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"n-Soliton" Solutions of the Classical Two Dimensional
Minkowskian O(3) Nonlinear σ -Model

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"n-Soliton" Solutions of the Classical Two Dimensional
Minkowskian $O(3)$ Nonlinear σ -Model

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Abstract: Using the correspondence between the σ -Model and the sine-Gordon field theory, those solutions of the σ -Model are calculated which correspond to the n-soliton solutions of the sine-Gordon equation.

the polar representation of a two dimensional vector field ψ . Applying a Bäcklund transformation yields another field $\hat{\psi}$, and $\hat{\psi}$ is determined by two linear, ordinary differential equations $\hat{\psi}_\xi = A\psi$, $\hat{\psi}_\eta = B\hat{\psi}$. A and B are known explicitly and are essentially functionals of α . The two equations are compatible iff α obeys the s.-G. equation. The solution can be described by a Wronski matrix W . [8]

The Bäcklund transformation depends on a parameter a . For $a = 1$ ($i^2 = -1$) the above Wronski matrix W is intimately related to the normalized field n^a corresponding to the s.-G. field α . Modulo some trivial factors, W is just the spin $\frac{1}{2}$ representation of the rotation R belonging to the field n^a , as described above. This situation is sketched in fig. 1.1 :

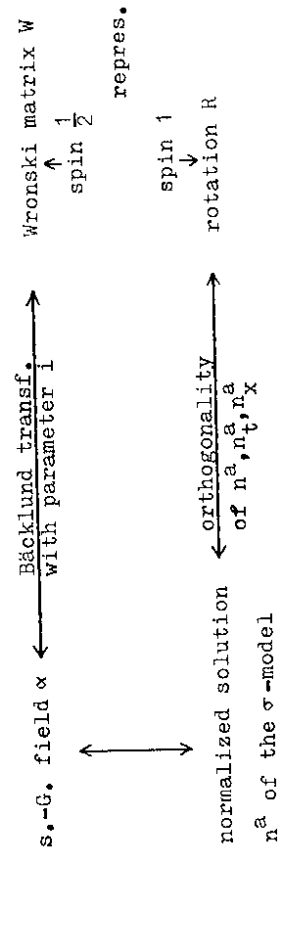


fig. 1.1

Therefore, if the effect of this Bäcklund transformation on the s.-G. field α is known explicitly then so is the normalized field n^a , too. As the so called n-soliton solutions of the s.-G. model are given explicitly [9] and as Bäcklund transformations interpolate between n- and (n+1)-soliton solutions [10], any normalized

1. Introduction

Recently there has been some progress concerning the nonlinear σ -model. All solutions of the two dimensional Minkowskian and Euclidean $O(3)$ nonlinear σ -model have been found that are symmetric under a $SO(2)$ subgroup of the conformal group [1]. An infinite set of conserved charges is given in [2]. In [3] a set of S-matrices is presented which are supposed to belong to the quantized $O(n)$ models.

As for the Euclidean version, the instanton solutions have been found and it was pointed out that the σ -model might be a training area for the non Abelian gauge theories [4]. These instantons are the only solutions with finite action [5]. The structure of the instanton plasma in the σ -model has been investigated in [6].

Here we present all solutions of the two dimensional Minkowskian $O(3)$ model that correspond to the n-soliton solutions of the sine-Gordon (s.-G.) equation.

The $O(3)$ field q^a ($a=1,2,3$) can be reduced to a normalized one n^a by a conformal transformation of the light cone coordinates. This normalized field has constant energy density $\frac{1}{2}$ and momentum density vanishing [7]. The field n^a and its time derivative n^a_t and space derivative n^a_x are an orthogonal dreibein field. Therefore, the field n^a can be represented by a corresponding field R^b of rotation matrices. The normalized fields are, modulo constant rotations, in one-to-one correspondence to the solutions of the s.-G. equation [7].

Let us represent the s.-G. field α as the 4-fold argument of

field of the σ -model corresponding to a n-soliton solution of the s.-G. equation can be calculated.

Section 2 is on the transition from the s.-G. field to the normalized field n^a and the associated field R of rotations. Section 3 is on Bäcklund transformations of the s.-G. field.

In section 4 the normalized field n^a corresponding to any n-soliton solution of the s.-G. equation is constructed.

2. Correspondence between s.-G. and normalized fields and their associated rotations

The normalized field $n^a(\xi, \eta)$ obeys the equation of motion:

$$(2.1.a) \quad n_{\xi\xi}^a + (n_{\xi\eta}^b n_{\eta\xi}^a) n^a = 0, \quad n_{\eta\eta}^a = 1$$

and the normalization condition:

$$(2.1.b) \quad n_{\xi\xi}^a n_{\eta\eta}^a = n_{\xi\eta}^a n_{\eta\xi}^a = 1$$

or, in space-time coordinates $t = \xi + \eta, x = \xi - \eta$:

$$(2.2.a) \quad n_{tt}^a - n_{xx}^a + (n_t^a n_t^a - n_x^a n_x^a) n^a = 0, \quad n^a n^a = 1$$

$$(2.2.b) \quad n_t^a n_t^a + n_x^a n_x^a = 1, \quad n_t^a n_x^a = 0$$

Any solution $q^a(\xi, \eta)$ of the equation of motion (2.1.a) can be written as

$$(2.3) \quad q^a(\xi, \eta) = n^a(a(\xi), b(\eta))$$

where n^a is a normalized solution and $a(\xi)$ and $b(\eta)$ are suitably chosen functions. The energy E of q^a is equal to

$$(2.4) \quad E = \frac{1}{2} \int_{-\infty}^{+\infty} d\tau (a'^2(\tau) + b'^2(\tau))$$

Therefore, if the set of normalized solutions n^a is known then so is the set of all solutions q^a , via (2.3). [6]

To any normalized field n^a belongs a field α :

$$(2.5) \quad \cos \alpha = n_{\xi\xi}^a n_{\eta\eta}^a, \quad \sin \alpha = \epsilon^{abc} n_{\xi}^a n_{\xi}^b n_{\eta}^c$$

which obeys the s.-G. equation:

$$(2.6) \quad \alpha_{\xi\xi\eta} + \sin \alpha = 0$$

Conversely, to any solution α of (2.6) a field n^a exists such that (2.5) and (2.1) hold. [7]

The dreibein field

$$(2.7) \quad R_a^1 = n^a, \quad R_a^2 = \frac{n_t^a}{\cos \frac{\alpha}{2}}, \quad R_a^3 = \frac{n_x^a}{\sin \frac{\alpha}{2}}$$

is an orthonormal system, because of (2.2) and (2.5). Hence, R_a^b is a field of rotation matrices. It obeys the two ordinary, linear differential equations:

$$(2.8.a) \quad R_{\xi} = \left(-I_1 \frac{\alpha_{\xi}}{2} + I_2 \sin \frac{\alpha}{2} - I_3 \cos \frac{\alpha}{2} \right) R$$

$$(2.8.b) \quad R_{\eta} = \left(I_1 \frac{\alpha_{\eta}}{2} - I_2 \sin \frac{\alpha}{2} - I_3 \cos \frac{\alpha}{2} \right) R$$

where I_1, I_2, I_3 denote the 3-dimensional generators of rotation with $[I_1, I_2] = I_3, \dots$. The equations (2.8.a, b) are compatible iff α obeys the s.-G. equation [11].

3. Bäcklund transformations and polar representation

In this section we give a brief review of that part in [8] which is relevant for us. Our notation corresponds to that in

[8], 1502, as follows:
 Our notation : $\xi, \eta, \alpha, \psi^1, \psi^2, a, \wedge$
 Notation in [8]: $x, -t, \varphi, f, g, -a, \prime, \prime$

The polar representation of a column vector $(\psi^1, \psi^2)^T$ ("T" denotes transposition and ψ^1 and ψ^2 are complex numbers) is determined by

$$(3.1) \quad (\psi^1, \psi^2)^T \leftrightarrow \psi^1 + \psi^2 (-i\sigma_2) = \rho \exp\left(\frac{\alpha}{4} (-i\sigma_2)\right)$$

where σ_2 is the second Pauli matrix. In the following we shall use the symbol ψ for the column vector $(\psi^1, \psi^2)^T$ and the associated matrix $\psi^1 + \psi^2 (-i\sigma_2)$, as well.

Let $\psi(\xi, \eta)$ and $\hat{\psi}(\xi, \eta)$ be two fields with a polar decomposition as described in (3.1). Then, for any complex constant a , the following identities hold:

$$(3.2.a) \quad \frac{1}{\hat{\psi}^T \psi} \left(\hat{\psi}_\xi^T \psi - \hat{\psi} \psi_\xi^T + \frac{a^{-1}}{2} \hat{\psi}^T \psi \right) = \\ + \frac{(-i\sigma_2)}{2} \left(\left(\frac{\hat{\alpha} + \alpha}{2} \right)_\xi - a^{-1} \sin\left(\frac{\hat{\alpha} - \alpha}{2}\right) \right),$$

$$\frac{1}{\hat{\psi}^T \psi} \left(\hat{\psi}_\eta \psi - \hat{\psi} \psi_\eta - \frac{a}{2} \hat{\psi}^T \psi^T \right) = \\ (3.2.b) \quad \left(\ln \frac{\hat{\alpha}}{\alpha} \right)_\eta - \frac{a}{2} \cos\left(\frac{\hat{\alpha} + \alpha}{2}\right) + \\ + \frac{(-i\sigma_2)}{2} \left(\left(\frac{\hat{\alpha} - \alpha}{2} \right)_\eta + a \sin\left(\frac{\hat{\alpha} + \alpha}{2}\right) \right)$$

The fields α and $\hat{\alpha}$ are connected by the Bäcklund transformation with parameter a iff:

$$(3.3.a) \quad \left(\frac{\hat{\alpha} + \alpha}{2} \right)_\xi - a^{-1} \sin\left(\frac{\hat{\alpha} - \alpha}{2}\right) = 0 \\ (3.3.b) \quad \left(\frac{\hat{\alpha} - \alpha}{2} \right)_\eta + a \sin\left(\frac{\hat{\alpha} + \alpha}{2}\right) = 0$$

This is obviously the case if, and only if, there exists a complex valued field $f(\xi, \eta)$ such that

$$(3.4.a) \quad \frac{1}{\hat{\psi}^T \psi} \left(\hat{\psi}_\xi^T \psi - \hat{\psi} \psi_\xi^T + \frac{a^{-1}}{2} \hat{\psi}^T \psi \right) = f_\xi \\ (3.4.b) \quad \frac{1}{\hat{\psi}^T \psi} \left(\hat{\psi}_\eta \psi - \hat{\psi} \psi_\eta - \frac{a}{2} \hat{\psi}^T \psi^T \right) = f_\eta$$

Thus, (3.4) describes the Bäcklund transformations in the new ψ -language. One can look upon (3.4) as a system of ordinary, linear differential equations for the column vector $\hat{\psi}$:

$$(3.5.a) \quad \hat{\psi}_\xi = \left((f + \ln f)_\xi + \left(\frac{-\alpha\xi}{2} \right) \left(\frac{\sigma_2}{2I} \right) + \frac{1}{a} \sin\left(\frac{\alpha}{2}\right) \left(\frac{-\sigma_1}{2I} \right) \right) \hat{\psi} \\ - \frac{1}{a} \cos\left(\frac{\alpha}{2}\right) \left(\frac{\sigma_3}{2I} \right) \hat{\psi},$$

$$\hat{\psi}_\eta = \left((f + \ln f) \gamma + \frac{\alpha_1}{2} \left(\frac{\sigma_2}{2i} \right) + ia \sin \left(\frac{\alpha}{2} \right) \left(\frac{-\sigma_1}{2i} \right) + ia \cos \left(\frac{\alpha}{2} \right) \left(\frac{\sigma_3}{2i} \right) \right) \hat{\psi} \quad (3.5.b)$$

The compatibility of this system is equivalent to the statement that α obeys the s.-G. equation. The solution of this system then can be described by a Wronski matrix $W(\{f, \eta\})$.

To transform W into a suitable form we introduce

$$(3.6.) \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and consider $\frac{1}{f} \exp(-f) Q^+ W Q$. It is easy to show that

$$\begin{aligned} & \left(\frac{1}{f} \exp(-f) Q^+ W Q \right)_f = \\ & \left(-\frac{\alpha f}{2} \frac{\sigma_1}{2i} + \frac{1}{a} \sin \left(\frac{\alpha}{2} \right) \frac{\sigma_2}{2i} - \frac{1}{a} \cos \left(\frac{\alpha}{2} \right) \frac{\sigma_3}{2i} \right) \left(\frac{1}{f} \exp(-f) Q^+ W Q \right), \\ & \left(\frac{1}{f} \exp(-f) Q^+ W Q \right)_\eta = \\ & \left(\frac{\alpha \eta}{2} \frac{\sigma_1}{2i} + ia \sin \left(\frac{\alpha}{2} \right) \frac{\sigma_2}{2i} + ia \cos \left(\frac{\alpha}{2} \right) \frac{\sigma_3}{2i} \right) \left(\frac{1}{f} \exp(-f) Q^+ W Q \right) \end{aligned} \quad (3.7.a)$$

4. Construction of the normalized field n^a corresponding to any n-soliton solution of the s.-G. equation

Notice that in the case $a = i$ the matrix $\frac{1}{f} \exp(-f) Q^+ W Q$ can be chosen to be the spin $\frac{1}{2}$ representation of R , as can be seen by comparing (3.7) and (2.8).

In this section the spin $\frac{1}{2}$ representation of R and the normalized field n^a belonging to a n-soliton field α are given explicitly, via the Bäcklund transformation with parameter $a = i$.

A two dimensional vector representation $\psi = (\psi^1, \psi^2)^T$ of any n-soliton solution $\alpha = 4 \arctan(\psi^2/\psi^1)$ is available [9], 1046. The notation in [9] corresponds to ours as follows:

Our notation : $\alpha, \quad m_k, \quad \lambda_k$

Notation in [9] : $-u, \quad -4im_k, \quad -4i \xi_k$

The vector representation $\psi(\{f, \eta; n, m_1, \dots, m_n, \lambda_1, \dots, \lambda_n\})$ of the n-soliton solution α with the parameters $m_1, \dots, m_n, \lambda_1, \dots, \lambda_n$ is:

$$(4.1) \quad \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \det(1-iML \exp(f\Lambda - \frac{\eta}{\Lambda})) + \det(1+iML \exp(f\Lambda - \frac{\eta}{\Lambda})) \\ \frac{1}{i} \det(1-iML \exp(f\Lambda - \frac{\eta}{\Lambda})) - \frac{1}{i} \det(1+iML \exp(f\Lambda - \frac{\eta}{\Lambda})) \end{bmatrix}$$

M, L, Λ are $n \times n$ -matrices:

$$(4.2.a) \quad M = \text{diag}(m_1, \dots, m_n),$$

$$(4.2.b) \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n),$$

$$(4.2.c) \quad L_{ik} = \frac{1}{\lambda_i + \lambda_k}, \quad (i, k = 1, \dots, n)$$

The reality of α is guaranteed by the requirement that m_i and λ_i are either real or can be combined to conjugate complex pairs.

The Bäcklund transform $\hat{\psi} = \psi(\{f, \eta; \hat{n}_1, \hat{m}_1, \dots, \hat{m}_n, \hat{\lambda}_1, \dots, \hat{\lambda}_n\})$, as determined by (3.4), can be described by the new parameters (see appendix):

$$(4.7) \quad \frac{1}{\xi} \exp(-f) Q^+ W Q = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} (u+v) \exp(i\frac{t}{2}), (\bar{u}-\bar{v}) \exp(-i\frac{t}{2}) \\ (-u+v) \exp(i\frac{t}{2}), (\bar{u}+\bar{v}) \exp(-i\frac{t}{2}) \end{bmatrix}$$

The normalized field n^a is just the first row of the corresponding spin 1 representation:

$$(4.8.a) \quad n^1 = (uv \exp(it) + \bar{u}\bar{v} \exp(-it)),$$

$$(4.8.b) \quad n^2 = \frac{1}{i} (uv \exp(it) - \bar{u}\bar{v} \exp(-it)),$$

$$(4.8.c) \quad n^3 = 1 - 2 |u|^2$$

To summarize: Let $\alpha = 4 \arctan(\psi^2/\psi^1)$ be a real n-soliton solution of the s.-G. equation where ψ^1 and ψ^2 are given by (4.1). Then the associated spin $\frac{1}{2}$ representation of R in (2.8) and the corresponding normalized field n^a are given by (4.7) and (4.8) where u and v are the functions in (4.5). Solutions $q^a(\xi, \eta)$ of the equation of motion (2.1.a) (n^a replaced by q^a) with energy finite can be obtained from n^a in (4.8) by the substitution (2.3) where $\int_{-\infty}^{\infty} dx (a'^2(x) + b'^2(x)) < \infty$.

Acknowledgement: I would like to thank H. Lehmann for bringing this problem to my attention.

$$(4.3.a) \quad \hat{n} = n + 1,$$

$$(4.3.b) \quad \hat{m}_i = m_i \left(\frac{a^{-1} + \lambda_i}{a^{-1} - \lambda_i} \right), \quad (i = 1, \dots, n), \quad \hat{m}_{n+1} \text{ arbitrary},$$

$$(4.3.c) \quad \hat{\lambda}_i = \lambda_i, \quad (i = 1, \dots, n), \quad \hat{\lambda}_{n+1} = a^{-1}$$

where the function f in (3.4) is chosen to be $f = \frac{a^{-1}}{2} \xi - \frac{a}{2} \eta$. A Wronski matrix of (3.5) can be found by choosing the two linear independent $\hat{\psi}$ -s:

$$(4.4) \quad W = \frac{1}{\sqrt{2}} \left(\hat{\psi} \Big|_{\hat{m}_{n+1}=2}, \hat{\psi} \Big|_{\hat{m}_{n+1}=-2} \right)$$

To simplify the formulas we use the abbreviations:

$$(4.5.a) \quad u = \frac{1}{\sqrt{2}} \det(1 - iM(\frac{-1+\lambda}{-1-\lambda})L \exp(f\Lambda - \frac{\eta}{\lambda})) \cdot \left| \det^{-1}(1 - iML \exp(f\Lambda - \frac{\eta}{\lambda})) \right|,$$

$$(4.5.b) \quad v = \frac{1}{\sqrt{2}} \det(1 + iM(\frac{-1+\lambda}{-1-\lambda})L \exp(f\Lambda - \frac{\eta}{\lambda})) \cdot \left| \det^{-1}(1 - iML \exp(f\Lambda - \frac{\eta}{\lambda})) \right|$$

When α is real the relation holds:

$$(4.6) \quad |u^2| + |v^2| = 1$$

Using (4.4), $f = \frac{a^{-1}}{2} \xi - \frac{a}{2} \eta$ and $\xi = \left| \det(1 - iML \exp(f\Lambda - \frac{\eta}{\lambda})) \right|$ a spin $\frac{1}{2}$ representation of R in (2.8) is equal to:

Appendix

ψ and $\hat{\psi}$ as determined by (4.1), (4.2) and (4.3) shall be shown to be connected by (3.4) where f is equal to $f = \frac{\alpha-1}{2} \xi - \frac{\alpha}{2} \eta$:

$$(A.1.a) \quad \hat{\psi}_f \psi^T - \hat{\psi} \psi_f^T + \frac{\alpha-1}{2} (\hat{\psi}^T \psi - \hat{\psi} \psi^T) = 0,$$

$$(A.1.b) \quad \hat{\psi}_\eta \psi - \hat{\psi} \psi_\eta - \frac{\alpha}{2} (\hat{\psi}^T \psi^T - \hat{\psi} \psi) = 0$$

Using the determinant formula [12]

$$(A.2) \quad \det \left(\frac{1}{\lambda_i + \lambda_k} \right) = \prod_i \frac{1}{2\lambda_i} \cdot \prod_{i < k} \left(\frac{\lambda_i - \lambda_k}{\lambda_i + \lambda_k} \right)^2$$

we find the following expressions for $\psi = \psi^1 - i\sigma_2 \psi^2$ and $\hat{\psi}$:

$$(A.3.a) \quad \psi = \sum_{\mathbb{I} \subseteq \{1, \dots, n\}} (i\sigma_2)^{|\mathbb{I}|} \cdot \prod_{i \in \mathbb{I}} \frac{m_i}{2\lambda_i} \cdot \prod_{\substack{i < k \\ i, k \in \mathbb{I}}} \left(\frac{\lambda_i - \lambda_k}{\lambda_i + \lambda_k} \right)^2 \cdot \exp \left[\sum_{i \in \mathbb{I}} \left(\xi \lambda_i - \frac{\eta}{\lambda_i} \right) \right],$$

$$(A.3.b) \quad \hat{\psi} = \sum_{\hat{\mathbb{I}} \subseteq \{1, \dots, (n+1)\}} (i\sigma_2)^{|\hat{\mathbb{I}}|} \cdot \prod_{i \in \hat{\mathbb{I}}} \frac{\hat{m}_i}{2\hat{\lambda}_i} \cdot \prod_{\substack{i < k \\ i, k \in \hat{\mathbb{I}}}} \left(\frac{\hat{\lambda}_i - \hat{\lambda}_k}{\hat{\lambda}_i + \hat{\lambda}_k} \right)^2 \cdot \exp \left[\sum_{i \in \hat{\mathbb{I}}} \left(\hat{\xi} \hat{\lambda}_i - \frac{\hat{\eta}}{\hat{\lambda}_i} \right) \right]$$

where the $\hat{m}_i, \hat{\lambda}_i$ are given by (4.3). Insertion of (A.3) into (A.1) yields:

$$\begin{aligned} & \hat{\psi}_f \psi^T - \hat{\psi} \psi_f^T + \frac{\alpha-1}{2} (\hat{\psi}^T \psi - \hat{\psi} \psi^T) = \\ & = \sum_{\substack{\mathbb{I} \subseteq \{1, \dots, (n+1)\} \\ \mathbb{I} \subseteq \{1, \dots, n\}}} (-1)^{|\mathbb{I}|} \left[\sum_{i \in \mathbb{I}} \lambda_i^{-1} \sum_{i \in \mathbb{I}} \lambda_i^{-1} \cdot \frac{1 - (-1)^{|\mathbb{I}|}}{2} \right]. \end{aligned}$$

$$\cdot \prod_{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}} (\alpha^{-1} \lambda_i) \cdot \prod_{i \in \hat{\mathbb{I}} - \mathbb{I} \cap \mathbb{I}} (\alpha^{-1} \lambda_i) \cdot \prod_{i < k} (\lambda_i - \lambda_k)^2 \cdot \prod_{i, k \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}}$$

$$\cdot \prod_{i < k} (\lambda_i - \lambda_k)^2 \cdot \prod_{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}} (\lambda_i + \lambda_k)^2 \cdot \prod_{k \in \hat{\mathbb{I}} - \mathbb{I} \cap \mathbb{I}}$$

$$(A.4.a) \quad \cdot (i\sigma_2)^{|\mathbb{I}| + |\hat{\mathbb{I}}|} \cdot \prod_{i \in \mathbb{I}} \frac{m_i}{2\lambda_i (\alpha^{-1} \lambda_i)} \exp \left(\xi \lambda_i - \frac{\eta}{\lambda_i} \right) \cdot$$

$$\cdot \prod_{i \in \hat{\mathbb{I}}} \frac{\hat{m}_i}{2\hat{\lambda}_i (\alpha^{-1} \lambda_i)} \exp \left(\xi \lambda_i - \frac{\eta}{\lambda_i} \right) \cdot$$

$$\cdot \prod_{i \in \mathbb{I} \cap \hat{\mathbb{I}}} (\alpha^{-1} \lambda_i) (\alpha^{-1} \lambda_i) \cdot \prod_{i < k} \left(\frac{\lambda_i - \lambda_k}{\lambda_i + \lambda_k} \right)^4 \cdot \prod_{i, k \in \mathbb{I} \cap \hat{\mathbb{I}}}$$

$$\cdot \prod_{i < k} (\lambda_i + \lambda_k)^{-2} \cdot \prod_{i \in \mathbb{I} \cup \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} \left(\frac{\lambda_i - \lambda_k}{\lambda_i + \lambda_k} \right)^2 \cdot \prod_{i, k \in \mathbb{I} \cup \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}}$$

and we have to show that (A.5) vanishes. Notice that the terms which are not explicitly written down are constant if $\mathbb{I} \cup \hat{\mathbb{I}} = \text{const.}$ and $\mathbb{I} \cap \hat{\mathbb{I}} = \text{const.}$. (A.4.a) gives nothing new as can be seen by dividing (A.5) with

$$\prod_{i \in \mathbb{I} \cup \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (\lambda_i a^{-1}) \cdot \prod_{\substack{i < k \\ i, k \in \mathbb{I} \cup \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}}} (\lambda_i \lambda_k)^2$$

and substituting $a \rightarrow a^{-1}$, $\lambda_i \rightarrow \lambda_i^{-1}$.

It is sufficient only to consider the cases $\mathbb{I} \cap \hat{\mathbb{I}} = \emptyset$ in (A.5). To write the formulas in a more symmetric form the projection operators

$$(A.6) \quad \mathcal{P}_\lambda : (\mathcal{P}_i \{ (\lambda_1, \dots, \lambda_i, \dots, \lambda_i, \dots) \} = \{ (\lambda_1, \dots, -\lambda_i, \dots) \})$$

are introduced and the condition (A.5) is equivalent to

$$(A.7) \quad \sum_{\hat{\mathbb{I}} \subseteq \{i_1, \dots, (n+1)\}} \prod_{i \in \hat{\mathbb{I}}} \mathcal{P}_i \cdot \left[- \sum_{i \in \mathbb{I} \cup \hat{\mathbb{I}}} \frac{\lambda_i}{\lambda_i} + a \frac{1 - (-1)^{|\mathbb{I}| + |\hat{\mathbb{I}}|}}{2} \right] \cdot \prod_{\substack{i < k \\ i, k \in \mathbb{I} \cup \hat{\mathbb{I}}}} (\lambda_i - \lambda_k)^2 \cdot \prod_{i \in \mathbb{I} \cup \hat{\mathbb{I}}} (a^{-1} + \lambda_i) = 0$$

or

$$(A.8) \quad \prod_{i \in \text{const.} - \{n+1\}} (1 + \mathcal{P}_i) \cdot \left[- \sum_{i \in \text{const.}} \frac{\lambda_i}{\lambda_i} + a \cdot \frac{1 - (-1)^{|\text{const.}|}}{2} \right] \cdot \prod_{i < k \in \text{const.}} (\lambda_i - \lambda_k)^2 \cdot \prod_{i \in \text{const.}} (a^{-1} + \lambda_i) = 0$$

The cases $(n+1) \in \text{const.}$ and $(n+1) \notin \text{const.}$ must be distinguished but, in any case, (A.8) holds iff the following identity is true:

$$(A.9) \quad \prod_{\lambda=1}^m (1 + \mathcal{P}_\lambda) \cdot \left[- \sum_{\lambda=1}^m \frac{1}{\lambda_\lambda} + a \frac{1 - (-1)^m}{2} \right] \cdot \prod_{i < k=1}^m (\lambda_i^{-1} + \lambda_k) \cdot \prod_{i < k=1}^m (\lambda_i - \lambda_k)^2 = 0$$

$$\hat{\psi}_\eta \psi - \hat{\psi}^T \psi \eta - \frac{\alpha}{2} (\hat{\psi}^T \psi^T - \hat{\psi} \psi) = \sum_{\hat{\mathbb{I}} \subseteq \{i_1, \dots, (n+1)\}} \left[- \sum_{i \in \hat{\mathbb{I}}} \frac{\lambda_i}{\lambda_i} + \sum_{i \in \mathbb{I}} \frac{\lambda_i}{\lambda_i} + a \frac{1 - (-1)^{|\hat{\mathbb{I}}|}}{2} \right] \cdot \prod_{i \in \mathbb{I} \cup \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (\lambda_i a^{-1}) \cdot \prod_{\substack{i < k \\ i, k \in \mathbb{I} \cup \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}}} (\lambda_i \lambda_k)^2$$

$$(A.4.b) \quad \prod_{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}} (a^{-1} + \lambda_i) \cdot \prod_{i \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (a^{-1} + \lambda_i) \cdot \prod_{i < k} (\lambda_i - \lambda_k)^2 \cdot \prod_{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}} (\lambda_i + \lambda_k)^2 \cdot \prod_{k \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (\lambda_i + \lambda_k)^2$$

... (as in (A.4.a))

Collecting in (A.4.b) the terms for which $\mathbb{I} \cup \hat{\mathbb{I}} = \text{const.}$ and $\mathbb{I} \cap \hat{\mathbb{I}} = \text{const.}$ leads to:

$$(A.5) \quad \sum_{\hat{\mathbb{I}} \subseteq \{i_1, \dots, (n+1)\}} \left[- \sum_{i \in \hat{\mathbb{I}}} \frac{\lambda_i}{\lambda_i} + \sum_{i \in \mathbb{I}} \frac{\lambda_i}{\lambda_i} + a \frac{1 - (-1)^{|\mathbb{I}| + |\hat{\mathbb{I}}|}}{2} \right] \cdot \prod_{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}} (a^{-1} + \lambda_i) \cdot \prod_{i \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (a^{-1} + \lambda_i) \cdot \prod_{i < k} (\lambda_i - \lambda_k)^2 \cdot \prod_{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}} (\lambda_i + \lambda_k) \cdot \prod_{i < k} (\lambda_i + \lambda_k)^2 \cdot \prod_{k \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (\lambda_i + \lambda_k)^2 \cdot \prod_{k \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (\lambda_i - \lambda_k)^2 \cdot \dots = 0$$

We prove (A.9) by induction. It is true for $m = 1$. Assume it holds for $m-1$. The left hand side of (A.9) obviously is a polynomial in a^{-1} of degree $\leq m-1$. Therefore it suffices to show that (A.9) vanishes at the points $a^{-1} = \lambda_i$ ($i = 1, \dots, m$), e.g. $a^{-1} = \lambda_m$:

$$\begin{aligned} & \prod_{i=1}^{m-1} (1 + \mathcal{P}_i) (1 + \mathcal{P}_m) \cdot \left[- \sum_{i=1}^m \frac{1}{\lambda_i} + a \frac{1 - (-1)^m}{2} \right] \cdot \prod_{i=1}^m (a + \lambda_i)^{-1} \cdot \prod_{i < k=1}^m (\lambda_i - \lambda_k)^2 = \\ & = 2^{-1} \cdot \prod_{i=1}^{m-1} (\lambda_i - a^{-1}) \cdot \prod_{i=1}^{m-1} (1 + \mathcal{P}_i) \cdot \left[- \sum_{i=1}^{m-1} \frac{1}{\lambda_i} - a \frac{1 - (-1)^{m-1}}{2} \right] \cdot \\ & \quad \cdot \prod_{i=1}^{m-1} (\lambda_i - a^{-1}) \cdot \prod_{i < k=1}^{m-1} (\lambda_i - \lambda_k)^2 = 0 \end{aligned}$$

by assumption.

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