DEUTSCHES ELEKTRONEN-SYNCHROTRON **DESY**

DESY 79/40
July 1979

COMPUTATION OF QUANTUM FLUCTUATIONS OF QUARK FIELDS IN AN ARBITRARY YANG-MILLIAN INSTANTON BACKGROUND

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Abstract

We explicitly compute the determinant of the Dirac operator in a general multi-instanton background field.

1. Introduction

Recent studies [1)2)] in two-dimensional non-linear σ-models suggest [3)] that the Yang-Mills instanton gas is dense and that the infrared divergencies present in the dilute gas approximation disappear, when this is properly taken into account. To give this intriguing idea substance, one has to include dense multi-instanton configurations in the semi-classical treatment of the functional integral and is thus led to the problem to compute the determinant of the fluctuation operator in a general multi-instanton background field. We here derive a formula for the determinant of the corresponding Dirac operator, hoping that the determinant of the gluon fluctuation operator can then be computed in a comparatively simple step exploiting supersymmetry [4)].

We use the same general method to calculate determinants as in the two-dimensional CP^{n-1} models [2)]. Thus, we first compute the variation $\delta\Gamma$ of

$$\Gamma = \ln \det' \not{D} \; ; \qquad \not{D} : \text{Dirac operator}$$

with respect to the parameters of the instanton background field. This is possible, because

$$\delta\Gamma = \mathrm{Tr}\{\delta\not{A}\, S\},$$

S being the known [5)6)] Green's function of $\not{D}$. Integrating $\delta\Gamma$, one obtains Γ up to an integration constant, which is finally computed by considering an especially simple instanton configuration, where all the eigenvalues of $\not{D}$ can be calculated explicitly. Of course, to make Γ well defined, its ultra-

violet and infrared divergencies must be regularized. This will be achieved by introducing an appropriate set of Pauli-Villars regulator fields and by projecting the Dirac operator onto a spherical space-time with adjustable world radius R.

Since the Green's function S is already known, the computation of $\delta\Gamma$ is straightforward, except that the projection onto a spherical space-time causes some calculations to be rather lengthy. The main difficulty, however, is to integrate $\delta\Gamma$, a step, which involves a lot of guesswork. For the special case of 't Hooft's instanton solutions [8], this problem simplifies and it has been solved by Brown & Creamer [7].

Our article is organized as follows. To fix notations, the general instanton solution due to Atiyah, Hitchin, Drinfeld and Manin [9)10)] for the gauge group Sp(r) is reviewed in sect. 2. These particular gauge groups are chosen for technical convenience. In fact, no serious limitations arise from this choice, since Sp(r) contains an SU(r) subgroup (also, Sp(1) = SU(2)). In sect. 3 we define the Dirac operator on S^4, discuss its zero modes and the Green's function S. $\delta\Gamma$ is computed subsequently (sect. 4) and integrated in sect. 5. Finally, the integration constant is evaluated in sect. 6 and our results are summarized and commented in the concluding sect. 7.

2. Description of the general Sp(r) instanton solutions [9)10)5)6)]

Sp(r) can be considered to be the subgroup of U(n), n = 2r, of all those unitary

matrices u with [+)]

$$u^{+} = J u^{T} J^{T} , \tag{1}$$

where J is the anti-symmetric n x n-matrix

$$J = \begin{vmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{vmatrix} \tag{2}$$

Correspondingly, Sp(r) gauge potentials A_μ are special U(n) gauge fields, i.e. they are anti-hermitian nxn-matrices having the additional reality property

$$A_\mu^{+} = J A_\mu^{T} J^{T} \tag{3}$$

In particular, Sp(r) instanton solutions are special cases of U(n) instanton fields. To explicitly describe these, we need some spinor algebra.

(Multi-) spinors are tensors $\xi_{A_1 \dots A_\ell , A'_1 \dots A'_m}$ with $A_i = 1,2$ and $A'_j = 1,2$.

[+)] T and + denote transposition and hermitian conjugation, respectively. Greek indices $\mu, \nu, \dots$ run from 0 to 3, capital Latin indices (spinor indices) A,B,... A',B'... from 1 to 2. The totally anti-symmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ is normalized such that $\varepsilon_{0123} = +1$ and repeated indices are always summed over.

They carry a representation of the spin covering SU(2) x SU(2) of the euclidean Lorentz group SO(4):

$$[(u,v)\cdot\xi]_{A_1\dots A_\ell, A'_1\dots A'_m} = u_{A_1B_1}\dots u_{A_\ell B_\ell}\,\bar{v}_{A'_1B'_1}\dots\bar{v}_{A'_mB'_m}\,\xi_{B_1\dots B_\ell, B'_1\dots B'_m}$$

(4)

$$(u,v)\in SU(2)\times SU(2)$$

($\bar{v}_{A'B'}$ is the complex conjugate of $v_{A'B'}$). Spinor indices can be raised and lowered, for example

$$\xi^A = \xi_B\,\epsilon^{BA}\ ;\quad \xi_A = \epsilon_{AB}\,\xi^B$$

(5)

$$\epsilon_{AB} = \epsilon^{AB} = -\epsilon_{BA}\ ;\quad \epsilon_{12} = 1$$

Contraction of an upper and a lower index of the same type (i.e. primed or unprimed) is an SU(2) x SU(2) invariant operation. Note that

$$\xi_A\,\eta^A = -\,\xi^A\eta_A$$

The adjoint ξ^+_A of a spinor ξ_A is defined by

$$(6)\qquad \xi^+_1 = -\,\bar{\xi}_2\ ;\quad \xi^+_2 = \bar{\xi}_1$$

(analogous formulae are understood for spinors $\eta_{A'}$). The merit of this construction is that the adjoint spinors transform in the same way under SU(2) x SU(2) as the original spinors. In particular,

$$(7)\qquad (\xi,\eta) = \xi^+_A\,\eta^A$$

is an invariant scalar product.

The spinor components $x_{AA'}$ of the position vector x_μ are defined by

$$x_{AA'} = x_\mu (e_\mu)_{AA'}$$

(8)

$$e_0 = \mathbb{1} \; ; \quad e_a = -i\sigma^a$$

where σ^a , a = 1,2,3, are the Pauli matrices. The covariant matrices e_μ satisfy the following useful identities:

$$(9) \quad e_\mu^+ = \epsilon\, e_\mu^T\, \epsilon^T$$

$$(10) \quad Tr\{e_\mu^+ e_\nu\} = 2\delta_{\mu\nu} \; ; \quad (e_\mu)_{AA'}(e_\mu)_{BB'} = 2\epsilon_{AB}\epsilon_{A'B'}$$

Now we are well prepared to describe the U(n) gauge potentials A_μ giving rise to self-dual field strengths $F_{\mu\nu}$. They can be represented in the form

$$(11) \quad A_\mu = V^+ \partial_\mu V \; ; \quad V^+ V = 1$$

where v is an (n+2k) x n complex matrix. Here, k denotes the instanton number, viz.

$$(12) \quad k = -\frac{1}{16\pi^2}\int d^4x \; Tr\left(F_{\mu\nu}{}^*F_{\mu\nu}\right) \; ; \quad {}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$$

v is the solution matrix of a set of 2k complex linear equations

$$(13)\quad v^{+}\Delta_{A'} = 0 \qquad (A' = 1,2)$$

with $\Delta_{A'}$ a spinor of (n+2k) x k matrices depending linearly on x_{μ} :

$$(14)\quad \Delta_{A'} = a_{A'} + b^{A}x_{AA'}$$

$a_{A'}$ and b_{A} are constant (n+2k) x k matrices, which parametrize the instanton solution. They are not unconstrained, however, but must be chosen such that

$$(15)\quad \Delta^{+}_{A'}\Delta_{B'} = \epsilon_{A'B'}\,\ell^{-1} \qquad \text{for all } x.$$

Here, $\Delta^{+}_{A'}$ is the adjoint of $\Delta_{A'}$ in the sense of spinors (cp. eq. (6)), viz.

$$(16)\quad \Delta^{+}_{1} = -\bar{\Delta}^{T}_{2} \;; \quad \Delta^{+}_{2} = \bar{\Delta}^{T}_{1}$$

Noting

$$(17)\quad \Delta^{+}_{A'} = a^{+}_{A'} + b^{+A}x_{AA'} ,$$

it follows that eq. (15) holds if and only if

$$(18a)\quad a^{+}_{A'}a_{B'} + a^{+}_{B'}a_{A'} = 0$$

$$(18b)\quad a^{+}_{A'}b_{A} + b^{+}_{A}a_{A'} = 0$$

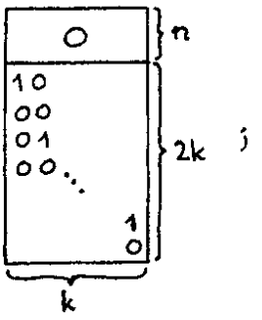
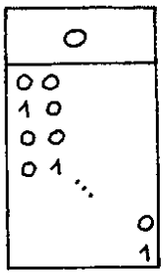
$$(18c)\quad b^{+}_{A}b_{B} + b^{+}_{B}b_{A} = 0$$

Thus, given a pair of parameter matrices $a_{A'}$, b_A satisfying these quadratic constraints, the gauge potential A_μ constructed via eqs. (13) and (11) has self-dual curvature. It will be non-singular provided the kxk matrix f^{-1} defined by eq. (15) is invertible for all x.

Two sets $(a_{A'}, b_A)$ and $(\hat{a}_{A'}, \hat{b}_A)$ of parameter matrices related by

$$a_{A'} = U\hat{a}_{A'}K \; ; \quad b_A = U\hat{b}_A K \qquad (U \in U(n+2k);\; K \in Gl(k,\mathbb{C}))$$

give rise to the same instanton solution. This invariance can be exploited to transform b_A to its "normal form":

(19) $b^1 =$  ; $b^2 =$ 

We shall henceforth assume that b_A is in its normal form.

U(n) instanton potentials A_μ, which are reducible to Sp(r), come from $a_{A'}$'s having the reality property (cp. eq. (2))

$$(20a) \qquad a^+_{A'} = a^T_{A'} J^T$$

The analogous equation for b_A,

$$(20b) \qquad b^+_A = b^T_A J^T \; ,$$

holds automatically, when b_A is in the normal form (19). Eqs. (20) imply in particular that f is a real symmetric kxk matrix for Sp(r) instantons.

3. The Dirac operator on S^4 and its Green's function

Analogous to the CP^{n-1} case [2], one could set up a manifestly O(5) covariant formalism to deal with Dirac fields and gauge potentials on S^4. We here prefer, however, to work with explicit stereographic coordinates x_μ instead. Thus, points r_α , $\alpha = 0,\dots,4$, $r_\alpha r_\alpha = R^2$, of S^4 are parametrized by x_μ according to

$$(21) \qquad r_\mu = \frac{2R^2 x_\mu}{R^2 + x^2} \quad ; \quad r_4 = R\,\frac{R^2 - x^2}{R^2 + x^2}$$

In these coordinates, the natural metric on S^4 is

$$(22) \qquad g_{\mu\nu} = \Omega\,\delta_{\mu\nu} \quad ; \quad \Omega = 4R^4 (R^2 + x^2)^{-2}$$

and the Dirac operator reads [+)]

$$(23) \qquad D = 2\,\Omega^{-1/2} i\gamma_\mu D_\mu \quad ; \quad D_\mu = \partial_\mu + A_\mu$$

[+)] Many authors prefer a normalization of Dirac fields different from ours, corresponding to a Dirac operator

$$\hat{D} = \Omega^{-3/4}\, D\, \Omega^{3/4} = 2\,\Omega^{-5/4} i\gamma_\mu D_\mu\, \Omega^{3/4}$$

Both formulations are completely equivalent of course.

The γ -matrices are defined by

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^+ & 0 \end{pmatrix}$$

The gauge potential A_μ is taken to be any Sp(r) k-instanton solution as described above. In particular, the Dirac fields, on which D acts, are in the quark representation of the color group. The factor of 2 in eq. (23) is added to insure that D formally approaches the usual flat space Dirac operator in the infinite volume limit $R \to \infty$.

A characteristic property of the Dirac operator is its off-diagonal shape

$$D = \begin{pmatrix} 0 & T^+ \\ T & 0 \end{pmatrix} \tag{24}$$

where

$$\begin{aligned} T &= 2\,\Omega^{-1/2}\, i\, e_\mu^+\, D_\mu \\ T^+ &= 2\,\Omega^{-1/2}\, i\, e_\mu\, D_\mu \end{aligned} \tag{25}$$

Writing $(e_\mu^+)^{A'A}$ for the matrix elements of e_μ, we see that T maps spinors ψ_A of negative chirality onto spinors $\chi^{A'}$ of positive chirality. These two spaces V_- and V_+ are equipped with the natural O(5) invariant scalar products

$$\begin{aligned} (\psi, \varphi) &= \int d^4x\; \Omega^{1/2}\; \psi_A^+\, \varphi^A \\ (\chi, \xi) &= \int d^4x\; \Omega^{1/2}\; \chi_{A'}^+\, \xi^{A'} \end{aligned} \tag{26}$$

(contraction of color indices is understood).

Correspondingly, T^+ is the adjoint of T mapping V_+ into V_-.

T^+ has no zero modes, but T has exactly k, viz. [5)11)]

$$\psi^i_{\alpha,A} = (v^+ b_A f)_{\alpha i} \tag{27}$$

Here, i = 1,...,k labels the solutions, α = 1,...,n is the color index and A = 1,2 the spinor index. The projector P_o onto the zero modes (27) can be written as an integral operator

$$(P_o \psi)_A(x) = \int d^4y \, \Omega(y)^{1/2} \, P_o(x,y)_{AB} \psi^B(y) \tag{28}$$

where

$$P_o(x,y)_{AB} = \sum_{i,j} \psi^i_A(x) \, N^{-1}_{ij} \, \psi^j_B(y)^+ \tag{29}$$

$$N_{ij} = (\psi^i, \psi^j)$$

Color indices are to be contracted appropriately, of course, and adjoints are taken in the spinor sense, eq. (6).

The Green's function S of the Dirac operator D is off-diagonal as well:

$$S = \begin{pmatrix} 0 & S_- \\ S_+ & 0 \end{pmatrix} \tag{30}$$

Thus, S_- maps V_+ into V_- and S_+ is the adjoint of S_-. The defining equations

for S_- are

$$(31) \qquad TS_- = 1 \; ; \quad P_o S_- = 0$$

More explicitly, defining an integral kernel $S_-(x,y)_{AA'}$ by

$$(32) \quad (S_-\psi)_A(x) = \int d^4y \; \Omega(y)^{1/2} \, S_-(x,y)_{AA'} \, \psi^{A'}(y)$$

eq. (31) reads

$$T^{A'A} \, S_-(x,y)_{AB'} = \Omega^{-1/2} \, \delta^{A'}_{B'} \, \delta(x-y)$$

$$(33) \qquad \int d^4y \; \Omega(y)^{1/2} \, P_o(x,y)_{AB} \, S_-(y,z)^{B}{}_{B'} = 0$$

The solution of these equations is not difficult to guess:

$$S_-(x,y)_{AA'} = \hat{S}_-(x,y)_{AA'} - \int d^4z \; \Omega(z)^{1/2} \, P_o(x,z)_{AB} \, \hat{S}_-(z,y)^{B}{}_{A'}$$

$$(34) \qquad \hat{S}_-(x,y)_{AA'} = \frac{i}{2} \, (e_\mu)_{AA'} \, D_\mu \, G(x,y)$$

Here, $G(x,y)$ denotes the known scalar Green's function [5)6)]

$$(35) \quad G(x,y) = \frac{V^+(x) \cdot V(y)}{4\pi^2 (x-y)^2} \; ; \qquad D_\mu D_\mu \, G(x-y) = -\delta(x-y)$$

The simple solution (34) can be traced back to the simple conformal transformation properties of the Dirac operator. In particular, we fear that

it would be much harder to find the Green's function for the gluon fluctuation operator on S^4.

4. Computation of the variation of the determinant of the Dirac operator

Because γ_5 anti-commutes with D, its non-zero eigenvalues come in pairs $(\lambda, -\lambda)$. The overall sign of det D is therefore ambiguous. We remove this ambiguity by defining

$$\Gamma = \ln \det' D \overset{def}{=} \frac{1}{2} \ln \det' D^2 \tag{36}$$

The prime indicates that the zero modes of D should be omitted. To regularize the UV divergence of Γ we now add Pauli-Villars regulator fields with large masses M_i and alternating "metric" e_i, such that

$$\sum_{i=1}^{\nu} e_i = -1 \; ; \quad \sum_{i=1}^{\nu} e_i M_i^{2p} = 0 \qquad (p = 1, \ldots, \nu - 1) \tag{37}$$

When the number ν of regulator fields is greater than 3, the regularized determinant

$$\Gamma_{reg} = \frac{1}{2} Tr \left\{ \ln (D^2 + P_o) + \sum_{i=1}^{\nu} e_i \ln (D^2 + M_i^2) \right\} \tag{38}$$

is finite and perfectly well-defined. Eq. (38) can be given a somewhat simpler form noting that

$$D^2 = \begin{pmatrix} T^+ T & 0 \\ 0 & T T^+ \end{pmatrix}$$

and that the non-zero eigenvalues of T^+T and TT^+ are identical:

$$(39)\quad \Gamma_{reg} = k \sum_{i=1}^{\nu} e_i \ln M_i + Tr\{\ln(TT^+) + \sum_{i=1}^{\nu} e_i \ln(TT^+ + M_i^2)\}$$

The trace here is to be taken in the space V_+ of normalizable spinor fields $\chi_{A'}$.

We now consider a curve $A_\mu(x,s)$, $0 \leq s \leq 1$, of instanton solutions with $A_\mu(x,0) = A_\mu(x)$. Our aim is, to compute

$$\delta\Gamma_{reg} = \frac{d}{ds}\,\Gamma_{reg}\Big|_{s=0}$$

From eq. (39) it follows that

$$\delta\Gamma_{reg} = Tr\left\{\delta T\left[T^+(TT^+)^{-1} + \sum_{i=1}^{\nu} e_i\, T^+(TT^+ + M_i^2)^{-1}\right]\right\} + c.c.$$

where c.c. means complex conjugate. Noting $T^+(TT^+)^{-1} = S_-$ and defining $S_-^M(x,y)_{AA'}$ to be the integral kernel of $T^+(TT^+ + M^2)^{-1}$, this becomes

$$\delta\Gamma_{reg} = \int d^4y\, Tr_{color}\Big\{ 2i\,(e_\mu^+)^{A'A}\,\delta A_\mu(y) \times$$

(40)

$$\times\Big[S_-(x,y)_{AA'} + \sum_{i=1}^{\nu} e_i\, S_-^{M_i}(x,y)_{AA'}\Big]_{x=y}\Big\} + c.c.$$

Thus, to compute $\delta\Gamma_{reg}$ we must study the short distance behaviour of S_- and S_-^M. From the explicit form eqs. (34), (35) of S_- one derives that up to terms vanishing at $x = y$

$$S_-(x,y)_{AA'} = \frac{i}{2}(e_\mu)_{AA'}\Big\{D_\mu\big[U(x,y)\,(4\pi^2(x-y)^2)^{-1}\big]$$

(41)

$$-\frac{1}{24\pi^2}\big[3(D_\mu v^+)\,b^B f\,b_B^+ v + v^+ b^B f\,b_B^+(D_\mu v) + v^+ b^B \partial_\mu f\,b_B^+ v\big](y)\Big\}$$

$$-\int d^4z\,\Omega(z)^{1/2}\,P_0(y,z)_{AB}\,\hat{S}_-(z,y)^B{}_{A'}$$

Here, U(x,y) is the gauge parallel transporter along the straight line from y to x:

$$(42)\qquad U(x,y) = P\exp-\int_y^x dz_\mu\,A_\mu(z)\;; \qquad \text{P: path ordering}$$

On the other hand, the short distance behaviour of S_-^M can be calculated perturbatively (Appendix):

$$\sum_{i=1}^{\nu} e_i\,S_-^{M_i}(x,y)_{AA'} =$$

(43)

$$-\frac{i}{2}(e_\mu)_{AA'}\Big\{D_\mu\big[U(x,y)\,(4\pi^2(x-y)^2)^{-1}\big] - F_{\mu\nu}(y)\,y_\nu\,\frac{\Omega(y)^{1/2}}{24\pi^2R^2}\Big\}$$

Terms, which vanish in the limit where first $x \to y$ and then $M_i \to \infty$, have been neglected here.

When inserting eqs. (41) and (43) into eq. (40) we see that the short distance singularities cancel and we are left with

$$\delta\Gamma_{reg} = \delta\Gamma_1 + \delta\Gamma_2 + \frac{1}{6\pi^2}\int d^4x\ Tr\{\delta A_\mu(x)\ j_\mu(x)\}$$

$$\delta\Gamma_1 = -\int d^4y\, d^4z\ Tr\{2i(e_\mu^+)^{A'A}\delta A_\mu(y)\, P_o(y,z)_{AB}\,\Omega(z)^{1/2}\,\hat{S}_-(z,y)^B{}_{A'}\} + c.c.$$

$$\delta\Gamma_2 = -(6\pi^2R^2)^{-1}\int d^4y\ \Omega(y)^{1/2}\, y_\nu\ Tr\{\delta A_\mu(y)\, F_{\mu\nu}(y)\}$$

$$j_\mu = v^+\{b^A \overleftrightarrow{\partial_\mu} b_A^+\,\Delta^{A'} \overleftrightarrow{\partial_\mu} \partial_\mu\Delta_{A'} - \partial_\mu\Delta^{A'} \overleftrightarrow{\partial_\mu} \Delta_{A'}^+,\ b^A \overleftrightarrow{\partial_\mu} b_A^+\}v$$

$\delta\Gamma_1$ can be simplified as follows. Denote by $\hat{T}$ and $\hat{P}_o$ the operators T and P_o with Ω set equal to one. Then

$$\delta\Gamma_1 = -Tr\{\delta\hat{T}\,P_o\,\Omega^{1/2}\,\hat{T}^+(\hat{T}\hat{T}^+)^{-1}\} + c.c.$$

Now $\hat{T}P_o = 0$ so that

$$\delta\hat{T}\,P_o = -\hat{T}\,\delta P_o$$

Furthermore,

$$\hat{T}^+(\hat{T}\hat{T}^+)^{-1}\hat{T} = 1 - \hat{P}_o$$

and therefore

$$\delta\Gamma_1 = \mathrm{Tr}\{\delta P_0\, \Omega^{1/2}(1-\hat{P}_0)\} + \text{c.c.}$$

$$= \delta\,\mathrm{Tr}\left(\ln N - \ln \hat{N}\right)$$

(N is the zero mode matrix (29)).

$\delta\Gamma_2$ can also be integrated easily. Namely, using the field equation $D_\mu F_{\mu\nu} = 0$, we have

$$\delta\,\mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu}\} = 4\,\partial_\mu\,\mathrm{Tr}\{\delta A_\nu F_{\mu\nu}\}$$

so that

$$\delta\Gamma_2 = -\frac{1}{12\pi^2}\int d^4x\,(\partial_\mu \ln\Omega)\,\mathrm{Tr}\{\delta A_\nu F_{\mu\nu}\}$$

$$= \delta\left[\frac{1}{48\pi^2}\int d^4x\,\ln\Omega\;\mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu}\}\right]$$

The rest of $\delta\Gamma$ will be integrated in the next section. As a preliminary step towards this goal, we here derive an expression for δA_μ in terms of $\delta a_{A'}$, the variation of the instanton parameter matrix ($\delta b_A = 0$ since b_A is always assumed to be in its normal form (19)). Defining the projector

$$(44)\qquad P = vv^+ = 1 - \Delta^{A'} \frac{1}{f} \Delta^+_{A'},$$

it is easy to check that

$$(45)\qquad \delta A_\mu = v^+[\delta P, \partial_\mu P]v + D_\mu(v^+\delta v)$$

The term $D_\mu (v^+ \delta v)$ can be dropped, because $\delta \Gamma_{reg}$ is gauge invariant (or, equivalently, since $D_\mu j_\mu = 0$). Inserting eq. (44) into eq. (45) gives δA_μ as a function of $\delta a_{A'}$.

For ease of reference, we finally collect the results of this section to obtain

$$\delta\Gamma_{reg} = \delta \left\{ Tr(\ln N - \ln \hat{N}) + \frac{1}{48\pi^2} \int d^4x \ln \Omega \, Tr(F_{\mu\nu} F_{\mu\nu}) \right\}$$

$$+ \frac{1}{6\pi^2} \int d^4x \; Tr(\delta A_\mu j_\mu) \tag{46}$$

Here, the kxk matrices N and $\hat{N}$ are given by (cf. eq. (27))

$$N_{ij} = \int d^4x \; \Omega^{1/2} \, \psi_A^{i+} \psi^{jA}$$

$$\hat{N}_{ij} = \int d^4x \; \psi_A^{i+} \psi^{jA} \tag{47}$$

and, with $d_{A'} \stackrel{def}{=} \delta a_{A'}$,

$$\delta A_\mu = v^+ \{ d^{A'} \not{f} \partial_\mu \Delta^+_{A'} - \partial_\mu \Delta^{A'} \not{f} d^+_{A'} \} v$$

$$j_\mu = v^+ \{ b^A \not{f} b^+_A \, \Delta^{A'} \not{f} \partial_\mu \Delta^+_{A'} - \partial_\mu \Delta^{A'} \not{f} \Delta^+_{A'} \, b^A \not{f} b^+_A \} v \tag{48}$$

5. Integration of $\delta \Gamma_{reg}$

Of course, Γ_{reg} can be obtained by integrating $\delta \Gamma_{reg}$ along any path of

instanton solutions connecting A_μ with some standard configuration A^0_μ, thus giving Γ_{reg} essentially as a five dimensional integral. Such an expression is quite implicit, however, because it is difficult to display the integration paths. In this section we derive a formula for Γ_{reg} involving a five dimensional integral too, but with an integrand, which is an explicit rational function of the instanton parameters and the integration variables.

We first rewrite $Tr(\delta A_\mu j_\mu)$ in terms of the basic matrices b_A, Δ_A, and f. From eqs. (10), (44) and (48) one finds

$$Tr(\delta A_\mu j_\mu) = \tag{49}$$

$$2\,Tr\{d\not{f}\overline{b^+ P\, b}\not{f}\overline{b^+\Delta}\not{f}b^+ P + d\not{f}\overline{b^+ P b}\not{f}\Delta^+\overline{b\not{f}b^+ P}\} + c.c.$$

The lines connecting pairs of matrices imply that their spinor indices are contracted such that the index of the left matrix is in the upper position, e.g.

$$\overline{bfb}^+ = b^A f b^+_A$$

By adding a suitable divergence $\partial_\mu \Lambda_\mu$ one can express $Tr(\delta A_\mu j_\mu)$ through f alone:

$$Tr(\delta A_\mu j_\mu) + \partial_\mu \Lambda_\mu = \tag{50}$$

$$\tfrac{1}{4}T_1 - \tfrac{3}{4}T_2 + \tfrac{3}{4}T_3 + \tfrac{1}{4}T_4 + T_5 - \tfrac{1}{2}T_6 - T_7 + T_8 - 3T_9$$

Here,

$$\Lambda_\mu = -2\,\mathrm{Tr}\{d\xi\,\partial_\mu\Delta^+\Delta\xi\,b^+\Delta\xi\,b^+\Delta\xi\Delta^+\} \tag{51}$$

and the T_i's are the following traces:

$$T_1 = \epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\{\xi\delta\xi^{-1}\,\xi\partial_\mu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\,\xi\partial_\rho\xi^{-1}\,\xi\partial_\sigma\xi^{-1}\}$$

$$T_2 = \mathrm{Tr}\{\xi\delta\xi^{-1}\,\xi\partial_\mu\xi^{-1}\,\xi\partial_\mu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\}$$

$$T_3 = \mathrm{Tr}\{\xi\delta\xi^{-1}\,\xi\partial_\mu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\,\xi\partial_\mu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\}$$

$$T_4 = \mathrm{Tr}\{\xi\delta\xi^{-1}\,\xi\partial_\mu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\,\xi\partial_\mu\xi^{-1}\}$$

$$T_5 = \mathrm{Tr}\{\xi\partial_\mu\delta\xi^{-1}\,\xi\partial_\mu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\}$$

$$T_6 = \mathrm{Tr}\{\xi\partial_\mu\delta\xi^{-1}\,\xi\partial_\nu\xi^{-1}\,\xi\partial_\mu\xi^{-1}\,\xi\partial_\nu\xi^{-1}\}$$

$$T_7 = \mathrm{Tr}\{\xi\delta\xi^{-1}\,\xi\,\overline{b^+b}\,\xi\partial_\mu\xi^{-1}\,\xi\partial_\mu\xi^{-1}\}$$

$$T_8 = Tr\{\not{f}\delta\not{f}^{-1}\not{f}\partial_\mu\not{f}^{-1}\not{f}\overline{b^+b}\not{f}\partial_\mu\not{f}^{-1}\}$$

$$T_9 = Tr\{\not{f}\partial_\mu\delta\not{f}^{-1}\not{f}\partial_\mu\not{f}^{-1}\not{f}\overline{b^+b}\}$$

$$T_{10} = Tr\{\not{f}\delta\not{f}^{-1}\not{f}\overline{b^+b}\not{f}\overline{b^+b}\}$$

(T_{10} is added to the list for later convenience). We only know of a rather lengthy proof of eq. (5C). It goes as follows. First observe that using the rules

$$\delta\not{f}^{-1} = \tfrac{1}{2}(d^+_{A'}\Delta^{A'} + \Delta^+_{A'}d^{A'})$$

$$\partial_\mu\not{f}^{-1} = (e_\mu)_{AA'}b^{+A}\Delta^{A'}\ ; \qquad \Delta^+_{A'}b_A = -b^+_A\Delta_{A'}$$

all T_i's can be written as linear combinations of traces of products of

$$d_{A'}\not{f}b^+_A\ ,\ b_A\not{f}d^+_{A'}\ ,\ \Delta_{A'}\not{f}d^+_{B'}\ ;$$

$$\Delta_{A'}\not{f}\Delta^+_{B'}\ ,\ \Delta_{A'}\not{f}b^+_A\ ,\ b_A\not{f}\Delta^+_{A'}\ ,\ b_A\not{f}b^+_B\ ,$$

with all spinor indices contracted. All terms contain precisely one matrix $d_{A'}$ or $d^+_{A'}$ and four matrices b_A or b^+_A. Such traces are called basic. For example, the expansion of T_2 in basic traces is

$$T_2 = \frac{1}{2} Tr\{ d f b^+ \Delta f b^+ \Delta f b^+ \Delta f b^+ \Delta f \Delta^+ \}$$

$$+ \frac{1}{2} Tr\{ \Delta f d^+ \Delta f b^+ \Delta f b^+ \Delta f b^+ \Delta f b^+ \}$$

It is obvious from eqs. (49) and (51) that the left hand side of eq. (50) can also be written as a linear combination of basic traces. The two sides of eq. (50) do not yet match, however, because the basic traces are not linearly independent.

There are three sources for linear relations between basic traces. First, the reality property eq. (20) of Sp(r) instanton parameter matrices implies that all basic traces are real. Any trace involving $d^+_{A'}$ is therefore equal to another basic trace containing $d_{A'}$, e.g.

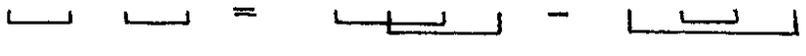
$$Tr\{ \Delta f d^+ \Delta f b^+ \Delta f b^+ \Delta f b^+ \Delta f b^+ \} =$$

$$Tr\{ d f b^+ \Delta f b^+ \Delta f b^+ \Delta f b^+ \Delta f \Delta^+ \}$$

A second set of relations between basic traces arises from the rule

$$\epsilon_{A'B'} \epsilon_{C'D'} = \epsilon_{A'C'} \epsilon_{B'D'} - \epsilon_{A'D'} \epsilon_{B'C'}$$

which reads graphically



Of course, the analogous rule holds for contractions of unprimed indices, too. Note also that

$$b^+_A b_B = \tfrac{1}{2} \epsilon_{AB} \overline{b^+ b}$$

A third type of relations is finally obtained from

$$d^+_{A'} \Delta_{B'} + \Delta^+_{A'} d_{B'} + d^+_{B'} \Delta_{A'} + \Delta^+_{B'} d_{A'} = 0$$

$$d^+_{A'} b_A + b^+_A d_{A'} = 0$$

(this is a consequence of eq. (15) resp. eq. (18b)). For example, using reality, we have

$$B_1 \overset{\text{def}}{=} \mathrm{Tr}\{\Delta^+ d \not{b}^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{}\}$$

$$= \mathrm{Tr}\{d^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{}\}$$

$$B_2 \overset{\text{def}}{=} \mathrm{Tr}\{\Delta^+ d \not{b}^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{}\}$$

$$= \mathrm{Tr}\{d^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{b}^+ \Delta \not{}\}$$

and therefore $B_1 + B_2 = 0$.

As a result of all these linear dependences, any basic trace can be represented as a linear combination of the elements of an irreducible set of 14 basic traces. When this is done for both sides of eq. (50), they turn out

to be identical.

Most of the right hand side of eq. (50) can be integrated quite easily, viz.

$$-\tfrac{3}{4}T_2 + \tfrac{3}{4}T_3 + \tfrac{1}{4}T_4 + T_5 - \tfrac{1}{2}T_6 - T_7 + T_8 - 3T_9 =$$

(52)

$$\delta\left(\tfrac{1}{4}D_1 - 5D_2\right) + \partial_\mu\left(\tfrac{1}{8}\Lambda^1_\mu - \tfrac{7}{16}\Lambda^2_\mu + \tfrac{1}{16}\Lambda^3_\mu - \tfrac{5}{4}\Lambda^4_\mu - \tfrac{7}{2}\Lambda^5_\mu\right)$$

$$D_1 = \mathrm{Tr}\{f\partial_\mu f^{-1} f\partial_\mu f^{-1} f\partial_\nu f^{-1} f\partial_\nu f^{-1}\}$$

$$D_2 = \mathrm{Tr}\{f^2\}$$

$$\Lambda^1_\mu = \mathrm{Tr}\{f\delta f^{-1} f\partial_\mu f^{-1} f\partial_\nu f^{-1} f\partial_\nu f^{-1}\}$$

$$\Lambda^2_\mu = \mathrm{Tr}\{f\delta f^{-1} f\partial_\nu f^{-1} f\partial_\mu f^{-1} f\partial_\nu f^{-1}\}$$

$$\Lambda^3_\mu = \mathrm{Tr}\{f\partial_\mu\delta f^{-1} f\partial_\nu f^{-1} f\partial_\nu f^{-1}\}$$

$$\Lambda^4_\mu = \mathrm{Tr}\{f\delta f^{-1} f^2\partial_\mu f^{-1}\}$$

$$\Lambda^5_\mu = \mathrm{Tr}\{f^2\partial_\mu\delta f^{-1}\}$$

Eq. (52) is proved straightforwardly by computing δD_i and $\partial_\mu \Lambda^i_\mu$, observing the rules

$$\delta f = - f \delta f^{-1} f \; ; \quad \partial_\mu f = - f \partial_\mu f^{-1} f \; ; \quad \partial_\mu \partial_\nu f^{-1} = 2 \delta_{\mu\nu} \; ,$$

and expressing the result as linear combinations of $T_2 \ldots T_{10}$ (as usual, we assume that b_A is in its normal form, in particular that $\overrightarrow{b^{+1}} b = -2$). For example,

$$\delta D_1 = - 2 T_2 - 2 T_4 + 4 T_5$$

$$\partial_\mu \Lambda^4_\mu = T_7 + \tfrac{1}{2} T_8 - \tfrac{1}{2} T_9 + 2 T_{10}$$

Summarizing eqs. (50) and (52) we obtain

$$(53) \quad \int d^4x \; Tr(\delta A_\mu j_\mu) = \int d^4x \left\{ \tfrac{1}{4} T_1 + \tfrac{1}{4} \delta D_1 - 5 \delta D_2 \right\}$$

The final step, the integration of T_1, requires a little detour. Let $M(\xi) \in Gl(k, \mathbb{C})$ be an arbitrary function of five real variables $\xi_0, \ldots, \xi_4$. Define

$$(54) \quad q(\xi) = \epsilon_{\alpha\beta\gamma\delta\lambda} Tr \{ M^{-1} \partial_\alpha M \; M^{-1} \partial_\beta M \; M^{-1} \partial_\gamma M \; M^{-1} \partial_\delta M \; M^{-1} \partial_\lambda M \}$$

$q(\xi)$ has a topological character in the sense that if M is varied, q changes by a local divergence only:

$$(55) \quad \delta q = 5 \, \partial_\alpha \, \epsilon_{\alpha\beta\gamma\delta\lambda} Tr \{ M^{-1} \delta M \; M^{-1} \partial_\beta M \; M^{-1} \partial_\gamma M \; M^{-1} \partial_\delta M \; M^{-1} \partial_\lambda M \}$$

An integrated form of this equation is

$$q = 5\,\partial_\alpha \int_0^1 dt\, \epsilon_{\alpha\beta\gamma\delta\lambda}\, \mathrm{Tr}\{K^{-1}\partial_t K\, K^{-1}\partial_\beta K \dots K^{-1}\partial_\lambda K\} \tag{56}$$

where $K(t,\xi)$, $0 \le t \le 1$, is any curve of invertible matrices such that $K(0,\xi)$ is diagonal and $K(1,\xi) = M(\xi)$.

To apply eq. (56) to the problem at hand, we identify ξ_μ with x_μ ($\mu = 0,\dots,3$), ξ_4 with any instanton parameter such that $\delta = \partial_4$, and M with f^{-1}. Then,

$$T_1 = \epsilon_{\mu\nu\rho\sigma}\, \mathrm{Tr}\{f\partial_4 f^{-1}\, f\partial_\mu f^{-1}\, f\partial_\nu f^{-1}\, f\partial_\rho f^{-1}\, f\partial_\sigma f^{-1}\} = q.$$

Furthermore, choosing

$$K_{ij} = (1-t)(1+x^2)\,\delta_{ij} + t\, f^{-1}_{ij},$$

which is strictly positive and therefore invertible for all t and x, eq. (56) becomes

$$T_1 = 5\delta \int_0^1 dt\, \epsilon_{\mu\nu\rho\sigma}\, \mathrm{Tr}\{K^{-1}\partial_t K\, K^{-1}\partial_\mu K \dots K^{-1}\partial_\sigma K\} + \partial_\mu \Sigma_\mu$$

Integrating over all space finally yields

$$\int d^4x\, \mathrm{Tr}(\delta A_\mu j_\mu) = \delta \int d^4x \left\{ \tfrac{1}{4} D_1 - 5 D_2 + k\,(1+x^2)^{-2} \right.$$
$$\left. + \tfrac{5}{4} \int_0^1 dt\, \epsilon_{\mu\nu\rho\sigma}\, \mathrm{Tr}\left[K^{-1}\partial_t K\, K^{-1}\partial_\mu K \dots K^{-1}\partial_\sigma K\right] \right\} \tag{57}$$

The term $k(1 + x^2)^{-2}$ has been added to make the integral absolutely convergent.

To sum up, we have shown that

$$\Gamma_{reg} - 2r\Gamma^0_{reg} = \alpha + Tr(\ln N - \ln \hat{N})$$

$$+ \frac{1}{48\pi^2}\int d^4x \ln\Omega \; Tr(F_{\mu\nu}F_{\mu\nu}) + \frac{1}{24\pi^2}\int d^4x \; I_1(x)$$

$$+ \frac{5}{24\pi^2}\int d^4x \int_0^1 dt \; I_2(t,x) \tag{58}$$

Here, Γ^0_{reg} equals Γ_{reg} for the case of no background field and one color only, α is a number, which is constant on each connected component of the instanton manifold, and the integrands I_1, I_2 are given by

$$I_1 = Tr\{f\partial_\mu f^{-1} f\partial_\mu f^{-1} f\partial_\nu f^{-1} f\partial_\nu f^{-1}\}$$

$$- 20\, Tr\{f^2\} + 4k(1+x^2)^{-2} \tag{59}$$

$$I_2 = \epsilon_{\mu\nu\rho\sigma} Tr\{K^{-1}\partial_t K \; K^{-1}\partial_\mu K \; K^{-1}\partial_\nu K \; K^{-1}\partial_\rho K \; K^{-1}\partial_\sigma K\} \tag{60}$$

6. Computation of α

Let A_μ be any Sp(r) instanton potential. Of course, A_μ can also be considered as an instanton configuration $\tilde{A}_\mu$ in $Sp(\tilde{r})$ for any $\tilde{r} \geq r$, viz.

(61) $$\tilde{A}_\mu = \begin{array}{|c|c|}\hline A_\mu & 0 \\ \hline 0 & 0 \\ \hline \end{array} \begin{matrix} \} 2r \\ \} 2(\tilde{r}-r) \end{matrix}$$

Denoting by $\tilde{\Gamma}_{reg}$ the log of the determinant of the corresponding Dirac operator, we obviously have

(62) $$\tilde{\Gamma}_{reg} - 2\tilde{r}\,\Gamma^0_{reg} = \Gamma_{reg} - 2r\,\Gamma^0_{reg}$$

i.e. the left hand side of eq. (58) is invariant, when A_μ is imbedded into a higher group. The same holds for N_{ij}, $\hat{N}_{ij}$ and the integrals on the right hand side of eq. (58) and must therefore be true for α , too. Since any k-instanton solution can be imbedded or reduced to Sp(k), it follows that it is sufficient to compute α for r = k.

The instanton manifold is essentially a convex space for r = k and is therefore connected (cf. the description of instantons given by Drinfeld & Manin [10]). α is hence the same for all k-instanton configurations. It can conveniently be computed by specializing eq. (58) to the case of a superposition of k one instanton solutions, each of them occupying one of the k commuting SU(2) subgroups on the diagonal of Sp(k). Then, $\alpha = k\,\alpha_1$ where α_1 is independent of k and can be calculated for k = 1.

The one instanton solution of size R centered at the origin has parameter matrices

$$a_1 = R\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad ; \quad a_2 = R\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$b^1 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad ; \quad b^2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{63}$$

Exploiting O(5) invariance, one finds that TT^+ has the following two series of eigenvalues (e.g. Ref. 12)

$$E_m^1 = 4R^{-2}(m+1)(m+4)$$

$$d_m^1 = \frac{1}{6}(m+2)(m+3)(2m+5) \tag{64a}$$

$$E_m^2 = 4R^{-2}(m+2)(m+3)$$

$$d_m^2 = \frac{1}{2}(m+1)(m+4)(2m+5) \tag{64b}$$

(E_m^i is the eigenvalue, d_m^i its multiplicity and $m = 0,1,2,\ldots$). In the vacuum case, there is only one series of eigenvalues

$$E_m^o = 4R^{-2}(m+2)^2$$

$$d_m^o = \frac{2}{3}(m+1)(m+2)(m+3). \tag{64c}$$

Γ_{reg} for the one instanton solution (63) can now be obtained using 't Hooft's

method [13] to compute the regularized sum (39) over all eigenvalues (64a,b):

$$\Gamma_{reg} = \sum_{i=1}^{\nu} e_i \ln(M_i R)\left[\frac{1}{24}(M_i R)^4 + \frac{1}{3}(M_i R)^2 + \frac{41}{45}\right] + \ln R$$

(65)

$$- \frac{4}{3}\zeta'(-1) - \frac{8}{3}\zeta'(-3) - \frac{41}{45}\ln 2 - \ln 3 + \frac{221}{540}$$

($\zeta(z)$ is Riemann's zeta function, Ref. 14, § 9.5). Similarly, from the series (64c) we get

$$\Gamma^0_{reg} = \sum_{i=1}^{\nu} e_i \ln(M_i R)\left[\frac{1}{48}(M_i R)^4 + \frac{1}{6}(M_i R)^2 + \frac{11}{90}\right]$$

(66)

$$+ \frac{4}{3}\zeta'(-1) - \frac{4}{3}\zeta'(-3) - \frac{11}{90}\ln 2 - \frac{1}{270}$$

The left hand side of eq. (58) thus becomes

$$\Gamma_{reg} - 2\Gamma^0_{reg} = \frac{2}{3}\sum_{i=1}^{\nu} e_i \ln M_i + \frac{1}{3}\ln R - 4\zeta'(-1)$$

(67)

$$- \frac{2}{3}\ln 2 - \ln 3 + \frac{5}{12}$$

It is a trivial matter to evaluate the right hand side of eq. (58) for the special configuration (63). Comparing with eq. (67), we then obtain α_1 and hence α :

$$\alpha = k\left\{\frac{2}{3}\sum_{i=1}^{\nu} e_i \ln M_i - 4\zeta'(-1) - \ln 2 + \frac{5}{12}\right\} \quad (68)$$

7. Summary and discussion

Our final formula for the regularized determinant (38) of the Dirac operator D (eq. (23)) in a general Sp(r) k-instanton solution is

$$\Gamma_{reg} - 2r\,\Gamma^{0}_{reg} = k\left\{\frac{2}{3}\sum_{i=1}^{\nu} e_i \ln M_i - 4\zeta'(-1) - \ln 2 + \frac{5}{12}\right\}$$

$$+ \mathrm{Tr}\left(\ln N - \ln \hat{N}\right) + \frac{1}{48\pi^2}\int d^4x\, \ln\Omega\, \mathrm{Tr}\left(F_{\mu\nu} F_{\mu\nu}\right) \tag{69}$$

$$+ \frac{1}{24\pi^2}\int d^4x\; I_1(x) \;+\; \frac{5}{24\pi^2}\int d^4x \int_0^1 dt\; I_2(t,x)$$

Here, Γ^{0}_{reg} is the determinant with vanishing background field, the kxk matrices N and $\hat{N}$ (eq. 47) are built from the zero modes (27) of D, and the integrands I_1, I_2 are

$$I_1 = \mathrm{Tr}\left\{f\,\partial_\mu f^{-1}\, f\,\partial_\mu f^{-1}\, f\,\partial_\nu f^{-1}\, f\,\partial_\nu f^{-1}\right\}$$

$$- 20\,\mathrm{Tr}\left\{f^2\right\} + 4k\,(1+x^2)^{-2} \tag{59}$$

$$I_2 = \epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left\{K^{-1}\partial_t K\, K^{-1}\partial_\mu K\, K^{-1}\partial_\nu K\, K^{-1}\partial_\rho K\, K^{-1}\partial_\sigma K\right\} \tag{60}$$

The real symmetric kxk matrix f(x) is defined in terms of the instanton parameters by eq. (15) and, finally,

$$K(t,x)_{ij} = (1-t)(1+x^2)\,\delta_{ij} + t\, f^{-1}(x)_{ij} \tag{70}$$

Eq. (69) simplifies in the infinite volume limit $R = \infty$:

$$\lim_{R\to\infty} (\Gamma_{reg} - 2r\,\Gamma^{o}_{reg}) = k\left\{\frac{2}{3}\sum_{i=1}^{\nu} e_i \ln M_i - 4\zeta'(-1) - \frac{2}{3}\ln 2 + \frac{5}{12}\right\}$$

(71)

$$+ \frac{1}{24\pi^2}\int d^4x\, I_1(x) + \frac{5}{24\pi^2}\int d^4x \int_0^1 dt\; I_2(t,x)$$

This is a tractable expression, although it is not completely explicit. Maybe, with sufficient ingenuity, one could calculate the integrals, but we did not attempt to do so. The integrand I_2 vanishes for all one- and two-instanton solutions. We checked that in general it contributes, when $k \geq 3$, thus indicating that two-instanton solutions are not generic multi-instanton configurations. I_2 also vanishes for 't Hooft's instanton solutions

$$(72)\qquad A_\mu = \frac{1}{2}\,\bar{\sigma}_{\mu\nu}\,\partial_\nu \ln\rho\;; \qquad \rho = 1 + \sum_{i=1}^{k}\frac{\lambda_i^2}{(x-a^i)^2}\;,$$

which are therefore rather special, too. For these solutions, eq. (71) can be written more compactly:

$$\lim_{R\to\infty} (\Gamma_{reg} - 2r\,\Gamma^{o}_{reg}) = k\left\{\frac{2}{3}\sum_{i=1}^{\nu} e_i \ln M_i - 4\zeta'(-1) - \frac{2}{3}\ln 2\right\}$$

(73)

$$+ \frac{1}{96\pi^2}\int d^4x\, \ln\tilde{\rho}\;\Box^2 \ln\tilde{\rho} + \frac{2}{3}\sum_{i=1}^{k}\ln\lambda_i + \frac{1}{3}\sum_{i<j}\ln(a^i-a^j)^2$$

where, following Belavin et al. [3)+)], $\tilde{\rho}$ is defined by

$$(74)\qquad \tilde{\rho} = \rho\prod_{i=1}^{k}(x-a^i)^2.$$

+) Our result, however, seems to disagree with their equation (7).

An equivalent form of eq. (73) has previously been found by Brown and Creamer [7].

The formulae for the determinants, which we have derived here, are strikingly similar to the corresponding ones in two dimensional non-linear σ-models [1)2)]. This suggests that the integrals over I_1 and I_2 have perhaps a universal geometrical meaning relating to the topological and metric properties of the instanton manifold.

To compute the determinant of the gluon fluctuation operator via supersymmetry, requires the quarks to be in the adjoint rather than the fundamental representation of the gauge group. In view of the tensor product formula for instantons [15], however, our result applies to this case as well. The ensuing exact instanton gas will be rather complicated and what its physical properties are, remains to be seen.

Acknowledgement

One of us (B.B.) thanks H. Krasemann and Z. Kunszt for discussions about computer problems.

Appendix: Short distance expansion of $S_-^M(x,y)_{AA'}$

S_-^M is defined by

$$(A1)\quad S_-^M(x,y)_{AA'} = 2\,\Omega(x)^{-1/2}\, i\,(e_\mu)_{AB'}\, D_\mu\, G^M(x,y)^{B'}{}_{A'}$$

where G^M is the Green's function for $TT^+ + M^2$:

$$(A2)\quad \{-\delta^{A'}_{B'}\, D_\mu D_\mu - (e_\mu^+ e_\nu)^{A'}{}_{B'}\,(\Omega^{1/2}\partial_\mu\Omega^{-1/2})\,D_\nu + \tfrac{1}{4}\Omega M^2\delta^{A'}_{B'}\}\, G^M(x,y)^{B'}{}_{C'} = \tfrac{1}{4}\Omega^{1/2}\,\delta(x-y)$$

We first derive the short distance expansion of $G^M(x,y)$ for $y = 0$ and shall later exploit O(5) covariance to extend the result to all y. Defining

$$U_\mu = 2A_\mu + e_\nu^+ e_\mu\,(\Omega^{1/2}\partial_\nu\Omega^{-1/2})$$

$$V = \partial_\mu A_\mu + A_\mu A_\mu + e_\mu^+ e_\nu\,(\Omega^{1/2}\partial_\mu\Omega^{-1/2})\,A_\nu$$

$$\varkappa = 1 - \tfrac{1}{4}\Omega$$

$$W = U_\mu\partial_\mu + V + \varkappa M^2$$

eq. (A2) reads (spinorial indices are suppressed)

$$(A3)\quad \{-\partial_\mu\partial_\mu + M^2 - W\}\, G^M(x,0) = \tfrac{1}{2}\,\delta(x).$$

The perturbative solution of this equation is

$$(A4)\quad G^M(x,0) = \frac{1}{2}\sum_{m=0}^{\infty} \int d^4x_1 \ldots d^4x_m\, G_0(x-x_1)W(x_1)G_0(x_1-x_2)\ldots W(x_m)G_0(x_m)$$

where the free propagator is

$$(A5)\quad G_0(z) = \int \frac{d^4p}{(2\pi)^4}\, e^{ipz}\,(p^2+M^2)^{-1}$$

Each term in the sum (A4) can be expanded for $x \to 0$. It is sufficient to calculate the expansion up to and including the order x_μ. Define

$$(A6)\quad C_m(x) = \int d^4x_1 \ldots d^4x_m\, G_0(x-x_1)\,W(x_1)\ldots\, W(x_m)\,G_0(x_m)$$

Expanding

$$(A7)\quad W(z) = W^0 + z_\mu W^1_\mu + z_\mu z_\nu W^2_{\mu\nu} + \ldots$$

the integral (A6) can be evaluated in p-space and reduces to a sum of Bessel functions. Most of the terms, however, do not contribute in the limit where first $x \to 0$ and then $M \to \infty$. In particular, for $m \geq 4$, $C_m(x)$ is once continuously differentiable at $x = 0$ and $C_m(0)$ as well as $\partial_\mu C_m(0)$ vanish for $M \to \infty$. Also, integrals involving W^ℓ, $\ell \geq 4$, need not be considered for the same reason. Thus, we are left with a finite number of integrals to compute and expand for $x \to 0$, $M \to \infty$. This calculation is lengthy but straightforward and we therefore merely quote the result:

$$\text{(A8)}\quad \sum_{j=1}^{\nu} e_j\, S_-^{M_j}(x,0)_{AA'} = -\frac{i}{2}(e_\mu)_{AA'} D_\mu\left[U(x,0)(4\pi^2x^2)^{-1}\right]$$

U(x,0) is the gauge parallel transporter from the origin to x along the straight line connecting the two (eq. (42)).

The O(5) covariant expression reducing to (A8) for y = 0 is

$$\sum_{j=1}^{\nu} e_j\, S_-^{M_j}(x,y)_{AA'} = -\frac{i}{2}\,\Omega(x)^{1/4}\left(e_\mu V^g(x,y)\right)_{AA'} \times$$

$$\text{(A9)}\quad \times D_\mu\Big\{U^g(x,y)\Big[\left(4\pi^2\Omega(x)^{1/2}(x-y)^2\Omega(y)^{1/2}\right)^{-1}$$

$$-(32\pi^2R^2)^{-1}\ln\left(\Omega(x)^{1/2}(x-y)^2\Omega(y)^{1/2}\right)\Big]\Big\}\,\Omega(y)^{3/4}$$

Here, $U^g(x,y)$ is the gauge parallel transporter along the (shorter) geodesic connecting y with x. Similarly, $V^g(x,y)$ is the spin parallel transporter along the same geodesic,

$$\text{(A10)}\quad V^g(x,y) = P\exp\Big\{-\int_y^x dz_\mu\,\bar{\sigma}_{\mu\nu}\,\frac{1}{4}\,\partial_\nu \ln\Omega\Big\}$$

where $\bar{\sigma}_{\mu\nu} = \frac{1}{2}\left(e_\mu^+ e_\nu - e_\nu^+ e_\mu\right)$. Noting that

$$\text{(A11)}\quad U^g(x,y) = U(x,y) + (x-y)^2(x-y)_\mu F_{\mu\nu}(y)\,y_\nu\,\frac{\Omega(y)^{1/2}}{12R^2} + O\left((x-y)^4\right)$$

(and an analogous formula for $V^g(x,y)$), another tedious but straightforward computation shows that eq. (A9) is equivalent to eq. (43) up to terms vanishing at x = y.

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