**DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY**

DESY 78/68
November 1978



# COMPARISON OF LATTICE GAUGE THEORIES WITH GAUGE GROUPS $Z_2$ AND SU(2)

by

G. Mack and V. B. Petkova

*II. Institut für Theoretische Physik der Universität Hamburg*

NOTKESTRASSE 85 · 2 HAMBURG 52

To be sure that your preprints are promptly included in the
HIGH ENERGY PHYSICS INDEX ,
send them to the following address ( if possible by air mail ) :

DESY
Bibliothek
Notkestrasse 85
2 Hamburg 52
Germany

Comparison of lattice gauge theories with gauge groups $Z_2$ and $SU(2)^+$

G.Mack and V.B.Petkova*

II. Institut für Theoretische Physik der Universität Hamburg

Abstract: We study a model of a pure Yang Mills theory with gauge group SU(2) on a lattice in Euclidean space. We compare it with the model obtained by restricting variables to $\mathbf{Z}_2$. An inequality relating expectation values of the Wilson loop integral in the two theories is established. It shows that confinement of static quarks is true in our SU(2) model whenever it holds for the corresponding $\mathbf{Z}_2$ -model. The SU(2) model is shown to have high and low temperature phases that are distinguished by a qualitatively different behavior of the t'Hooft disorder parameter.

+ Work supported by Deutsche Forschungsgemeinschaft

* permanent address: Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia 1113 / Bulgaria

## 1. Introduction and discussion of results.

We will study Euclidean gauge field theories on a hypercubic lattice $\Lambda\subset\mathbb{Z}^\nu$ in $\nu$ dimensions, $\nu$ = (2), 3, 4, with gauge groups G = SU(2) or $\Gamma = Z_2$. $\Gamma$ is the center of G, i.e. its elements commute with all elements of G. *)

We only consider pure Yang Mills theories without charged fields. The (random) variables of the theory are the socalled "string bit variables" $U[b]\in G$ resp. $\Gamma$ : A configuration $U$ is a map which assigns an element $U[b]\in G$ resp. $\Gamma$ to every directed link $b$ between nearest neighbour vertices x,y on the lattice. $U[b]\to U[b]^{-1}$ under reversal of direction of the link b.

If $C$ is an oriented path (with prescribed initial point) consisting of links $b_1 \dots b_n$ then we write

$$U[C] = U[b_n]\cdots U[b_1] \tag{1.1a}$$

In particular, a plaquette $p$ (=2-dimensional unit cell) has a boundary $\dot{p}\equiv\partial p$ consisting of four links $b_1\dots b_4$. So

$$U[\dot{p}] = U[b_4]\cdots U[b_1] \tag{1.1b}$$

$U[C]$ is called the parallel transporter around $C$. Functions taking values in $\Gamma$ will hence-forth be named $\gamma$ rather than $U$. The same notations (1.1) are used for them.

The standard model of an SU(2) lattice gauge theory is specified by the Euclidean action [1]

$$L(U) = \sum_p \mathcal{L}(U[\dot{p}]) \text{ with } \mathcal{L}(V)=\beta\chi(V) \text{ for } V\in SU(2) \tag{1.2}$$

Sum over $p$ is over all plaquettes $p$ in the lattice $\Lambda$. Their orientation is immaterial since $\mathcal{L}(V^{-1}) = \mathcal{L}(V)$.

$\chi$ is a character of SU(2) given by

$$\chi(V) = \mathrm{tr}\, V \qquad (V\in SU(2)) \tag{1.3}$$

-----

*) G is the group of all unitary unimodular complex 2 x 2 matrices, and $\Gamma$ may be considered as a subgroup of G consisting of diagonal matrices $\pm\mathbb{1}$. We shall not distinguish in notation between matrices $\gamma=\pm\mathbb{1}$ and numbers $\gamma=\pm1$.

Observables are (real) functions $F(U)$ of the random variables $U[b]$ Their expectation value in the standard model is

$$< F > = \int d\mu(U) F(U) \tag{1.4a}$$

$$d\mu(U) = \frac{1}{Z} e^{L(U)} \prod_b dU[b] \quad ; \quad Z = \int e^{L(U)} \prod_b dU[b] \tag{1.4b}$$

Integrations over $U[b]$ are always over G; $dU[b]$ is normalized Haar measure on G. [ Normalized means $\int_G dU[b] = 1$ ]. The product over $b$ runs over all links in the lattice.

The standard model has the following features:

I ) gauge invariance
II) in the limit $\beta \to \infty$ (or lattice spacing $\to 0$) it goes formally over into the usual SU(2) Yang Mills theory in continuous $\nu$- dimensional Euclidean space time
III)it satisfies Osterwalder Schrader positivity and admits a selfadjoint transfer matrix $T$

Condition III) guarantees that the model specifies a Euclidean quantum field theory (QFT). A positive definite Hilbert space of physical states may be reconstructed and a selfadjoint Hamiltonian can be defined [ 2 ].

Here we propose to study a modified model. It can be shown to have the same features I), II) and III) and is therefore a priori an equally good candidate for lattice approximation to a theory in continuous space time. It is obtained by a change in the measure $d\mu$ which amounts to restricting the admissible configurations $U$ .

Consider a 3-dimensional cell c (=cube) in the lattice $\Lambda$ . Its boundary $\partial c$ consists of six plaquettes $p \in \partial c$ . We restrict the admissible configurations by requiring that *

------

*One can interpolate between the standard model and our modified model by using action $L_\lambda(U) = \Sigma_p \mathcal{L}(U[\dot{p}]) + \Sigma_c \ln \frac{1}{2}\left[1 + \tanh \lambda \prod_{p\in\partial c} \chi(U[\dot{p}])\right]$ , $0 \le \lambda \le \infty$ and no constraint on the configurations. [ $\lambda^{-1}$ can be interpreted as a soliton decay constant in 3 dimensions.]

$$\prod_{p\in\partial c} \chi(U[\dot{p}]) \geq 0 \qquad \text{for all cubes } c \qquad (1.5)$$

If the $U[\dot{p}] \approx \mathbb{1}$ (as one assumes in the formal discussion of the continuum limit [1]) then $\chi(U[\dot{p}]) \approx +2$ and the requirement is satisfied.

The expectation value $\langle F \rangle = \int d\mu(U) F(U)$ of an observable is given by the new measure (with the same action $L(U)$ as before)

$$d\mu(U) = \frac{1}{Z} e^{L(U)} \left\{ \prod_c \theta\left( \prod_{p\in\partial c} \chi(U[\dot{p}]) \right) \right\} \prod_b dU[b]$$
$$Z = \int e^{L(U)} \left\{ \text{as above} \right\} \prod_b dU[b] \qquad (1.6)$$

Product over c is over all cubes in the lattice. The definition of the modified model also includes a specification of boundary conditions. They will be described in the text (Sect.2). In $\nu = 2$ dimensions there are no cubes c and the modified model is equal to the standard one.

We compare this model with the $\mathbb{Z}_2$ gauge theory which is obtained by restricting all variables to $\Gamma$ (same value of $\beta$ ! ) Evidently

$$\chi(\alpha V) = \chi(V)\alpha \qquad \text{for } \alpha = \pm 1 \in \Gamma \qquad (1.7)$$

In particular $\chi(\pm 1) = \pm 2$ . For variables $\gamma[b]$ in $\Gamma$ one has the identity

$$\prod_{p\in\partial c} \gamma[\dot{p}] = 1 \qquad \text{for} \quad \gamma[\dot{p}] \equiv \prod_{b\in\partial p} \gamma[b] \qquad (1.8)$$

As a result, inequalities (1.5) hold automatically when variables $U[\cdot]$ are restricted to $\Gamma$ , and the path measure reduces to that of the standard $\mathbb{Z}_2$ - model [3]

$$d\mu_0(\gamma) = \frac{1}{Z_0} e^{L(\gamma)} \prod_b d\gamma[b]$$
$$L(\gamma) = \sum_p 2\beta\gamma[\dot{p}] \qquad (1.9)$$
$$Z_0 = \int e^{L(\gamma)} \prod_b d\gamma[b]$$

$d\gamma[b]$ is normalized Haar measure on $\Gamma$ . Explicitly

$$\int d\gamma[b](\cdots) = \frac{1}{2} \sum_{\gamma[b]=\pm 1} (\cdots) \qquad (1.10)$$

Expectation values in this $\mathbb{Z}_2$ model will be denoted by $\langle\ \rangle_{\mathbb{Z}_2}$.

Consider a closed path $C$ which is boundary $C = \partial\Xi$ of a 2-dimensional surface $\Xi$ consisting of a certain number of plaquettes $p \in \Xi$. We study the expectation value of the "Wilson loop integral" $\chi(U[C])$. Let us introduce the auxiliary variables

$$\sigma[\dot{p}] = \operatorname{sign} \chi(U[\dot{p}]) = \pm 1 \tag{1.11}$$

The following inequalities will be shown to hold for our modified $SU(2)$-model

$$|\langle\chi(U[C])\rangle| \leq 2\langle\prod_{p\in\Xi}\sigma[\dot{p}]\rangle \leq 2\langle\gamma[C]\rangle_{\mathbb{Z}_2} \tag{1.12}$$

The express-ion in the middle is independent of the choice of $\Xi$ for a given $C$ because of (1.5). The factors 2 arise because $\chi(\pm 1) = \pm 2$. The expression on the right is (twice) the expectation value of the Wilson loop integral in the $\mathbb{Z}_2$-theory.

Let $|\Xi|$ = minimal area of $\Xi$ with boundary $C$. It is known from Wilson's work that static quarks are confined if

$$|\langle\chi(U[C])\rangle| \leq \text{const} \cdot e^{-\alpha|\Xi|} \qquad (\alpha > 0) \tag{1.13}$$

Inequality (1.12) shows that the Wilson criterium for confinement of static quarks is fulfilled for all values of the parameter $\beta$ for which the same is true in the $\mathbb{Z}_2$ gauge theory.

Inequalities (1.12) are proved by noting that the SU(2) model may be reinterpreted * as a $\mathbb{Z}_2$ gauge theory with fluctuating but positive coupling constants. Then Griffiths-Kelly-Sherman (GKS) inequalities [4] can be applied (Sects 2,3).

The result may also be expressed by saying that confinement of static quarks is already obtained by integrating out the variables associated with the center of the gauge group if $\beta < \beta_{co}$, $\beta_{co}$ = critical coupling constant in the $\mathbb{Z}_2$-model. [It is known [3,6] that confinement of static

-------

*Such reinterpretation is also possible for the standard model, but positivity of coupling constants fails there [5] quarks fails in $\mathbb{Z}_2$-models in $\nu = 3,4$ dimensions for $\beta$ *above* some finite $\beta_{co}$. In $\nu = 2$ dimensions $\beta_{co} = \infty$]. In 2 dimensions this result was known [7]; for $\nu = 3,4$ its validity was suggested by work of Yoneya [8] and later also by Glimm and Jaffe [9]; see also Foerster [10]. From now on we restrict attention to 3 and 4 dimensions.

Next we consider the behavior of the model in the limit of high and low temperatures $\beta^{-1} \to \infty$ resp. $0$. We show that the high temperature phase and the low temperature phase are distinguished by a qualitatively different behavior of the t'Hooft disorder parameter [11]. In 3 dimensions, the low temperature phase is also distinguished by the appearance of two QFT superselection sectors associated with solitons.
At high temperatures, the solitons condense into the vacuum; i.e. conservation of their charge is spontaneously broken.

The t'Hooft disorder parameter is the expectation value of the t'Hooft operator. It is defined as follows.

The QFT Hilbert space of physical states consists of wave functions $\Psi(\{U[b]\})$. They depend on variables $U[b]$ with links b in the (Euclidean) time t = o plane $\Sigma$. Their scalar product* is

$$(\Psi_1, \Psi_2) = \frac{1}{Z} \int \prod_{b\in\Sigma} dU[b]\ \bar{\Psi}_1(\{U[b]\}) \Psi_2(\{U[b]\}) \prod_{\dot{p}\in\Sigma} e^{\mathcal{L}(U[\dot{p}])} \tag{1.14}$$

The vacuum state is given by

$$\Omega(\{U[b]\}_{b\in\Sigma}) = \int \prod_{b>0} dU[b] \prod_{p>0} \exp \mathcal{L}(U[\dot{p}]) \tag{1.15}$$

$b>0$ resp. $p>0$ are all links resp. plaquettes in the half space $t>0$, excluding those in $\Sigma$. Note that $U[\dot{p}]$ may involve variables $U[b]$ for $b\in\Sigma$ (cp. (1.1b)). These are not integrated over; instead $\Omega$ depends on them. Observables $\mathcal{F}$ which are of the form described after Eq. (1.3) and which depend only on variables $U[b]$ with $b\in\Sigma$ act on states $\Psi$ as multiplication operators, and one has in this case

$$\langle \mathcal{F} \rangle = (\Omega, \mathcal{F}\Omega)$$

------

* A simpler formula for the scalar product is obtained if a factor $Z^{-1/2} \exp \sum_{p\in\Sigma} \mathcal{L}(U[\dot{p}])$ is incorporated in the wave function $\Psi$. For our purpose the choice (1.14) is however more convenient

Let $S$ be a set of links $b$. Then the t'Hooft operator $\mathcal{B}[S]$ is defined by specifying its action on states $\Psi$

$$(\mathcal{B}[S]\Psi)(\{U[b]\}) = \Psi(\{U[b]\sigma[b]^{-1}\}) \qquad (1.16a)$$

with

$$\sigma[b] = \begin{cases} -1 & \text{for } b \in S \\ 1 & \text{otherwise} \end{cases} \qquad (1.16b)$$

[For gauge groups with arbitrary center $\Gamma$, an operator $\mathcal{B}_\alpha[S]$ is defined for every $\alpha \in \Gamma$ by the same formula, but with $\sigma[b] = \alpha$ for $b \in S$ ]

Its expectation value is defined as

$$< \mathcal{B}[S] > = (\Omega, \mathcal{B}[S]\Omega) \qquad (1.17)$$

For plaquettes $p$ and links $b$ in the $\nu-1$ - dimensional lattice $\Sigma$ (or in the $\nu$-dimensional lattice $\Lambda$ ) we say that

$$p \in \hat{\partial} b \quad \text{if and only if} \quad b \in \partial p \qquad (1.18)$$

$\partial$ is the boundary operator.+ The definition of $\hat{\partial}$ extends to sets $S$ of links (1-chains) in $\Sigma$ etc. in a standard way*. For our purposes (where $\Gamma = \mathbb{Z}_2$ ) a simplified definition may be used: $\hat{\partial} S$ consists precisely of those plaquettes $p$ in $\Sigma$ which contain an odd number of links $b \in S$ in their boundary. In applications we are interested in $S$, $\hat{\partial} S$ of the form shown in Figs.1b,d.

As a consequence of its definition, the t'Hooft operator satisfies the following commutation relations with the multiplication operator $\chi(U[C])$ for closed paths $C$ in $\Sigma$ (t'Hooft algebra) [11]

$$\mathcal{B}[S]\chi(U[C]) = \xi\, \chi(U[C])\mathcal{B}[S] \quad ; \quad \xi = \pm 1 \,. \qquad (1.19)$$

Let $C = \partial \Xi$, $\Xi \subset \Sigma$. Then $\xi = -1$ if $\Xi$ contains an odd number of plaquettes in $\hat{\partial} S$ and $\xi = +1$ otherwise.

-----

* $\hat{\partial}$ is the boundary operator on the dual lattice of $\Sigma$ ( cp. ref. [3a])

+ footnote p. 12.

**Remark:** One may define electric field operators $E^i[b]$ $(i=1,2,3)$ acting on wave functions $\Psi$ by $E^i[b']\Psi(\{U[b]\}) = -i\beta^{-1}\frac{d}{ds}\Psi(\{U_s[b]\})_{s=0}$ where $U_s[b] = e^{-is\tau^i/2}U[b]$ if $b = b'$, and $U_s[b] = U[b]$ otherwise [21]. In this language $\mathcal{B}[S] = \exp -2\pi i\beta \sum_{b\in S} E^3[b]$, and the t'Hooft algebra follows from the canonical commutation relations of $E^i[b']$ with $U[b]$. If $S = \hat{\partial}Y$ (hence $\hat{\partial}S = \emptyset$) then $B[S] = 1$ on gauge invariant states by a Gauss law [5].

Let $|S|$ be the number of links in S and $|\hat{\partial}S|$ the number of plaquettes in $\hat{\partial}S$. We show that in our modified SU(2) model the t'Hooft disorder parameter obeys

$$<\mathcal{B}[S]> \leq \text{const} \cdot e^{-\alpha_1|S|} \qquad (\alpha_1 > 0) \tag{1.20}$$

in the low temperature phase ( $\beta \to \infty$ )

On the other hand, an argument analogous to that leading to inequalities (1.12) - but applied to the model obtained after performing a duality transformation (Sect.4) - implies

$$<\mathcal{B}[S]> \geq <\mathcal{B}[S]>_{\mathbb{Z}_2} \geq 0 \qquad \textit{always} \tag{1.21}$$

It follows that

$$<\mathcal{B}[S]> \geq \text{const}' \cdot e^{-\alpha_2|\hat{\partial}S|} \tag{1.22}$$

in the high temperature phase $(\beta \to 0)$

if we take it for granted that the same is true in the standard $\mathbb{Z}_2$- model. We are interested in S of the form shown in Figs.1b,d. In 4 dimensions, inequality (1.2o) is then an area law while (1.22) implies a perimeter law. t'Hooft has presented a plausibility argument that inequality (1.22) together with a mass gap is a sufficient condition for confinement of static quarks [11]. We see that this condition is not fu'lfilled in the low temperature phase of our model.

We present one more inequality which will help in interpreting our results. Let $T$ be any collection of $|T|$ plaquettes. Then the probability of finding $\chi(U[\dot{p}]) < 0$ at all plaquettes $p \in T$ is bounded in the low temperature phase by

$$< \prod_{p\in T} \theta\left(-\chi\left(U[\dot{p}]\right)\right) > \leq \mathcal{D}(\beta)^{|T|} \tag{1.23}$$

with $\mathcal{D}(\beta) \to 0$ as $\beta \to \infty$.

In 3 dimensions, the $\ell.h.s.$ can be interpreted as the probability of finding solitons at all $p \in T$. It tends to zero even for a single plaquette.

Remark: Result (1.23) remains true for the standard SU(2) model. The proof is the same. Therefore also the probability that inequality (1.5) fails at a given cube c tends to zero.

We will now turn to a discussion of the implications of our results for the question of confinement of static quarks. One cannot conclude from (1.2o) that static quarks are not confined in the low temperature phase. Something about possible mechanisms of confinement can be said, however.

Confinement of static quarks in the high temperature phase of the standard $\mathbb{Z}_2$-model can by explained by a vortex condensation mechanism. This has been know for some time ; see for instance Yoneya's paper [8] or ref. [9]. Let us briefly recapitulate. Let $T$ be a collection of plaquettes in the lattice which is closed in the sense that $\hat{\partial} T$ is empty - that is, T is a closed path ($\nu = 3$) or closed 2-dimensional surf-ace ($\nu = 4$) in the dual lattice of $\Lambda$ (see Fig.2). One says that there is a $\mathbb{Z}_2$- vortex located on $T$ if $\chi[\dot{p}] = -1$ for all $p \in T$. These vortices are the analog of Bloch walls (=Peierls countours) in Ising ferromagnets ( cp. be-low). One may define a free energy $F$ (chemical potential) of a vortex of extension $|T|$ by

$$\beta F = \beta E - S \tag{1.24}$$

with energy E given by $e^{-\beta E} = \langle \prod_{p \in T} \Theta(-\chi[\dot{p}]) \rangle$
and entropy $S = \ln$ (number of vortex configurations T of extension $|T|$)
(To eliminate translational degrees of freedom, $T$ is required to contain a given plaquette $p_0$ ). In the high temperature phase, vortices of large extension $|T|$ are abundant and confine static quarks. At low temperatures $\beta^{-1}$ however, the term $\beta E$ dominates, the free energy $F$ increases proportional to $|T|$ , and the probability of finding a vortex of length $|T|$ in the Gibbs state decreases exponentially with $|T|$ . As a result, these vortices are no longer able to produce an area law (1.13) for the Wilson loop.

Now we turn to our modified SU(2) model. It can be viewed as a $\mathbb{Z}_2$-model with fluctuating coupling constants (Sect 2). The $\mathbb{Z}_2$-vortices still exist of course. We will call them <u>thin</u> vortices because they are only one lattice spacing thick . Thereis such a thin vortex located at T if $\sigma[\dot{p}] \equiv \mathrm{sign}\,\chi(U[\dot{p}]) = -1$ for all plaquettes $p$ in $T$. Since the fluctuating coupling constants turn out to be always smaller than the fixed coupling constants of the corresponding standard $\mathbb{Z}_2$- model (see Sect 2), the energy E of a vortex is lowered. Consequently one expects that thin vortices can confine static quarks in our modified SU(2) model whenever the same is true in the corresponding $\mathbb{Z}_2$ model with which we compare. This matches with our result (1.12) and the discussion following it.

Yoneya has expressed the hope that renormalization of $\mathbb{Z}_2$ coupling constants in SU(2) theories could shift the critical point of the standard $\mathbb{Z}_2$- model to zero temperature and enable the $\mathbb{Z}_2$-vortices to confine static quarks at all temperatures. We see from inequality (1.23) that this hope does not materialize in our model (nor in the standard SU(2) model since (1.23) is also true there). The number of closed paths resp. surfaces $T$ of extension $|T|$ is bounded by $\exp \kappa |T|$, $\kappa$ a constant. Inserting into (1.24) we see that at low temperatures $\mathcal{F}$ still increases proportional to $|T|$; hence long and thin vortices are very rare.

In conclusion then, confinement of static quarks in the low temperature regime by a vortex condensation mechanism would have to involve thick vortices[++]. In contrast with thin vortices, thick vortices need not involve

---

[++] There is also another possible attitude that one can take in view of the different properties of the high and low temperature phase of our model.

Let $\beta_c$ be the maximal value of $\beta$ such that inequality (1.20) does not hold. One could try to construct a continuum theory by letting $\beta \nearrow \beta_c$. Some trace of asymptotic freedom might be preserved by choosing as a Lagrangean [5,3]

$$\mathcal{L}(U) = \beta\chi(U) + \alpha\chi_3(U) \qquad (\alpha > 0)$$

where $\chi = \chi_2$ and $\chi_3$ are the characters of the 2-and 3- dimensional representations of SU(2) respectively. The added term $\alpha\chi_3(U)$ depends on U only through the coset $\tilde{U} \equiv U\Gamma \in G/\Gamma$. It does not change our results. By letting $\alpha \to \infty$ one could force $\tilde{U}[\dot{p}] \approx 1$. Hopefully (?) $\beta_c$ tends to a critical point of the $\mathbb{Z}_2$-theory in this limit. The vector potential could be recovered from $\tilde{U}[b] = U[b]\Gamma$ rather than from $U[b]$ since the Lie algebra of G and of $G/\Gamma$ is the same. Such a theory would appear to have one coupling constant more than expected for QCD; the strength of the confining force (if it survives at all!) is not fixed by the gluon gluon coupling constant. Never-the-less the possibility deserves further investigation.

field configurations with $\chi(U[\dot{p}])$ very different from +2 for any plaquette $p$ .

The importance of thick vortices is also suggested by analogy with ferromagnets. In the work of Migdal, Polyakov, Kadanoff, t'Hooft and others [11,12] the idea is prevalent that one should try to generalize the mechanism which prevents spontaneous magnetization in 2-dimensional ferromagnets with a continuous (global) symmetry group $G$ to gauge theories+++

Spontaneous magnetization in ferromagnets is destroyed by the appearance of Bloch walls. In Ising ferromagnets they are the famous Peierls countours [13]. Spontaneous magnetization breaks down when there is a sizable probability of very large Bloch walls in the Gibbs state. The difference between 2-dimensional and $\geqslant$ 3-dimensional models is that in $\geqslant 3$ dimensions Bloch walls always have a thickness of only few lattice spacings. The free energy of such thin Block walls always increases with their size at low temperatures, even in two dimensions (In one dimension the situation is different) In two dimensions, however, the increase in free energy with size (length L) can be undone by increasing also their thickness $d$ so that the spin direction rotates smoothly across them. (The rate of falloff depends on how fast $d$ has to be increased with $L$ ) This is an old idea [14] A mathematically rigorous proof that it works has been given by Dobrushin and Shlosman [15] .

+++ The analog of the Wilson loop is the 2-point spin correlation function. An exponential fall off corresponds to confinement, where-as spontaneous magnetization would correspond to Debye screening i.e. absence of long range forces between quarks due to a screening mechanism.

Analogy between ferromagnets and gauge theories may be a useful guide. But it is not complete: 1) Consider a classical Heisenberg ferromagnet with 4-dimensional unit spins; they may be identified with variables $U[x] \in SU(2)$. For links $b=(x,y)$ let $U[\dot{b}]=U[x]U[y]^{-1}$. Then $\prod_{b\in\partial p} U[\dot{b}]=1$ for every plaquette $p$ (The product must be taken in the right order.) The analogous relation (1.8) is only true for gauge theories with Abelian gauge group.
2) In ferromagnets with continuous symmetry group, a mass gap alone is sufficient to produce exponential decay of two point spin correlation functions since the Goldstone theorem asserts that there is no spontaneous magnetization. In gauge theories the situation is not so simple. Higgs models with continuous gauge group $G$ and fundamental scalars that transform trivially (only) under a discrete subgroup $\Gamma$ of $G$ can behave similar to ferromagnets with discrete symmetry group $\Gamma$ ,rather than $G$ (cp.e.g.[22]).

According to t'Hooft, vortices in gauge theories should be substituted for Bloch walls in ferromagnets. This suggests again that thick vortices could be essential at low temperature.

In a second paper [23] we will derive a sufficient condition for confinement of static quarks by a vortex condensation mechanism. It admits vortices that are thick at all times at the cost of constraining them to a finite volume $\Lambda_i$ whose complement is not simply connected. It estimates the confining potential V(L) in terms of the change of free energy of a system enclosed in $\Lambda_i$ which is induced by a change of vorticity (= singular gauge transformation applied to boundary conditions on $\partial\Lambda_i$).

---

+(p.7). An n-chain with coefficients in $\mathbb{Z}_2$ is a formal sum $\Sigma_a \gamma_a a$ where $a$ are n-dimensional unit cells, and $\gamma_a \in \mathbb{Z}_2$. In this context it is customary to represent elements $\gamma_a$ of $\mathbb{Z}_2$ by 0, 1 rather than +1, -1, and to write group multiplication as + (addition modulo 2). The boundary of an n-cell is an n-1-chain, e.g. $\partial c = \Sigma p_i$ for a 3-cell c, with sum over the six plaquettes $p_i$ in the boundary of c. To every set S of n-cells a unique n-chain $\sum_{a \in S} a$ is associated, and vice versa (since $\gamma_a = 0,1$). Its boundary $\partial \Sigma_S a = \Sigma_S \partial a$.

## 2. $\mathbb{Z}_2$ gauge theory with fluctuating coupling constants.

To rewrite our modified SU(2) model, let us begin by considering a theory with gauge group $\Gamma = \mathbb{Z}_2$, and with space time dependent coupling constants, on the lattice $\Lambda$.

The action of such a theory is

$$L_K(\gamma) = \sum_p \mathcal{L}_p(\gamma[\dot{p}]) \tag{2.o}$$

$$\mathcal{L}_p(\pm 1) = \pm K_p \quad ; \quad K_p \geqslant 0$$

The coupling constants are given by a function $K$ which assigns a real $K_p \geqslant 0$ to every plaquette $p$. The path measure $d\mu_K(\gamma)$, partition function $Z_K$, and expectation value $\langle F \rangle_K$ of an observable $\mathcal{F}(\gamma)$ are given by

$$\begin{aligned} d\mu_K(\gamma) &= Z_K^{-1}\, e^{L_K(\gamma)} \prod_b d\gamma[b] \\ Z_K &= \int e^{L_K(\gamma)} \prod_b d\gamma[b] \\ \langle F \rangle_K &= \int d\mu_K(\gamma)\,\mathcal{F}(\gamma) \end{aligned} \tag{2.1}$$

(Haar measure $d\gamma$ on $\Gamma$ is defined by (1.1o)).

For lattice $\Lambda$ we take a $\nu$-dimensional hypercube of side length $2^{N_1}\times\ldots\times 2^{N_\nu}$. We impose mixed boundary conditions: cyclic boundary conditions for $\gamma[\dot{p}] = \prod_{b\in\partial p}\gamma[b]$, but not for the variables $\gamma[b]$ themselves. Then the variables $\gamma[\dot{p}]$ are constrained only by the requirement that

$$\prod_{p\in\partial c}\gamma[\dot{p}] = 1 \tag{2.2}$$

for every 3-cell $c$. Local* observables $\mathcal{F}$ are gauge invariant and may be considered as functions $\mathcal{F} = \mathcal{F}(\{\gamma[\dot{p}]\})$ of the gauge invariant variables $\gamma[\dot{p}]$. One may therefore write in place of Eqs (2.1)

$$\begin{aligned} d\mu_K(\gamma) &= Z_K^{-1} \prod_p d\gamma[\dot{p}] \left\{ \prod_c \delta\Big(\prod_{p\in\partial c}\gamma[\dot{p}]\Big)\right\} \exp \sum_p K_p\gamma[\dot{p}] \\ Z_K &= \int \prod_p d\gamma[\dot{p}] \left\{ \text{as above} \right\} \exp \sum_p K_p \gamma[\dot{p}] \\ \langle \mathcal{F} \rangle_K &= \int d\mu_K(\gamma)\,\mathcal{F}(\{\gamma[\dot{p}]\}) \end{aligned} \tag{2.3}$$

------

* Local means here that $\mathcal{F}$ depends only on $\gamma[b]$ with $b \in X \subset \Lambda$, where $X$ does not meet the boundary of $\Lambda$ and may be assumed to be topologically trivial. One should think of measurements in a finite region $X$ whereas the volume $|\Lambda|$ is taken to infinity.

Let us now return to our SU(2) model. We make a variable transformation

$$\mathcal{U}[b] = \mathcal{U}'[b]\gamma[b] \tag{2.4}$$

with $\mathcal{U}'[b] \in G, \gamma[b] \in \Gamma$. We choose $\gamma[b]$ such that

$$\gamma[\dot{p}] = \mathrm{sign}\, \chi(\mathcal{U}[\dot{p}]) \tag{2.5a}$$

or equivalently (since $\chi(\mathcal{U}[\dot{p}]) = \gamma[\dot{p}]\chi(\mathcal{U}'[\dot{p}])$)

$$\chi(\mathcal{U}'[\dot{p}]) \geq 0 \tag{2.5b}$$

As a consequence of the constraint (1.5), $\gamma[\dot{p}]$ defined by (2.5a) satisfies $\prod_{p\in\partial c}\gamma[\dot{p}] = 1$ for all 3-cells. Therefore a suitable function $\gamma = \gamma[b]$ exists.
It need not satisfy cyclic boundary conditions.
We consider the configuration $\mathcal{U}$ as given. We note that requirement (2.5) fixes $\mathcal{U}'$ and $\gamma$ up to a simultaneous gauge transformation in $\Gamma$ :
$\mathcal{U}'[b] \to \sigma[x]\,\mathcal{U}'[b]\,\sigma[y]^{-1}$; $\gamma[b] \to \sigma[x]^{-1}\gamma[b]\sigma[y]$ with $\sigma[\cdot] \in \Gamma$, for b= ( x , y ). As a result, we may sum over $\gamma[\dot{p}]$ and integrate over $\mathcal{U}'[b]$ subject to the constraints (2.2) and (2.5b) instead of integrating over $\mathcal{U}[b]$ subject to the constraint (1.5)

In the new variables, the Lagrangean (1.2) becomes

$$L = \sum_{p} K_p(\mathcal{U}')\gamma[\dot{p}] \tag{2.6a}$$

with

$$K_p(\mathcal{U}') = \beta\chi(\mathcal{U}'[\dot{p}]) \geq 0 \tag{2.6b}$$

Local observables may be reexpressed in terms of the new variables.

$$F(\mathcal{U}) = F_{\mathcal{U}'}(\{\gamma[\dot{p}]\}) \tag{2.7}$$

Eq. (2.6a) is recognized as action of a $\mathbb{Z}_2$ gauge theory with fluctuating but nonnegative coupling constants $K_p$ that depend on the random variables $\mathcal{U}'$ . Partition function and expectation value of local observables may therefore by expressed in terms of those of the $\mathbb{Z}_2$-theory with coupling constants $K = K(U')$

$$Z = \int d\rho\,(U') \tag{a}$$

$$\langle F \rangle = Z^{-1} \int d\rho\,(U') \langle F_{U'} \rangle_{K(U')} \tag{b}$$

$$d\rho(U') = \prod_b dU'[b] \prod_p \theta\left(\chi\,(U'[\dot{p}])\right) Z_{K(U')} \tag{c}$$

(2.8)

As our boundary conditions we choose cyclic boundary conditions for $U'[b]$ and for $\gamma[\dot{p}]$ , but not for $\gamma[b]$ themselves. If we use formulae (2.3) for the $\mathbb{Z}_2$ -theory, variables $\gamma[b]$ do not enter and exact translation invariance is preserved. In terms of the variables of the original SU(2) model, these boundary conditions mean that we impose cyclic boundary conditions on cosets $\bar{U}[b] \equiv U[b]\Gamma \in G/\Gamma$ and on sign $\chi(U[\dot{p}])$ only, but not on $U[b]$ themselves. This is a mixture of free and cyclic boundary conditions.

## 3. Inequalities

In this section we shall derive the inequalities (1.12) between correlation functions in our modified SU(2) model and the $\mathbb{Z}_2$-model.

The $\mathbb{Z}_2$- model (1.9) is a special case of the models considered at the beginning of Sec.2.

$$d\mu_0(\gamma) = d\mu_{K^0}(\gamma) \quad ; \quad \langle F \rangle_{\mathbb{Z}_2} = \langle F \rangle_{K^0} \quad ; \quad Z_0 = Z_{K^0} \tag{3.o}$$
$$\text{with} \quad K_p^0 = 2\beta \qquad\qquad \text{for all } p$$

If we compare this with the fluctuating coupling constants $K_p(u')$ given by. Eq. (2.6b) we see that

$$0 \leq K_p(u') \leq K_p^0 = 2\beta \tag{3.1}$$

These inequalities will translate into inequalities for the correlation functions by the Griffiths Kelly Sherman (GKS) inequalities [4]

We are interested in the Wilson loop integral. Under the change of variables (2.4),(2.7)

$$\chi(u[c]) = \chi(u'[c])\,\gamma[c] \tag{3.2}$$

If $C = \partial\Xi$ is boundary of the surface (2-chain) $\Xi$ then

$$\gamma[c] = \prod_{p\in\Xi} \gamma[\dot{p}] \tag{3.3}$$

The expectation value becomes by Eqs (2.8)

$$\langle \chi(u[c]) \rangle = \frac{1}{Z} \int d\rho(u')\, \chi(u'[c]) \langle \prod_{p\in\Xi} \gamma[\dot{p}] \rangle_{K(u')}$$

$\rho$ is a positive measure, and $|\chi(u')| \leq \chi(\mathbb{1}) = 2$. Therefore it follows that

$$|\langle \chi(u[c]) \rangle| \leq \frac{2}{Z} \int d\rho(u') \,|\langle \prod_{p\in\Xi} \gamma[\dot{p}] \rangle_{K(u')}| \tag{3.4}$$

Now we apply the GKS inequalities to the expectation values $\langle\ \rangle_K$. The first and second Griffiths inequality give respectively

I) $\quad \langle \gamma[c] \rangle_K \geq 0 \qquad\qquad$ (3.5 I)

II) $\quad \langle \gamma[c] \rangle_K \leq \langle \gamma[c] \rangle_{K^0} \qquad\qquad$ (3.5 II)

when inequalities (3.1) hold.

Inserting this in (3.4) gives

$$|<\chi(u[c])>| \leq \frac{2}{Z}\int d\rho(u')<\prod_{p\in\Xi}\gamma[\dot{p}]>_{K(u')} = 2<\prod_{p\in\Xi}\gamma[\dot{p}]>$$

$$\leq \frac{2}{Z}<\gamma[c]>_{K^{o}}\int d\rho(u') = 2<\gamma[c]>_{\mathbb{Z}_2}.$$

We used Eq. (3.3) and, in the last equation, Eqs. (2.8) and (3.o). These are the desired inequalities (1.11).

For the benefit of the reader, let us recall the assertions of the GKS inequalities [3,4] . Consider $N$ sites b with variables $\sigma_b = \pm 1$ attached. Let $\mathcal{B} = \{R, S, \ldots\}$ be a family of subsets of sites and write $\sigma_R = \prod_{b\in R}\sigma_b$ for $R \in \mathcal{B}$. Define

$$< \sigma_R >_K = \Big[\sum_{\{\sigma_b=\pm1\}}\sigma_R \exp\big(\sum_{S\in\mathcal{B}} K_S\sigma_S\big)\Big]\Big[\sum_{\{\sigma_b=\pm1\}}\exp\big(\sum_{S\in\mathcal{B}}K_S\sigma_S\big)\Big]^{-1} \quad (3.6)$$

Then if $K_R \geq 0$ for all $R$ in $\mathcal{B}$

$$\begin{aligned} &\mathrm{I}) \quad < \sigma_R >_K \geq 0 \\ &\mathrm{II}) \quad < \sigma_R\sigma_S >_K \geq < \sigma_R >_K < \sigma_S >_K \end{aligned} \quad (3.7)$$

From the second inequality it follows that

$$\frac{\partial}{\partial K_S} < \sigma_R >_K \geq 0 \quad (3.8)$$

In our application, the sites are links on the lattice $\Lambda$ , and $\mathcal{B}$ consists of all closed paths. The variables $\sigma_b = \gamma[b] \in \mathbb{Z}_2$ . $K_S \neq 0$ only if $S$ is boundary of a plaquette. Inequality (3.8) implies (3.5 II).

## 4. Duality transformation

We consider again the $\mathbb{Z}_2$-theory with coupling constants $K_p$. We perform a duality transformation on it. In this way we derive new expressions for the expectation values $\langle\ \rangle_K$. They can be used in formulae (2.8) Therefore they are useful for our SU(2) model.

The duality transformation for $\mathbb{Z}_2$ gauge theories has been described in detail by Balian, Drouffe, and Itzykson [3,3a]. It amounts simply to a Fourier transformation on the abelian group $\Gamma$. The result can therefore also be stated (and derived) without having to introduce the dual lattice first.

The variables of the dual model take values in the dual group $\hat{\Gamma}$ = group of characters of unitary irreducible representations of $\Gamma$ . For $\Gamma = \mathbb{Z}_2$ there are two such characters , and $\hat{\Gamma} \approx \mathbb{Z}_2$ again. We identify them with numbers $\omega_0 = \pm 1$. The corresponding characters are functions on $\Gamma$ given by

$$\tilde{\omega}_0(\gamma) = \begin{cases} 1 & \text{if } \omega_0 = 1 \\ \gamma & \text{if } \omega_0 = -1 \end{cases} \qquad \text{for } \gamma = \pm 1 \in \Gamma \tag{4.1}$$

A variable $\omega[c]$ is assigned to every 3-cell c of the lattice . It takes values $\omega[c] = \pm 1$. It is convenient to use the coboundary operator $\hat{\partial}$ which is defined by saying that a 3-cell

$$c \in \hat{\partial} p \quad \text{if and only if } p \in \partial c \quad \text{etc.} \tag{4.2}$$

$p$ a plaquette. One writes accordingly

$$\omega[\hat{\partial} p] \equiv \prod_{c \in \hat{\partial} p} \omega[c] = \prod_{\substack{c \\ p \in \partial c}} \omega[c] \tag{4.3}$$

The action of the dually transformed model comes out as

$$\hat{L}_K(\omega) = \sum_p \hat{\mathcal{L}}_p(\omega[\hat{\partial} p]) \tag{4.4a}$$

Summation is still over all plaquettes of the (original) lattice The new Lagrangean $\hat{\mathcal{L}}_p$ is related to the old one by a Fourier transformation

$$e^{\hat{\mathcal{L}}_p(\omega_0)} = \int d\gamma\, e^{\mathcal{L}_p(\gamma)}\, \tilde{\omega}_0(\gamma) \tag{4.4b}$$

Explicitly, if the Lagrangean $\mathcal{L}_p$ of the original model is given by Eq. (2.o) then

$$\hat{\mathcal{L}}_p(\omega_o) = \hat{M}_p + \hat{K}_p \omega_o \quad , \quad \omega_o = \pm 1 \in \hat{\Gamma} \tag{4.4c}$$

$$\hat{K}_p = \tfrac{1}{2} \ln \coth K_p \geqslant 0 \tag{4.5}$$
$$\hat{M}_p = \tfrac{1}{2} \ln (\sinh K_p \cosh K_p)$$

The constants $\hat{M}_p$ must be kept since later on they will depend on other variables $\mathcal{U}'$. From formula (4.5) one sees that positivity of coupling constants $K_p \geqslant 0$ is essential to make the new coupling constants $\hat{K}_p$ come out real. If they were not real, the new path measure would not be positive, and so we would not have a statistical mechanical system.

To write down the new formula for the partition function $Z_K$ of Eq. (2.1) we introduce a Haar measure on $\hat{\Gamma}$ by

$$\int d\omega\, (\cdots) = \sum_{\omega = \pm 1} (\cdots) \tag{4.6}$$

In contrast with (1.1o) we do not normalize this measure. This saves us from having to write many factors of 2.

The new path measure is

$$d\hat{\mu}_K(\omega) = Z_K^{-1} \prod_c d\omega[c] \exp \hat{L}_K(\omega) \tag{4.7}$$

and the new expression for the partition function is

$$Z_K = \int \prod_c d\omega\,[c] \exp \hat{L}_K(\omega) \tag{4.8}$$

The action $\hat{L}_K$ was given in Eqs (4.4). Product over $c$ runs over all 3-cells in the lattice $\Lambda$ .

We will also need the dually transformed formula for the expectation value of a local observable $\mathcal{F}$ . Suppose $\mathcal{F}$ depends on gauge invariant variables $\gamma[\dot{p}]$ associated with plaquettes $p \in Y \subset \Lambda$. Let us write down its Fourier expansion

$$\mathcal{F}\left(\{\gamma[\dot{p}]\}_{p\in Y}\right) = \int \hat{\mathcal{F}}\left(\{\omega_p\}_{p\in Y}\right) \prod_{p\in Y} \left\{ \tilde{\omega}_p(\gamma[\dot{p}])^{-1} d\omega_p \right\} \tag{4.9}$$

Each varaible $\omega_p$ is summed over $\pm 1$ (cp.(4.6)) and $\tilde{\omega}(\gamma[\dot{p}])$ is defined by (4.1). Since $\mathcal{F}$ is only defined for variables $\gamma[\dot{p}]$ satisfying constraints (2.2), $\hat{\mathcal{F}}$ is not unique. The following formulae are true for any choice of $\hat{\mathcal{F}}$ .

The expression for the expectation value of $\mathcal{F}$ becomes after the duality transformation

$$\langle \mathcal{F} \rangle_K = Z_K^{-1} \int \prod_c d\omega[c] \prod_{p\in Y} d\omega'_p \hat{\mathcal{F}}(\{\omega'_p\}) \exp \sum_p \hat{\mathcal{L}}_p(\omega'_p \, \omega[\hat{\partial} p]) \qquad (4.10)$$

If $p$ is not in $Y$ , one must put $\omega'_p = 1$ in the argument of $\hat{\mathcal{L}}_p$ , otherwise $\omega'_p$ is summed over $\pm 1$ by (4.6).

To recognize the dual models as something familiar, one interprets plaquettes $p$ and 3-cells $c$ as elements ot the dual lattice (they become links and vertices resp. plaquettes and links in 3 resp. 4 dimensions, cp. [3] ). This is convenient, since $\hat{\partial}$ becomes the boundary operator on the dual lattice.

Then one sees (cp.ref [3] ) that in $\nu = 3$ dimensions the dual model is an Ising ferromagnet with space time dependent coupling constants $\hat{K}_p$ In $\nu = 4$ dimension it is again a $\mathbb{Z}_2$ gauge theory with new coupling constants $\hat{K}_p$ ; the variables $\omega[c]$ are analog to $\gamma[b]$ since c are links of the dual lattice. So they are gauge-variant.

A point to watch are the boundary conditions. The dual models have purely cycle boundary conditions on variables $\omega[c]$ . Thus in $\nu = 4$ dimensions the dual model differs from the original one not only in the coupling constants but also in the boundary conditions.

This comes about as follows: Starting point of the duality transformation is formula (2.3) which has exact translation invariance (and does not involve variables $\gamma[b]$). So it lives on a lattice $\Lambda$ in which opposite sides of the boundary may be identified to form a torus. The new variables $\omega[c]$ are assigned to cells of this toroidal lattice.

We shall now apply the result (4.1o) to find a new expression for the expectation value $\langle B[S] \rangle = (\Omega, B[S]\Omega)$ of the t'Hooft operator.

To every link b resp.plaquette $p$ in the $t=0$ plane $\Sigma$ there is a unique plaquette $p_b$ resp. cube $c_p$ in the halfspace $t>0$ which has b resp. $p$ in its boundary ( $p_b$ projects from b in the positive time direction, similarly for $c_p$ , compare Fig. 3 below).

From the definition (1.16) of $\mathcal{B}[S]$ and expression (1.15) for the wave function of the vacuum state it follows that

$$< \mathcal{B}[S] > = < \mathcal{F} > \tag{4.11a}$$

where $\mathcal{F}$ is a multiplication operator given by

$$\mathcal{F}\left(\{U[b]\}_{b\in\Sigma}\right) = \exp \sum_{b\in S} \left\{ \mathcal{L}\left(-U[\dot{p}_b]\right) - \mathcal{L}\left(U[\dot{p}_b]\right)\right\} \tag{4.11b}$$

We perform the variable transformation (2.4), (2.7), considering $\mathcal{F}$ as a function $\mathcal{F}_{U'}$ of the variables $\gamma[b]$ of the $\mathbb{Z}_2$-theory. This gives

$$\begin{aligned} \mathcal{F}_{U'}\left(\{\gamma[b]\}_{b\in\Sigma}\right) &= \exp\left\{-2\sum_{b\in S} K_{p_b}(U')\gamma[\dot{p}_b]\right\} \\ &= \prod_{\substack{p=p_b \\ b\in S}} \left[\cosh 2K_p(U') - \gamma[\dot{p}] \sinh 2K_p(U')\right] \end{aligned} \tag{4.12}$$

The Fouriertransform of this is $\hat{F} = \prod\left[\delta_{\omega_p,1}\cosh 2K_p - \delta_{\omega_p,-1}\sinh 2K_p\right]$ One inserts this into (4.10) and carries out the $\omega_p'$- summations. Upon use of Eq. (4.5) relating $\hat{K}_p$ to $K_p$ , the result simplifies to

$$<\mathcal{F}_{U'}>_K = <\prod_{b\in S}\omega[\hat{\partial}p_b]>$$

But

$$\prod_{b\in S}\omega[\hat{\partial}p_b] = \prod_{b\in S}\prod_{c\in\hat{\partial}p_b}\omega[c] = \prod_{p\in\hat{\partial}S}\omega[c_p] \tag{4.13}$$

All other factors $\omega[c] (=\pm 1)$ cancel out, cp. the definition of $\hat{\partial}S$ given in the introduction. Using Eq. (2.8) we obtain our final result

$$<\mathcal{B}[S]> = Z^{-1}\int d\rho(U') < \prod_{p\in\hat{\partial}S}\omega[c_p] >_{K(U')} \tag{4.14}$$

($c_p$ was defined before Eq. (4.11a)).
The expectation value $< >_{K(U')}$ is to be computed with the measure (4.7), viz.

$$\begin{aligned} d\hat{\mu}_K(\omega) &= Z_K^{-1}\prod_c d\omega[c]\exp\sum_p\left\{\hat{M}_p + \hat{K}_p\,\omega[\hat{\partial}p]\right\} \\ \hat{M}_p &= \tfrac{1}{2}\ln\left[\sinh K_p\cosh K_p\right] \\ \hat{K}_p &= \tfrac{1}{2}\ln\coth K_p \\ K_p &\equiv K_p(U') = \beta\chi(U'[\dot{p}]) \geq 0 \end{aligned} \tag{4.15}$$

with $Z_K$ from Eq. (4.8). We used Eqs. (4.4a,c) and (4,5).

For a choice of $S$ as in Fig. 1, the r.h.s of (4.14) is the spin correlation function of the Ising model ($\nu = 3$) resp. the expectation value of the Wilson loop integral of the $\mathbb{Z}_2$ gauge theory ($\nu = 4$) with fluctuating coupling constants into which our model goes by the duality transformations.

Inequalities (3.1) for the old coupling constants $K_p$ imply

$$0 \leq \hat{K}_p^{\circ} \leq \hat{K}_p(u') \tag{4.16}$$

where $\hat{K}_p^{\circ} = \frac{1}{2} \ln \coth 2\beta$ (for all $p$) is the coupling constant for the dually transformed $\mathbb{Z}_2$-model (1.9). Inequalities (1.21) follow from (4.16) by applying GKS inequalities to expressions (4.14). (Note that the duality transformation has reversed the direction of the inequalities).

The wave functions $\hat{\Psi}$ in the QFT Hilbert space of physical states of the dually transformed model are related to wave functions $\Psi$ of the original model by a Fourier transformation.

$$\Psi(\{u'[b], \gamma[\dot{p}]\}) = \int \prod_{p\in\Sigma} \{d\omega_p \, \tilde{\omega}_p(\gamma[\dot{p}])\} \hat{\Psi}(\{u'[b], \omega_p\}_{b,p\in\Sigma}) \tag{4.17}$$

For the ground state wave function $\hat{\Omega}$ an explicit formula analogous to Eq.(1.15) can be given, but it will not be used in this paper. $\hat{\Omega}$ depends on variables $u'[b], \omega_p \equiv \omega[c_p]$ with links b and plaquettes p in the $t = 0$ plane $\Sigma$.

## 5. Osterwalder Schrader positivity

Our model satisfies Osterwalder Schrader (OS ) positivity for reflections in $\nu-1$ - dimensional hyperplanes $\Sigma$ of the lattice just like the standard model. The proof is the same in both cases (see e.g. [2] )
OS-positivity implies chessboard estimates for correlation functions. They will be used in the next section to prove convergence of cluster expansions, and later on in the proof of inequalities (1.23).

We consider time reflections $\theta$, i.e. reflections by the "t=0 plane" $\Sigma$. With periodic boundary conditions it consists of the union $\Sigma$ of the $\nu-1$-dimensional hyperplanes $x^\nu = 0$ and $x^\nu = 2^{N_\nu - 1}$

$$\theta x = (x^1 \dots x^{\nu-1}, -x^\nu) \quad \text{for} \quad x = (x^1 \dots x^\nu) \tag{5.1}$$

We consider gauge invariant observables $F$ . In the language of Sect.2 ( $\mathbb{Z}_2$-theory with fluctuating coupling constants) they are (real) functions

$$F = F\left(\{U'[b], \gamma[\dot{p}]\}\right) \tag{5.2}$$

Time reflections act on them according to

$$\theta F\left(\{U'[b], \gamma[\dot{p}]\}\right) = F\left(\{U'[\theta b], \gamma[\theta\dot{p}]\}\right) \tag{5.3}$$

Osterwalder Schrader positivity is the statement that

$$\langle (\theta F) F \rangle \geqslant 0 \tag{5.4}$$

for all real observables $F$ that depend only on variables $U'[b]$, $\gamma[\dot{p}]$ with b and $p$ in the halfspace $t \geqslant 0$ (viz $0 \leqslant x^\nu \leqslant 2^{N_\nu - 1}$ ). The halfspace contains $\Sigma$ .

OS-positivity (5.4) implies a Schwartz inequality

$$|\langle (\theta F) G \rangle| \leqslant \langle (\theta F) F \rangle^{1/2} \langle (\theta G) G \rangle^{1/2} \tag{5.5}$$

provided $F, G$ only depend on variables in one halfspace , e.g. $t \geqslant 0$ . $\theta F$, $\theta G$ will then depend on the variables in the other halfspace.

Because of invariance under lattice translations and rotations by $90^{\circ}$ one can use reflections $\Theta$ in any $\nu - 1$ dimensional hyperplane ( $x^{i}$ = integer constant) in the lattice.

Let $\mathcal{P}_{2h}$ be the set of all plaquettes in the lattice $\Lambda$ which lie in an "even horizontal" 2-dimensional plane ( $x^3 = 2n^3$, (resp. $x^3 = 2n^3$, $x^4 = 2n^4$), $n_i$ = integer, for $\nu = 3$ (resp. 4)). We consider products of observables of the following special form ( cp. Eq. (1.1b))

$$\prod_{p \in \mathcal{P}_{2h}} \mathcal{F}_p \left( \mathcal{U}'[\dot{p}], \gamma[\dot{p}] \right) \tag{5.6}$$

Choosing a $\nu - 1$ dimensional hyperplane of the form $x^j = n^j$ ( $j \in (1 \ldots \nu)$ , $n^j$ odd if $j$=3 or 4) by which to reflect, such an expression is of the form $(\Theta F)G$ where $\mathcal{F}, G$ meet the requirements stated after (5.4). Therefore one can apply inequality (5.5). Repeating the procedure with different hyperplanes of the form just mentioned, one arrives at the following socalled chessboard estimates

$$\left| \left\langle \prod_{p \in \mathcal{P}_{2h}} \mathcal{F}_p \left( \mathcal{U}'[\dot{p}], \gamma[\dot{p}] \right) \right\rangle \right| \leq \prod_{p' \in \mathcal{P}_{2h}} \left\langle \prod_{p \in \mathcal{P}_{2h}} \mathcal{F}_{p'} \left( \mathcal{U}'[\dot{p}], \gamma[\dot{p}] \right) \right\rangle^{1/|\mathcal{P}_{2h}|} \tag{5.7}$$

$|\mathcal{P}_{2h}|$ is the number of plaquettes in $\mathcal{P}_{2h}$ $[\,|\mathcal{P}_{2h}| = |\mathcal{P}|/2^{\nu-3}\nu(\nu-1)$ if $|\mathcal{P}|$ is the number of all plaquettes in $\Lambda$ ] .

For a comprehensive introduction to Osterwalder Schrader positivity, chessboard estimates and their uses, the reader is referred to ref. [16] . A nice and readable outline is also found in [17] .

## 6. Cluster expansions

We are interested in the behavior of the expectation value of the t'Hooft operator in the SU(2) model at low temperatures, i.e. when $\beta \to \infty$.

For this purpose we use the model in the form obtained after the duality transformation. The random variables are then $U'[b] \in G$, $\omega[c] \in \Gamma$; they are associated with links b resp. 3-dimensional cubes c.

For any set $X$ of such cubes, let

$$\omega[X] = \prod_{c \in X} \omega[c] \tag{6.1a}$$

With this notation we have according to Eq.(4.14)

$$\mathcal{B}[S] = \langle \omega[\hat{\partial} S] \rangle \tag{6.1b}$$

where

$$\langle \omega[X] \rangle = \frac{1}{Z} \int d\rho(U') \langle \omega[X] \rangle_{\kappa(U')} \tag{6.2a}$$

$$\langle \omega[X] \rangle_{\kappa} = \int d\hat{\mu}_{\kappa}(\omega)\, \omega[X] \tag{6.2b}$$

Measures and coupling constants are given by Eqs (4.15), (2.8c) and (4.8).

We note that as $\beta \to \infty$, the coupling constants $\hat{\kappa}_p(U') \to 0$ for almost all values of $U'[\dot{p}]$ (These are subject to constraints (2.5b). This suggests to use cluster expansions.

They are based on writing

$$e^{\hat{\kappa}_p(U')[\omega_0+1]} = f_p(\omega_0)+1 \qquad \text{for} \quad \omega_0 = \pm 1 \in \hat{\Gamma} \tag{6.3}$$

and expanding in products of f's. $f_p$ also depends on $U'[\dot{p}]$ ; we will not indicate this dependence explicitly.

Our strategy is as follows: We use cluster expansions for $\langle \omega[X] \rangle_{\kappa}$ on a finite lattice $\Lambda$ . They are finite sums and no convergence problem arises therefore. Then we integrate over the fluctuating coupling constants (i.e. over $U'$ ) term by term. The result is an expansion for

$\langle\omega[X]\rangle$ and still a finite sum. By using chessboard estimates we derive bounds on the individual terms in this sum which are uniform in the volume $|\Lambda|$ . They show that the expansion continues to converge in the infinite volume limit, for large enough $\beta$ , and they also allow to estimate $\langle\mathcal{B}[S]\rangle$. This produces the bound (1.2o).
Upon inserting decomposition (6.3), the expectation value $\langle\omega[X]\rangle_K$ becomes

$$\langle\omega[X]\rangle_K = \frac{1}{\tilde{Z}_K}\int\prod_c d\omega[c]\,\omega[X]\prod_p\left\{f_p(\omega[\hat{\partial}p])+1\right\} \qquad (6.4)$$

$$\tilde{Z}_K = \int\prod_c d\omega[c]\prod_p\left\{f_p(\omega[\hat{\partial}p])+1\right\} = Z_K\left[\prod_p e^{\hat{M}_p-\hat{K}_p}\right]^{-1} \qquad (6.5)$$

The cluster expansion for $\tilde{Z}_K\langle\omega[X]\rangle_K$ is derived in the standard manner by expanding the product over $p$ of $f_p + 1$ and then partially resumming by using factorization properties of the integrals that arise (cp. e.g. refs. 2,18) The result is as follows:

$$\tilde{Z}_K\langle\omega[X]\rangle_K = \sum_\Delta \tilde{Z}_K[\Lambda\setminus\bar{\Delta}]\int\prod_{c\in\bar{\Delta}}d\omega[c]\,\omega[X]\prod_{p\in\Delta}f_p(\omega[\hat{\partial}p]) \qquad (6.6)$$

Summation is over all sets $\Delta$ of plaquettes with the following property. Let $\bar{\Delta}$ be the union of all open 3-cells c which contain a plaquette $p\in\Delta$ in their boundary. Then $\bar{\Delta}$ is required to contain $X$ , and each connected component of $\bar{\Delta}\cup\Delta$ must intersect $X$ . $\tilde{Z}_K[\Lambda\setminus\bar{\Delta}]$ is the partition function for the lattice $\Lambda\setminus\bar{\Delta}$ . It is defined by Eq. (5.5), but with product over $c$ restricted to cubes not in $\bar{\Delta}$ , and product over $p$ restricted to plaquettes $p$ such that none of the cubes $c\in\hat{\partial}p$ is in $\bar{\Delta}$ .

We will now carry out the integration over fluctuating coupling constants. In order to get rid of the $\Theta$ -function in expression (2.8c) it is convenient to adopt the convention

$$\left.\begin{aligned}\hat{M}_p(u') &= -\infty\\ \hat{K}_p(u') &= 0\end{aligned}\right\}\quad \text{if } \chi(u[\dot{p}])<\infty$$

Otherwise they remain given by Eqs. (4.15). This convention preserves $\hat{K}_p\geq 0$. From (6.2a)

$$\langle\omega[X]\rangle = \sum_\Delta \frac{1}{Z}\int\prod_b dU'[b]\exp\sum_{p'}\left\{\hat{M}_{p'}(u')-\hat{K}_p(u')\right\} \qquad (6.7)$$
$$\tilde{Z}_{K(u')}[\Lambda\setminus\bar{\Delta}]\int\prod_{c\in\bar{\Delta}}d\omega[c]\,\omega[X]\prod_{p\in X}\left\{e^{\hat{K}_p(u')(\omega[\hat{\partial}p]+1)}-1\right\}$$

We have inserted the explicit expressions (2.8c), (6,3) for $d\rho(U')$ and $f_p$. We also used relation (6.5) between $Z_K$ and $\tilde{Z}_K$.

Next we will derive a bound on the individual terms in the sum (6.7) over $\Delta$. Let us write it as

$$< \omega[X] > = \sum_{\Delta} I_\Delta \tag{6.8}$$

We will show that

$$| I_\Delta | \leq C(\beta)^{|\Delta|} \tag{6.9}$$

with $C(\beta)$ independent of $|\Lambda|$ and

$$C(\beta) \to 0 \qquad as \quad \beta \to \infty \tag{6.10}$$

Inserting the integral representation of $\tilde{Z}_K$ in (6.7), the individual terms in the sum may be written in the form

$$\begin{aligned} I_\Delta &= \frac{1}{Z} \int \prod_b dU'[b] \prod_c d\omega[c]\, \omega[X] \\ &\qquad \prod_p \left[ F_p'(\omega[\hat{\partial} p], u'[\dot{p}]) \exp\left\{ \hat{M}_p(u') + \hat{K}_p(u')\omega[\hat{\partial} p] \right\} \right] \\ &= < \omega[X] \prod_p F_p'(\omega[\hat{\partial} p], u'[\dot{p}]) > \end{aligned} \tag{6.11}$$

with

$$F_p' = \begin{cases} 1 - \exp\left\{-(\omega[\hat{\partial} p]+1)\hat{k}_p(u')\right\} & \text{for } p \in \Delta \\ 1 & \text{for } p \in \Lambda \setminus \bar{\Delta} \\ \exp\left\{-(\omega[\hat{\partial} p]+1)\hat{k}_p(u')\right\} & \text{otherwise} \end{cases} \tag{6.12}$$

For future use we define

$$F(U'[\dot{p}]) = 1 - \exp(-2\hat{K}_p)$$

$\hat{K}_p$ depends on $U'[\dot{p}]$ by (4.15). We have $0 \leq F(U'[\dot{p}]) \leq F_p'$ for $p \in \Delta$
We note furthermore that $|\omega[X]| = 1$ and $0 \leq F_p' \leq 1$ for all $p$.
Therefore we may estimate

$$| I_\Delta | \leq < \prod_{p \in \Delta_{2h}} F(u'[\dot{p}]) > \tag{6.13}$$

$\Delta_{2h} = \Delta \cap P_{2h}$ is the set of even horizontal plaquettes in $\Delta$, cp. end of Sect.5. We have the freedom of choice what we call even and what we call horizontal. Therefore we may assume that the number $|\Delta_{2h}|$ of plaquettes in $\Delta$ satisfies

$$|\Delta_{2h}| \geqslant |\Delta| / 2^{\nu-3}\nu(\nu-1) \tag{6.14}$$

We note that $\mp$ depends only on variables $U'$ which are unaffected by the duality transformation. Therefore it does not matter whether one computes the expectation value of the r.h.s. of Eq. (6.13) with the measure obtain-ed after the duality transformation, or with the original measure.Consequen-tly, the chessboard estimates (5.7) of sect. 5 apply. They give

$$|I_\Delta| \leq \left< \prod_{p\in P_{2h}} \mp(U'[\dot{p}]) \right>^{|\Delta_{2h}|/|P_{2h}|} \tag{6.15}$$

It remains to estimate the expectation value

$$\left< \prod_{p\in P_{2h}} \mp(\cdots) \right> = \frac{1}{Z} \int \prod_b dU'[b] \prod_c d\omega[c] \exp \sum_p \{\hat{M}_p + \hat{K}_p \omega[\hat{\partial} p]\} \cdot \\ \cdot \prod_{p\in P_{2h}} \mp(U'[\dot{p}]) \tag{6.16}$$

$Z$ = (same integral with the last factor in the integrand omitted.)

To bound $Z$ from below, we restrict the variables $\omega[c]$ to 1 and the $U'[b]$ - integrations to a region in $G$ such that $\chi(U'[\dot{p}]) \geqslant 2e^{-\vartheta}$. Let $\tau_\vartheta$ be the volume of the subset of SU(2) such that this is true ($\tau_\vartheta < 1$).

From Eqs (4.15)

$$\exp(\hat{M}_p + \hat{K}_p) = \cosh\beta\chi(U'[\dot{p}]) \tag{6.17}$$

Therefore

$$Z \geqslant [\tau_\vartheta \cosh 2\beta e^{-\vartheta}]^{|P|} \tag{6.18}$$

for any $\vartheta > 0$. To bound the numerator from above, we determine the maximum of the integrand. For $p \in P_{2h}$ we have

$$\exp(\hat{M}_p + \hat{K}_p \omega[\hat{\partial} p]) \leq \exp(\hat{M}_p + \hat{K}_p) \leq \cosh 2\beta$$

from (6.17). For $p \in \mathcal{P}_{2h}$

$$\exp(\hat{M}_p + \hat{K}_p)F(U'[\dot{p}]) = e^{\hat{M}_p + \hat{K}_p}[1 - e^{-2\hat{K}_p}]$$
$$= \cosh\beta\chi_p\,[1 - \tanh\beta\chi_p] = e^{-\beta\chi_p} \leq 1$$

by (4.15). Thus

$$\text{numerator} \leq 2^{n|P|}\,[\cosh 2\beta]^{|P| - |P_{2h}|} \quad ; \quad n = 2^{\nu-2}/6$$

The factors of 2 come because each $\omega[c]$ is summed over 2 values, $n|P|$ is the number of 3 dimensional cubes in the lattice. Since $|P_{2h}| = |P|/r$, $r = 2^{\nu-3}\nu(\nu-1)$

$$\text{numerator} \leq 2^{n|P|}\,[\cosh 2\beta]^{s|P|} \quad ; \left( s = 1 - \tfrac{1}{r} < 1 \quad , \quad r = 2^{\nu-3}\nu(\nu-1) \right) \tag{6.18'}$$

Let us define

$$C(\beta) = \min_{\vartheta} \left( \frac{2^n[\cosh 2\beta]^s}{\tau_\vartheta \cosh(2\beta e^{-\vartheta})} \right) \tag{6.19}$$

Since $s<1$, $C(\beta)$ satisfies condition (6.9). Estimate (6.9) follows from inequalities (6.18) and (6.18').

Convergence of the cluster expansion in the infinite volume limit follows now from an estimate on the number $N(|\Delta|)$ of terms in the sum (6.8) with given number $|\Delta|$ of plaquettes in $\Delta$. The estimate obtains as a corollary to the solution of the Königsberg bridge problem+ [19] in essentially the same way as in ref. [18]. One finds that there exists a constant $c$ such that

$$N(|\Delta|) \leq c^{|\Delta| + |X|} \tag{6.20}$$

To estimate $\langle B[S]\rangle = \langle\omega[\hat{\partial}S]\rangle$ we need to find the first nonvanishing term in its expansion. For this it is sufficient to find the first nonvanishing term in expansion (6.6) for $\langle\omega[\hat{\partial}S]\rangle_K$.

In four resp. three dimensions this amounts to the same mathematical problem already solved in determining the leading term in the high temperature expansion for the Wilson loop of a $\mathbb{Z}_2$ gauge theory [2], resp. spin correlation function of an Ising ferromagnet, cp. end of Sect. 4. The result is that $|\Delta| \geq |S|$ if $S$ has minimal extension for given $\hat{\partial}S$ (as in Fig. 1).

-----

+One considers plaquettes $p$ as islands, and cubes $c$ as collections of 5! bridges joining pairs of plaquettes in the boundary of c. In addition one imagines a set $I$ of $|X|$ extra islands which provide for a closed path of bridges consisting of one bridge out of every cube c in $X$. Then the set $I \cup \Delta$ of islands is connected by bridges.

An alternative proof goes as follows. Let $Z$ a union of 3-dimensional cubes. Its boundary $\partial Z$ consists of plaquettes. Let it be chosen so that $\partial Z$ "winds around" $\hat{\partial} S$ in the sense that $\partial Z$ contains precisely one of the plaquettes $p_b$ , $b \in S$ ( $p_b$ projects from link $b \in S$ in time direction) To every $p_b$ there is a suitable $Z$ such that $\partial Z$ contains $p_b$ . An example for a 3-dimensional lattice is shown in Fig. 3. We make a variable transformation $\omega[c] \to -\omega[c]$ for $c$ in $Z$ . Then $\omega[\hat{\partial} p] \to -\omega[\hat{\partial} p]$ if $p \in \partial Z$, otherwise $\omega[\hat{\partial} p]$ remains unchanged. Thus $\omega[\hat{\partial} S] = \prod_{b \in S} \omega[\hat{\partial} p_b]$ (see Eq. (4.13)) changes sign, since exactly one factor has $p_b \in \partial Z$ . Suppose now that $\Delta$ does not have any plaquette in common with $\partial Z$. Then the integrand of $I_\Delta$ is odd under the change of variables and so the integral vanishes. If $\Delta$ has fewer plaquettes than (the minimal choice of) S one can always find a suitable $Z$ such that $\partial Z$ does not interesect $\Delta$ . For S as shown in Fig. 1 this is obvious. Therefore all integrals $I_\Delta$ with $|\Delta| < |S|$ are zero

In the analogous argument for the high temperature expansion of the Wilson loop the analog of $\partial Z$ is the location of a thin vortex as described in the introduction.

We are only interested in sets S such that $|\hat{\partial} S| \leq S$ . The bound (1.2o) on $\langle B[S] \rangle$ follows then immediately from the identification of the leading term in its expansion, together with the bounds (6.2o) and (6.9). One has $|\hat{\partial} S| \leq |S| \leq |\Delta|$, hence $c^{|\Delta| + |X|} \leq c^{2|\Delta|}$ in (6.2o), and convergence of the cluster expansion holds for a range of values of $\beta$ which is independent of $S$ ( so long as $|\hat{\partial} S| \leq |S|$ ).

## 7. Solitons in three dimensions

Gauge invariant wave functions of QFT states depend on $Z_2$-variables $\gamma[b]$ only through gauge invariant "field strengths" $\gamma[\dot{p}] = \prod_{b\in\partial p} \gamma[b]$ Let us define operators $\omega_p$ acting on such states according to

$$\omega_{p'} \Psi(\{u'[b], \gamma[\dot{p}]\}) = \Psi(\{u'[b], \gamma[\dot{p}]\sigma_p\})$$

$$\sigma_p = -1 \quad \text{if } p = p' \qquad \text{and } +1 \text{ otherwise}$$

In terms of the original variables $U[b] = U'[b]\gamma[b]^{-1}$, this means that $\chi(U[\dot{p}]) \rightarrow -\chi(U[\dot{p}])$. We say that configuration $U$ contains a (bare) soliton at $p$ if $\chi(U[\dot{p}]) < 0$ . Thus $\omega_p$ creates or destroys a soliton. Soliton number is conserved modulo 2 because of constraint (1.5). If a soliton enters a cube c through plaquette $p$ , it has to leave it again through another plaquette.

From the definition (1.16a) of $B[S]$ it follows that

$$B[S] = \prod_{p\in\hat{\partial}S} \omega_p$$

This is so because every plaquette $p$ not in $\hat{\partial}S$ contains an even number of links in S and so the substitution (1.16) does not affect $\gamma[\dot{p}]$ .

We see that $B[S]$ is a product of two soliton creation-annihilation operators if S is of the form Fig. 1b.

Under the duality transformation of Sect.4, the model goes over into an Ising ferromagnet with fluctuating coupling constants. Operators $\omega_p$ become multiplication with spin variables $\omega[c_p]$ of this ferromagnet and $\langle B[S]\rangle$ is a 2-point spin correlation function. Convergence of cluster expansion of Sect. 6 shows that the model will behave like an ordinary Ising model at high temperatures when our $\beta \rightarrow \infty$ . Thus there is no breaking of the symmetry under reversal of all spins. The QFT Hilbert space of states decomposes therefore into superselection sectors that are even or odd under the reversal of all spins $\omega[c] \rightarrow -\omega[c]$ .The multiplication operators $\omega[c_p]$ = (soliton creation operators $\omega_p$ ) make transitions between them.

Conversely, if $\beta \rightarrow 0$ the model has spontaneous magnetization. i.e. $\langle \omega_p \rangle \neq 0$ in the pure phases of the system. This follows from inequalities (1.21) and the fact that the ordinary 3 dimensional Ising model has spontaneous magnetization at low temperatures. In other words, the solitons condense into the vacuum.

Thin vortices may be interpreted as "world lines" of soliton pairs in Euclidean space time.

## 8. Inequalities (1.23)

In this section we want to prove inequalities (1.23). The argument will be the same as used in the modern version of the Peierls argument for ferromagnets [16,17,20] . Let $\Delta$ be any set of plaquettes and $\Delta_{2h}$ the set of even horizontal ones among them, as in Sect. 6. We may assume that inequality (6.14) holds. Then we have

$$\begin{aligned} \langle \prod_{p\in\Delta} \Theta(-\chi_p) \rangle &\leq \langle \prod_{p\in\Delta_{2h}} \Theta(-\chi_p) \rangle \\ &\leq \langle \prod_{p\in\mathcal{P}_{2h}} \Theta(-\chi_p) \rangle^{|\Delta_{2h}|/|\mathcal{P}_{2h}|} \end{aligned} \tag{8.1}$$

by chessboard estimates (5.7). We wrote $\chi_p \equiv \chi(U[\dot{p}]) = \chi(U'[\dot{p}])\gamma[\dot{p}]$ . It remains to estimate

$$\langle \prod_{p\in\mathcal{P}_{2h}} \Theta(-\chi_p) \rangle = \frac{1}{Z} \int \prod_b dU'[b]\, d\gamma[b] \prod_{p\in\Lambda} \left\{ \Theta\left(\chi(U'[\dot{p}])\, e^{K_p(U')\gamma[\dot{p}]}\right\} \cdot \right. \\ \cdot \prod_{p\in\mathcal{P}_{2h}} \Theta(-\gamma[\dot{p}]) \tag{8.2}$$

We have used definition (2.5a) and the form of the path measure derived in Sect.2. $Z$ is given by the same integral without the last factor.

The partition function is bounded below by restricting all $\gamma[b]$ to +1 and integrations over $U'$ such that $\chi(U'[\dot{p}]) \geq 2e^{-\vartheta}$ as in Sect. 6. This gives

$$Z \geq 2^{-n|\mathcal{P}|} \left[ \tau_\vartheta \exp(2\beta e^{-\vartheta}) \right]^{|\mathcal{P}|} \tag{8.3}$$

$n|\mathcal{P}|$ = No. of links in $\Lambda$. The numerator is estimated by estimating the maximum of the integrand. If $p \in \mathcal{P}_{2h}$

$$\Theta(-\gamma[\dot{p}])\, e^{K_p(U')\gamma[\dot{p}]} \leq 1 \qquad \text{since} \qquad K_p \geq 0 \, .$$

If $p \notin \mathcal{P}_{2h}$

$$e^{K_p(U')\gamma[\dot{p}]} \leq e^{2\beta}$$

This gives

$$\text{numerator} \leq [e^{2\beta}]^{|\mathcal{P}|-|\mathcal{P}_{2h}|} = [e^{2\beta}]^{s|\mathcal{P}|} , \qquad (s<1) \tag{8.4}$$

with $s$ as in (6.19). Let

$$D(\beta) = \min_{\vartheta} \left[ \frac{e^{2\beta s}\, 2^n}{\tau_\vartheta \exp(2\beta e^{-\vartheta})} \right] \tag{8.5}$$

Since $\vartheta$ can be made small, $D(\beta) \to 0$ as $\beta \to \infty$ .

It follows from inequalities (8.3), (8.4) that $\langle \prod_{p\in \mathcal{P}_{2h}} \Theta(-\chi_p) \rangle \leq \mathcal{D}(\beta)^{|\mathcal{P}|}$. Since $|\Delta_{2h}|/|\mathcal{P}_{2h}| \geq |\Delta|/|\mathcal{P}|$ [by (6.14), and expression for $|\mathcal{P}_{2h}|$ following Eq.(5.7)] , the estimate (1.23) follows then from (8.1).

Acknowledgement: The authors are indebted to M. Lüscher for discussions and to the Deutsche Forschungsgemeinschaft for financial support.

## List of References


1. K.Wilson, Phys.Rev. D 1o (1974) 2445
2. K.Osterwalder, E.Seiler, Ann.Phys. 11o (1978) 44o
3. B.Balian, J.M.Drouffe, C.Itzykson, Phys.Rev. D 1o (1974) 3376; D 11 (1975) 2o98

3a. J.M.Drouffe, Phys. Rev. D 18 (1978) 1174
J.M.Drouffe, C.Itzykson, Phys.Rep. 38C (1978) 133

3b. J.M.Drouffe, (to appear)

4. R.B.Griffiths, J.Math.Phys. 8 (1967) 978
D.G.Kelly and S.Sherman, J.Math.Phys. 9 (1968) 466
J.Ginibre, Comm.Math.Phys. 16 (197o) 31o

5. G.Mack, Phys.Lett. B 78 (1978) 263
6. G.Gallavotti, F.Guerra and S.Miracle-Solé, in Mathematical Problems in Theoretical Physics, ed. G.Dell-Antonio, J.Doplicher and G.Jona-Lasinio, Lecture Notes in Physics, Vol.8o (1978) 436, Springer, Heidelberg.
7. G.Mack, Confinement of Static Quarks in Two Dimensional Lattice Gauge Theories, preprint DESY 78/4o (to appear in Commun.Math. Phys.)
8. T.Yoneya, Z(N) Topological Excitations in Yang-Mills Theories: Duality and confinement, preprint City College of New York,1978
9. J.Glimm, A.Jaffe, Charges, Vortices and Confinement, preprint Harvard, 1978
10. D.Foerster, Phys.Lett. 76B (1978) 597; 77B (1978) 211
11. G. t'Hooft, Nucl.Phys. 138B (1978) 1
12. A.M.Polyakov, Phys.Lett. B 59 (1975) 79; Nucl.Phys. B 12o (1977)429;
A.A.Migdal, Zh.Eksp.Teor.Fiz. 69 (1975) 81o; 69 (1975) 1457
L.Kadanoff, Rev.Mod.Phys. 49 (1977) 267
13. R.Peierls, Proc.Camb.phil. Soc.32 (1936) 477
D.Ruelle, Statistical Mechanics: Rigorous results, Benjamin,New York (1969)
14. M.Fisher, J.Appl, Phys.38 (1967) 981
15. R.L.Dobrushin, S.B. Shlosman, Comm.Math.Phys. 42 (1975) 31
16. J.Fröhlich, R.Israel, E.H.Lieb and B.Simon, Comm.Math.Phys. 62 (1978) 1.

17. B.Simon, New Rigorous Existence Theorems for Phase Transitions in Model Systems., preprint Princeton 1977

18. J.Glimm, A.Jaffe, T.Spencer, in Constructive Quantum Field Theory, G.Velo and A.Wightman, Springer Lecture Notes in Physics,Vol.25(1973)

19. L.Euler, Solutio problematis ad geometriam situs pertinentis, Commentarii Academiae Petropolitanae 8 (1736) 128; engl.transl.: Scientific American Nr. 7 (1953) 66

2o. E.H.Lieb, in Mathematical Problems in Theoretical Physics, ed. G.Dell-Antonio, J.Doplicher and G.Jona-Lasinio, Springer Lecture Notes in Physics, Vol.8o (1978) 59

21. J.Kogut and L.Susskind, Phys.Rev. D 11 (1975) 395
M.Lüscher, Absence of spontaneous gauge symmetry breaking,DESY 77/16

22. M.Einhorn and R.Savits, Phase transitions in the abelian Higgs model preprint Fermilab-Pub- 77/97 - THY (Oct.77)

23. G.Mack, V.B.Petkova, Sufficient condition for confinement of static quarks by a vortex condensation mechanism, DESY 78/69

**Figure captions.**

Fig.1. Argument S of t'Hooft operator.
Sets S of links (heavy lines), and plaquettes in $\hat{\partial}S$ (squares)
Figs. 1a,b are for a $\nu-1=2$ dimensional lattice
Figs. 1c,d are for a $\nu-1=3$ dimensional lattice

Fig.2. Support T of a thin vortex.
A set T of plaquettes in a 3-dimensional lattice $\Lambda$ which is closed in the sense that $\hat{\partial}T = \emptyset$ . That is, every 3-cell in the lattice has an even number of plaquettes $p \in T$ in its boundary.

Fig.3. Illustration to Sect.4 (before (4.11)) and end of Sect.6. Links $b \in S$ (heavy lines), plaquettes $p_b$ projecting from them in time direction, plaquettes in $\hat{\partial}S$ ($p_1$ and $p_2$ in the figure), cubes projecting from them ($c_1$ and $c_2$ in the figure; $c_i \equiv c_{p_i}$ ). In addition, a closed surface $\partial Z$ is shown which contains exactly one plaquette (hatched) with $b \in S$. Drawing for 3 dimensions.

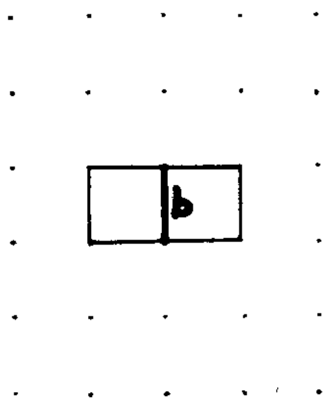


Fig. 1a

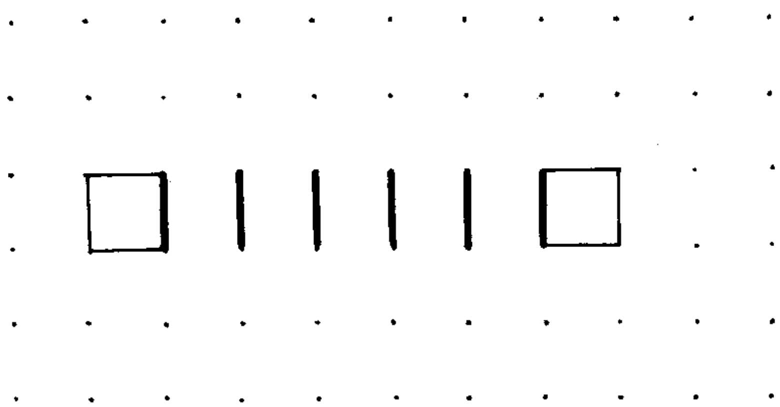

Fig. 1b

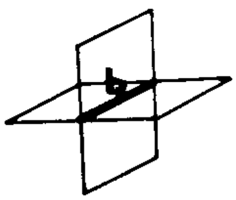


Fig. 1c

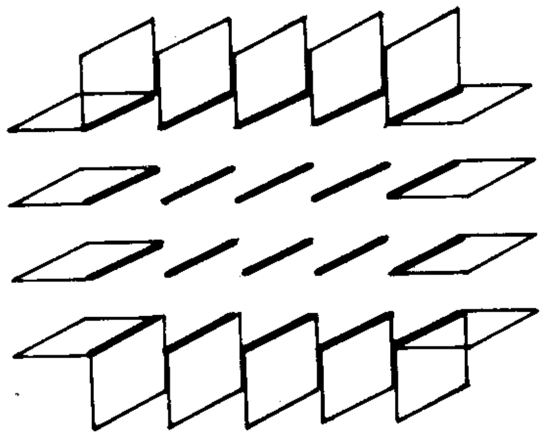

Fig. 1d

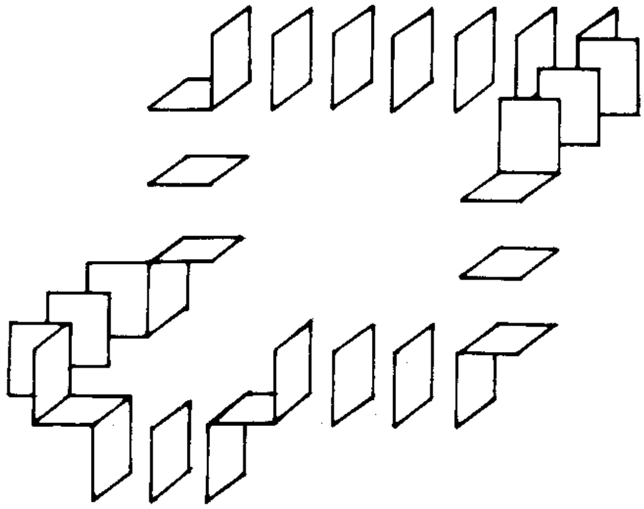

Fig. 2

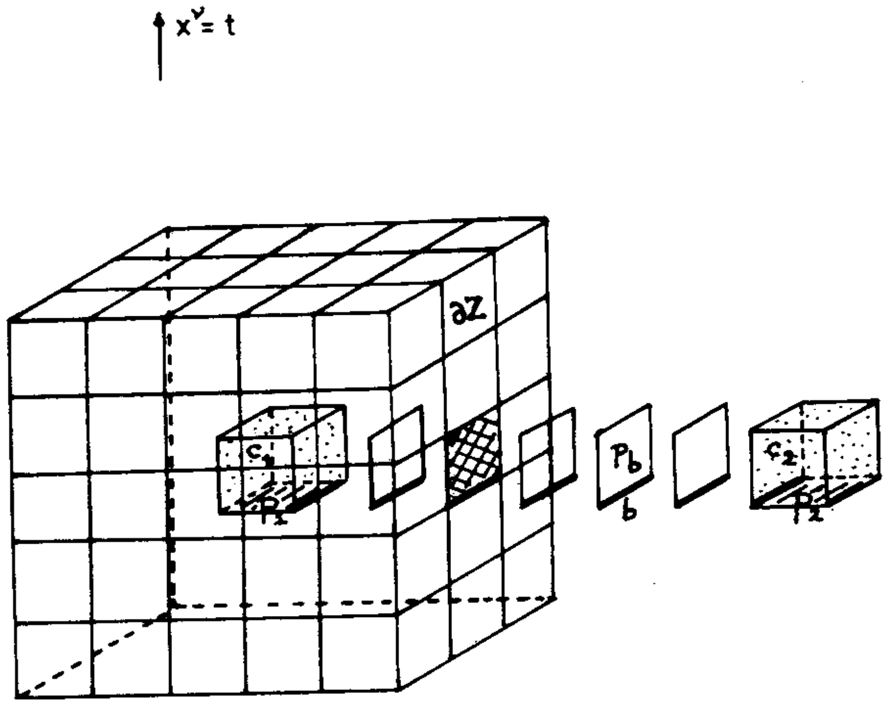


Fig. 3