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# Scattering Amplitudes of the Gross-Neveu and Nonlinear $\sigma$-Models in Higher Orders of the $\frac{1}{N}$-Expansion

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Abstract

The exact S-matrices proposed by Alexander and Alexey Zamolodchikov for the nonlinear $\sigma$-model and Gross-Neveu model are verified to order $\frac{1}{N^2}$ perturbation theory. This provides a good check of the nature of the bound state spectrum.

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The Gross-Neveu (GN) and nonlinear $\sigma$-models (NLS) in two dimensions are described by the Lagrangians

$$\mathcal{L}^{GN} = \sum_{j=1}^{N} \bar{\psi}_j \, i\partial\!\!\!/ \, \psi_j + \frac{1}{2} g \left( \sum_{j=1}^{N} \bar{\psi}_j \psi_j \right)^2$$

$$\mathcal{L}^{NLS} = \frac{1}{2} \sum_{j=1}^{N} (\partial_\mu n_j)^2 \qquad \text{with} \qquad g \sum_{j=1}^{N} n_j^2 = 1 \, .$$

Exact S-matrices for these models were recently proposed by Zamolodchikov and Zamolodchikov [1,2] who analysed the factorization constraints [3] for the case of scattering of an O(N) N-plet of massive particles. Their arguments for identifying the S-matrices obtained by the factorization condition to those of the models given by $\mathcal{L}^{GN}$ and $\mathcal{L}^{NLS}$ relied essentially on a check on lowest order of the $\frac{1}{N}$-expansion. Shortly later it was recognized that the quantum NLS- [4,5] and GN-models [6] possess infinite sets of conservation laws which imply [7] the factorization equations.

In the present note we calculate up to $\frac{1}{N^2}$ the S-matrices of the GN- and NLS-models. Because of the ambiguity in the solution of the factorization equations (which is related to the spectrum), our calculation is a nontrivial check for the correctness of the spectrum of the GN- and NLS-models which is exhibited by the chosen S-matrices. Especially the rich particle spectrum of the GN-model, as determined in the semiclassical approximation [8], is confirmed.

Consider the elastic scattering of an O(N) isovector N-plet of particles $P_i$ of mass m. The S-matrix elements are given by

$$\begin{aligned} {}^{out}\langle P_j(\tilde{p}_1) P_\ell(\tilde{p}_2) | P_i(p_1) P_k(p_2) \rangle^{in} \\ = {}_{ik}S_{j\ell}(\theta,N)\, \delta(\tilde{p}_1' - p_1')\, \delta(\tilde{p}_2' - p_2') \pm {}_{ik}S_{\ell j}(\theta,N)\, \delta(\tilde{p}_1' - p_2')\, \delta(\tilde{p}_2' - p_1') \end{aligned} \tag{1}$$

with $_{ik}S_{jl}(\theta,N) = \sigma_1(\theta,N)\delta_{ik}\delta_{jl} + \sigma_2(\theta,N)\delta_{ij}\delta_{kl} + \sigma_3(\theta,N)\delta_{il}\delta_{jk}$

where $\theta$ the rapidity variable is given by

$$p_1 p_2 = m^2 \operatorname{ch}\theta$$

and the +(-) in (1) refers to bosons (fermions), respectively.

For special models, such as the NLS and the GN, the S-matrix factorizes in terms of two-particle scattering matrices and the S-matrix fulfills severe constraints [3]. Indeed, as Zamolodchikov and Zamolodchikov [1] showed, the amplitude $\sigma_3$ is simply related to $\sigma_2$ (remember crossing: $\sigma_1(i\pi-\theta) = \sigma_3(\theta)$ ) by:

$$\sigma_3(\theta,N) = -\frac{2\pi i}{N-2}\,\frac{\sigma_2(\theta,N)}{\theta} \tag{2}$$

And the general solution of $\sigma_2$ is given by

$$\sigma_2(\theta,N) = \left[\prod_{k=1}^{L}\frac{\operatorname{sh}\theta + i\sin\alpha_k}{\operatorname{sh}\theta - i\sin\alpha_k}\right]\sigma_2^{(0)}(\theta,N)$$

where the real parameters $\alpha_k$ correspond to poles in the physical plane. The minimal solution is given by

$$\sigma_2^{(0)}(\theta,N) = Q(\theta,N)\,Q(i\pi-\theta,N)$$

with

$$Q(\theta,N) = \frac{\Gamma\left(\frac{1}{N-2}-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}-\frac{i\theta}{2\pi}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\frac{1}{N-2}-\frac{i\theta}{2\pi}\right)}$$

For $\frac{1}{N}$-perturbation calculations it is more convenient to cast the solution into the form

$$\ln\sigma_2^{(0)}(\theta,N) = -\int_0^\infty\frac{dt}{t}\,\frac{\operatorname{ch}\frac{t}{4}\left(1+\frac{2i\theta}{\pi}\right)}{\operatorname{ch}\frac{t}{4}}\left(1-e^{-\frac{t}{N-2}}\right) \qquad \text{for } 0<\operatorname{Im}\theta<\pi$$



In the O(N) NLS-model no bound states are expected and, hence,

$$\sigma_2^{NLS}(\theta,N) = \sigma_2^{(0)}(\theta,N)$$

was proposed [1].

Assuming for the U(N) GN-model the qualitative nature of the rich bound state spectrum which was obtained in the semiclassical analysis [8], the exact S-matrix is proposed [2] to be given by

$$\sigma_2^{GN}(\theta,2N) = \frac{\operatorname{sh}\theta + i\sin\frac{\pi}{N-1}}{\operatorname{sh}\theta - i\sin\frac{\pi}{N-1}}\;\sigma_2^{(0)}(\theta,2N)$$

We expand the amplitudes to order $\frac{1}{N^2}$ and obtain for the T-matrix elements

$$T^{NLS}(\theta,N) = 4\operatorname{sh}\theta\left(\sigma_2^{NLS}(\theta,N)-1\right) = -\frac{8\pi i}{N} + \frac{1}{N^2}\left(\chi(\theta)-16\pi i\right) + O(N^{-3}) \tag{3a}$$

$$T^{GN}(\theta,N) = 4\operatorname{sh}\theta\left(\sigma_2^{GN}(\theta,2N)-1\right) = \frac{4\pi i}{N} + \frac{1}{4N^2}\left(\chi(\theta)+16\pi i\right) + O(N^{-3}) \tag{3b}$$

where

$$\chi(\theta) = 2\operatorname{sh}\theta\left[\int_0^\infty dt\,t\,\frac{\operatorname{ch}\frac{t}{4}\left(1+\frac{2i\theta}{\pi}\right)}{\operatorname{ch}\frac{t}{4}} - \frac{4\pi^2}{\operatorname{sh}^2\theta}\right]$$

which has the behaviour

$$\chi(\theta) \approx \frac{16\pi^2}{\theta} \quad \text{as} \quad \theta\to 0\,.$$

at threshold. Since the linearity relation (2), which is a consequence of the conversation laws [4-6], relates $\sigma_3$ to $\sigma_2$, it is sufficient to calculate $T^{GN}$ and $T^{NLS}$ defined in eq. (3).

To first order $\frac{1}{N}$ only the tree diagram (fig. 1) contributes and one obtains:

$$T_{tree}^{NLS}(\theta,N) = -\frac{1}{N}\,8\pi i$$

$$T_{tree}^{GN}(\theta,N) = \frac{1}{N}\,4\pi i$$

in agreement with (3).

In second order $\frac{1}{N^2}$ a variety of graphs contribute (fig. 2). Of these only the box diagrams 2(a) and 2(b) give energy dependent contributions. Due to the asymptotic $(\log k^2)^{-1}$ behaviour of the propagator

$$D^{GN}(k^2) = -\frac{2\pi i}{N} \frac{\operatorname{th}\frac{\phi}{2}}{\phi} \qquad \text{where } k^2 = -4m^2 \operatorname{sh}^2\frac{\phi}{2}$$

$T^{GN}_{Box}$ $(\theta,N)$ is convergent. $T^{NLS}_{Box}$ $(\theta,N)$ diverges as the ultraviolet cut-off parameter $\Lambda \to \infty$. But again due to $(\log k^2)^{-1}$ factor in

$$D^{NLS}(k^2) = \frac{8\pi i}{N} m^2 \frac{\operatorname{sh}\phi}{\phi}$$

it is sufficient to make only one subtraction. First we can show

$$T^{NLS}_{Box} - 4\, T^{GN}_{Box} = \text{const.}$$

Hence, it is sufficient to check only $T^{GN}$ in detail to obtain agreement for $T^{NLS}$ up to a constant. We calculate $T^{GN}_{Box}$ $(\theta,N)$ by introducing the dispersion relation

$$\left[D^{GN}(k^2)\right]^2 = \frac{(2\pi)^2}{N^2} \int_{-\infty}^{\infty} d\phi \left[\frac{\phi}{(\phi^2+\pi^2)^2} \frac{\operatorname{ch}^3\frac{\phi}{2}}{\operatorname{sh}\frac{\phi}{2}} + \frac{1}{\pi^2}\delta(\phi)\right] \frac{4m^2}{k^2 - 4m^2 \operatorname{ch}^2\frac{\phi}{2} + i\varepsilon}$$

and then performing the k-integration. We find

$$T^{GN}_{Box}(\theta,N) = \frac{1}{N^2}\Big\{\frac{1}{4}\chi(\theta) - 16 i\Big[\frac{1}{\pi^2}\ln 2 + \qquad (4)$$

$$+ \int_0^\infty d\phi \frac{\phi}{(\phi^2+\pi^2)^2} \operatorname{ch}^2\frac{\phi}{2}\left(2\operatorname{cth}\frac{\phi}{2}\ln(2\operatorname{ch}\frac{\phi}{2}) - \phi\right)\Big]\Big\} + O(N^{-3})$$

reproducing the energy dependent term in (3b).

Finally we evaluate the constant contribution coming from diagrams (2c) and (2d). They are seperately divergent but their sum is convergent:

$$T^{GN}_{2c+2d}(\theta,N) = \frac{8\pi i}{N^2}\left\{1 + \int_0^\infty d\phi \frac{\operatorname{cth}\frac{\phi}{2}}{\phi^2+\pi^2}\left[\frac{\phi}{2} - \operatorname{cth}\frac{\phi}{2}\ln\operatorname{ch}\frac{\phi}{2}\right]\right\} + O(N^{-3}) \qquad (5)$$



The final contribution comes from the (finite) $Z_2^2$ factor multiplying the one-particle irreducible 4-point function

$$(Z_2^2 - 1)\, T^{GN}_{tree}(\theta,N) = -\frac{16\pi i}{N^2}\left\{\frac{1}{4} + \int_0^\infty d\phi \frac{\operatorname{ch}^2\frac{\phi}{2}}{\phi^2+\pi^2}\left[\frac{\phi}{2}\operatorname{cth}\phi - \ln(2\operatorname{ch}\frac{\phi}{2})\right]\right\} + O(\frac{1}{N^3}) \qquad (6)$$

The $\phi$-integrations can be done by means of Laplace transformations.

Summing up the contributions (4), (5) and (6) we reproduce the Zamolodchikov prediction. Details of the present investigation [9] and the calculation of the form factors [10] will be published elsewhere.

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Figure Captions

Figure 1) : Tree graph contribution to $\sigma_2$

Figure 2) : Contributions in order $\frac{1}{N^2}$ to $\sigma_2$

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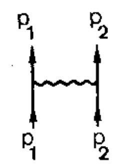


Fig.1

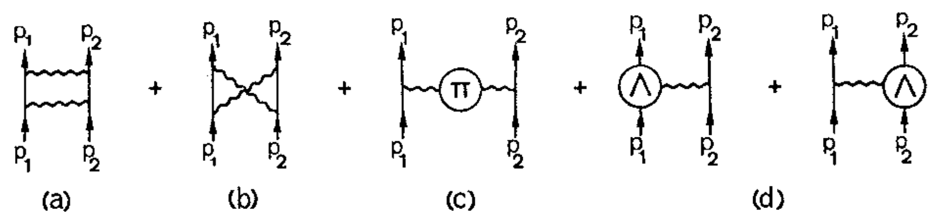


Fig.2