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COMMENTS ON THE COMPUTATION OF QUANTUM FLUCTUATIONS OF GLUONS IN A MULTI INSTANTON BACKGROUND

by

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Abstract

Relying on previous results the computation of quantum fluctuations of gluons in a Yang - Mills multi instanton background field is explained.

I. Introduction

Let the Dirac operator in an external Yang - Mills gauge field be

$$\not{D} = \begin{pmatrix} 0 & D^+ \\ D & 0 \end{pmatrix} \quad , \qquad (1)$$

where $\quad D = i e^+_\mu D_\mu \; , \quad D^+ = i e_\mu D_\mu \; , \quad (e_\mu) = (1, -i\sigma_j)$

(σ_j are the Pauli matrices) and $D_\mu = 1_n \partial_\mu + A_\mu$ the covariant derivative. For definiteness we use a SU(n) gauge group (the extension to other gauge groups is straightforward). By $\not{D}_0$ we denote the same operator without gauge field.

Recently formulae for the determinant of the Dirac operator in a multi instanton background field A_μ of topological charge k were derived [1]. In flat space the result for the regularized determinant with zero modes omitted reads+)

$$det' \frac{\not{D}}{\not{D}_0} \overset{def}{=} \sqrt{det' \frac{\not{D}^2}{\not{D}_0^2}} = e^{\Gamma} \quad , \qquad (2.a)$$

where

$$\begin{aligned} \Gamma &= \Gamma_{reg} - \Gamma^0_{reg} \\ &= k \left\{ -\frac{2}{3} \ln \mu - 4\zeta'(-1) - \frac{2}{3} \ln 2 + \frac{5}{12} \right\} \\ &\quad + \frac{1}{24\pi^2} \int d^4x \, I_1(x) + \frac{1}{24\pi^2} \int d^4x \int_0^1 dt \, I_2(t,x) \qquad (2.b) \end{aligned}$$

+) The results of I were derived for a Sp(r) gauge group. By imbedding SU(n) into Sp(n) it is quite straightforward to show that (2) holds.

and $$I_1 = \mathrm{Tr}\{ f\partial_\mu f^{-1} f\partial_\mu f^{-1} f\partial_\nu f^{-1} f\partial_\nu f^{-1}\} - 5\,\mathrm{Tr}\{ f\, \underbrace{b^+ b}\, f\, \underbrace{b^+ b}\} + 4k\,(1+x^2)^{-2} , \qquad (3.a)$$

$$I_2 = \varepsilon_{\mu\nu\varrho\sigma}\,\mathrm{Tr}\{ K^{-1}\partial_t K K^{-1}\partial_\mu K K^{-1}\partial_\nu K K^{-1}\partial_\varrho K K^{-1}\partial_\sigma K \} \qquad (3.b)$$

with $$K(t,x) = \frac{1}{2}(t-1)(\lambda^2+x^2)\underbrace{b^+ b} + t\, f^{-1} . \qquad (3.c)$$

The k x k matrix f is related to the instanton parameter matrices of the Atiyah - Drinfield - Hitchin - Manin (ADHM) construction by means of

$$f^{-1} = -\frac{1}{2}\,\underbrace{\Delta^+\Delta} , \quad \Delta_{A'} = a_{A'} + b^A x_{AA'} . \qquad (3.d)$$

Here $x = (x_{AA'})$ is the quaternionic representation of the position vector. The $(n+2k) \times k$ matrices $a_{A'}$, b^A ($A' = 1,2$; $A = 1,2$) are the instanton parameter matrices and $\underbrace{\quad}$ stands for contraction of spinorial indices, e.g. $\underbrace{\Delta^+\Delta} = \Delta^{+A'}\Delta_{A'}$. Indices are raised and lowered by $\xi_A = \varepsilon_{AB}\xi^B$ and the adjoint is in the sense of spinors, i.e. $\Delta_1^+ = -\bar{\Delta}_2^T$ and $\Delta_2^+ = \bar{\Delta}_1^T$.
Γ_{reg} is defined by

$$\Gamma_{reg} = \frac{1}{2}\mathrm{Tr}\left\{ \ln(\not{D}^2 + P_0) + \sum_{i=1}^{\nu} e_i \ln(\not{D}^2 + M_i^2) \right\} ,$$

where P_0 is the projector onto the zero modes of $\not{D}$ and $\ln\mu$ is related to the Pauli - Villars regulators by $\ln\mu = -\sum_{i=1}^{\nu} e_i \ln M_i$.
For more details of the notation cf. I.

In flat space the fluctuation operator for gluons (i.e. of the vector fields in a pure SU(n) Yang - Mills theory) is trivially related to the Dirac operator in the adjoint SU(n) representation and a corresponding relation between the determinants of

these operators has been assumed in the literature [2]. This conjecture may be supported by an investigation of the gluon determinant on the shere [3]. We therefore study in the present note the quantum fluctuation of the Dirac operator in the adjoint SU(n) representation.

The adjoint SU(n) representation can be trivially imbedded into the fundamental SU(n^2) representation. It follows that special instanton parameter matrices $\tilde{a}_{A'}$, $\tilde{b}^A$ of the fundamental SU(n^2) representation, from which a corresponding $\tilde{f}$ can be constructed by (3.d), describe the adjoint SU(n) representation. For the determinant of the Dirac operator formulae (2) and (3) remain true with f replaced by $\tilde{f}$ and K replaced by a corresponding $\tilde{K}$. The matrix $\tilde{f}$ has already been constructed by Corrigan, Goddard and Templeton [4] in their investigation of Green functions in a multi instanton field of a tensor product representation. In section II we give the explicit formulae for $\tilde{f}$ in the adjoint SU(n) representation. The thus obtained formulae for the determinant of the Dirac operator in the adjoint SU(n) representation are illustrated in section III by an explicit evaluation for the one - instanton case and a discussion of the 't Hooft solutions in section IV.
Equivalent formulae can be obtained along the lines of [1] by using the Green function [4] of the adjoint representation. This has been done by Jack [5] for tensor product representations. For completeness we also give his formulae for the adjoint SU(n) representation at the end of section II. These

formulae cannot be used for the one-instanton case, because a new integration constant has still to be determined. They **are**, however, more promising than the first formulae for trying a further evaluation. This is illustrated for the 't Hooft solutions in section IV. Both formulae suffer at the time from the lack of an unconstrained parametrization of the SU(n) instantons. This is the reason, why the instanton gas of the nonlinear σ- and CP^N-models [6] is still much more explicit than that of the Yang-Mills theory.

II. Determinant of the Dirac operator in the adjoint SU(n) representation

Let us consider the Hilbert spaces H_- (H_+) of spinors ψ_A of negative ($\psi_{A'}$ of positive) chirality which take values in the SU(n) Lie algebra. These two spaces are equipped with the scalar product

$$(\psi,\phi) = \int d^4x \, \langle \psi_A^+ , \phi^A \rangle , \qquad \psi , \phi \in H_- , \qquad (4.a)$$

$$(\psi,\phi) = \int d^4x \, \langle \psi_{A'}^+ , \phi^{A'} \rangle , \qquad \psi , \phi \in H_+ , \qquad (4.b)$$

where $\langle A , B \rangle = - \frac{1}{2} \, Tr \, (A \cdot B)$

for matrices of the SU(n) Lie algebra.

The Dirac operator in the adjoint SU(n) representation is given by

$$\not{D} = \begin{pmatrix} 0 & \nabla^+ \\ \nabla & 0 \end{pmatrix} , \qquad (5)$$

where $\nabla = i e_\mu^+ \nabla_\mu \; , \quad \nabla^+ = i e_\mu \nabla_\mu$

and $\nabla_\mu = \partial_\mu + [A_\mu , \cdot] \; .$

∇ maps negative onto positive chirality spinors and ∇^+ vice versa. With respect to the scalar product (4) ∇^+ is the adjoint of ∇ .

We may trivially imbed the adjoint SU(n) representation into the fundamental $SU(n^2)$ representation by defining the new gauge field

$$\tilde{A}_\mu = A_\mu \otimes 1_n + 1_n \otimes A_\mu \; .$$

The gauge field $\tilde{A}_\mu$ is a multi instanton solution of topological charge $\tilde{k} = 2nk$.

Let $a(1)_{A'} = a_{A'}$ and $b(1)^A = b^A$ be the instanton parameter matrices of the fundamental SU(n) representation, then the instanton parameter matrices of the conjugate SU(n) representation are given by

$$a(2)_{A'} = \varepsilon_{A'B'} \overline{a(1)}_{B'} \quad \text{and} \quad b(2)^A = \varepsilon^{AB} \overline{b(1)}^B \; , \qquad (6)$$

where $\varepsilon_{12} = \varepsilon^{12} = 1$. Because all instanton fields are obtained by the ADHM-construction, there have to be instanton parameter matrices $\tilde{a}_{A'}$, $\tilde{b}^A$ which describe $\tilde{A}_\mu$. Therefore the determinant of the Dirac operator in the adjoint SU(n) representation+) is obtained by replacing in (3) b by $\tilde{b}$, f by $\tilde{f}$ and corresponding-

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+) It is easily seen that the singlett part does not contribute.

ly K by

$$\tilde{K} = \frac{1}{2}(t-1)(\lambda^2 + x^2)\ \underline{\tilde{b}^+\tilde{b}} + t\,\tilde{f}^{-1} \ .$$

In the present notation the recipe [4] for constructing $\tilde{f}$ is given in the following. Up to a transformation $\tilde{f} \longrightarrow K^+\tilde{f}K$ with an element $K \in Gl(\tilde{k},\mathbb{C})$ $\tilde{f}$ is given by

$$\tilde{f}^{-1} = \frac{1}{2}\, Z^+\Omega^{-1}Z \ , \qquad (7)$$

where $$Z_{rs} = Tr\,(c_r^+ c_s) + Tr\,(d_r^+ d_s) \ , \qquad (8.a)$$

$$\Omega_{rs} = 4\left\{ Tr\,(c_r^+ c_s \bar{f}) + Tr\,(d_r^+ d_s f) - Tr\,(\bar{f} c_r^+ \underline{\Delta f \Delta^+} c_s)\right\} \ . \qquad (8.b)$$

The matrices c_s, d_s ($s = 1,\ldots,\tilde{k}$) are a basis of $(n+2k)\times k$ matrices, which are defined by the relation

$$(\Delta^+_{A'}(1)\,c)_{ij} = (\Delta^+_{A'}(2)\,d)_{ji} \ , \qquad A' = 1,2 \ , \qquad (9)$$

where $$\Delta^+_{A'}(i) = a^+(i)_{A'} + b^+(i)^A x_{AA'} \ , \qquad i = 1,2 \ .$$

$\underline{\tilde{b}^+\tilde{b}}$ is obtained by means of

$$\underline{\tilde{b}^+\tilde{b}} = -\,Z^+\Omega_b^{-1}Z \ , \qquad (10.a)$$

with $$(\Omega_b)_{rs} = -8\left\{ Tr\,(c_r^+ c_s\,(\underline{b^+b})^{-1}) + Tr\,(d_r^+ d_s\,(\underline{b^+b})^{-1}) \right.$$
$$\left. +\,2\ Tr\,((\underline{b^+b})^{-1} c_r^+\, \underline{b\,(\underline{b^+b})^{-1}\, b^+}\, c_s)\right\} \ . \qquad (10.b)$$

In the next section we will illustrate the obtained formulae for special examples.

Equivalent formulae can be derived along the lines of [1], by using the Green function [4] for tensor product repre-

sentations. This has been done by Jack [5]. For the Dirac operator in the adjoint SU(n) representation his result would become

$$\det{}'\,\frac{\nabla\!\!\!/}{\nabla\!\!\!/_0} \overset{def}{=} \sqrt{\det{}'\,\frac{\nabla\!\!\!/^2}{\nabla\!\!\!/_0^2}} = e^{\tilde{\Gamma}} \ , \qquad (11.a)$$

with

$$\tilde{\Gamma} = 2n\Gamma - 2\ \ln\det\left[\frac{1}{4} M\,(\underline{b^+b} \otimes \underline{b^+b})\right]$$
$$+ \frac{1}{8\pi^2}\int d^4x\ \ln\det\left[-\tfrac{1}{2} f\,\underline{b^+b}\right] \Box^2 \ln\det\left[-\tfrac{1}{2} f\,\underline{b^+b}\right] + \tilde{k}\cdot c \ . \qquad (11.b)$$

Here c is an unknown constant and M is the conformal invariant matrix first introduced in [4] and given by

$$M^{-1} = \frac{1}{4}\ \underline{a^+(1)\,a(1)} \otimes \underline{b^+(2)\,b(2)} + \frac{1}{4}\ \underline{b^+(1)\,b(1)} \otimes \underline{a^+(2)\,a(2)}$$
$$-\ \underline{a^+(1)\,b(1) \otimes a^+(2)\,b(2)} \ . \qquad (11.c)$$

III. The one - instanton case

We now evaluate explicitly the one - instanton case, using (2), (3) with f replaced by $\tilde{f}$, and (7), (8). We only consider the SU(2) one - instanton solution, the SU(n) one - instanton solution can be obtained by trivial imbedding.

The SU(2) one - instanton solution is given by

$$A_\mu = \frac{1}{2}\,\bar{\sigma}_{\mu\nu}\,\partial_\nu \ln \rho \ , \qquad \rho = 1 + \frac{\lambda^2}{(x-y)^2}$$

and $\bar{\sigma}_{\mu\nu} = e^+_\mu e_\nu - \delta_{\mu\nu}$. y is the position and $\lambda > 0$ the scale size of the instanton. The corresponding instanton parameter matrices of the ADHM construction are

$$a_{A'} = \begin{pmatrix} \lambda\,\delta_{1A'} \\ \lambda\,\delta_{2A'} \\ -y_{1A'} \\ -y_{2A'} \end{pmatrix} , \qquad b^A = \begin{pmatrix} 0 \\ 0 \\ \delta^{1A} \\ \delta^{2A} \end{pmatrix} . \tag{12.a}$$

The indices of y refer to it's quaternionic representation. The matrices $\Delta_{A'} = a_{A'} + b^A x_{AA'}$ are

$$\Delta_{A'} = \begin{pmatrix} \lambda\,\delta_{1A'} \\ \lambda\,\delta_{2A'} \\ (x-y)_{1A'} \\ (x-y)_{2A'} \end{pmatrix} , \tag{12.b}$$

and it follows $f^{-1} = \lambda^2 + (x-y)^2$. In the following we exploit euclidean covariance and choose $y = o$. The full formulae are obtained by the substitution $x \longrightarrow x - y$.

A convenient basis for the matrices (c_r, d_r), defined by (9) is

$$c_r = \begin{pmatrix} \delta_{r1} \\ \delta_{r2} \\ \delta_{r3} \\ \delta_{r4} \end{pmatrix} \quad \text{and} \quad d_r = \begin{pmatrix} -\delta_{r2} \\ \delta_{r1} \\ -\delta_{r4} \\ \delta_{r3} \end{pmatrix} . \tag{13}$$

Using formulae (8), (1o) it is not difficult to compute $\tilde{f}$ and $\underbrace{\tilde{b}^+\tilde{b}}$. The result is convenietly expressed in the form of 2 x 2 matrices of quaternions:

$$\tilde{f} = 2\,f^2 \begin{pmatrix} \lambda^2 + 2x^2 & -\lambda x^+ \\ -\lambda x & 2\lambda^2 + x^2 \end{pmatrix} , \tag{14.a}$$

$$\tilde{f}^{-1} = \frac{1}{4} \begin{pmatrix} 2\lambda^2 + x^2 & \lambda x^+ \\ \lambda x & \lambda^2 + 2x^2 \end{pmatrix} , \tag{14.b}$$

$$\underbrace{\tilde{b}^+ b} = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} . \tag{14.c}$$

Remember that the gauge field corresponding to $\tilde{f}$ has topological charge $\tilde{k} = 4$.

We use these formulae to calculate the integrals in (2). $\tilde{K}$ is obtained from (3.c). After some calculation the traces (cf. (3.a), (3.b)) become

$$\tilde{I}_1 = \frac{64x^{12} + 352\lambda^2 x^{10} + 784\lambda^4 x^8 + 896\lambda^6 x^6 + 544\lambda^8 x^4 + 160\lambda^{10} x^2 + 16\lambda^{12}}{(x^2 + \lambda^2)^8} - \frac{80x^4 + 240\lambda^2 x^2 + 170\lambda^4}{(x^2 + \lambda^2)^4} + \frac{16}{(1 + x^2)^2}$$

and

$$\tilde{I}_2 = (c_5 x^{10} + c_4 x^8 + c_3 x^6 + c_2 x^4 + c_1 x^2 + c_o) \cdot \tilde{N}^{-1} ,$$

where

$$\begin{aligned} c_5 &= 4\lambda^6 t^4 , \\ c_4 &= 4\lambda^8 t^4 (-t^2 + t + 5) , \\ c_3 &= \lambda^{10} t^4 (t^4 - 2t^3 - 15t^2 + 16t + 40) , \\ c_2 &= \lambda^{12} t^4 (3t^4 - 6t^3 - 21t^2 + 24t + 40) , \\ c_1 &= \lambda^{14} t^4 (3t^4 - 6t^3 - 13t^2 + 16t + 20) , \\ c_o &= \lambda^{16} t^4 (t^4 - 2t^3 - 3t^2 + 4t + 4) , \end{aligned}$$

$$\tilde{N} = \frac{2}{9}\,(x^2+\lambda^2)^5(x^2+z^2)^5 \quad ,$$

$$z^2 = \frac{1}{2}\,\lambda^2\,(1+t)(2-t) \quad .$$

Doing even more calculations the integrals are obtained to be

$$\frac{1}{24\pi^2}\int d^4x\,\tilde{I}_1 \;=\; -\frac{181}{72} + \frac{4}{3}\,\ln\lambda \quad , \tag{15.a}$$

$$\frac{1}{24\pi^2}\int d^4x\int_0^1 dt\,\tilde{I}_2 \;=\; 2\,\ln 2 \;-\; \frac{11}{8} \quad . \tag{15.b}$$

It is amazing to note that in this formulation the 5 - dimensional integral already contributes to the one - instanton fluctuations. Collecting all contributions we arrive at

$$\tilde{\Gamma} = -\frac{8}{3}\ln\mu - 16\,\zeta'(-1) - \frac{2}{3}\ln 2 + \frac{4}{3}\ln\lambda - \frac{20}{9} \quad . \tag{16}$$

This result is consistent with previous literature [7]. It is an easy matter to evaluate (11) for the one - instanton case. Comparison with (16) determines the constant c to be

$$c = 0 \quad . \tag{17}$$

IV. The 't Hooft solution

The evaluation of $\tilde{\Gamma}$ using (7) - (10) is straightforward but very cumbersome as is already seen for the 't Hooft solution

$$A_\mu = \frac{1}{2}\,\bar{\sigma}_{\mu\nu}\,\partial_\nu \ln\rho \quad , \qquad \rho = 1 + \sum_{i=1}^{k}\frac{\lambda_i^2}{(x-y^i)^2}$$

in the two - instanton case (k = 2).

Then

$$a_{A'} = \begin{pmatrix} \lambda_1\delta_{1A'} & \lambda_2\delta_{1A'} \\ \lambda_1\delta_{2A'} & \lambda_2\delta_{2A'} \\ -y^1_{1A'} & 0 \\ -y^1_{2A'} & 0 \\ 0 & -y^2_{1A'} \\ 0 & -y^2_{2A'} \end{pmatrix} \quad , \quad b^A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \delta^{1A} & 0 \\ \delta^{2A} & 0 \\ 0 & \delta^{1A} \\ 0 & \delta^{2A} \end{pmatrix} \quad . \tag{18}$$

From (9) follows

$$c = \begin{pmatrix} \gamma_1 & \gamma_3 \\ \gamma_2 & \gamma_4 \\ \gamma_5 & u_1 \\ \gamma_6 & u_2 \\ u_1 & \gamma_7 \\ u_2 & \gamma_8 \end{pmatrix} \quad , \quad d = \begin{pmatrix} -\gamma_2 & -\gamma_4 \\ \gamma_1 & \gamma_3 \\ -\gamma_6 & -u_2 \\ \gamma_5 & u_1 \\ -u_2 & -\gamma_8 \\ u_1 & \gamma_7 \end{pmatrix} \tag{19.a}$$

with $u_A = \frac{1}{(y^1-y^2)^2}\left\{ (y^1-y^2)_{A1'}\,(\lambda_1\gamma_3-\lambda_2\gamma_1) + (y^1-y^2)_{A2'}\,(\lambda_1\gamma_4-\lambda_2\gamma_2) \right\}$ (19.b)

and arbitrary γ_i. Setting $\gamma_i = \delta_{is}$ we get a basis of matrices c_s and d_s in which the 8 x 8 matrices $\tilde{f}$, $\tilde{f}^{-1}$ and $\underbrace{\tilde{b}^+\tilde{b}}$ can be expressed as 4 x 4 matrices of quaternions. The results are stated in appendix II. They are much more complicated than the corresponding f, f^{-1} in the fundamental representation.

For the evaluation of $\tilde{\Gamma}$ in the multi instanton case formula (11) is more convenient. Casting b^A into normal form, i.e. $\underbrace{b^+b} = -2$, we have (cf. formula (5.16) of [5] or (73) of I)

$$\Gamma = -k \left\{ \frac{2}{3} \ln \mu + 4 \zeta'(-1) + \ln 2 \right\}$$

$$+ \frac{1}{3} \ln \det M_S^{-1} + \frac{1}{96\pi^2} \int d^4x \ln \det f^{-1} \Box^2 \ln \det f^{-1} \qquad (20.a)$$

(where $\det M_S^{-1}$ is given by (26)) and

$$\tilde{\Gamma} = 4\Gamma + 2 \ln \det M^{-1} + \frac{1}{8\pi^2} \int d^4x \ln \det f^{-1} \Box^2 \ln \det f^{-1} . \qquad (20.b)$$

For the k - instanton 't Hooft solution we generalize (18) and find

$$f^{-1}_{ij} = \lambda_i \lambda_j + (x - y^i)^2 \delta_{ij} , \qquad i,j = 1,\dots,k \qquad (21)$$

and $$M^{-1}_{il,jm} = \lambda_i \lambda_j \delta_{lm} + \lambda_l \lambda_m \delta_{ij} + (y^i - y^l)^2 \delta_{ij} \delta_{lm} , \quad i,j,l,m = 1,\dots,k \qquad (22)$$

Noting that $\hat{f}^{-1}_{ij} \overset{def}{=} \frac{\lambda_i}{\sqrt{(x-y^i)^2}} \cdot \frac{\lambda_j}{\sqrt{(x-y^j)^2}} + \delta_{ij}$

is a unit matrix plus a matrix of rank 1 with

$$\det \hat{f}^{-1} = \rho$$

we have $$\det f^{-1} = \rho \prod_{i=1}^{k} (x - y^i)^2 = \tilde{\rho} . \qquad (23)$$

M^{-1} can be factorised into one part M_S^{-1} which acts only on the symmetric part of the tensor product space and another part M_A^{-1} which acts on the antisymmetric part of the tensor product space.

$$M^{-1}_{S\, il,jm} = M^{-1}_{il,jm} + M^{-1}_{il,mj} - M^{-1}_{il,jj} \delta_{jm} , \qquad i \geq l ,\ j \geq m \qquad (24.a)$$

$$M^{-1}_{A\, il,jm} = M^{-1}_{il,jm} - M^{-1}_{il,mj} , \qquad i > l ,\ j > m \qquad (24.b)$$

and $$\det M^{-1} = \det M_S^{-1} \cdot \det M_A^{-1} . \qquad (24.c)$$

Inserting (22) yields

$$M^{-1}_{S\, il,jm} = (1-\delta_{il})(1-\delta_{jm})[\lambda_i\lambda_j\delta_{lm} + \lambda_l\lambda_m\delta_{ij} + \lambda_i\lambda_m\delta_{lj} + \lambda_l\lambda_j\delta_{im} + (y^i-y^l)^2\delta_{ij}\delta_{lm}]$$
$$+ (1-\delta_{il})\delta_{jm}[\lambda_i\lambda_j\delta_{lj} + \lambda_l\lambda_j\delta_{ij}] + \delta_{il}\delta_{jm}[2\lambda_i^2\delta_{ij}]$$
$$+ \delta_{il}(1-\delta_{jm})[2\lambda_i\lambda_j\delta_{im} + 2\lambda_i\lambda_m\delta_{ij}] ,$$
$$k \geq i \geq l \geq 1 , \quad k \geq j \geq m \geq 1 , \qquad (25.a)$$

$$M^{-1}_{A\, il,jm} = \lambda_i\lambda_j\delta_{lm} + \lambda_l\lambda_m\delta_{ij} - \lambda_i\lambda_m\delta_{lj} - \lambda_l\lambda_j\delta_{im} + (y^i-y^l)^2\delta_{ij}\delta_{lm} ,$$
$$k \geq i > l \geq 1 , \quad k \geq j > m \geq 1 . \qquad (25.b)$$

By the elementary row transformation for $i > 1$

$$\text{row } il \longrightarrow \text{row } il - \frac{\lambda_l}{2\lambda_i} \text{row } ii - \frac{\lambda_i}{2\lambda_l} \text{row } ll$$

M_S^{-1} is brought to triangular form and we read off

$$\det M_S^{-1} = 2^k \prod_{i=1}^{k} \lambda_i^2 \prod_{r>s} (y^r - y^s)^2 . \qquad (26)$$

The computation of $\det M_A^{-1}$ is not so easy. For a complete result see appendix I. Here we only give a formula, which is exact for $k \leq 3$. For $k > 3$ it describes the leading behaviour when the instantons are far apart from each other compared to their scale size λ_i.

$$\det M_A^{-1} = R(k) \prod_{r>s} (y^r - y^s)^2 \qquad (27.a)$$

with $$R(k) = 1 + \frac{1}{2} \sum_{i,j=1}^{k}{}' \frac{\lambda_i^2 + \lambda_j^2}{(y^i - y^j)^2} + \frac{1}{2} \sum_{i,j,l=1}^{k}{}' \frac{(\lambda_i^2 + \lambda_j^2 + \lambda_l^2)\, \lambda_j^2}{(y^i - y^j)^2 (y^j - y^l)^2}$$
$$+ \frac{1}{8} \sum_{i,j,l,m=1}^{k}{}' \frac{(\lambda_i^2 + \lambda_j^2)(\lambda_l^2 + \lambda_m)^2}{(y^i - y^j)^2 (y^l - y^m)^2} + \cdots , \qquad (27.b)$$

The prime on the multiple summations means that terms where two indices are equal should be omitted. Note that n-fold sums only contribute for $k \geq n$.

Collecting our results we arrive at

$$\Gamma = -k\left[\frac{2}{3}\ln\mu + 4\zeta'(-1) + \frac{2}{3}\ln 2\right] + \frac{2}{3}\sum_{i=1}^{k}\ln\lambda_i + \frac{1}{3}\sum_{r>s}\ln(y^r-y^s)^2 + \frac{1}{96\pi^2}\int d^4x\,\ln\tilde{g}\;\Box^2\ln\tilde{g} \quad , \qquad (28.a)$$

$$\tilde{\Gamma} = 4\Gamma + \frac{1}{8\pi^2}\int d^4x\,\ln\tilde{g}\;\Box^2\ln\tilde{g} + 2k\ln 2 + 4\sum_{i=1}^{k}\ln\lambda_i + 4\sum_{r>s}\ln(y^r-y^s)^2 + 2\ln R(k)$$

$$= -4k\left[\frac{2}{3}\ln\mu + 4\zeta'(-1) + \frac{1}{6}\ln 2\right] + \frac{20}{3}\sum_{i=1}^{k}\ln\lambda_i + \frac{16}{3}\sum_{r>s}\ln(y^r-y^s)^2 + \frac{1}{6\pi^2}\int d^4x\,\ln\tilde{g}\;\Box^2\ln\tilde{g} + 2\ln R(k) \quad . \qquad (28.b)$$

These formulae complete the early work of Brown and Creamer [8].

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Appendix I. Evaluation of $\det M_A^{-1}$

From $\quad M_A^{-1}(k)_{il,jm} = \lambda_i\lambda_j\delta_{lm} + \lambda_l\lambda_m\delta_{ij} - \lambda_i\lambda_m\delta_{lj} - \lambda_l\lambda_j\delta_{im} + (y^i - y^l)\delta_{ij}\delta_{lm}$,

$$k \geq i > l \geq 1 \;, \quad k \geq j > m \geq 1 \;,$$

we see

1) $\mathrm{Det}M_A^{-1}(k)$ is a polynomial in $(y^r - y^s)^2$ and λ_i^2 which is symmetric under the interchange of any two indices. Each term of the polynomial contains n factors $(y^r - y^s)^2$ and $\hat{k} - n$ factors λ_i^2 with $\hat{k} = \frac{1}{2}k(k-1)$ the dimension of the tensor product space on which $M_A^{-1}(k)$ acts.

2) If all $(y^r - y^s)^2$ vanish, the rank of M_A^{-1} is $k-1$, as one sees by choosing as an orthogonal basis $\omega^{(r,s)}$ for the tensor product space on which M_A^{-1} acts

$$\omega^{(r,s)}_{jm} = \lambda_j^{(r)}\lambda_m^{(s)} - \lambda_m^{(r)}\lambda_j^{(s)} \quad , \qquad r > s \;,$$

with $\quad \sum_{j=1}^{k}\lambda_j^{(r)}\lambda_j^{(s)} = \delta^{rs}\sum_{j=1}^{k}(\lambda_j^{(r)})^2 \quad$ and $\quad \lambda_j^{(1)} = \lambda_j$.

Thus for $\det M_A^{-1}(k)$ to be nonvanishing, it is necessary that at least $\hat{k} - (k-1)$ different $(y^r - y^s)^2$ are unequal zero. It follows that each term in the polynomial must contain at least $\hat{k} - (k-1)$ factors $(y^r - y^s)^2$, i.e. at most $k-1$ factors λ_i^2.

Now for $\lambda_k = 0$ $\det M_A^{-1}(k)$ factorizes according to

$$\underbrace{\hat{k}\left\{\begin{array}{|c|}\hline \\ M_A^{-1}(k)\Big|_{\lambda_k=0} \\ \\ \hline\end{array}\right.}_{\hat{k}} \qquad \underbrace{\begin{array}{|c|}\hline M_A^{-1}(k-1) \\ \hline 0 \\ \hline\end{array}}_{\widehat{k-1}}\underbrace{\left.\begin{array}{c|}\hline 0 \\ \hline \rho^{-1}(y^k,k-1) \\ \hline\end{array}\right.}_{k-1}\begin{array}{l}\}\ \widehat{k-1} \\ \}\ k-1\end{array}$$

which means, see (23):

$$det\ M_A^{-1}(k) = det\ M_A^{-1}(k-1)\cdot\tilde{g}(y^k, k-1) + \lambda_k^2\cdot g$$

with unknown g. Because of the symmetry property 1) we can repeat this procedure for any λ_i, $i = 1,\dots,k-1$. Comparing the different expressions for $\det M_A^{-1}(k)$ found in this way, we see that $\det M_A^{-1}(k)$ is determined up to a term of the form $\hat{g}\cdot\prod_{i=1}^{k}\lambda_i^2$. But this term must vanish because it contains too many factors of λ_i^2 according to 2).

Setting $$det\ M_A^{-1}(k) = R(k)\cdot\prod_{i>j}(y^i - y^j)^2$$

we find that R(k) is determined by the following requirements:

1) R(k) is a dimensionless polynomial in λ_i^2 and $(y^r - y^s)^{-2}$ which is symmetric under the interchange of any two indices. The highest power of λ_i^2 is $k-1$.

2) $$R(k)\Big|_{\lambda_k=0} = R(k-1)\cdot g(y^k, k-1) = R(k-1)\left\{1 + \sum_{j=1}^{k-1}\frac{\lambda_j^2}{(y^k - y^j)^2}\right\}.$$

3) $$R(1) = 1 .$$

The solution is conveniently stated in graph theoretical language [9].

R(k) can be expressed as the sum of all possible unoriented graphs with k vertices (labelled $r = 1,\dots,k$) and an arbitrary number of links, such that no closed pathes are formed. There is no restriction on the number of links attached to one vertex, but the total number of links in a graph is less than k due to the absense of closed loops. The graphs are evaluated according to the following rules:

- each vertex r gives a factor λ_r^{-2},
- each link (connecting vertex r with vertex s) gives a factor $\lambda_r^2\,\lambda_s^2(y^r - y^s)^{-2}$,
- each set S of connected vertices gives a factor $\sum_{i\in S}\lambda_i^2$.

Immediate consequences of these rules are

- each isolated vertex gives just a factor 1,
- each vertex r with $n \geq 1$ links attached to it gives a factor $\lambda_r^{2(n-1)}$,
- the contribution of a connected part of a graph is dimensionless.

One easily sees that requirements 1) and 3) are met. Realising that for $\lambda_k = 0$ only those graphs give a nonvanishing contribution, which have not more than one link attached to vertex k, one also sees that requirement 2) is fulfilled.

The formula (27.b) given in the text was gained by adding all graphs with not more than two links. The numerical factors in front of the sums compensate double counting.

Appendix II. The matrix elements of $\tilde{f}$, $\tilde{f}^{-1}$, and $\underbrace{\tilde{b}^{+}\tilde{b}}$ in the two - instanton case

We choose our basis of matrices c_r, d_r (cf. (19)) in such a way that the matrix elements of $\tilde{f}$, $\tilde{f}^{-1}$ and $\underbrace{\tilde{b}^{+}\tilde{b}}$ have some useful relations among each other:

1) $\tilde{f}$, $\tilde{f}^{-1}$, $\underbrace{\tilde{b}^{+}\tilde{b}}$ are selfadjoint (for all choices of c_r, d_r).

2) They can be expressed as 4 x 4 matrices of quaternions and we use the convention

$$x^{+} \cdot x = x \cdot x^{+} = x_\mu x_\mu 1_2 = x^2 \quad .$$

3) Let m be one of the matrices $\tilde{f}$, $\tilde{f}^{-1}$, $\underbrace{\tilde{b}^{+}\tilde{b}}$. Then the quaternionic matrix elements

$$m\,(\lambda_1, \lambda_2, y^1, y^2)_{ij} \; , \qquad i,j = 1, \dots, 4 \; ,$$

fulfill

$$m\,(\lambda_1, \lambda_2, y^1, y^2)_{2r,2s} = m\,(\lambda_2, \lambda_1, y^2, y^1)_{2r-1,2s-1} \; ,$$

$$m\,(\lambda_1, \lambda_2, y^1, y^2)_{2r,2s-1} = m\,(\lambda_2, \lambda_1, y^2, y^1)_{2r-1,2s}$$

for r = 1,2 ; s = 1,2 .

Therefore it is sufficient that here we only list the quaternionic matrix elements m_{11}, m_{12}, m_{13}, m_{14}, m_{33}, m_{34}. Using the following abbreviations

$$\tilde{\varrho} = (x-y^1)^2 (x-y^2)^2 + \lambda_1^2 (x-y^2)^2 + \lambda_2^2 (x-y^1)^2 \; ,$$

$$\sigma = (y^1-y^2)^2 + 2\lambda_1^2 + 2\lambda_2^2 \; ,$$

$$\tau = (y^1-y^2)^2 + \lambda_1^2 + \lambda_2^2 \; ,$$

we obtain

1. Matrix elements of $\tilde{f}$:

$$\begin{aligned} \sigma^2 \tilde{\varrho}^2 \tilde{f}_{11} = \; & 2\lambda_2^2 (\tilde{\varrho} + \lambda_1^2\lambda_2^2)(y^1-y^2)^2 \left[(y^1-y^2)^2 - 2(x-y^1)^2 - 2(x-y^2)^2 \right] \\ & + 2\sigma (y^1-y^2)^2 (\lambda_2^2 + (x-y^2)^2) \left[\tilde{\varrho} + (x-y^1)^2 (x-y^2)^2 \right] \\ & + 4(\sigma - 2\lambda_2^2)\, \tilde{\varrho} \left[\tilde{\varrho} - (x-y^1)^2 (x-y^2)^2 \right] \\ & + 4\lambda_1^2 \sigma (x-y^1)^2 (x-y^2)^4 \; , \end{aligned}$$

$$\begin{aligned} \sigma^2 \tilde{\varrho}^2 \tilde{f}_{12} = & -2\lambda_1\lambda_2 (\tilde{\varrho} + \lambda_1^2\lambda_2^2)(y^1-y^2)^2 \left[(y^1-y^2)^2 - 2(x-y^1)^2 - 2(x-y^2)^2 \right] \\ & - 2\lambda_1\lambda_2 \sigma \tilde{\varrho} (y^1-y^2)^2 + 8\lambda_1\lambda_2 \tilde{\varrho} \left[\tilde{\varrho} - (x-y^1)^2 (x-y^2)^2 \right] \\ & - 4\lambda_1\lambda_2 \sigma (x-y^1)^2 (x-y^2)^2 (x-y^2)^{+} (x-y^1) \; , \end{aligned}$$

$$\begin{aligned} \sigma \tilde{\varrho}^2 \tilde{f}_{13} = & -2\lambda_1 (y^1-y^2)^2 (\lambda_2^2 + (x-y^2)^2)^2 (x-y^1)^{+} \\ & - 4\lambda_1^3 (x-y^2)^4 (x-y^1)^{+} + 4\lambda_1^3 \lambda_2^2 (x-y^2)^2 (y^1-y^2)^{+} \; , \end{aligned}$$

$$\sigma \tilde{g}^2 \tilde{f}_{14} = 2\lambda_1\lambda_2^2 (y^1-y^2)^2 (\lambda_1^2 + 2(x-y^1)^2)(x-y^2)^+$$
$$- 4\lambda_1\lambda_2^2 (x-y^1)^4 (x-y^2)^+ + 4\lambda_1\lambda_2^4 (x-y^1)^2 (y^1-y^2)^+ \ ,$$

$$\tilde{g}^2 \tilde{f}_{33} = 2(\lambda_2^2 + (x-y^2)^2)(\tilde{g} + \lambda_1^2 (x-y^2)^2) \ ,$$

$$\tilde{g}^2 \tilde{f}_{34} = -2\lambda_1^2\lambda_2^2 (x-y^1)(x-y^2)^+ \ .$$

2. Matrix elements of $\tilde{f}^{-1}$ (Note that $\tilde{f}^{-1}$ is indeed a quadratic polynomial in x):

$$4\tau (y^1-y^2)^2 \tilde{f}^{-1}_{11} = (\sigma-\lambda_1^2)(\sigma-2\lambda_1^2)(x-y^1)^2 + 2\lambda_1^2\lambda_2^2 (x-y^2)^2$$
$$+ 2\lambda_1^2 (\sigma-\lambda_1^2)(y^1-y^2)^2 + 4\lambda_1^2\lambda_2^4 \ ,$$

$$4\tau (y^1-y^2)^2 \tilde{f}^{-1}_{12} = -\lambda_1\lambda_2 (\sigma-2\lambda_1^2)(x-y^1)^2 - \lambda_1\lambda_2(\sigma-2\lambda_2^2)(x-y^2)^2$$
$$+ \lambda_1\lambda_2 (y^1-y^2)^4 - 4\lambda_1^3\lambda_2^3 - \lambda_1\lambda_2\,\sigma\,(x-y^2)^+(x-y^1) \ ,$$

$$4\tau (y^1-y^2)^2 \tilde{f}^{-1}_{13} = \lambda_1(\sigma-\lambda_1^2)(y^1-y^2)^2(x-y^1)^+ - 2\lambda_1^3\lambda_2^2 (y^1-y^2)^+ \ ,$$

$$4\tau (y^1-y^2)^2 \tilde{f}^{-1}_{14} = -\lambda_1\lambda_2^2 (y^1-y^2)^2 (x-y^1)^+ - 2\lambda_1\lambda_2^2(\tau-\lambda_1^2)(y^1-y^2)^+ \ ,$$

$$4\tau\, \tilde{f}^{-1}_{33} = 2\tau (x-y^1)^2 + \lambda_1^2(\tau-\lambda_2^2) \ ,$$

$$4\tau\, \tilde{f}^{-1}_{34} = \lambda_1^2\lambda_2^2 \ .$$

3. Matrix elements of $\underbracket{\tilde{b}^+\tilde{b}}$:

$$2\tau (y^1-y^2)^2\, \underbracket{\tilde{b}^+\tilde{b}}_{11} = -\sigma^2 + \lambda_1^2 (3\sigma - 2\lambda_1^2 - 2\lambda_2^2) \ ,$$

$$2\tau (y^1-y^2)^2\, \underbracket{\tilde{b}^+\tilde{b}}_{12} = \lambda_1\lambda_2 (3\sigma - 2\lambda_1^2 - 2\lambda_2^2) \ ,$$

$$\underbracket{\tilde{b}^+\tilde{b}}_{33} = -1 \ ,$$

$$\underbracket{\tilde{b}^+\tilde{b}}_{13} = \underbracket{\tilde{b}^+\tilde{b}}_{14} = \underbracket{\tilde{b}^+\tilde{b}}_{34} = 0 \ .$$

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