and  $dU(b)$ ,  $d\sigma(b)$  are the Haar measures on the gauge groups.

High temperature i.e. small  $\beta$  expansions are obtained by the well known method of cluster expansion<sup>16,3</sup>. It is convenient to expand in terms of the coefficients  $c_j$  of the series

$$\exp \frac{\beta}{2} \text{tr } U = N \left( 1 + \sum_{j \neq 0} (2j+1) c_j(\beta) \chi_j(U) \right) = N(1+h(U)) \quad (4 a)$$

$$\exp \beta \sigma = N' (1 + k \sigma) = N'(1+h'(\sigma)) \quad (4 b)$$

where  $\chi_j$ ,  $j = 1/2, 1, 3/2, \dots$  are the irreducible characters of  $SU(2)$  and  $N, N'$  are normalization factors. Explicitely

$$c_j(\beta) = I_{2j+1}(\beta)/I_1(\beta), \quad I_n = \text{modified Bessel functions} \quad (5 a)$$

$$k = \tanh \beta \quad (5 b)$$

For  $\ln Z$  one obtains expansions in terms of clusters. A cluster  $C = (X_1^{n_1}, X_2^{n_2}, \dots, X_i^{n_i})$  consists of a connected set of polymers  $X_1$  with multiplicities  $n_1$ . Polymers are closed surfaces of plaquettes, which are allowed to possess branching lines. The cluster

expansion is

$$\ln Z = \sum_C b(C) \prod_i \frac{(A(X_i))^{n_i}}{n_i!} \quad (6)$$

$$A(X_i) = \int \prod_b dU(b) \prod_{p \in X_i} h(U(p)) \quad \text{for } SU(2)$$

where  $b(C)$  are certain combinatorial coefficients and  $A(X_i)$  are the so-called activities of polymers. Details will be published elsewhere<sup>18</sup>.

An interesting quantity to study is the free energy of vortices per unit length resp. area<sup>14,17</sup>. It is defined in the following way. Consider a finite lattice  $L$  of sidelengths  $a_{l_i}$  in  $d=3$  or 4 dimensions. With periodic boundary conditions one gets a partition function  $Z_+$ . A quantum of magnetic vortex flux perpendicular to the  $x_1$ - $x_2$ -plane can be introduced in  $L$  by applying a singular gauge transformation<sup>13,14,17</sup> leading to twisted boundary conditions. The corresponding partition function is denoted  $Z_-$ . The free energy of a vortex per unit length resp. area is defined by

$$f(l_1, l_2) = \lim_{l_3, l_4 \rightarrow \infty} \left( \prod_{i=3}^d l_i^{-1} \right) (\ln Z_+ - \ln Z_-) \quad (7)$$

It is a suitable quantity for the characterization of different phases. In the confining phase  $f$  should obey an area law<sup>14</sup>

$$f(l_1, l_2) \sim \exp(-\alpha a^2 l_1 l_2) \quad (8)$$

In the high temperature expansion of  $f$  only a special kind of clusters survive. They contain an odd number of polymers which include a surface cutting  $L$  like the  $x_1$ - $x_2$ -plane. The expansion leads to an area law as expected.

$$f(l_1, l_2) = 2 \exp(-\alpha a^2 l_1 l_2) \quad (9)$$

We proved that the coefficient  $\alpha$  is in the high temperature expansion equal to the string tension<sup>18</sup>, i.e. the corresponding coefficient in the area law for the Wilson loop expectation value<sup>1,3</sup>. It gives the slope of the linear potential between static quarks. A direct expansion for the string tension can be set up. We calculated the series up to order  $\beta^{12}$ . The results are :

with  $u = c_{\frac{1}{2}}$  ,  $v = c_1$  ,  $w = c_{\frac{3}{2}}$  ;  
for SU(2) in d=3 dimensions

$$\begin{aligned} a^2 \alpha = & -\ln u - 2u^4 + 4u^6 - 6u^4 v - 10u^8 + 4u^{10} + 24u^8 v - 36u^6 v^2 - 6v^5 \\ & -(602/3)u^{12} + 204u^{10} v - 36u^8 v^2 - 24u^6 v^3 - 24u^4 v^4 + 18v^6 \\ & - 8u^9 w - 12v^5 w u^{-1} \end{aligned} \quad (10)$$

for SU(2) in d=4 dimensions

$$\begin{aligned} a^2 \alpha = & -\ln u - 4u^4 + 8u^6 - 12u^4 v - 56u^8 + 120u^{10} - 168u^8 v - 72u^6 v^2 - 12v^5 \\ & -(4588/3)u^{12} + 888u^{10} v - 396u^8 v^2 - 48u^6 v^3 - 48u^4 v^4 + 36v^6 \\ & - 48u^9 w - 24v^5 w u^{-1} \end{aligned} \quad (11)$$

for  $Z_2$  in d=3 dimensions

$$a^2 \alpha = -\ln k - 2k^4 - 2k^6 - 10k^8 - 16k^{10} - (242/3)k^{12} \quad (12)$$

for  $Z_2$  in d=4 dimensions

$$a^2 \alpha = -\ln k - 4k^4 - 4k^6 - 56k^8 - 144k^{10} - (3616/3)k^{12} . \quad (13)$$

The string tension for SU(2) in d=3 dimensions has been calculated previously<sup>9</sup> up to order  $\beta^{14}$ . Our results agree with theirs.

The case of SU(2) in d=4 dimensions is most interesting. It can be compared with the Monte Carlo simulations of Creutz<sup>8</sup>. In fig.1 we have plotted  $a^2 \alpha$  according to (11) and the points given in ref.8. The high temperature series is in very good agreement with the Monte Carlo data in the intermediate coupling region between  $\beta = 1.6$  and  $\beta = 2.25$ . At  $\beta = 2.25$  the curve matches very nicely the weak coupling fit of Creutz. Above  $\beta = 2.3$  the string tension seems to follow the weak coupling asymptotic freedom behaviour. This cannot be expected to be reproduced by the high temperature series (11), which in fact goes through zero near  $\beta = 2.4$ .

In conclusion we see that high temperature expansions yield accurate values of the string tension at strong and intermediate coupling and their prediction joins smoothly to the expected behaviour at weak coupling.

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Figure caption

Fig. 1      The string tension  $\propto$  times the lattice spacing squared as a function of  $\beta = 4/g^2$  for SU(2) lattice gauge theory in  $d=4$  dimensions. The solid line represents the result of high temperature expansions up to order  $\beta^{12}$ . The dots are results of a Monte Carlo calculation of Creutz<sup>8</sup>. The dashed lines are the lowest order high temperature curve and the weak coupling fit of ref.8 .

