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Equation of Motion for String Operators in Quantum Chromodynamics

by

H. Suura

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Equation of motion for string operators in quantum chromodynamics *

H. Suura +

Deutsches Elektronen-Synchrotron DESY, Hamburg

Abstract

I derive from the QCD Lagrangian differential laws describing motions and interactions of an infinite set of string operators - locally gauge-invariant color-singlet operators. By truncating the set, I obtain a q-$\bar{q}$ wave equation with a confinement potential, and also a jet-fragmentation equation which describes splitting of a q-$\bar{q}$ string and creation of $I = 0$ vector mesons. I argue for the validity of the perturbative treatment of the string operators.

+ On leave from the University of Minnesota, Minneapolis, Minn. 55455

I. Introduction and Summary

In quantum chromodynamics (QCD) all physical states, including the vacuum, are invariant under local gauge transformations. Only locally gauge invariant operators have non-vanishing matrix elements between the physical states. Such operators must involve a line integral of the vector potential as a phase and are called string operators to indicate their path dependence. There have been some attempts to formulate the motion of such operators [(1)]. In my previous paper [(2)] (referred to as I in the following) I considered a differential equation for a q-$\bar{q}$ string (quark and antiquark operators connected by a string) defined on a t-plane. In the present paper I formulate the dynamics of a more general set of string operators, eliminating altogether the propagators of quarks and gluons. The virtue of defining the string operators on a t = const plane is that their time development is completely determined by the QCD Lagrangian, or by the equations of motion of the field operators. Since the gauge symmetry is preserved in QCD, a string operator may split with time into two or more strings, or convert itself into another kind, but it will never split into two disconnected objects, each having a color. Thus the QCD Lagrangian leads to an infinite set of first order differential equations in time for an infinite set of string operators.

A string operator consists of end points (quark or antiquark) and vertices (electric or magnetic fields), connected by strings. The equations of motion of the string operators show that the end points and the vertices move according to the free Dirac and Maxwell equations, respectively, providing the basis for the parton model. The interaction manifests itself in creation of new vertices along the string and also conversion of field vertices into quark currents and densities. Whenever a part of a string operator forms a gauge independent sub-string it will split from the parent string. Thus, two or more field vertices can form a gauge invariant subset and will be emitted as a glue ball. Also, if a vertex is converted into the quark current, the original string either splits at this vertex creating two new end points, or emits a $I = 0$ vector meson without splitting itself. The string splitting is a realization of the conjecture by Kogut and Susskind [(3)]. The observation that there are two processes in a jet fragmentation should have a direct physical consequence.

In order to deal with an infinite system of string operators and their equations of motion I will make two approximations in the present paper. First, I will neglect all string operators with two or more vertices. With this truncation I obtain a closed set of equations. Second, I will consider only straight strings to connect end points and vertices. Such operators alone do not form a closed system in general, but they do in the truncated set of equations. The equations of motion of the string operators involve a linear confinement potential, which is the energy of the electric flux represented by the string. It is naturally there, because I am dealing with an instantaneous motion of the string. Whether or not such a configuration can be maintained over a longer period of time as assumed here is the central problem of the confinement. In other words, the truncation will not be valid if there exist infinitely many almost degenerate configurations into which the retained strings can transform. I have two observations which may shed some light on this question.

The first observation is about a possible essential difference between the quantum electrodynamics (QED) and QCD. In QED one can define a gauge invariant string operator with a finite cross section (a sausage-shaped string which converges at both ends to a point which is a quark or an antiquark) as shown in I. In QCD the same does not appear to be possible [4], because the only way a color spin is carried from one end to the other in a gauge invariant way seems to be along a strictly linear path. Thus in QED a string of a small but finite cross section tends to spread with time without some outside pressure, because energetically that is more favorable. In QCD, if the above conjecture is correct, the situation is entirely different. A linear string may change shape or split but it stays linear. A large change in shape is energetically less favorable because the electric energy of a linear string is proportional to its length.

The second observation is about an analogy with infrared photons in QED. Just as emission of soft quanta gives rise to an infinite degeneracy, creation of soft vertices may give the same kind of degeneracy in the string configuration. In QED a charged particle must be dressed with the proper infrared field and any real transition is accompanied by emission of an infinite number of soft quanta.

Yet the Klein-Nishina formula remains meaningful unless the energy resolution is extremely good, thanks to the smallness of the fine structure constant. Also the hydrogen spectrum is accounted for including the fine structure without considering the effect of the proper infrared field. The treatment of the q-$\bar{q}$ spectrum and the string fragmentation discussed in the present paper is very similar in nature to the above two examples in QED. In order for the present formulation to be valid, the coupling constant that govern the vertex creation must be small, even though the running coupling constant for the infrared gluon emission may be large as is commonly believed [5].

II. Equations of motion for string operators

I summarize first the field equations in QCD, which I use to derive the equations of motion for the string operators. The process may be interpreted in the sense of the correspondence principle. The standard QCD Lagrangian for a color triplet quark field q(x) of mass m interacting with an octet of gluon vector potentials $A^a_\mu(x)$ (a = 1,2,...8) is given by

$$L = \int d^3x \left\{ -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{q} \gamma_\mu \left(i\partial^\mu - g \frac{\lambda_a}{2} A^{a\mu} \right) q - m \bar{q} q \right\}, \tag{2.1}$$

where the field strength $F^a_{\mu\nu}$ are defined by

$$F^a_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f_{abc} A_\mu^a A_\nu^b . \tag{2.2}$$

f_{abc} is the structure constants of SU(3). In the following I take a gauge

$$A_o^a = 0. \tag{2.3}$$

In this gauge the electric field is given by

$$E_i^a = F_{oi}{}^a = -\dot{A}^{ai}, \tag{2.4}$$

and the magnetic field by

$$B_i^{\,a} = -\frac{1}{2}\,\varepsilon_{ijk}\,F_{jk}^{\,a}$$
$$= \left(\vec{\nabla}\times\vec{A}^a + \frac{1}{2}\,g\,f_{abc}\,\vec{A}^b\times\vec{A}^c\right)_i . \tag{2.5}$$

As is customary I introduce for an octet $\{\vec{A}^a\}$ a matrix operator $\vec{A}$ by

$$\vec{A} = g\,\frac{\lambda_a}{2}\,\vec{A}^a, \tag{2.6}$$

and similarly the field strength $\vec{E}$ and $\vec{B}$. The field equations which follow from the Lagrangian (2.1) in the gauge (2.3) are the equivalent of the Ampère-Maxwell equation

$$\vec{D}_A \times \vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}, \tag{2.7}$$

and the Dirac equation

$$i\,\frac{\partial q}{\partial t} = \left[\vec{\alpha}\cdot(-i\vec{\nabla} - \vec{A}) + \beta m\right] q . \tag{2.8}$$

The quark current $\vec{j}^a$ is defined by

$$\vec{j}^{\,a}(x) = q^{+}(x)\,\vec{\alpha}\,\frac{\lambda_a}{2}\,q(x), \tag{2.9}$$

and the matrix $\vec{j}(x)$ in the same way as (2.6). The covariant derivative $\vec{D}_A$ in (2.7) is defined by

$$\vec{D}_A\, O = \vec{\nabla} O + [\vec{A}, O], \tag{2.10}$$

for any octet matrix O. The conservation law for the color charge

$\rho^a = q^+ \frac{\lambda^a}{2} q$ follows from (2.8) and can be written as

$$\vec{D}_A \cdot \vec{j} + \dot{\rho} = 0. \tag{2.11}$$

From the definitions of $\vec{E}$ and $\vec{B}$ given by (2.4) and (2.5) there follow two equations

$$\vec{D}_A \times \vec{\vec{E}} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{2.12}$$

and

$$\vec{D}_A \cdot \vec{B} = 0 \tag{2.13}$$

As is well known, an equivalent of the Gauss' law does not hold as a field equation in the present gauge. Instead one obtains from (2.7) and (2.11)

$$\frac{\partial}{\partial t} [\vec{D}_A \cdot \vec{\vec{E}} - \rho] = 0. \tag{2.14}$$

Thus,

$$G = \vec{D}_A \cdot \vec{\vec{E}} - \rho \tag{2.15}$$

is a constant of motion, and is a generator for local gauge transformations. Namely, if I define for a set of time-independent infinitesimal c-number functions $\omega^a(x)$ a unitary operator

$$U(\omega) = \exp\left\{ \frac{2i}{g^2} \, tr^c \int \omega(x)\, G(x)\, d^3x \right\}, \tag{2.16}$$

where $\omega = g(\lambda_a/2)\omega^a$, and tr^c means the trace with respect to color spins, then U generates a time-independent infinitesimal gauge transformation

$$\vec{A}' = U\vec{A}U^{-1} = \vec{A} - \vec{\nabla}\omega + [\omega, \vec{A}], \tag{2.17}$$

and

$$q' = U q U^{-1} = (1 - i\omega) q . \tag{2.18}$$

Since all the physical states are assumed to be locally gauge invariant a physical state Ψ must satisfy

$$G(x)\Psi = (\vec{D}_A \cdot \vec{E}(x) - \rho(x))\Psi = 0 . \tag{2.19}$$

Hence, only locally gauge invariant operators have non-vanishing matrix elements between two physical states.

Now I derive the equations of motion for such locally gauge invariant operators, which I call string operators. To study a q-$\bar{q}$ system, I consider a q-$\bar{q}$ string defined by

$$q(1,2) = tr^c [U(2,1)\, q(1)\, q^+(2)]. \tag{2.20}$$

The quark field q(x) has three sets of indices, spinor, color spin and flavor components, and $q(1)q^+(2)$ in the above equation must be regarded as a matrix in these three set of indices. The string part $U(2,1)$ is given by

$$U(2,1) = P \exp\left(i \int_1^2 \vec{A} \cdot d\vec{x} \right), \tag{2.21}$$

where P means the ordering along the integration path, which is taken to be a straight line. More explicitly, I may use a definition by an infinite product,

$$U(2,1) = \lim_{N \to \infty} \prod_{n=1}^{N} \exp\left[i \vec{A}(\xi_n) \cdot \vec{\Delta} \right], \tag{2.22}$$

where $\vec{\Delta} = (\vec{x}_2 - \vec{x}_1)/N$ and $\vec{\xi}_n = n\vec{\Delta} + \vec{x}_1$. The product is ordered from right to left with increasing n. Under the gauge transformation (2.17) and (2.18), q(1,2) is invariant. The equation of motion for q(1,2) was essentially derived in I, and is

$$\begin{aligned} i \dot{q}(1,2) &= (H_D + k^2 r)\, q(1,2) \\ &+ \int_1^2 d\vec{x} \cdot \left[q_{\vec{E}}(1,2;x) + q_{\vec{B}}(1,2;x) \times \vec{\alpha}_S \right]. \end{aligned} \tag{2.23}$$

Here $r = |\vec{x}_2 - \vec{x}_1|$ and H_D is the Dirac Hamiltonian for the quark and the anti-quark,

$$H_D = -i \vec{\alpha}_L \cdot \vec{\nabla}_1 + \beta_L m - i \vec{\alpha}_R \cdot \vec{\nabla}_2 - \beta_R m, \tag{2.24}$$

where $\vec{\alpha}_{L(R)}$ means $\vec{\alpha}$ operating from the left (right) on q(1,2). $\vec{\alpha}_S$ is an abbreviation

$$\vec{\alpha}_S = r^{-1} \left(|\vec{x} - \vec{x}_2| \vec{\alpha}_L + |\vec{x} - \vec{x}_1| \vec{\alpha}_R \right). \tag{2.25}$$

In deriving (2.23), the Dirac equation (2.8) has been used. The interaction term $-i\vec{\alpha}_L \cdot \vec{A}$ has been cancelled by a term from $-i\vec{\alpha}_L \cdot \vec{\nabla}_1 U(2,1)$. The latter derivative yields also a line integral involving the magnetic field $\vec{B}$ (the last term in Eq. (2.23)), which arises from the shift of the integration path when the end point 1 or 2 is moved. $q_{\vec{E}(\vec{B})}(1,2;x)$ is a gauge invariant string operator with an electric (magnetic) vertex at $\vec{x}$, and naively is defined by

$$q_{\vec{E}}(1,2;x) = \mathrm{tr}^c \left[U(2,x)\,\vec{E}(x)\, U(x,1)\, q(1)\, q^\dagger(2) \right], \tag{2.26}$$

and similarly for $q_{\vec{B}}(1,2;x)$. $U(2,x)$ and $U(x,1)$ are defined as in (2.21) with a straight integration path, but in general the vertex $\vec{x}$ need not be on the straight line connecting 1 and 2. The term involving $q_{\vec{E}}(1,2;x)$ in (2.23) obviously arises from the time derivative of $U(2,1)$. That the Eq. (2.26) needs a refinement can be seen by taking the time derivative of $\dot{U}(2,1)$ defined by the infinite product (2.22),

$$i\dot{U}(2,1) = \lim_{N\to\infty} \sum_{n=1}^{N} \cdots e^{i\vec{A}(\xi_{n+1})\cdot\vec{\Delta}} \left[-\dot{\vec{A}}(\xi_n)\cdot\vec{\Delta} - \right.$$

$$\left. -\frac{i}{2}\left\{ \dot{\vec{A}}(\xi_n)\cdot\vec{\Delta}, \vec{A}(\xi_n)\cdot\vec{\Delta} \right\}_+ + \cdots \right] e^{i\vec{A}(\xi_{n-1})\cdot\vec{\Delta}} \cdots$$

I move $\dot{\vec{A}}(\xi_n)$ to the left of $\vec{A}(\xi_n)$, and use a commutation relation

$$[A_i^a(\xi_n), \dot{A}_j^b(\xi_{n'})] = i\,\delta_{ab}\,\delta_{ij}\,\delta_{nn'}/\Delta^3,$$

which corresponds to the standard canonical commutation relation

$$[A_i^a(x), \dot{A}_j^b(x')] = i\,\delta_{ab}\,\delta_{ij}\,\delta^3(x-x'). \tag{2.27}$$

Thus, I obtain

$$i\dot{U}(2,1) = k^2 r\, U(2,1) + \lim_{N\to\infty} \sum_{n=1}^{\infty} \cdots e^{i\vec{A}(\xi_{n+1})\cdot\vec{\Delta}}\, \vec{E}(\xi_n)\cdot\vec{\Delta}\, e^{i\vec{A}(\xi_n)\cdot\vec{\Delta}} \cdots, \tag{2.28}$$

where

$$k^2 = \frac{1}{2}\frac{g^2}{\Delta^2} C = \frac{1}{2} g^2 C\, \delta^2(0). \tag{2.29}$$

C is a Casimir operator $C = \sum_a (\lambda a/2)^2 = 4/3$. $1/\Delta^2$ is equal to $\delta^2(0)$ as indicated in (2.29), which can be interpreted as the inverse cross section of the string. Since k^2 must be finite, the unrenormalized coupling constant g^2 must vanish. With the string energy $k^2 r$ explicitly extracted in (2.23), the precise meaning of the definition (2.26) must be given by the second term of Eq. (2.28). To denote the continuous limit of this term I may introduce a notation of the open and closed ended string like $U[x,1)$ and $U[2,x)$, and write instead of (2.26)

$$q_{\vec{E}}(1,2;x) = \mathrm{tr}^c \left[U[2,x)\, \vec{E}(x)\, U[x,1)\, q(1)\, q^\dagger(2) \right]. \tag{2.26'}$$

Admittedly, the notion of the open and closed ended string may not be well founded mathematically. Nevertheless the recognition of the difference between (2.26) and (2.26') seems to play a vital role in the later development. For simplicity, however, I will use a loose notation (2.26) in the following, keeping in mind the precise definition (2.26'). An equivalent qualification of Eq. (2.26) is to understand that $\vec{E}^a(x)$ in $\vec{E}(x)$ stands to the left of everything else. The difference between this and (2.26) will be

$$g^2 C \int_x^2 \delta^3(x-x')\,dz' = \frac{1}{2} g^2 C\, \delta^2(0) = k^2,$$

where I have chosen the line $1 \rightarrow 2$ as the z-axis. One may wonder why $\vec{E}^a(x)$ is moved to the left instead of to the right, which would produce $-k^2 r$. Later I will consider matrix elements of $q(1,2)$ and $q_{\vec{E}(\vec{B})}(1,2;x)$ between a hadron ket and the vacuum bra, in which case moving $\vec{E}^a(x)$ to the left proves to be the right step.

Next I investigate the equation of motion of a q-$\bar{q}$ string with one vertex $q_{\vec{E}(\vec{B})}(1,2;x)$, or $q\bar{q}g$ string. Instead of $\vec{E}$ and $\vec{B}$ it is convenient to consider the vertices

$$\vec{F}_{\pm}(x) = \vec{E}(x) \pm i\,\vec{B}(x),$$

and the corresponding operator $q_{\vec{F}_{\pm}}(1,2;x)$ defined like in (2.26), or more precisely (2.26'). The time derivatives of $q(1)q^+(2)$, $U(2,x)$ and $U(x,1)$ can be treated in the same way as in the case of $q(1,2)$. They produce

$$\begin{aligned}
&\left(H_D + k^2\,|\vec{x}_1-\vec{x}| + k^2\,|\vec{x}_2-\vec{x}|\right) q_{\vec{F}_{\pm}}(1,2;x) \\
&+\left(\int_1^x + \int_x^2\right) \mathrm{tr}^c \Big\{ P\Big[U(2,x)\,\vec{F}_{\pm}(x)\,U(x,1)\,\cdot \\
&\cdot\left(\vec{E}(x')\cdot d\vec{x}' + \vec{B}(x')\times \vec{\alpha}_I(x,x')\cdot d\vec{x}'\right) q(1)\,q^+(2)\Big]\Big\},
\end{aligned} \tag{2.30}$$

where

$$\vec{\alpha}_I(x,x') = \begin{cases} \vec{\alpha}_L\,|\vec{x}-\vec{x}'|/|\vec{x}-\vec{x}_1|, & \vec{x}' \in [\vec{x}_1, \vec{x}] \\ \vec{\alpha}_R\,|\vec{x}-\vec{x}'|/|\vec{x}-\vec{x}_2|, & \vec{x}' \in [\vec{x}, \vec{x}_2] \end{cases} \tag{2.31}$$

The P symbol in (2.30) means that $\vec{E}(x')$ and $\vec{B}(x')$ be placed at its position along the path $1 \to \vec{x} \to 2$. As stated before, $\vec{x}$ need not be on the straight line $1 \to 2$ and when taking $\vec{\nabla}_1$ and $\vec{\nabla}_2$ of $q_{\vec{F}}(1,2;x)$, $\vec{x}$ is to be held fixed. The q-$\bar{q}$ string with two vertices can further be reduced. The operator involving only two vertices and no end points,

$$g_{\vec{F}_\pm \vec{F}_\pm}(x,x') = tr^c\left[U(x',x)\vec{F}_\pm(x)\,U(x,x')\vec{F}_\pm(x')\right], \tag{2.32}$$

is a gauge invariant tensor, which may be called a gg string. It represents a glue ball consisting two gluons. The two-vertex strings in (2.30) can be a product of a gg string and a q-$\bar{q}$ string. For instance the operator involving the vertices $\vec{F}_\pm(x)$ and $\vec{E}(x')$ can be written, after taking $\vec{x}$ on the line $1 \to 2$, which is chosen to be z-axis, as

$$\int_1^2 P\left[\, g_{\vec{F}_\pm E_3}(x,x')\right] dz' \cdot q(1,2)$$
$$+\int_1^2 tr^c\left\{P\left[U(2,1)\vec{F}_\pm(x)E_3(x')\,q(1)\,q^+(2)\right]\right\}_{irred.} dz' \tag{2.33}$$

The second term is a genuine $q\bar{q}gg$ operator, which cannot be reduced into a product of strings. The P symbol in the first integral is the ordering of the two vertices along the integration path $1 \to 2$. It should be recognized that because of the definition (2.26') I have been able to move the operator $g_{\vec{F}_\pm E_3}$ to the left of q(1,2) without producing an extra term from the commutation of $\vec{E}^a(x)$ with $\vec{A}^a(x)$.

The time derivative of $\vec{F}_\pm$ is, from (2.7) and (2.12),

$$i\dot{\vec{F}}_\pm = \pm \vec{D}_A \times \vec{F}_\pm - i\vec{j}. \tag{2.34}$$

Again the covariant derivative of $\vec{F}_\pm(x)$ is converted to a simple derivative of $q_{\vec{F}_\pm}(1,2;x)$ with respect to $\vec{x}$, plus a correction term coming from the shift of the integration path of $U(x,1)$ and $U(2,x)$. Hence

$$\begin{aligned} & i\, tr^c \left[U(2,x)\, \dot{\vec{F}}_\pm(x)\, U(x,1)\, q(1)\, q^+(2) \right] \\ & = \pm \vec{\nabla} \times q_{\vec{F}_\pm}(1,2;x) \\ & \pm i \left(\int_1^x + \int_x^2 \right) \tau(x,x')\, tr^c \left\{ P\left[U(2,x) \vec{F}_\pm(x) \cdot \right.\right. \\ & \left.\left. \cdot U(x,1) \times \left(\vec{B}(x') \times d\vec{x}' \right) q(1)\, q^+(2) \right] \right\} \\ & - ig\, tr^c \left[U(2x) \frac{\lambda^a}{2} \left(q^+ \vec{\alpha} \frac{\lambda^a}{2} q \right) U(x,1)\, q(1)\, q^+(2) \right], \end{aligned} \tag{2.35}$$

where

$$\tau(x,x') = \begin{cases} |\vec{x}' - \vec{x}_1| / |\vec{x} - \vec{x}_1| , & \vec{x}' \in [\vec{x}_1, \vec{x}] \\ |\vec{x}' - \vec{x}_2| / |\vec{x} - \vec{x}_2| , & \vec{x}' \in [\vec{x}, \vec{x}_2]. \end{cases} \tag{2.36}$$

The last term of (2.35) is equal to

$$\frac{i}{6} g \vec{j}_s(x)\, q(1,2) - \frac{i}{2} g\, q(1,x)\vec{\alpha}\, q(x,2), \tag{2.37}$$

where $\vec{j}_s(x) = q^+(x)\vec{\alpha}\, q(x)$, if we use a Fierz identity

$$\left(\frac{\lambda a}{2}\right)_{\alpha\beta}\left(\frac{\lambda a^T}{2}\right)_{\gamma\delta} = \frac{1}{2}\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{1}{6}\delta_{\alpha\beta}\delta_{\gamma\delta}. \tag{2.38}$$

The two terms in (2.37) represent the creation of I = 0 vector meson from a $q\bar{q}$ string and the splitting of a q-$\bar{q}$ string into two q-$\bar{q}$ strings. Both will contribute to the fragmentation of a $q\bar{q}$ jet. Collecting (2.30), (2.33), (2.35) and (2.37) I obtain the equation of motion for a $q\bar{q}g$ string operator, after setting $\vec{x}$ on the line $1 \to 2$,

$$\begin{aligned} i\dot{q}_{\vec{F}_\pm}(1,2;x) &= \left[H_D + k^2\lambda \pm \vec{\nabla}_x \times\right] q_{\vec{F}_\pm}(1,2;x) \\ &+ \frac{i}{6} g \vec{j}_s(x)\, q(1,2) - \frac{i}{2} g\, q(1,x)\vec{\alpha}\, q(x,2) \\ &+ \int_1^2 dz'\, \vec{f}_\pm(x,x')\; q(1,2) \end{aligned}$$

+ (irreducible $q\bar{q}gg$ operators), (2.39)

where the glue ball operator $\vec{f}_\pm$ is defined by

$$\begin{aligned} \vec{f}_\pm(x,x') = P\Big\{ & g_{\vec{F}_\pm E_3}(x,x') + \varepsilon_{3jk}\, g_{\vec{F}_\pm B_j}(x,x') \times \\ & \times \alpha_I(x,x')_k \pm i\,\tau(x,x')\left[g_{F_{\pm 3}\vec{B}}(x,x') - \vec{n}_3\, g_{F_{\pm i} B_i}(x,x')\right]\Big\}. \end{aligned} \tag{2.40}$$

$\vec{n}_3$ is a unit vector in z-direction. $\vec{\alpha}_I(x,x')$ and $\tau(x,x')$ are defined by (2.31) and (2.36) respectively. Figure 1 shows schematically the time development of q(1,2) and $q_{\vec{F}_\pm}(1,2;x)$.

III. Confinement wave equation for q-$\bar{q}$ systems

Equations (2.23) and (2.39) are the coupled equations for q(1,2) and $q_{\vec{F}_\pm}(1,2;x)$, but they are not closed even if the irreducible $q\bar{q}gg$ terms are neglected because of the unknown glue ball operator $g_{\vec{F}_\pm\vec{F}_\pm}(1,2;x)$ in (2.39). A third equation for $g_{\vec{F}_\pm\vec{F}_\pm}$ is necessary to have a closed set. In this paper, I will neglect glue ball emissions, which are represented by products of a glue ball operator g and a q-$\bar{q}$ operator q(1,2). Also by considering a q-$\bar{q}$ system with some flavor quantum number (for instance a u-$\bar{d}$ system) I need not consider the conversion of a q-$\bar{q}$ system into a glue ball, which are represented by products of a glue ball operator g and the vacuum expectation value of q(1,2). This process is certainly important in discussing the spectrum of I = 0, J = 0 mesons. I have to retain products of the vacuum expectation value of $g_{\vec{F}_\pm\vec{F}_\pm}$ and a q-$\bar{q}$ operator q(1,2). I will also neglect the fragmentation terms which may be considered higher order terms as far as the spectrum is concerned. I introduce c-number wave functions

$$\left.\begin{aligned} &\chi(1,2) = (\Psi_0,\ q(1,2)\,\Psi), \\ &\text{and} \\ &\chi_{\vec{F}_\pm}(1,2;x) = (\Psi_0,\ q_{\vec{F}_\pm}(1,2;x)\,\Psi), \end{aligned}\right\} \tag{3.1}$$

which satisfy, after the above mentioned approximations to (2.23) and (2.39),

$$(W - H_D - k^2 r)\,\chi(1,2)$$
$$= \int_1^2 d\vec{x}\cdot\left[\,\chi_{\vec{E}}(1,2;x) + \chi_{\vec{B}}(1,2;x)\times\vec{\alpha}_{S'}\right] \tag{3.2}$$

and

$$(W - H_D - k^2 r \mp \vec{\nabla}_x\times)\,\chi_{\vec{F}_\pm}(1,2;x)$$
$$= I_\pm(1,2;x)\,\chi(1,2). \tag{3.3}$$

Here W is the energy eigenvalue of the state Ψ, and

$$I_\pm(1,2;x) = \int_1^2 \langle \vec{f}_\pm(x,x')\rangle_0\, dz', \tag{3.4}$$

with $\vec{f}_\pm(x,x')$ given by (2.40). In the following I will neglect the velocity dependent terms (terms involving $\vec{\alpha}$) throughout.

At a large distance r, the gluon momentum $\vec{\nabla}_x$ and the kinetic energy $W - H_D$ could be neglected against $k^2 r$, so that

$$\chi_{\vec{E}}(1,2;x) \sim -\frac{1}{2k^2 r}\left[\vec{I}_+(1,2;x) + \vec{I}_-(1,2;x)\right]\chi(1,2). \tag{3.5}$$

Introducing (3.5) into (3.2), I obtain

$$(W - H_D - k^2 r)\,\chi(1,2) = U(r)\,\chi(1,2), \tag{3.6}$$

with

$$U(r) \underset{r\to\infty}{\longrightarrow} -\frac{1}{k^2 r} \iint_1^2 dz\, dz'\, h(x, x'). \tag{3.7}$$

Here

$$\begin{aligned} h(x,x') &= \frac{1}{2}\, \vec{n}_3 \cdot [\vec{f}_+(x,x') + \vec{f}_-(x,x')] \\ &= \langle P[g_{E_3E_3}(x,x') + g_{B_jB_j}(x,x') - g_{B_3B_3}(x,x')]\rangle_0. \end{aligned} \tag{3.8}$$

I have neglected terms proportional to $|x'-x|$ in (2.40), which prove to be unimportant. From the translational invariance of h(x,x') I can write (3.7) as

$$U(r) \underset{r\to\infty}{\longrightarrow} -\frac{2}{k^2 r}\left[r\int_c^r dz - \int_c^r dz\, z\right] h(z,0).$$

Hence if $h(z,0)$ falls off faster than z^{-2} for large z, then

$$U(r) \to -\frac{2}{k^2}\int_c^\infty dz\, h(z,0) + O(1/r). \tag{3.9}$$

For comparison, $\langle E_3(x)E_3(x') + \vec{B}(x)\cdot\vec{B}(x') - B_3(x)B_3(x')\rangle_0$ behaves like $|x-x'|^{-4}$ everywhere for free transversal fields. Then the upper limit of integration (3.9) is convergent, but the lower limit is not. I expect that $\langle g_{\vec{F}_\pm \vec{F}_\pm}(x,x')\rangle_0$ falls off at least as fast, because the phase factor would act destructively. For a small distance, however, the phase factor should be negligible, and from the asymptotic freedom [(6)]

I expect the same singularity $1/z^4$ at $z = 0$, which would lead to an infinite constant in U. Actually, this difficulty is due to the unjustified transition from Eq. (3.3) to (3.5), and can be removed easily. The z^{-4} singularity of g's leads to an infinite constant term of the form $\vec{n}_3 \Lambda^3$ (Λ : cutoff) already on the right hand side of (3.4) and hence of (3.3). Since this term is independent of $\vec{x}$, the use of Eq. (3.5) would lead to $\vec{x}$-independent $\chi_{\vec{E}}(1,2;x)$ which cannot be correct. Eq. (3.3) tells that there is a term $-i\vec{\nabla} \times \chi_{\vec{B}}(1,2;x)$ on the left hand side. The constant term $\vec{n}_3 \Lambda^3$ should be matched with $\chi_{\vec{B}} \sim \vec{n}_3 \times (\vec{x}-\vec{x}_1) \cdot \Lambda^3$, which however does not contribute to (3.2) where $\vec{x}-\vec{x}_1$ is on the z-axis. Thus, a subtraction should be made like

$$h(x,x') \to h(x,x') - \frac{\lambda}{|\vec{x}-\vec{x}'|^4}$$

and λ should be chosen so that there will be no term of order Λ^3 in (3.9).

At a short distance r, the gluon momentum $\vec{\nabla}_x$ will be large on the left hand side of Eq. (3.3). Hence

$$-\vec{\nabla}_x \times \chi_{\vec{E}}(1,2;x) \sim \frac{1}{\alpha}\left[\vec{I}_+(1,2;x) - \vec{I}_-(1,2;x)\right]\chi(1,2)$$

$$= i\int_1^2 dz' P\left\{\langle g_{\vec{B}E_3}(x,x')\rangle_0 + i\tau(x,x')\langle g_{E_3\vec{B}}(x,x')\rangle_0\right\}. \tag{3.11}$$

As $r \to 0$, I can make a replacement

$$\langle P[g_{\vec{B}E_3}(x,x')]\rangle_0 \to \langle P[\vec{B}(x)\cdot E_3(x')]\rangle_0$$

$$= -\frac{i}{\alpha} g^2 C\, \vec{\nabla} \times \left[\vec{n}_3\, \varepsilon(z-z')\, \delta^3(x-x')\right], \tag{3.12}$$

where the last expression was obtained using the free fields for $\vec{E}(x)$ and $\vec{B}(x)$. Introducing (3.12) into (3.11), I obtain

$$\chi_{\vec{E}}(1,2;x) \sim \left\{ -g^2 C \, \vec{n}_3 \int_1^2 \varepsilon(z-z') \, \delta^3(x-x') \, dz' + \vec{\nabla} y(x) \right\} \chi(1,2). \tag{3.13}$$

To determine the unknown function y, I invoke the Gauss' law (2.19). Using the exact expression for $q_{\vec{E}}(1,2;x)$ as given by (2.26'), I notice that $U[2,x)$ can be moved all the way to the right of all the operators in the trace, because $U[2,x)$ commutes with $\vec{E}^a(x)$ in $\vec{E}(x)$. Thus, in (3.1) $\vec{E}(x)$ can be considered to operate directly on the vacuum to the left. Hence, I may use (2.19) and find

$$\vec{\nabla} \cdot \chi_{\vec{E}}(1,2;x) = \left(\Psi_0, \mathrm{tr}^c [U(2,x) \rho(x) U(x,1) q(1) q^\dagger(2)] \Psi \right) \sim 0. \tag{3.14}$$

The last step is an approximation consistent with neglecting the fragmentation terms in going from (2.39) to (3.3). Combining (3.13) and (3.14) I have

$$\nabla^2 y(x) = g^2 C \frac{\partial}{\partial z} \int_1^2 \varepsilon(z-z') \, \delta^3(x-x') \, dz'$$

$$= g^2 C \left[\delta^3(x-x_2) + \delta^3(x-x_1) \right]$$

Hence

$$y(x) = -\frac{g^2 C}{4\pi} \left[\frac{1}{|\vec{x}-\vec{x}_2|} + \frac{1}{|\vec{x}-\vec{x}_1|} \right]. \tag{3.15}$$

Introducing this back into (3.13) I find

$$U(r) = \int_1^2 \chi_{\vec{E}}(1,2;x)\cdot d\vec{x} = 0.$$

The mystery of the disappearance of the Coulomb potential can be solved by again going back to (2.26') which must be used in defining $\chi_{\vec{E}}(1,2;x)$ Then a careful analysis shows that in Eq. (3.11) only the integration region $z' \lessgtr z$ gives non-vanishing contributions. Hence, instead of (3.12) I should have

$$\langle P[g_{\vec{B}E_3}(x,x')]\rangle_0 \to -\frac{i}{2} g^2 C\, \vec{\nabla} \times [\vec{n}_3\, \theta(z-z')\, \delta^3(x-x')]. \qquad (3.16)$$

Going through the same argument I obtain, instead of (3.13),

$$\chi_{\vec{E}}(1,2;x) \sim \left\{ -g^2 C\, \vec{n}_3 \int_1^2 \theta(z-z')\, \delta^3(x-x')\, dz' + \vec{\nabla} y(x) \right\} \chi(1,2), \qquad (3.17)$$

and instead of (3.15)

$$y(x) = -\frac{g^2 C}{4\pi} \frac{1}{|\vec{x}-\vec{x}_1|}\,. \qquad (3.18)$$

Finally from (3.15) I obtain

$$U(r) = -k^2 r - \frac{g^2 C}{4\pi}\left(\frac{1}{r} - \Lambda\right), \qquad (3.19)$$

where the constant term represents the electrostatic self-energy.

Thus, I have recovered the Coulomb potential, but I have yet to consider the charge renormalization. Since the first term of (3.19) is already finite any multiplicative renormalization constant must be finite. If the same renormalization multiplies the second term, then the Coulomb potential would vanish because $g^2 \sim 0$. Hence in order to obtain a non-vanishing Coulomb potential, the two tems in (3.19) must be renormalized in different ways. I do not know how this could be done. Summarizing, I have established a wave equation for a q-$\bar{q}$ system, which is

$$\left[i \frac{\partial}{\partial t} - H_D - V(r) \right] \chi(1,2) = 0, \tag{3.20}$$

where

$$V(r) \to k^2 r \qquad (r \to \infty).$$

The behavior of V(r) as $r \to 0$ is not very clear. The conventional wisdom will tell us to take as the electrostatic part of V(r)

$$V(r) \to -\frac{\alpha_s}{r} \qquad (r \to 0),$$

where α_s is the running coupling constant. This would require different renormalizations for two terms in (3.19). The solutions of Eq. (3.20) for the light-quark systems have been given by D.A. Geffen and myself. [7]

IV. Fragmentation equation

The string operators can be used to represent various jets if we use momentum representations for the end points and the vertices. Thus, q(1,2) describes a q-$\bar{q}$ jet, $q_{\vec{F}}(1,2;x)$ a q-$\bar{q}$-gluon jet, and $g_{FF}(x,x')$ a two-gluon jet, and so forth. The equations of motion for the string operators as derived in section II, can then be used to obtain transition probabilities for a jet to fragment into different channels. For example, the coupled equations (2.23) and (2.39) describe the fragmentation of a q-$\bar{q}$ jet through splitting, emission of a I = 0 vector meson, or emission of a glue ball. Just to illustrate how this is done, I will retain only the splitting term in (2.39). Assuming the splitting vertex does not move very fast I get from (2.39)

$$q_{\vec{E}}(1,2;x) \sim -\frac{i}{2}\, g\left(i\frac{\partial}{\partial t}-H_D-k^2\lambda\right)^{-1} q(1,x)\,\vec{\alpha}\, q(x,2), \tag{4.1}$$

and $q_{\vec{B}}(1,2;x) \sim 0$. The singularity of the denominator should be avoided in the ordinary way. Introducing (4.1) into (2.23),

$$\begin{aligned}&\left(i\frac{\partial}{\partial t}-H_D-k^2\lambda\right) q(1,2)\\ &= -\frac{ig}{2}\int_1^2 d\vec{x}\cdot\left(i\frac{\partial}{\partial t}-H_D-k^2\lambda\right)^{-1} q(1,x)\,\vec{\alpha}\, q(x,2).\end{aligned} \tag{4.2}$$

The right hand side represents a string-splitting interaction. I infer the matrix element M for the parent string to split into two to be

$$\begin{aligned}M = -\frac{ig}{2}\int_{-\infty}^{\infty} dt \iint d^3x_1\, d^3x_2 \int_1^2 d\vec{x}\cdot u^+(1,2)\, \times\\ \times \left(i\frac{\partial}{\partial t}-H_D-k^2\lambda\right)^{-1} u(1,x)\,\vec{\alpha}\, u(x,2).\end{aligned} \tag{4.3}$$

where u(1,2), u(1,χ) and u(χ,2) are the time-dependent c-number wave functions of the parent and two daughter strings. There are certain problems in this equation.

It develops that if u(1,2) is taken to be a product of two plane waves then the transition probability is inversely proportional to the linear dimension of the space. The difficulty is not essential because any q-$\bar{q}$ state of energy E has a natural boundary given by $r = E/k^2$ due to the confining potential[7]. How the renormalization of the coupling constant g can be done is a problem as already mentioned at the end of section III. Finally, the non-covariant character of (4.3) may not be just apparent. In spite of these difficulties, (4.3) may be used to evaluate relative branching ratios for fragmentation into different flavor states. Similarly to Eq. (4.3), the matrix element for creation of a I = 0 vector meson V is given by

$$M = \frac{ig}{\hbar} f_V \int_{-\infty}^{\infty} dt \iint d^3x_1\, d^3x_2 \int_1^2 d\vec{X} \cdot u^+(1,2) \times \times \left(i\frac{\partial}{\partial t} - H_D - k^2 \lambda \right)^{-1} \vec{V}(x)\, u'(1,2). \tag{4.4}$$

Here u(1,2) and u'(1,2) are the wave functions of the original and resultant strings, and $\vec{V}(x)$ is the wave function for the vector meson. f_V is the coupling of $\vec{V}(x)$ to the current $\vec{j}_s(x)$.

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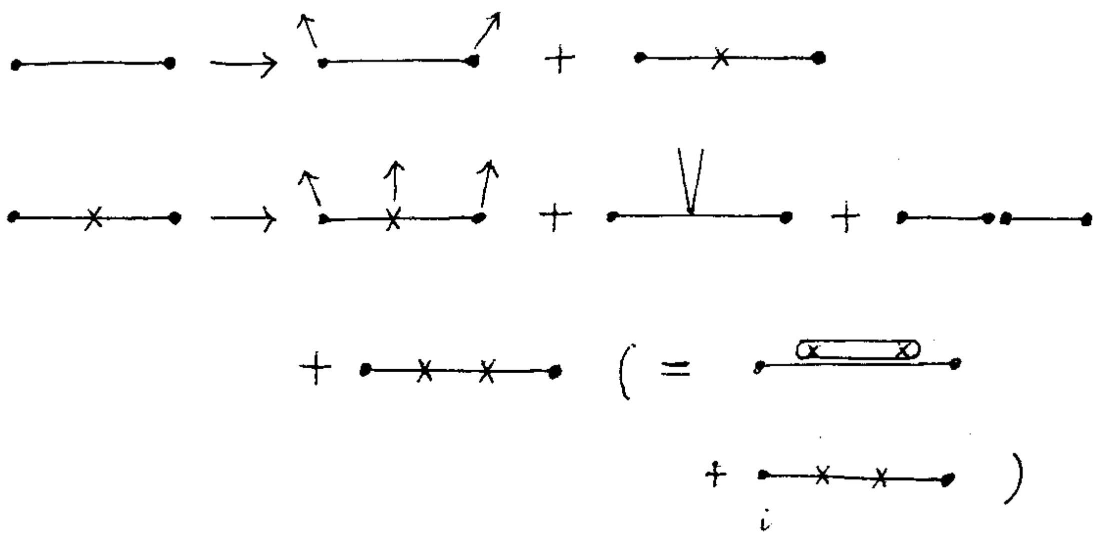

Fig. 1

Diagramatical representation of the time development of a q-$\bar{q}$ string and a q-$\bar{q}$-gluon string as given by equations (2.23) and (2.39), respectively. The end points (dots) represent a quark and an antiquark and the vertices (crosses) the gluon field. The diagrams for a q-$\bar{q}$-gluon string represent kinetic energies, creation of a vector meson, splitting of the string and creation of a second vertex in that order. The last reduces to the sum of emission of a glue ball and an irreducible $q\bar{q}gg$ operator.